Key polynomials and generating sequences of valuations centered in $k[x, y, z]$

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Assumptions

- k is a field
- $R = k[x, y, z]$ and $m_R = (x, y, z)R$
- $\nu : k(x, y, z) \to \mathbb{Q}$ is a k-valuation
- (V, m_V) is the valuation ring of ν
- $im(\nu) = G_\nu \subset \mathbb{Q}$
- $R \subset V$ and $m_R \subset m_V$
- $V/m_V = k$

Generating sequences

For $\gamma \in G_\nu$ let $I_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}$.

A (possibly infinite) sequence $\{Q_i\}$ of elements of R is a *generating sequence* of ν if for every $\gamma \in G_\nu$ the ideal I_γ is generated by the set

$$\{\prod_i {Q_i}^{b_i} \mid b_i \in \mathbb{Z}_{\geq 0}, \sum_i b_i \nu(Q_i) \geq \gamma\}.$$

Key polynomials

Let K be a field and μ be a valuation on $K(z)$ with value group $\Gamma \subset \bar{\Gamma}$.

A MacLane key polynomial $\varphi \in K[z]$ for μ is a monic polynomial together with the value $\gamma_1 \in \bar{\Gamma}$ such that $\gamma_1 > \mu(\varphi)$ and if $f \in K[z]$ has an expansion

$$f = f_n\varphi^n + f_{n-1}\varphi^{n-1} + \cdots + f_1\varphi + f_0$$

where $f_i \in K[z]$ and $deg_z f_i < deg_z \varphi$, then $\mu_1(f) = \min\{\mu(f_i) + i\gamma_1\}$ is a valuation on $K[z]$ satisfying $\mu_1(f) \geq \mu(f)$ for all $f \in K[z]$.

The map μ_1 is called an augmented valuation and denoted $\mu_1 = [\mu; \mu_1(\varphi) = \gamma_1]$.

Complete sequences of key polynomials

Let K be a field and μ be a valuation on $K(z)$, denote by μ_0 the Gauss valuation on $K(z)$ with $\mu_0(a) = \mu(a)$ for $a \in K$.

The sequence of key polynomials $\{\varphi_i\}_{i\geq 1}$ with values $\{\gamma_i\}_{i\geq 1}$ is called *complete* if the sequence of augmented valuations $\{[\mu_{i-1}; \mu_i(\varphi_i) = \gamma_i]\}_{i\geq 1}$ satisfies the following conditions $\varphi_1 = z$ and for every $f \in K[z]$ there exists $i \geq 1$ such that $\mu(f) = \mu_i(f)$.

Dimension 2

Let $P_0 = x$ and $P_1 = y$. For all $i \geq 0$ set $\beta_i = \nu(P_i)$,

$$G_i = \sum_{j=0}^{i} \beta_j \mathbb{Z}, \quad S_i = \sum_{j=0}^{i} \beta_j \mathbb{Z}_{\geq 0}, \quad q_i = [G_i : G_{i-1}]$$

Then $q_i\beta_i \in S_{i-1}$ and there exist unique $n_{i,0} \geq 0$ and $0 \leq n_{i,j} < q_j$ such that $q_i\beta_i = \sum_{j=0}^{i-1} n_{i,j}\beta_j$

Since $\nu(P_i^{q_i}) = \nu(\prod_{j=0}^{i-1} P_j^{n_{i,j}})$, the residue of $P_i^{q_i} / \prod_{j=0}^{i-1} P_j^{n_{i,j}}$ is $\lambda_i \in k^*$. Set

$$P_{i+1} = P_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} P_j^{n_{i,j}}$$

Let $G = \bigcup_{i=0}^{\infty} G_i$ and $S = \bigcup_{i=0}^{\infty} S_i$.

Dimension 2

Denote $\nu_0 = \nu|_{k(x,y)}$.

$\{P_i\}_{i\geq 0}$ is a generating sequence of ν_0 in $k[x,y]$.

$\{P_i\}_{i\geq 1}$ is a complete sequence of key polynomials for ν_0.

Extension to Dimension 3

Let $Q_1 = z$. For all $i > 0$ set $\bar{\gamma}_i = \nu(Q_i)$,

$$\bar{H}_i = \sum_{j=1}^{i} \bar{\gamma}_j \mathbb{Z}, \quad \bar{U}_i = \sum_{j=1}^{i} \bar{\gamma}_j \mathbb{Z}_{\geq 0}, \quad \bar{s}_i = [(\bar{H}_i + G) : (\bar{H}_{i-1} + G)],$$

Since $\bar{s}_i \bar{\gamma}_i \in \bar{H}_{i-1} + G$ there exists j s.t. $\bar{s}_i \bar{\gamma}_i \in \bar{H}_{i-1} + G_j$

$$\bar{m}_i = \max(\bar{m}_{i-1}, \quad \min\{j \in \mathbb{Z}_{\geq 0} \mid \bar{s}_i \bar{\gamma}_i \in (\bar{H}_{i-1} + G_j)\})$$

Problem: $\bar{s}_i \bar{\gamma}_i \in (\bar{U}_{i-1} + S_{\bar{m}_i})$ does not have to be true

Set $\bar{r}_{i,0} = \min\{r \in \mathbb{Z} \mid r\beta_0 + \bar{s}_i \bar{\gamma}_i \in (\bar{U}_{i-1} + S_{\bar{m}_i})\}$

Extension to Dimension 3

Now $\bar{r}_{i,0}\beta_0 + \bar{s}_i\bar{\gamma}_i \in (\bar{U}_{i-1} + S_{\bar{m}_i})$ and there exist unique $0 \le \bar{n}_{i,0}$, $0 \le \bar{n}_{i,j} < q_j$ and $0 \le \bar{l}_{i,j} < \bar{s}_j$ such that

$$\bar{r}_{i,0}\beta_0 + \bar{s}_i\bar{\gamma}_i = \bar{n}_{i,0}\beta_0 + \sum_{j=1}^{\bar{m}_i} \bar{n}_{i,j}\beta_j + \sum_{j=1}^{i-1} \bar{l}_{i,j}\bar{\gamma}_j$$

Since $\nu(x^{\bar{r}_{i,0}}Q_i^{\bar{s}_i}) = \nu(\prod_{j=0}^{\bar{m}_i} P_j^{\bar{n}_{i,j}} \prod_{j=1}^{i-1} Q_j^{\bar{l}_{i,j}})$, the residue of $x^{\bar{r}_{i,0}}Q_i^{\bar{s}_i}/(\prod_{j=0}^{i-1} P_j^{\bar{n}_{i,j}} \prod_{j=1}^{i-1} Q_j^{\bar{l}_{i,j}})$ is $\bar{\mu}_i \in k^*$. Set

$$Q_{i+1} = x^{\bar{r}_{i,0}}Q_i^{\bar{s}_i} - \bar{\mu}_i \prod_{j=0}^{\bar{m}_i} P_j^{\bar{n}_{i,j}} \prod_{j=1}^{i-1} Q_j^{\bar{l}_{i,j}}$$

Dimension 3

$$P_0 = x, \quad P_1 = y$$

$$P_{i+1} = P_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} P_j^{n_{i,j}}$$

$$Q_1 = z$$

$$Q_{i+1} = x^{\bar{r}_{i,0}} Q_i^{\bar{s}_i} - \bar{\mu}_i \prod_{j=0}^{\bar{m}_i} P_j^{\bar{n}_{i,j}} \prod_{j=1}^{i-1} Q_j^{\bar{l}_{i,j}}$$

Set $$\bar{Q}_i = x^{-d_i} Q_i$$

Dimension 3

Theorem
$\{\bar{Q}_i\}_{i\geq 1}$ is a sequence of key polynomials for ν.

Remark
If every $\bar{s}_i = 1$ then $\{z - \bar{Q}_i\}_{i\geq 1}$ is a pseudo-convergent sequence for which z is a limit.

Dimension 3

Theorem
$\{\bar{Q}_i\}_{i\geq 1}$ is a sequence of key polynomials for ν.

Theorem
If infinitely many $\bar{s}_i$ are greater than 1 then $\{\bar{Q}_i\}_{i\geq 1}$ is a complete sequence of key polynomials for ν.

Remark
If every $\bar{s}_i = 1$ then $\{z - \bar{Q}_i\}_{i\geq 1}$ is a pseudo-convergent sequence for which z is a limit.

Ex: Noncomplete sequence

Suppose k is a field of characteristic $p \neq 2$

$$\begin{aligned} P_0 &= x & \beta_0 &= 1 \\ P_1 &= y & \beta_1 &= p - \frac{1}{p} \\ P_2 &= y^p + x^{p^2-1} & \beta_2 &= p^2 - \frac{1}{p^2} \\ P_{i+1} &= P_i^p + x^{p^{i+1}-p^{i-1}} P_{i-1} & \beta_{i+1} &= p^{i+1} - \frac{1}{p^{i+1}} \end{aligned}$$

$$\begin{aligned} Q_1 &= z & \bar{\gamma}_1 &= 1 - \frac{1}{p} \\ Q_2 &= x^{p-1} z - y & \bar{\gamma}_2 &= p - \frac{1}{p^3} \\ Q_3 &= x^{p^3-p} Q_2 - P_3 & \bar{\gamma}_2 &= p^3 - \frac{1}{p^5} \\ Q_{i+1} &= x^{p^{2i-1}-p^{2i-3}} Q_i - P_{2i-1} & \bar{\gamma}_{i+1} &= p^{2i-1} - \frac{1}{p^{2i+1}} \end{aligned}$$

Extension to Dimension 3: #2

Let $T_1 = z$. For all $i > 0$ set $\gamma_i = \nu(T_i)$,

$$H_i = \sum_{j=1}^{i} \gamma_j \mathbb{Z}, \quad U_i = \sum_{j=1}^{i} \gamma_j \mathbb{Z}_{\geq 0}, \quad s_i = [(H_i + G) : (H_{i-1} + G)],$$

$$m_i = \max(m_{i-1}, \quad \min\{j \in \mathbb{Z}_{\geq 0} \mid s_i \gamma_i \in (H_{i-1} + G_j)\})$$

Problem: $s_i \gamma_i \in (U_{i-1} + S_{m_i})$ does not have to be true

We look for all possible $\tau \in (U_i + S_{m_i})$ such that $\tau + s_i \gamma_i \in (U_{i-1} + S_{m_i})$ and $\tau + s_i \gamma_i - b \notin (U_{i-1} + S_{m_i})$ for any $b \in (U_i + S_{m_i})$

Extension to Dimension 3: #2

For every τ as above set

$$T_{i,\tau} = T_i^{c_i s_i} \prod_{j=0}^{m_i} P_j^{a_j} \prod_{j=0}^{i-1} T_j^{c_j} - \mu_\tau \prod_{j=0}^{m_i} P_j^{n_{\tau,j}} \prod_{j=0}^{i-1} T_j^{l_{\tau,j}}$$

with $\quad \nu(T_i^{c_i s_i} \prod_{j=0}^{m_i} P_j^{a_j} \prod_{j=0}^{i-1} T_j^{c_j}) = \nu(\tau \prod_{j=0}^{m_i} P_j^{n_{\tau,j}} \prod_{j=0}^{i-1} T_j^{l_{\tau,j}}),$

$$\mu_\tau = (T_i^{c_i s_i} \prod_{j=0}^{m_i} P_j^{a_j} \prod_{j=0}^{i-1} T_j^{c_j})/(\prod_{j=0}^{m_i} P_j^{n_{\tau,j}} \prod_{j=0}^{i-1} T_j^{l_{\tau,j}}) \text{ in } V/m_V$$

Example

Suppose k is a field.

$$
\begin{aligned}
P_0 &= x & \beta_0 &= 1 \\
P_1 &= y & \beta_1 &= 1\frac{1}{3} \\
P_2 &= y^3 - x^4 & \beta_2 &= 4 + \frac{1}{9} \\
P_3 &= P_2^3 - x^{11} P_1 & \beta_3 &= 12\frac{1}{3} + \frac{1}{27} \\
P_{i+1} &= P_i^3 - x^{11 \cdot 3^{i-2}} P_{i-1} & \beta_{i+1} &= 3\beta_{i-1} + \frac{1}{3^{i+1}}
\end{aligned}
$$

$$
\begin{aligned}
Q_1 &= z & \bar{\gamma}_1 &= 1\frac{2}{3} \\
Q_2 &= xz - y^2 & \bar{\gamma}_2 &= 3\frac{1}{5} \\
Q_3 &= Q_2^5 - x^{16} & \bar{\gamma}_3 &= 16 + \frac{1}{25} \\
Q_{i+1} &= Q_i^5 - x^{77 \cdot 5^{i-3}} Q_{i-1} & \bar{\gamma}_{i+1} &= 5\bar{\gamma}_i + \frac{1}{5^i}
\end{aligned}
$$

Example

$$T_1 = z, \qquad \gamma_1 = 1\frac{2}{3}$$

$$T_2 = xT_1 - y^2 = Q_2, \qquad \gamma_2 = 3\tfrac{1}{5}$$

$$T_3 = yT_1 - x^3 = x^{-1}(P_2 + yQ_2), \qquad \gamma_3 = 3\tfrac{1}{9}$$

$$T_4 = T_1^2 - x^2y = x^{-2}(yP_2 + 2y^2Q_2 + Q_2^2), \qquad \gamma_4 = 3\tfrac{1}{3} + \tfrac{1}{9}$$

$$T_5 = T_2^5 - x^{16} = Q_3, \qquad \gamma_6 = 16\tfrac{1}{25}$$

$T_6 = xT_3 - P_2 = yQ_2 = yT_2,$ redundant

$T_7 = y^2T_3 - P_2T_1 = x^3Q_2 = x^3T_2,$ redundant

Theorem

$\{P_i\}_{i\geq 0} \cup \{T_i\}_{i>0}$ *is a generating sequence of* ν.

Theorem

$\{P_i\}_{i\geq 0} \cup \{T_i\}_{i>0}$ is a generating sequence of ν.

Corollary

If $\{z-\bar{Q}_i\}_{i\geq 1}$ is a pseudo-convergent sequence of algebraic type then there exists i such that T_i is not fixed by $\{z-\bar{Q}_i\}_{i\geq 1}$.

Ex: Noncomplete sequence

Suppose k is a field of characteristic $p = 3$

$$P_0 = x \qquad \beta_0 = 1$$

$$P_1 = y \qquad \beta_1 = 2\tfrac{2}{3}$$

$$P_2 = y^3 + x^8 \qquad \beta_2 = 8\tfrac{8}{9}$$

$$P_{i+1} = P_i^3 + x^{8\cdot 3^{i-1}} P_{i-1} \qquad \beta_{i+1} = 3^{i+1} - \frac{1}{3^{i+1}}$$

$$Q_1 = z \qquad \bar{\gamma}_1 = \tfrac{2}{3}$$

$$Q_2 = x^2 z - y \qquad \bar{\gamma}_2 = 2\tfrac{26}{27}$$

$$Q_3 = x^{24} Q_2 - P_3 \qquad \bar{\gamma}_2 = 26\tfrac{242}{243}$$

$$Q_{i+1} = x^{8\cdot 3^{2i-3}} Q_i - P_{2i-1} \qquad \bar{\gamma}_{i+1} = 3^{2i-1} - \frac{1}{3^{2i+1}}$$

Ex: Noncomplete sequence

$$T_1 = z \qquad \gamma_1 = \frac{2}{3}$$

$$T_2 = x^2 T_1 - P_1 = Q_2, \qquad \gamma_2 = 2\tfrac{26}{27}$$

$$T_3 = P_1 T_1^2 + x^4 = x^{-4}(2P_1^2 Q_2 + P_1 Q_2^2 + P_2), \qquad \gamma_3 = 4\tfrac{8}{27}$$

$$T_4 = P_1^2 T_1 + x^6 = x^{-2}(P_1^2 Q_2 + P_2), \qquad \gamma_4 = 6\tfrac{8}{27}$$

$$T_5 = T_1^3 + x^2 = \frac{P_{2k}}{x^{3^{2k}-3}} + \frac{Q_{k+1}^3}{x^{3^{2k}-3}}, \qquad \gamma_5 \geq 3 - \frac{1}{3^{2k}}$$

Thank you!