PERRON TRANSFORMS AND HIRONAKA'S GAME

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Introduction

I will talk about my first paper in mathematics, written by me and my advisor Josnei Novacoski. The title is Perron transforms and Hironaka's game. These subjects appear in the combinatorial part of proofs of local uniformization, which is the local form of resolution of a singularity.
In our paper we present a matricial result that generalizes Hironaka's game and Perron transforms simultaneously. We also show how one can deduce the various forms in which the algorithm of Perron appears in proofs of local uniformization from our main result.

The paper was mainly based on three papers:

1) *A solution to Hironaka's Polyhedra Game*, M. Spivakovsky (1983).
In this paper the author present the Hironaka's Polyhedra game. The game is a two player game, and the author shows that the first player always has a winning strategy.

2) *Abhyankar places admit local uniformization in any characteristic*, H. Knaf and F.-V. Kuhlmann (2005).
Our interest in this paper is Lemma 4.2, which is the central question about Perron transforms. It is about ordered abelian groups and I will present it below.

3) *Defect and local uniformization*, S.D. Cutkosky and H. Mourtada (2017).
In this paper the authors explicit how Perron transforms appear in local uniformization proofs.

I will first present the Hironaka's polyhedra game. Next, I will present the main result of our paper, which is equivalent to the first player having a winning strategy in Hironaka's game. Then, I will deduce Lemma 4.2 of Knaf and Kuhlmann's paper from our main result.

Hironaka's (polyhedra) game

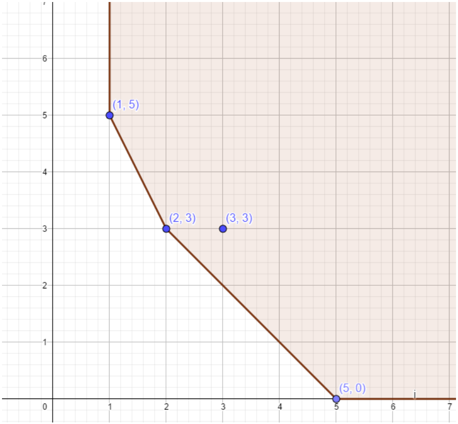
The structure of the game is:
• It is a two player game, with the players $\mathcal{P}_1$ and $\mathcal{P}_2$.
• It starts with a fixed positive $n \in \mathbb{N}$, and a finite non-empty subset $\mathcal{V} \subset \mathbb{N}^n$.
In the game, the set $\mathcal{V}$ represents a polyhedra. This polyhedra is the positive convex hull of $\mathcal{V}$ in $\mathbb{R}^n$, that is, the set $\{v + {\mathbb{R}_+}^n \mid v \in \mathcal{V}\}$.

$\mathbb{R}_+$ is the set of the non-negative real numbers.

Hironaka's (polyhedra) game

Example

$\mathcal{V} = \{(1,5),(2,3),(5,0),(3,3)\} \subset \mathbb{N}^2$. The polyhedra represented by $\mathcal{V}$ is in red in the plane. Note that, in this case, the polyhedra has three vertices.



Hironaka's (polyhedra) game

A round of the game consists on the following steps:

1) $\mathcal{P}_1$ chooses a non-empty subset J of $\{1, \cdots, n\}$.
2) $\mathcal{P}_2$ chooses $j \in J$.
3) Each element

$$v = (x_1, \cdots, x_n) \in \mathcal{V}$$

is replaced by

$$v' = (x_1, \cdots, x_{j-1}, \sum_{i \in J} x_i, x_{j+1}, \cdots, x_n),$$

forming a new set $\mathcal{V}'$.

Hironaka's (polyhedra) game

Finally

4) If the popyhedra represented by $\mathcal{V}'$ is a ortant, that is, has just one vertex, then the game ends and $\mathcal{P}_1$ wins. If not, the steps 1 to 4 are redone with $\mathcal{V}'$ instead of $\mathcal{V}$.

Note that a set $\mathcal{V}$ defines a ortant if, and only if, there is $v \in \mathcal{V}$ such that $v \leq w$ componentwise, for every $w \in \mathcal{V}$.

In his paper, Spivakovsky proved that there is a winning strategy for player $\mathcal{P}_1$, that is, there is a choice for player $\mathcal{P}_1$, at each step, such that after finitely many rounds, the resulting set is an ortant.

The matrix $A_{J,j}$

Now I will present our main theorem.

Before presenting it, we need to introduce a class of matrices.
For a non-empty subset J of $\{1, \cdots, n\}$ and $j \in J$, we define the matrix $A_{J,j} = (a_{rs})$ by

$$a_{rs} = \begin{cases} 1 & \text{if } r = s \text{ or if } r = j \text{ and } s \in J \\ 0 & \text{otherwise} \end{cases} .$$

The matrix $A_{J,j}$

Example

$J = \{1,3,4\} \subset \{1,2,3,4\}$ and $j = 3$.

$$A_{J,j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $A_{J,j}$

The determinant of $A_{J,j}$ is always equal to 1.
As a linear map,

$$A_{J,j}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_{j-1}, \sum_{i \in J} \alpha_i, \alpha_{j+1} \cdots, \alpha_n).$$

That is, $A_{J,j}$ acts in a vector in the same way as a round of Hironaka's game acts in the elements of $\mathcal{V}$, where $\mathcal{P}_1$ chooses J and $\mathcal{P}_2$ chooses $j \in J$.

Main theorem

We translate the question of whether $\mathcal{P}_1$ has a winning strategy in the Hironaka's game by our main theorem:

Theorem 1 Let $\alpha, \beta \in \mathbb{N}^n$. Then there exist $l \in \mathbb{N}$ and finite sequences $J_1, \ldots, J_l$ of subsets of $\{1, \cdots, n\}$ and $j_1, \ldots, j_l$ such that, for $k = 1, \cdots, l$,

$$J_k \text{ is chosen in function of the set } \{\alpha, \beta, J_1, \cdots, J_{k-1}, j_1, \cdots, j_{k-1}\},$$

$$j_k \text{ is randomly assigned in } J_k$$

and

$$A\alpha \leq A\beta \text{ or } A\beta \leq A\alpha$$

componentwise, where

$$A = A_{J_l, j_l} \cdots A_{J_1, j_1}.$$

Main theorem

We see that our choice of J_k represents $\mathcal{P}_1$'s move in Hironaka's game, and "j_k is randomly assigned in J_k" represents $\mathcal{P}_2$'s move. The matrix A represents the final result of the game, where $\mathcal{P}_1$ wins.

Here, the set of vertices has just two elements, α and β. However this is enough, because a matrix $A_{J,j}$ preserves componentwise inequalities, and then we may do induction to solve the general case, with more vertices.

Perron transforms

Now we present Perron transforms. The theorem below is Lemma 4.2 of Knaf and Kuhlmann's paper.

Theorem 2 Let Γ be a finitely generated ordered abelian group and $\alpha_1, \ldots, \alpha_l \in \Gamma$ positive elements. Then there exists a basis $\mathcal{B}$ of Γ, formed by positive elements, such that

$$\alpha_1, \ldots, \alpha_l \in \langle \mathcal{B} \rangle_+,$$

Where $\langle \mathcal{B} \rangle_+ = \{n_1\gamma_1 + \cdots + n_l\gamma_l \mid n_i \in \mathbb{N} \text{ and } \gamma_i \in \mathcal{B}\}$.

Knaf and Kuhlmann use this result as an important step to prove that every Abhyankar valuation admits local uniformization in any characteristic.

To deduce theorem 2 from our main theorem, we define Perron transforms. Consider Γ a finitely generated ordered abelian group and $\mathcal{B} = \{\gamma_1, \cdots, \gamma_n\}$ a basis of Γ formed by positive elements, (i.e., $\Gamma = \gamma_1\mathbb{Z} \oplus \ldots \oplus \gamma_n\mathbb{Z}$).

Definition A *simple Perron transform* on $\mathcal{B}$ is a new basis $\mathcal{B}' = \{\gamma'_1, \ldots, \gamma'_n\}$ of Γ, obtained in the following way: let $J \subseteq \{1, \ldots, n\}$ and $j \in J$ such that $\gamma_j \leq \gamma_i$ for all $i \in J$. Then

$$\gamma'_i = \begin{cases} \gamma_i - \gamma_j & \text{if } i \in J \setminus \{j\} \\ \gamma_i & \text{otherwise} \end{cases}.$$

A Perron transform is a basis obtained by performing finitely many simple Perron transforms starting from $\mathcal{B}$.

A Perron transform $\mathcal{B}'$ in $\mathcal{B}$ has the following three properties:

1) $\mathcal{B}'$ is a basis of Γ formed by positive elements.
2) $\langle \mathcal{B} \rangle_+ \subseteq \langle \mathcal{B}' \rangle_+$.
3) If $\mathcal{B}'$ is a simple Perron transform of $\mathcal{B}$ represented by J and j, and $[v]_{\mathcal{B}} = (x_1, \cdots, x_n) \in \mathbb{N}^n$ then

$$[v]_{\mathcal{B}'} = (x_1, \ldots, x_{j-1}, \sum_{i \in J} x_i, x_{j+1}, \ldots, x_n) = A_{J,j} \, [v]_{\mathcal{B}} \, .$$

Then, by 3), when we perform a simple Perron transform, we rewrite a vector in the new basis like a round of Hironaka's game change a vertex, where we choose J and $j \in J$ is given by the rule $\gamma_j \leq \gamma_i$ for all $i \in J$.

To proof theorem 2, by item 1) and 2) above, we start with a basis $\mathcal{B}$ of Γ, and perform perron transforms to include all $\alpha_1, \cdots, \alpha_l$. By item 2) $(\langle \mathcal{B} \rangle_+ \subseteq \langle \mathcal{B}' \rangle_+)$, we may consider $l = 1$, and the result follows easy by induction. Then, we rewrite theorem 2 as follows:

Theorem 2 Let Γ be a finitely generated ordered abelian group, $\mathcal{B}$ a basis of Γ formed by positive elements and $\alpha \in \Gamma$ a positive element. Then there exists a Perron transform $\mathcal{B}'$ of $\mathcal{B}$ such that $\alpha \in \langle \mathcal{B}' \rangle_+$.

Proof Note that $\alpha \in \langle \mathcal{B}' \rangle_+$ if, and only if, α has non-negative coordinates on the basis $\mathcal{B}'$. Write

$$\alpha = \alpha_+ - \alpha_-,$$

where $[\alpha_+]_{\mathcal{B}}$ and $[\alpha_-]_{\mathcal{B}}$ have non-negative coordinates. We apply Theorem 1 to the vectors $[\alpha_+]_{\mathcal{B}}$ and $[\alpha_-]_{\mathcal{B}}$.

By Theorem 1, there is a matrix

$$A = A_{J_l,j_l} \dots A_{J_1,j_1}$$

such that $j_1, \dots, j_l$ is given so that A is the change-of-basis matrix of a Perron transform $\mathcal{B}'$ on $\mathcal{B}$, and one of the following componentwise inequalities holds:

$$A[\alpha_-]_{\mathcal{B}} \leq A[\alpha_+]_{\mathcal{B}} \text{ or } A[\alpha_+]_{\mathcal{B}} \leq A[\alpha_-]_{\mathcal{B}}$$

Since $\mathcal{B}'$ is formed by positive elements and α is positive, the equation

$$\alpha = \alpha_+ - \alpha_-$$

ensures that

$$[\alpha_+]_{\mathcal{B}'} = A[\alpha_+]_{\mathcal{B}} \geq A[\alpha_-]_{\mathcal{B}} = [\alpha_-]_{\mathcal{B}'}$$

componentwise. Then $[\alpha]_{\mathcal{B}'}$ has non-negative coordinates, and therefore

$$\alpha \in \langle \mathcal{B}' \rangle_+.$$

Thank you!

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