Product of Complete Ideals

Jugal Verma

Joint work with Parangama Sarkar
Indian Institute of Technology Bombay

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Outline

1. Integral closure of ideals and Zariski's Product Theorem
2. Results of Lipman and Cutcosky about complete ideals in a rational singularity
3. Reid-Roberts-Vitulli Theorem about complete monomial ideals
4. Rees' theorem about joint reductions and normal Hilbert polynomials
5. Rees's Theorem $\Rightarrow$ Lipman's generalisation of ZPT to 2-dimensional rational singularities and Huneke's result on vanishing of $\overline{e}_2(I)$
6. Joint reduction vectors of multi-graded filtrations of ideals
7. Effect of filter-regular sequences on vanishing of graded components of the local cohomology modules of multi-graded modules over standard multi-graded rings
8. Local cohomological interpretation of joint reduction vectors of multi-graded filtrations of ideals
9. A generalisation of the Reid-Roberts-Vitulli Theorem for completeness of power products of $\mathfrak{m}$-primary ideals in analytically unramified local rings
10. Futoshi Hayasaka's theorem about the product of complete monomial ideals

Integral Closure of Ideals

1. Let R be a commutative ring, I an ideal of R.
2. An element $a \in R$ is called **integral over** I, if there exist $a_i \in I^i$ for $i = 1, 2, \ldots, n$ so that
$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$
3. The **integral closure** of $I = \overline{I} = \{a \in R \mid a \text{ is integral over } I\}$.
4. An ideal I is called **complete** or **integrally closed** if $\overline{I} = I$.
5. **O. Zariski**, *Polynomial ideals defined by infinitely near base points*, American Journal of Mathematics (1938), 151-204.
6. **Theorem:** Let $R = k[X, Y]$ be the polynomial ring k is an algebraically closed field of characteristic zero. Then product of complete ideals in R is complete. Moreover any complete ideal is uniquely written as a product of simple complete ideals upto their reordering.
7. This was generalised to two-dimensional regular local rings in Appendix 5 of the Volume II of *Commutative Algebra* by Oscar Zariski and Pierre Samuel.

Lipman's Results about complete ideals (1969,1978)

1. **Definition :** A two-dimensional normal local ring $(R, \mathbf{m})$ is said to have a **rational singularity** if there exists a desingularisation X of Spec R for which $H^1(X, \mathcal{O}_X) = 0$.
2. **Examples of rational singularities:** (1) Any 2-dimensional complete local UFD with algebraically closed residue field.
   (2) A 2-dimensional normal local domain birationally dominating a 2-dimensional regular local ring.
3. **Theorem: (Lipman, 1969)** Let R be a 2-dimensional local ring with a rational singularity. Then
   (1) Product of complete ideals is complete.
   (2) If the completion of R is a UFD then every complete ideal factors as a product of simple complete ideals uniquely.
4. **Definition: (Lipman, 1978)** A two-dimensional Noetherian local ring $(R, \mathbf{m})$ is called **pseudo-rational** if it is normal, analytically unramified and for every birational proper map $W \to$ Spec R where W is normal, we have $H^1(W, \mathcal{O}_W) = 0$.

Normal Hilbert Polynomials : Rees' Approach

1. **Definition:** For any $\mathfrak{m}$-primary ideal I in an analytically unramified local ring $(R, \mathfrak{m})$ of dimension d, the **normal Hilbert function** $\overline{H}(I, n) = \lambda(R/\overline{I^n})$ for large n, is given by the **normal Hilbert polynomial**
$$\overline{P}(I, x) = \overline{e}_0(I)\binom{x+d-1}{d} - \overline{e}_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d \overline{e}_d(I),$$
for some integers $\overline{e}_0(I), \overline{e}_1(I), \ldots, \overline{e}_d(I)$.
2. These integers are called the **normal Hilbert coefficients** of I.
3. **Definition:** **(Rees)** A 2-dimensional local normal analytically unramified ring $(R, \mathfrak{m})$ is pseudo-rational iff $\overline{e}_2(I) = 0$ for all $\mathfrak{m}$-primary ideals I.
4. **Theorem** **(Lipman, Rees)** Product of $\mathfrak{m}$-primary complete ideals is complete in two-dimensional pseudo-rational local rings.
5. **Definition:** **(Rees)** Let $I_1, I_2, \ldots, I_d$ be $\mathfrak{m}$-primary ideals of a d-dimensional local ring $(R, \mathbf{m})$. Let $\mathcal{F}(\mathbf{n})$ be a filtration of ideals such that for all $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$, $I_1^{n_1} I_2^{n_2} \ldots I_d^{n_d} \subseteq \mathcal{F}(\mathbf{n})$.
6. **Definition:** A set of elements $x_i \in I_i$ for $i = 1, 2, \ldots, d$ is called a joint reduction of $\mathcal{F}(\mathbf{n})$ if for all large $\mathbf{n} \in \mathbb{N}^n$
$$\mathcal{F}(\mathbf{n}) = x_1\mathcal{F}(\mathbf{n} - e_1) + x_2\mathcal{F}(\mathbf{n} - e_2) + \cdots + x_d\mathcal{F}(\mathbf{n} - e_d).$$

Rees' Theorem, 1981

1. **Lemma:** Let $(R, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension 2 with infinite residue field and let I, J be $\mathfrak{m}$-primary ideals. Then there exists a joint reduction (a, b) of $\{\overline{I^r J^s}\}$ satisfying the conditions:

$$(a) \cap \overline{I^r J^s} = a\overline{I^{r-1} J^s} \text{ for all } r > 0 \text{ and } (b) \cap \overline{I^r J^s} = b\overline{I^r J^{s-1}} \text{ for all } s > 0.$$

2. **Definition:** We say (a, b) is a **good joint reduction** of $\{\overline{I^r J^s}\}$ if (a, b) satisfies the above equations. Such joint reductions exist if $|R/\mathfrak{m}| = \infty$.
3. **Theorem: (Rees, 1981)** Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension 2. Let I, J be $\mathfrak{m}$-primary ideals and let (a, b) be a good joint reduction of $\{\overline{I^r J^s} \mid r, s \geq 0\}$.
4. Then $\overline{e}_2(IJ) \leq \overline{e}_2(I) + \overline{e}_2(J)$ and

$$\overline{e}_2(IJ) = \overline{e}_2(I) + \overline{e}_2(J) \iff \overline{I^r J^s} = a\overline{I^{r-1} J^s} + b\overline{I^r J^{s-1}} \text{ for all } r, s > 0, \quad (1)$$

5. **Definition:** If the equation (1) is satisfied for all $(r, s) \geq (p, q) \in \mathbb{N}^2$, then we say that **$(p, q)$ is a normal joint reduction vector** of the filtration $\{\overline{I^r J^s}\}$.

Consequences of Rees' Theorem

1. **Theorem:(Rees,1981)** Let $(R, \mathfrak{m})$ be a two-dimensional pseudo-rational local ring and let I, J be complete ideals in R. Then IJ is complete.
2. **Proof:** Since $\overline{e}_2(I) = 0$ for all $\mathfrak{m}$-primary ideals in a pseudo-rational local ring, Rees' theorem gives that $\overline{IJ} = a\overline{J} + b\overline{I}$ for all $\mathfrak{m}$-primary ideals I and J. Therefore $\overline{IJ} \subseteq \overline{I}\ \overline{J}$. Since $\overline{I}\ \overline{J} \subseteq \overline{IJ}$ in any ring, $\overline{IJ} = \overline{I}\ \overline{J}$. Hence IJ is complete if I and J are so.
3. **Theorem: (Huneke, 1987)** Let $(R, \mathfrak{m})$ be a two-dimensional analytically unramified Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal. Then

$$\overline{e}_2(I) = 0 \iff \overline{I^n} = (x, y)\overline{I^{n-1}} \text{ for } n \geq 2 \text{ and for any reduction } (x, y) \text{ of } I.$$

In particular, if I is complete and $\overline{e}_2(I) = 0$ then I^n is complete for all $n \geq 1$.

Proof: Let $\overline{e}_2(I) = 0$. Put $I = J$ and $r = 1,\ s = n - 1$ in Rees' Theorem to get both the conclusions. We can calculate the normal Hilbert polynomial to see that $\overline{e}_2(I) = 0$

Theorems of Cutcosky (1990)

1. **Theorem:** Let $(R, \mathbf{m})$ be a 2-dimensional excellent normal local domain with algebraically closed residue field $k = R/\mathbf{m}$. Then the following are equivalent:

   (1) R has a rational singularity.
   (2) Product of complete ideals in R is complete.
   (3) Product of complete $\mathbf{m}$-primary ideals is complete.
   (4) If I is a complete $\mathbf{m}$-primary ideal then I^2 is complete.

2. **Theorem:** Let k be a field of characteristic not equal to 3. Set $R(k) = k[[x, y, z]]/(x^3 + 3y^3 + 9z^3)$. Then

   (1) $R(k)$ is a normal local domain and it is **not** a rational singularity.
   (2) Product of complete ideals is complete in $R(\mathbb{Q})$.
   (3) There exists a complete $\mathbf{m}$-primary ideal whose square is not complete if k has positive characteristic or if k is algebraically closed.

The RRV Theorem about complete monomial Ideals

1. **Definition:** Let $X \subset R = k[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field k.

$$\exp(X) = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \mid x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \in X\}.$$

2. The **Newton Polyhedron** of a monomial ideal I is defined to be

$$NP(I) = \text{ convex hull } \exp(I)$$

3. **Theorem:** **(B. Teissier, 1975)** Let I be a monomial ideal of R. Then the integral closure of I is also a monomial ideal and

$$\exp(\bar{I}) = NP(I) \cap \mathbb{N}^n.$$

4. **Theorem:** **(L. Reid-L. G. Roberts-M. Vitulli, 2002)** Suppose that I is a **monomial ideal** in the polynomial ring $k[x_1, x_2, \ldots, x_d]$. Then

$$I, I^2, \ldots, I^{d-1} \text{ are complete } \implies I^n \text{ is complete for all } n.$$

p_g-ideals (Okuma-Watanabe-Yoshida, 2014)

1. **Definition:** Let $(R, \mathfrak{m}, k)$ be a 2-dimensional excellent normal local domain where k is algebraically closed. A complete $\mathfrak{m}$-primary ideal I is called a p_g-ideal if $\overline{e}_2(I) = 0$.
2. **Theorem:** **(Okuma-Watanabe-Yoshida, 2014)** Let I and J be p_g-ideals of R. Then IJ is also a p_g ideal. Moreover the Rees algebra of such ideals is a Cohen-Macaulay normal domain with minimal multiplicity at its maximal homogeneous ideal.
3. **Remark:** This result also follows from Rees' Theorem since for such ideals there are $a \in I$ and $b \in J$ so that $IJ = aJ + bI = \overline{IJ}$ is complete.
4. Moreover $r(I) = 1$ which implies that $\mathcal{R}(I)$ is CM normal domain.
5. It can then be proved that $r(\mathfrak{m}, It) = 1$ which shows that $\mathcal{R}(I)$ has minimal multiplicity.

Multi-graded filtrations of ideals

1. $(R, \mathfrak{m})$ denotes a local ring of dimension d with infinite residue field and $\mathbf{I} = I_1, \ldots, I_s$ denotes a sequence of $\mathfrak{m}$-primary ideals of R.
2. Put $e = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^s$ and for all $i = 1, \ldots, s$, $e_i = (0, \ldots, 1, \ldots, 0)$ denotes the i^{th} vector in the standard basis of $\mathbb{Q}^s$.
3. For $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, we write $\mathbf{I}^{\mathbf{n}} = I_1^{n_1} \cdots I_s^{n_s}$.
4. For $s \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, put $|\alpha| = \alpha_1 + \cdots + \alpha_s$.
5. Define $\mathbf{m} = (m_1, \ldots, m_s) \geq \mathbf{n} = (n_1, \ldots, n_s)$ if $m_i \geq n_i$ for all $i = 1, \ldots, s$.
6. The phrase "for all large $\mathbf{n}$," means $\mathbf{n} \in \mathbb{N}^s$ and $n_i \gg 0$ for all $i = 1, \ldots, s$.
7. A set of ideals $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded **$\mathbf{I}$ − filtration** if for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^s$,

   (i) $\mathbf{I}^{\mathbf{n}} \subseteq \mathcal{F}(\mathbf{n})$, (ii) $\mathcal{F}(\mathbf{n})\mathcal{F}(\mathbf{m}) \subseteq \mathcal{F}(\mathbf{n}+\mathbf{m})$ (iii) if $\mathbf{m} \geq \mathbf{n}$, $\mathcal{F}(\mathbf{m}) \subseteq \mathcal{F}(\mathbf{n})$.

Multi-graded admissible filtrations of ideals

1. Let $t_1, \ldots, t_s$ be indeterminates. For $\mathbf{n} \in \mathbb{Z}^s$, we put $\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_s^{n_s}$ and denote the $\mathbb{N}^s$-graded **Rees ring of** $\mathcal{F}$ by $\mathcal{R}(\mathcal{F}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} \mathcal{F}(\mathbf{n})\mathbf{t}^{\mathbf{n}}$
2. For an $\mathbb{N}^s$-graded ring $S = \bigoplus_{\mathbf{n} \geq \mathbf{0}} S_{\mathbf{n}}$, put $S_{++} = \bigoplus_{\mathbf{n} \geq e} S_{\mathbf{n}}$
3. For $\mathcal{F} = \{\mathbf{I}^{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^s}$, Put $\mathcal{R}(\mathcal{F}) = \mathcal{R}(\mathbf{I})$, $\mathcal{R}'(\mathcal{F}) = \mathcal{R}'(\mathbf{I})$ and $\mathcal{R}(\mathbf{I})_{++} = \mathcal{R}_{++}$.
4. The **associated multi-graded ring of** $\mathcal{F}$ **with respect to** $\mathcal{F}(e)$ is the ring

$$G(\mathcal{F}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} \frac{\mathcal{F}(\mathbf{n})}{\mathcal{F}(\mathbf{n} + e)}$$

5. **Definition:** (Rees) A $\mathbb{Z}^s$-graded $\mathbf{I} = (I_1, \ldots, I_s)$-filtration $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ of ideals in R is called an $\mathbf{I} = (I_1, \ldots, I_s)$**-admissible filtration** if $\mathcal{R}(\mathcal{F})$ is a finite $\mathcal{R}(\mathbf{I})$-module.
6. Two main examples of admissible filtrations are
(i) the $\mathbf{I}$-adic filtration $\{\mathbf{I}^{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^s}$ in any ring and
(ii) the filtration $\{\overline{\mathbf{I}^{\mathbf{n}}}\}_{\mathbf{n} \in \mathbb{Z}^s}$ in an analytically unramified local ring.

Joint reductions of multi-graded filtrations of ideals

1. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be a $\mathbb{Z}^s$-graded $\mathbf{I}$-admissible filtration of ideals in R. Let $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ such that $|\mathbf{q}| = \dim R = d \geq 1$. The set

$$\mathcal{J}_{\mathbf{q}}(\mathcal{F}) = \{a_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$$

2. is called a **joint reduction of** $\mathcal{F}$ of type $\mathbf{q}$ if there exists an $\mathbf{m} \in \mathbb{N}^s$ such that for all $\mathbf{n} \geq \mathbf{m}$ we have

$$\sum_{i=1}^{s}\sum_{j=1}^{q_i} a_{ij}\mathcal{F}(\mathbf{n} - e_i) = \mathcal{F}(\mathbf{n}).$$

3. The vector $\mathbf{m}$ is called a **joint reduction vector** of $\mathcal{F}$ with respect to the joint reduction $\mathcal{J}_{\mathbf{q}}(\mathcal{F})$.

4. **Question:** How to detect joint reduction vectors using graded components of the local cohomology modules of the Rees algebra $\mathcal{R}(\mathcal{F})$?

Filter-regular sequences and joint reductions

1. **Definition:** **(N. V. Trung)** Suppose $R = \bigoplus_{\mathbf{n}\in\mathbb{N}^s} R_{\mathbf{n}}$ is a standard $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$, and $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^s} M_{\mathbf{n}}$ is a finitely generated $\mathbb{Z}^s$-graded R-module. A homogeneous element $a \in R$ is called M-**filter-regular** if for all large $\mathbf{n}$
$$(0 :_M a)_{\mathbf{n}} = 0$$
2. Let $a_1, \ldots, a_r \in R$ be homogeneous elements. Then $a_1, \ldots, a_r$ is called an M-**filter-regular sequence** if a_i is $M/(a_1, \ldots, a_{i-1})M$-filter-regular for all $i = 1, \ldots, r$.
3. **Theorem:** Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R and $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ such that $|\mathbf{q}| = d$ Then there exists a joint reduction of $\mathcal{F}$ of type $\mathbf{q}$ such that images of its elements in $G(\mathbf{I})$ form a $G(\mathcal{F})$-filter-regular sequence.

Eero Hyry's condition $H_{\mathbf{m}}$

1. **Theorem:(E. Hyry)** Let S be a $\mathbb{Z}$-graded ring defined over a local ring $(R, \mathfrak{m})$. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal of S. Let M be a finitely generated $\mathbb{Z}$-graded S-module and $n_0 \in \mathbb{Z}$. Then $[H^i_{\mathcal{M}}(M)]_n = 0$ for all $n \geq n_0$ and $i \geq 0$ $\iff [H^i_{(\mathfrak{a}, S_+)}(M)]_n = 0$ for all $n \geq n_0$ and $i \geq 0$.
2. **Definition:** Let R be a standard $\mathbb{N}^s$-graded ring and $\mathbf{m} \in \mathbb{Z}^s$. We say that a finitely generated $\mathbb{Z}^s$-graded R-module M satisfies **Hyry's condition** $\mathrm{H}_{\mathbf{m}}$ if

$$[H^i_{R_{++}}(M)]_{\mathbf{n}} = 0 \text{ for all } i \geq 0 \text{ and } \mathbf{n} \geq \mathbf{m}.$$

3. **Theorem:** Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$, $R_{e_i} \neq 0$ for all $i = 1, \ldots, s$ and $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. Let $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$. Suppose $[H^i_{\mathcal{M}}(M)]_{\mathbf{n}} = 0$ for all $i \geq 0$ and $\mathbf{n} \in \mathbb{Z}^s$ such that $n_k > a_k$ for at least one $k \in \{1, \ldots, s\}$. Then M satisfies Hyry's condition $\mathrm{H}_{\mathbf{a}+\mathbf{e}}$.

Filter-regular elements and Hyry's condition

1. Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n} \in \mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R.
2. **Lemma:** If $\mathcal{R}(\mathcal{F})$ satisfies $\mathrm{H}_{\mathbf{m}}$ then $G(\mathcal{F})$ satisfies $\mathrm{H}_{\mathbf{m}}$.
3. **Lemma:** Let $R = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} R_{\mathbf{n}}$ be a standard $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\mathbf{0}}, \mathfrak{m})$ and $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^s} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^s$-graded R-module. Suppose M satisfies Hyry's condition $\mathrm{H}_{\mathbf{m}}$. Let $a \in R_{e_j}$ be M-filter-regular. Then M/aM satisfies $\mathrm{H}_{\mathbf{m}+e_j}$.
4. **Theorem:** Suppose $G(\mathcal{F})$ satisfies $\mathrm{H}_{\mathbf{m}}$. Let $\mathbf{q} \in \mathbb{N}^s$ such that $|\mathbf{q}| = d$ and $\mathcal{J} = \{a_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$ be a joint reduction of $\mathcal{F}$ of type $\mathbf{q}$ such that $a_{11}^*, \ldots, a_{1q_1}^*, \ldots, a_{s1}^*, \ldots, a_{sq_s}^*$ is a $G(\mathcal{F})$-filter-regular sequence where a_{ij}^* is the image of a_{ij} in $G(\mathbf{I})_{\mathbf{e_i}}$ for all i and j. Then $\mathbf{m} + \mathbf{q}$ is a joint reduction vector of $\mathcal{F}$ with respect to $\mathcal{J}$. In other words

$$\mathcal{F}(\mathbf{n}) = \sum_{i=1}^{s} \sum_{j=1}^{q_i} a_{ij} \mathcal{F}(\mathbf{n} - e_i) \text{ for all } \mathbf{n} \geq \mathbf{m} + \mathbf{q}.$$

Product of complete ideals in any dimension (Sarkar-Verma)

1. **Theorem:** (E. Hyry) Let $(R, \mathfrak{m})$ be a local ring of dimension d and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^s}$ be an $\mathbf{I}$-admissible filtration of ideals in R. If $\mathcal{R}(\mathcal{F})$ is CM then it satisfies Hyry's condition $\mathrm{H}_{\underline{0}}$.
2. **Theorem:** Let $(R, \mathfrak{m})$ be an analytically unramified local ring of dimension $d \geq 2$ and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\overline{\mathbf{I}^{\mathbf{n}}}\}$ and $\mathcal{R}(\mathcal{F})$ satisfy Hyry's condition $\mathrm{H}_{\mathbf{0}}$. Suppose $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ with $1 \leq |\mathbf{n}| \leq d-1$. Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ where $|\mathbf{n}| \geq d$.
3. **Proof:** We use induction on $|\mathbf{n}|$. By given hypothesis the result is true for $|\mathbf{n}| \leq d-1$. Suppose $|\mathbf{n}| \geq d$. Let $\mathbf{m} \in \mathbb{N}^s$ such that $\mathbf{m} \leq \mathbf{n}$ and $|\mathbf{m}| = d$.
4. Consider the filtration $\mathcal{F} = \{\overline{\mathbf{I}^{\mathbf{n}}}\}_{\mathbf{n}\in\mathbb{Z}}$. Then there exists a joint reduction $\{a_{ij} \in I_i : j = 1, \ldots, m_i; i = 1, \ldots, s\}$ of $\mathcal{F}$ of type $\mathbf{m}$ such that

$$\overline{\mathbf{I}^{\mathbf{r}}} = \sum_{i=1}^{s}\sum_{j=1}^{m_i} a_{ij}\overline{\mathbf{I}^{\mathbf{r}-\mathbf{e_i}}} \text{ for all } \mathbf{r} \geq \mathbf{m}. \quad \text{Hence} \quad \overline{\mathbf{I}^{\mathbf{n}}} = \sum_{i=1}^{s}\sum_{j=1}^{m_i} a_{ij}\overline{\mathbf{I}^{\mathbf{n}-\mathbf{e_i}}}.$$

5. As $\overline{\mathbf{I}^{\mathbf{n}-\mathbf{e_i}}}$ are complete for all i by induction hypothesis, $\mathbf{I}^{\mathbf{n}}$ is also complete.

The RRV Theorem for products of monomial ideals

1. **Theorem:** Let $R = k[X_1, \ldots, X_d]$ and let $\mathfrak{m}$ be its maximal homogeneous ideal. Let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary monomial R-ideals. Suppose $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $1 \leq |\mathbf{n}| \leq d-1$. Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$.
2. **Proof:** If $d = 1$ then R is a PID and hence normal. Therefore every ideal is complete since principal ideals in normal domains are complete.
3. Let $d \geq 2$. Since $I_1, \ldots, I_s$ are monomial ideals, $\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay. Let $W = R \setminus \mathfrak{m}$. Then $S = W^{-1}\overline{\mathcal{R}}(\mathbf{I})$ is Cohen-Macaulay. We have

$$W^{-1}\overline{\mathcal{R}}(\mathbf{I}) = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} W^{-1}\overline{\mathbf{I}^{\mathbf{n}}} = \bigoplus_{\mathbf{n} \in \mathbb{N}^s} (\overline{W^{-1}(\mathbf{I}^{\mathbf{n}})}) = \overline{\mathcal{R}}(W^{-1}I_1, \ldots, W^{-1}I_s).$$

4. Therefore S satisfies Hyry's condition $H_{\underline{0}}$.
5. $W^{-1}(\mathbf{I}^{\mathbf{n}})$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ such that $|\mathbf{n}| \geq 1$.
6. Since $\mathfrak{m}$ is the maximal homogeneous ideal of R and $W^{-1}\left(\overline{\mathbf{I}^{\mathbf{n}}}/\mathbf{I}^{\mathbf{n}}\right) = 0$, $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$.

Normality of power products of monomial ideals

- **Theorem: (Pooja Singla, 2007)** Let I be a monomial ideal of analytic spread ℓ in the polynomial ring $K[X_1, X_2, \ldots, X_d]$ over a field K. Suppose that I^n is complete for all $n \leq \ell - 1$ then I^n is complete for all n.

Recent results of Futoshi Hayasaka (2017)

- **Theorem:** Let $(R, \mathbf{m})$ be an analytically unramified local ring of dimension d. Let $I_1, I_2, \ldots, I_s$ be ideals of positive height in R.
- Let the integral closure of the multi-Rees algebra $\mathcal{R}(I_1, I_2, \ldots, I_s)$ in the polynomial ring $R[t_1, t_2, \ldots, t_s]$ be Cohen-Macaulay. Let $\ell = \ell(I_1 I_2 \ldots I_s)$.
- Suppose that $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n} \in \mathbb{N}^s$ where $|\mathbf{n}| \leq \ell - 1$.
- Then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n}$.
- **Corollary:** Let $I_1, I_2, \ldots, I_s$ are monomial ideals in $K[X_1, X_2, \ldots, X_d]$ and $\ell = \ell(I_1 I_2 \ldots I_s)$. Suppose that $I_1^{n_1} I_2^{n_2} \ldots, I_s^{n_s}$ is complete for all $\mathbf{n} = (n_1, n_2, \ldots, n_s) \in \mathbb{N}^s$ where $|\mathbf{n}| \leq \ell - 1$ then $\mathbf{I}^{\mathbf{n}}$ is complete for all $\mathbf{n}$.