Finiteness of integral closure of complete local rings

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Introduction

- We denote by $(A, \mathfrak{m}_A)$ a noetherian local ring and by $\kappa(A)$ its residue field.
- For a field k, $k^{[[n]]} := k[[X_1, \ldots, X_n]]$ denotes the formal power series ring in n indeterminates over k.
- If $\mathfrak{p} \in \operatorname{Spec} A$, the residue field of A at $\mathfrak{p}$, denoted by $\kappa(\mathfrak{p})$, is defined as $\kappa(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = Q(A/\mathfrak{p})$.
- All homomorphisms between local rings are local homomorphism.
- Let A be a complete Noetherian local ring. Then it is excellent. Hence for every finite extension L of its quotient field $K = Q(A)$, the integral closure of A in L is a finitely generated A-module.

The following example is due to Abhyankar.

- Let X, Y be indeterminates over $\mathbb{C}$. For any positive integer n, let $A := \mathbb{C}[[X, X(e^Y - 1), X(e^{Y^2} - 1), \dots, X(e^{Y^n} - 1)]] \subseteq \mathbb{C}[[X, Y]]$. Then $\dim A = n + 1$, but $\dim B = 2$.

Theorem

Let $A \subseteq B$ be Noetherian rings with A being a catenary local ring. Suppose that the maximal ideal of A is contained in the Jacobson radical of B. If the induced map Spec $B \to$ Spec A is surjective, then dim $A \leqslant$ dim B.

Theorem

Let $A \subseteq B$ be finite dimensional Noetherian domains with B being catenary, and $P \subseteq B$ a prime ideal satisfying ht $(P \cap A) <$ ht P. Suppose that $J_1, J_2, \ldots, J_n$ are prime ideals in B such that P is not contained in the union $\bigcup_{i=1}^{n} J_i$. Then there exists a nonzero element $x \in P \setminus \bigcup_{i=1}^{n} J_i$ satisfying $xB \cap A = (0)$.

- Let $A \subseteq B$ be Noetherian local domains with B being regular and $\dim A < \dim B$. Then we can find a part of a regular system of parameters $z_1, z_2, \ldots, z_m \in B$, with $m := \dim B - \dim A$, such that $(z_1, z_2, \ldots, z_m) \cap A = (0)$.
  In particular, if A is a complete local ring over k satisfying $A \subseteq k^{[[n]]}$, then A can be embedded in $k^{[[\dim A]]}$.

- Let $A \subseteq B$ be Noetherian local domains with $\dim A < \dim B$. If B is catenary then there exists a prime ideal $\mathfrak{p} \subseteq B$ such that $\operatorname{ht} \mathfrak{p} = \dim B - \dim A$ and $\mathfrak{p} \cap A = (0)$.

Main Result

Theorem

If $A \subseteq B$ are complete local domains such that $\kappa(A) = \kappa(B) = \mathbb{C}$, the field of complex numbers, and the natural map Spec $B \to$ Spec A is surjective. Then the integral closure of A in B is a finite A-module.

Proof of main result

It is enough to show that any increasing sequence of complete local domains $A_0(= A) \subseteq A_1 \subseteq A_2 \subseteq \dots$, where each A_i is a finite A-module and $\cup_i A_i \subseteq B$, is eventually stationary.

We apply induction on dim A.

(i) If dim $A = 0$ or dim $A = 1$. Then B is a finite module over A.

(ii) If dim $A = 2$. Then we can further assume that dim $B = 2$.

- We can further consider that A and B are normal domains.
- We can find a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $\operatorname{ht}\mathfrak{p} = 1$. Then $A_{\mathfrak{p}}$ is a DVR.
- Let $\pi \in A_{\mathfrak{p}}$ be a uniformizing parameter and $Q_1, Q_2, \dots, Q_n$ the prime ideals of B, lying over $\mathfrak{p}$. Also $\operatorname{ht} Q_i = 1$ for all i.

- Consider $A_\mathfrak{p} \subset B_\mathfrak{p}$. Then for $i = 1, \ldots, n$, we define

$$e_i := ord_{B_{Q_i}}(\pi) \text{ and } \; f_i := [\kappa(Q_i) : \kappa(\mathfrak{p})]$$

.

- Let $K = Q(A)$ and $L = Q(B)$.
If F/K be a finite field extension such that $F \subseteq L$.
We can show that $[F : K] \leqslant \sum_{i=1}^{n} e_i f_i$, thereby proving that the algebraic closure of K in L is finite.

Flenner's Theorem

Theorem

Let k be a field of characteristic zero and A an excellent normal local k-domain of dimension $m \geqslant 3$ with $\kappa(A) = k$. If $Q := (a_1, .., a_m)$ is an $\mathfrak{m}_A$-primary ideal, then there exists a non-empty Zariski open set $U \subseteq \mathbb{A}_k^m$ such that for any $(c_1, \ldots, c_m) \in U$, the linear combination $\Sigma c_i a_i \in A$ is a prime element in A.

- Let $A_0 \subseteq A_1 \subseteq \cdots$ be an increasing sequence excellent normal local domains of dimension $m \geqslant 3$ such that $k := \kappa(A_0)$ is an uncountable field of characteristic zero and each A_i is a finite A_0-module. If every A_i contains its residue field $\kappa(A_i)$, then there exists a non zero prime element $x \in A_0$ such that x is a prime in all A_i.

(iii) Let dim $A = r$, for some $r \geq 3$. We can take $A = \kappa(A)^{[[r]]}$ and B to be a normal domain.

- Each A_i, being an excellent Henselian local domain, has a finite normalization which is also local. Consequently, the sequence $A \subseteq A_1 \subseteq A_2 \subseteq \ldots$ eventually stabilizes if and only if the induced sequence of normalizations $A \subseteq \bar{A}_1 \subseteq \bar{A}_2 \subseteq \ldots$ stabilizes eventually. So we can replace A_i by $\bar{A}_i$.
- Using Flenner's result we can find a prime element $x \in A$, which remains a prime in every A_i.
- $xA_i = xB \cap A_i$ for all i.
- Therefore, going modulo x, we get an induced sequence of equicharacteristic complete local domains $A_0/xA_0 (= A/xA) \subseteq A_1/xA_1 \subseteq A_2/xA_2 \subseteq \ldots$, such that each A_i/xA_i is a finite A/xA-module and $\bigcup_i A_i/xA_i \subseteq B/xB$.
- As dim $A/xA <$ dim A, by induction hypothesis, the sequence $A/xA \subseteq A_1/xA_1 \subseteq A_2/xA_2 \subseteq \ldots$ eventually stabilizes.
- Therefore the original sequence $A \subseteq A_1 \subseteq A_2 \subseteq \ldots$ also stabilizes eventually.

Theorem

If $A \subseteq B$ are complete local domains such that $\kappa(A)$ is an uncountable field of characteristic zero, $\kappa(A) \subseteq \kappa(B)$ is a finite extension and the image of Spec B contains all prime ideals of co-height two in A, then the integral closure of A in B is a finite A-module.

References

S. Abhyankar, *Two notes on formal power series*, Proc. Amer. Math. Soc. **7** (1956), 903–905.

Sagnik Chakraborty, Rajendra Vasant Gurjar and Dibyendu Mondal, *Some results about complete and analytic local rings*, Preprint.

H. Flenner, *Die Sätze von Bertini für lokale Ringe*, Math. Ann. **229(2)** (1977), 97–11.

Thank You for Your Attention