NOTES ON EXTREMAL AND TAME VALUED FIELDS

FRANZ-VIKTOR KUHLMANN

Abstract. We extend the characterization of extremal valued fields given in [1] to the missing case of valued fields of mixed characteristic with perfect residue field. This leads to a complete characterization of the tame valued fields that are extremal. The key to the proof is a model theoretic result about tame valued fields in mixed characteristic. Further, we prove that in an extremal valued field of finite p-degree, the images of all additive polynomials have the optimal approximation property. This fact can be used to improve the axiom system that is suggested in [5] for the elementary theory of Laurent series fields over finite fields. Finally we give examples that demonstrate the problems we are facing when we try to characterize the extremal valued fields with non-perfect residue fields.

1. Introduction

A valued field (K, v) with valuation ring $\mathcal{O}$ and value group vK is called **extremal** if for every multi-variable polynomial $f(X_1, \ldots, X_n)$ over K the set

$$\{v(f(a_1, \ldots, a_n)) \mid a_1, \ldots a_n \in \mathcal{O}\} \subseteq vK \cup \{\infty\}$$

has a maximal element. For the history of this notion, see [1]. In that paper, extremal fields were characterised in several special cases, but some cases remained open. In the present paper we answer the question stated after Theorem 1.2 of [1] to the positive, thereby removing the condition of equal characteristic from the theorem. Thus, the theorem now reads:

Theorem 1.1. *Let (K, v) be a nontrivially valued field. If (K, v) is extremal, then it is algebraically complete and*

(i) *vK is a $\mathbb{Z}$-group, or*
(ii) *vK is divisible and Kv is large.*

Conversely, if (K, v) is algebraically complete with divisible value group and large perfect residue field, then (K, v) is extremal.

Note that a valued field (K, v) is called **algebraically complete** if every finite algebraic extension (L, v) satisfies

$$[L : K] \;=\; (vL : vK)[Lv : Kv]\,, \tag{1}$$

where Lv, Kv denote the respective residue fields. Every algebraically complete valued field (K, v) is **henselian**, i.e., v admits a unique extension to its algebraic closure $\tilde{K}$ (which we will again denote by v). Also, every algebraically complete valued field (K, v) is **algebraically maximal**, that is, does not admit proper algebraic immediate extensions (L, v) (**immediate** means that $vL = vK$ and $Lv = Kv$).

Date: May 15, 2014.

Further, (K,v) is a **tame field** if it is henselian and $\tilde{K}$ is equal to the ramification field of the extension $(\tilde{K}|K,v)$. All tame fields are algebraically complete (cf. [9, Lemma 3.1]).

A field K is **large** if every smooth curve over K which has a K-rational point, has infinitely many such points. For more information about large fields, see [11], [7] and [1].

We were not able to cover the mixed characteristic case in the converse because a corresponding analogue of the Ax–Kochen–Ershov Principle stated in Theorem 4.2 of [1] was not known. In fact, we will show below (Theorem 1.5) that it is false. However, we can do with lesser tools that *are* known. After all, at least the corresponding Ax–Kochen–Ershov Principle for elementary extensions has been proved in [9]:

Theorem 1.2. *If $(L|K,v)$ is an extension of tame fields such that $vK \prec vL$ and $Kv \prec Lv$, then $(K,v) \prec (L,v)$.*

This theorem enables us to prove:

Theorem 1.3. *Take a nontrivially valued tame field (K,v) and two ordered abelian groups Γ and Δ such that $\Gamma \prec vK$ and $\Gamma \prec \Delta$. Then there exist two tame fields (K',v) and (L,v) with $vK' = \Gamma$, $vL = \Delta$, $Kv = K'v = Lv$, $(K',v) \prec (K,v)$ and $(K',v) \prec (L,v)$. In particular, $(K,v) \equiv (L,v)$.*

If vK is nontrivial and divisible and Δ is any nontrivial divisible ordered abelian group, then we can take $\Gamma = \mathbb{Q}$ to obtain that $\Gamma \prec vK$ and $\Gamma \prec \Delta$ since the elementary class of nontrivial divisible ordered abelian groups is model complete. Thus, Theorem 1.3 yields the following result:

Corollary 1.4. *If (K,v) is a nontrivially valued tame field with divisible value group and Δ is any nontrivial divisible ordered abelian group, then there is a tame field $(L,v) \equiv (K,v)$ with $vL = \Delta$ and $Lv = Kv$.*

The Ax–Kochen–Ershov Principle

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \quad\Longrightarrow\quad (K,v) \equiv (L,v) \tag{2}$$

holds for all tame valued fields of equal characteristic (see [9, Theorem 1.4]). But it is easy to see that it cannot hold in the mixed characteristic case. One can construct two algebraic extensions (L,v) and (L',v') of $(\mathbb{Q},v_p)$, where v_p is the p-adic valuation on $\mathbb{Q}$, both having residue field $\mathbb{F}_p$, such that:

1) L does not contain $\sqrt{p}$ and vL is the p-divisible hull of $(v_p p)\mathbb{Z}$,
2) L' contains $\sqrt{p}$ and $v'L'$ is the p-divisible hull of $(v_p\sqrt{p})\mathbb{Z} = \frac{1}{2}(v_p p)\mathbb{Z}$.

Then $vL \simeq v'L'$ and hence $vL \equiv v'L'$, but $(L,v) \not\equiv (L',v')$.

One could hope, however, that this problem vanishes when one strenthens the conditions by asking that vL and $v'L'$ are equivalent over $v_p\mathbb{Q}$ (and Lv and $L'v'$ are equivalent over $\mathbb{Q}v_p$). But the problem remains:

Theorem 1.5. *For every odd prime p there exist two tame fields (L,v) and (L',v'), both algebraic over $\mathbb{Q}$, with $Lv = \mathbb{F}_p = L'v'$ and $vL = v'L'$ the p-divisible hull of $\frac{1}{2}(v_p p)\mathbb{Z}$, but $(L,v) \not\equiv (L',v')$.*

Note that $(L,v) \equiv (L',v')$ if and only if they are equivalent over $(\mathbb{Q},v_p)$, and this in turn holds if and only if we have the equivalence

$$(L,v)_\delta \equiv (L',v')_\delta \quad \text{over} \quad (\mathbb{Q},v_p)_\delta$$

of their amc structures of level δ, for all $\delta \in (v_p p)\mathbb{Z}$ (see [4, Corollary 2.4]). But this fact is of little use for the proof of Corollary 1.4 since it is by no means clear how to construct an extension of $(\mathbb{Q}, v_p)$ whose amc structures of level δ are equivalent to those of (K, v).

The improvement in Theorem 1.1 yields a corresponding improvement of Proposition 5.3 from [1]:

Proposition 1.6. *Take a valued field (K, v) with perfect residue field. Assume that v is the composition of two nontrivial valuations: $v = w \circ \overline{w}$. Then (K, v) is extremal with divisible value group if and only if the same holds for (K, w) and $(Kw, \overline{w})$.*

This follows from Theorem 1.1 by means of the following facts:

1) (K, v) is algebraically complete if and only (K, w) and $(Kw, \overline{w})$ are.

2) The value group of (K, v) is divisible if and only those of (K, w) and $(Kw, \overline{w})$ are.

3) If $(Kw, \overline{w})$ is algebraically complete with divisible value group and perfect residue field, then it is perfect by Theorem 3.2 and Lemma 3.1 of [9], and large by [7, Proposition 16].

It should be noted that the condition on the value groups cannot be dropped without a suitable replacement, even when all residue fields have characteristic 0. Indeed, if the value group of (K, w) is a $\mathbb{Z}$-group and $\overline{w}$ is nontrivial, then the value group of (K, v) is neither divisible nor a $\mathbb{Z}$-group and (K, v) cannot be extremal.

Tame fields of positive residue characteristic $p > 0$ are algebraically complete, and by [9, Theorem 3.2], they have p-divisible value groups which consequently are not $\mathbb{Z}$-groups. On the other hand, by the same theorem all algebraically complete valued fields with divisible value group and perfect residue field are tame fields. Therefore, in the case of positive residue characteristic and value groups that are not $\mathbb{Z}$-groups, the above Theorem 1.1 is in fact talking about tame fields:

Theorem 1.7. *A tame field of positive residue characteristic is extremal if and only if its value group is divisible and its residue field is large.*

Again, we see that we know almost everything about tame fields (with the exception of quantifier elimination in the case of equal characteristic), but almost nothing about non-perfect valued fields. As shown in [1], there are some algebraically complete valued fields with value group a $\mathbb{Z}$-group and a finite residue field that are extremal, and others that are not. In particular, the Laurent series field $\mathbb{F}_q((t))$ over a finite field $\mathbb{F}_q$ with q elements is extremal.

Since it is a longstanding open question whether $\mathbb{F}_q((t))$ has a decidable elementary theory, it is important to search for a complete recursive axiomatization. Such an axiomatization was suggested in [5], using the elementary property that the images of additive polynomials have the optimal approximation property (see Section 3 for the definition of this notion). For the case of $\mathbb{F}_q((t))$, this was proved in [2]. At first sight, extremality seems to imply the optimal approximation property for the images of additive polynomials. But the latter uses inputs from the whole field while the former restricts to inputs from the valuation ring. However, we will prove in Section 3:

Theorem 1.8. *If (K,v) is an extremal field of characteristic $p>0$ with $[K:K^p]<\infty$, then the images of all additive polynomials have the optimal approximation property.*

Since the elementary property of extremality is more comprehensive and easier to formulate than the optimal approximation property, it is therefore a good idea to replace the latter by the former in the proposed axiom system for $\mathbb{F}_q((t))$. We also note that every extremal field is algebraically complete by Theorem 1.1. So we ask:

Open problem: Is the following axiom system for the elementary theory of $\mathbb{F}_q((t))$ complete?

1) (K,v) is an extremal valued field of positive characteristic,
2) vK is a $\mathbb{Z}$-group,
3) $Kv=\mathbb{F}_q$.

In order to obtain the assertion of Theorem 1.8 in the case of algebraically complete perfect fields of positive characteristic (which are exactly the tame fields of positive characteristic), one does not need the assumption that the field be extremal. Indeed, S. Durhan recently proved in [3]:

Theorem 1.9. *If (K,v) is a tame field of positive characteristic, then the images of all additive polynomials have the optimal approximation property.*

Since there exist tame fields of positive residue characteristic whose residue field is not large, this together with Theorem 1.7 shows that a valued field of positive characteristic need not be extremal even when the images of all additive polynomials have the optimal approximation property.

Finally, let us point out that we still do not have a complete characterization of extremal fields:

Open problem: Take a valued field (K,v) of positive residue characteristic. Assume that vK is a $\mathbb{Z}$-group or that vK is divisible and Kv is a non-perfect large field. Under which additional assumptions do we obtain that (K,v) is extremal?

Additional assumptions are indeed needed, as we will show in Section 4:

Proposition 1.10. *a) There are algebraically complete valued fields (K,v) of positive characteristic and value group a $\mathbb{Z}$-group that are extremal, and others that are not.*

b) There are algebraically complete valued fields (K,v) of mixed characteristic with value group a $\mathbb{Z}$-group that are extremal, and others that are not.

c) There are algebraically complete nontrivially valued fields (K,v) of positive characteristic with divisible value group and non-perfect large residue field that are not extremal.

d) There are algebraically complete valued fields (K,v) of mixed characteristic with divisible value group and non-perfect large residue field that are not extremal.

As parts c) and d) indicate, we do not know the answer to the following question:

Open problem: Is there any extremal field with divisible value group and non-perfect large residue field?

Acknowlegements.
The ideas contained in this paper were conceived at a 2 hour seminar talk the author gave to the logic group at the University of Wroclaw in Poland. The audience was arguably the most lively and inspiring the author has ever witnessed. He would like to thank this group for the great hospitality.

The author would like to thank Koushik Pal for proofreading the paper.

During this research, the author was partially supported by a Canadian NSERC grant and a sabbatical grant from the University of Saskatchewan.

2. Proof of Theorems 1.1, 1.3 and 1.5

As a preparation, we need a few basic facts about tame fields. For the following lemma, see [9, Lemma 3.7]:

Lemma 2.1. *Take a tame field (L, v). If K is a relatively algebraically closed subfield of L such that $Lv|Kv$ is algebraic, then (K, v) is a tame field, vL/vK is torsion free, and $Lv = Kv$.*

We derive:

Lemma 2.2. *Take a tame field (K, v) and an ordered abelian group $\Gamma \subset vK$ such that vK/Γ is torsion free. Then there exists a tame subfield (K', v) of (K, v) with $vK' = \Gamma$ and $K'v = Kv$.*

Proof. Denote the prime field of K by K_0 and note that $k_0 := K_0v$ is the prime field of Kv. Take a maximal system γ_i, $i \in I$, of elements in Γ rationally independent over vK_0. Choose elements $x_i \in K$ such that $vx_i = \gamma_i$, $i \in I$. Further, take a transcendence basis t_j, $j \in J$, of Kv over its prime field, and elements $y_j \in K$ such that $y_jv = t_j$ for all $j \in J$. For $K_1 := K_0(x_i, y_j \mid i \in I, j \in J)$ we obtain from [9, Lemma 2.2] that $vK_1 = vK \oplus \bigoplus_{i\in I} \gamma_i\mathbb{Z}$ $K_1v = k_0(t_j \mid j \in J)$, so that Γ/vK_1 is a torsion group and $Kv|K_1v$ is algebraic.

Now we take K' to be the relative algebraic closure of K_1 in K. Then by Lemma 2.1, (K', v) is a tame field with vK/vK' torsion free and $K'v = Kv$. Since $\Gamma \subseteq vK$ and Γ/vK_2 is a torsion group, we have that $\Gamma \subseteq vK'$. Since vK/Γ is torsion free, we also have that $vK' \subseteq \Gamma$, so that $vK' = \Gamma$. □

Lemma 2.3. *Take a tame field (K, v) and an ordered abelian group Δ containing vK such that Δ is p-divisible, where p is the characteristic exponent of Kv. Then there exists a tame extension field (L, v) of (K, v) with $vL = \Delta$ and $Lv = Kv$.*

Proof. By Theorem 2.14 of [6] there is an extension (K_1, v) of (K, v) such that $vK_1 = \Delta$ and $K_1v = Kv$. We take (L, v) to be a maximal immediate algebraic extension of (K_1, v); then (L, v) is algebraically maximal. Since $vL = vK_1 = \Delta$ is p-divisible, and $Lv = K_1v = Kv$ is perfect by [9, Theorem 3.2] applied to (K, v), it follows from the same theorem that (L, v) is a tame field. □

Now we can give the
Proof of Theorem 1.3: By Lemma 2.2 we find a tame subfield (K', v) of (K, v) with $vK' = \Gamma$ and $K'v = Kv$. Since $\Gamma \prec vK$ by assumption, it follows from Theorem 1.2 that $(K', v) \prec (K, v)$.

By Lemma 2.3 we find a tame extension field (L, v) of (K', v) with $vL = \Delta$ and $Lv = K'v$. Since $vK' = \Gamma \prec \Delta = vL$ by assumption, it follows again from Theorem 1.2 that $(K', v) \prec (L, v)$. □

Theorem 1.3 is the key to the

Proof of Theorem 1.1: In view of Theorem 1.2 of [1], we only have to show that if (K,v) is a tame field with divisible value group and large residue field, then (K,v) is extremal. Every trivially valued field is extremal, so we may assume that (K,v) is nontrivially valued. We apply Corollary 1.4 with $\Delta=\mathbb{R}$ to obtain a tame field $(L,v)\equiv(K,v)$ with value group $vL=\mathbb{R}$. By the proof of Theorem 1.2 in [1], this field is extremal. Since extremality is an elementary property, also (K,v) is extremal. □

We turn to the

Proof of Theorem 1.5: We extend the p-adic valuation v_p of $\mathbb{Q}$ to some valuation v on the algebraic closure of $\mathbb{Q}$. We take ϑ to be a root of the polynomial $p\cdot(X^p-X)^2-1$ and η to be a root of $p\cdot(X^p-X-1)^2-1$ over $\mathbb{Q}$. Then $v(\eta^p-\eta)=v(\vartheta^p-\vartheta)=-\frac{vp}{2}$. This yields that $v\eta=v\vartheta=-\frac{vp}{2p}$. By the fundamental inequality "$n\geq e\cdot f$", we have that

$$2p\geq[\mathbb{Q}(\eta):\mathbb{Q}]\geq(v\mathbb{Q}(\eta):v\mathbb{Q})\cdot[\mathbb{Q}(\eta)v:\mathbb{Q}v]\geq 2p\cdot[\mathbb{Q}(\eta)v:\mathbb{Q}v]\geq 2p\,,$$

so equality holds everywhere with $[\mathbb{Q}(\eta)v:\mathbb{Q}v]=1$ and we find that $v\mathbb{Q}(\eta)=(v\eta)\mathbb{Z}=\frac{vp}{2p}\mathbb{Z}$ and $\mathbb{Q}(\eta)v=\mathbb{Q}v$. Similarly, one shows that $v\mathbb{Q}(\vartheta)=(v\vartheta)\mathbb{Z}=\frac{vp}{2p}\mathbb{Z}$ and $\mathbb{Q}(\vartheta)v=\mathbb{Q}v$.

Now we choose algebraic extensions (L_1,v) of $(\mathbb{Q}(\eta),v)$ and (L_1',v') of $(\mathbb{Q}(\vartheta),v)$ such that $vL_1=v'L_1'$ is the p-divisible hull of $v\mathbb{Q}(\eta)=v\mathbb{Q}(\vartheta)$ and hence of $\frac{1}{2}(v_pp)\mathbb{Z}$, and $L_1v=\mathbb{Q}(\eta)v=\mathbb{Q}v_p=\mathbb{Q}(\vartheta)v=L_1'v'$; this is possible by [6, Theorem 2.14].

Now we take (L,v) to be a maximal immediate algebraic extension of (L_1,v) and (L',v') to be a maximal immediate algebraic extension of (L_1',v'). Then by [9, Theorem 3.2], (L,v) and (L',v') are tame fields. Their value groups and residue fields are as in the assertion of Theorem 1.5.

It remains to show that (L,v) and (L',v') are not elementarily equivalent. Assume they were; then the two sentences

$$\exists X\; p\cdot(X^p-X)^2=1 \quad\text{and}\quad \exists X\; p\cdot(X^p-X-1)^2=1$$

would both hold in (L,v). So there would be $a,b\in L$ such that

$$(a^p-a)^2=\frac{1}{p} \quad\text{and}\quad (b^p-b-1)^2=\frac{1}{p}\,.$$

Therefore, $b^p-b=\frac{\varepsilon}{\sqrt{p}}+1$ with $\varepsilon=1$ or -1. Since p is odd, replacing a by $-a$ if necessary, we can assume that $a^p-a=\frac{\varepsilon}{\sqrt{p}}$. But then,

$$\begin{aligned}(b-a)^p-(b-a) &= b^p-b-(a^p-a)+\sum_{i=1}^{p-1}\binom{p}{i}b^i(-a)^{p-i}\\ &= \frac{\varepsilon}{\sqrt{p}}+1-\frac{\varepsilon}{\sqrt{p}}+\sum_{i=1}^{p-1}\binom{p}{i}b^i(-a)^{p-i}\\ &= 1+\sum_{i=1}^{p-1}\binom{p}{i}b^i(-a)^{p-i} \qquad (3)\end{aligned}$$

Similarly as for η and ϑ, we have that $va=vb=-\frac{vp}{2p}$. For $1\leq i\leq p-1$, $vb^i(-a)^{p-i}=-\frac{vp}{2}$, and the binomial coefficient is divisible by p, so the sum lies in the valuation ideal and the residue of (3) is 1. It follows that $v(b-a)=0$ and

$(b-a)v$ is a root of the irreducible polynomial $X^p - X - 1$ over $\mathbb{F}_p$. But this contradicts the fact that by construction, $Lv = \mathbb{F}_p$. □

3. Additive polynomials over extremal fields

We start by introducing a more precise notion of extremality. Take a valued field (K, v), a subset S of K, and a polynomial f in n variables over K. Then we say that (K, v) **is S-extremal with respect to** f if the set $vf(S^n) \subseteq vK \cup \{\infty\}$ has a maximum. We say that (K, v) **is S-extremal** if it is S-extremal with respect to every polynomial in any finite number of variables. With this notation, (K, v) being extremal means that it is $\mathcal{O}$-extremal, where $\mathcal{O}$ denotes the valuation ring of (K, v).

A subset S of a valued field (K, v) has the **optimal approximation property** if for every $z \in K$ there is some $y \in S$ such that $v(z-y) = \max\{v(z-x) \mid x \in S\}$. A polynomial $h \in K[X_1, \ldots, X_n]$ is called a **p-polynomial** if it is of the form $f + c$, where $f \in K[X_1, \ldots, X_n]$ is an additive polynomial and $c \in K$. The proof of the following observation is straightforward:

Lemma 3.1. *The images of all additive polynomials over (K, v) have the optimal approximation property if and only if K is K-extremal with respect to all p-polynomials over K.*

We will work with ultrametric balls

$$B_\alpha(a) \;:=\; \{b \in K \mid v(a-b) \geq \alpha\}\;,$$

where $\alpha \in vK$ and $a \in K$. Observe that $\mathcal{O} = B_0(0)$. We note:

Proposition 3.2. *Take $\alpha, \beta \in vK$ and $a, b \in K$. Then (K, v) is $B_\alpha(a)$-extremal if and only if it is $B_\beta(b)$-extremal. In particular, (K, v) is $B_\alpha(a)$-extremal if and only if it is extremal.*

Proof. It suffices to prove that "$B_\alpha(a)$-extremal" implies "$B_\beta(b)$-extremal". Take a polynomial f in n variables. If $c \in K$ is such that $vc = \beta - \alpha$, then the function $y \mapsto c(y-a) + b$ establishes a bijection from $B_\alpha(a)$ onto $B_\beta(b)$. We set $g(y_1, \ldots, y_n) := f(c(y_1 - a) + b, \ldots, c(y_n - a) + b)$. It follows that $f(B_\beta(b)^n) = g(B_\alpha(a)^n)$, whence $vf(B_\beta(b)^n) = vg(B_\alpha(a))^n$. Hence if (K, v) is $B_\alpha(a)$-extremal with respect to g, then it is $B_\beta(b)$-extremal with respect to f. This yields the assertions of the proposition. □

A valued field (K, v) of characteristic $p > 0$ is called **inseparably defectless** if every finite purely inseparable extension $(L|K, v)$ satisfies equation (1) (note that the extension of v from K^p to L is unique). This holds if and only if every finite subextension of $(K|K^p, v)$ satisfies equation (1).

If (K, v) is inseparably defectless with $[K : K^p] < \infty$, then for every $\nu \geq 1$, the extension $(K|K^{p^\nu}, v)$ has a **valuation basis**, that is, a basis of elements $b_1, \ldots, b_m$ that are **valuation independent** over K^{p^ν}, i.e.,

$$v(c_1 b_1 + \ldots + c_m b_m) \;=\; \min_{1 \leq i \leq m} vc_i b_i$$

for all $c_1, \ldots, c_m \in K^{p^\nu}$.

Note that every algebraically complete valued field is in particular inseparably defectless. By Theorem 1.1, every extremal field is algebraically complete and hence inseparably defectless.

Proposition 3.3. *Take an inseparably defectless valued field (K,v) with $[K:K^p]<\infty$ and an additive polynomial f in n variables over K. Then for some integer $\nu\geq 0$ there are additive polynomials $g_1,\ldots,g_m\in K[X]$ in one variable such that*
a) $f(K^n)=g_1(K)+\ldots+g_m(K)$,
b) all polynomials g_i have the same degree p^ν,
c) the leading coefficients $b_1,\ldots,b_m$ of $g_1,\ldots,g_m$ are valuation independent over K^{p^ν}.

Proof. The proof can be taken over almost literally from Lemma 4 of [2]. One only has to replace the elements $1,t,\ldots,t^{\delta_i-1}$ from that proof by an arbitrary basis of $K|K^{\delta_i}$. □

The following theorem is a reformulation of Theorem 1.8 of the Introduction.

Theorem 3.4. *Assume that (K,v) is an extremal field of characteristic $p>0$ with $[K:K^p]<\infty$. Then it is K-extremal w.r.t. all p-polynomials and therefore, the images of all additive polynomials have the optimal approximation property.*

Proof. Take a p-polynomial h in n variables over K, and write it as $h=f+c$ with f an additive polynomial in n variables over K and $c\in K$. We choose additive polynomials $g_1,\ldots,g_m\in K[X]$ in one variable satisfying assertions a), b), c) of Proposition 3.3. Then $h(K^n)=g_1(K)+\ldots+g_m(K)+c$.

We write $g_i=b_iX^{p^\nu}+c_{i,\nu-1}X^{p^{\nu-1}}+\ldots+c_{i,0}X$ for $1\leq i\leq m$. Then we choose $\alpha\in vK$ such that

$$\alpha\;<\;\min\{0,\,vc-vb_i\,,\,vc_{i,k}-vb_i\mid 1\leq i\leq m\,,\,0\leq k<\nu\}\,.$$

Because $\alpha<0$, it then follows that for each a with $va\leq\alpha$,

$$vb_i+p^\nu va\;\leq\;vb_i+p^\nu\alpha\;\leq\;vb_i+\alpha\;<\;vc$$

and for $0\leq k<\nu$,

$$vb_i+p^\nu va\;\leq\;vb_i+p^\nu\alpha\;\leq\;vb_i+\alpha+p^k\alpha\;<\;vc_{i,k}+p^kva\,.$$

It then follows that

$$vg_i(a)\;=\;vb_i+p^\nu va\;\leq\;vb_i+p^\nu\alpha\;<\;vc\,. \tag{4}$$

On the other hand, if $va'\geq\alpha$, then $vb_i+p^\nu va'\;\geq\;vb_i+p^\nu\alpha$ and $vc_{i,k}+p^kva'\;\geq\;vc_{i,k}+p^k\alpha\;>\;vb_i+p^\nu\alpha$ for $0\leq k<\nu$. This yields that

$$vg_i(a')\;\geq\;vb_i+p^\nu\alpha\,. \tag{5}$$

Now take any $(a'_1,\ldots,a'_m)\in B_\alpha(0)^n$ and $(a_1,\ldots,a_m)\in K^n\setminus B_\alpha(0)^n$. So we have:

$$\min\{va_1,\ldots,va_m\}\;<\;\alpha\;\leq\;\min\{va'_1,\ldots,va'_m\}\,.$$

Since $b_1,\ldots,b_m$ are valuation independent over K^{p^ν}, we then obtain from (4) and (5) that

$$\begin{aligned} vh(a_1,\ldots,a_m) &= \min_{1\leq i\leq m} vb_i+p^\nu va_i\\ &< \min_{1\leq i\leq m} vb_i+p^\nu\alpha\;\leq\;vh(a'_1,\ldots,a'_m)\,.\end{aligned}$$

This proves that

$$vh(B_\alpha(0)^n)\;>\;vh(K^n\setminus B_\alpha(0)^n)\,.$$

Since (K, v) is extremal by assumption, Proposition 3.2 shows that $vh(B_\alpha(0)^n)$ has a maximal element, and the same is consequently true for $vh(K^n)$. This shows that (K, v) is K-extremal w.r.t. h, from which the first assertion follows. The second assertion follows by Lemma 3.1. □

4. More about extremal fields

It follows from [1, Theorem 5.1] that the Laurent series fields $(\mathbb{F}_p((t)), v_t)$ and the p-adic fields $(\mathbb{Q}_p, v_p)$ are extremal. The former have equal characteristic, the latter mixed characteristic. All of them have $\mathbb{Z}$ as their value group, which is a $\mathbb{Z}$-group.

In [5] a valued field extension (L, v) of $(\mathbb{F}_p((t)), v_t)$ is presented in which not all images of additive polynomials have the optimal approximation property. In [1] it is shown that (L, v) is not extremal, although it is algebraically complete and its value group vL is a $\mathbb{Z}$-group (of rank 2). It is also shown that for the nontrivial coarsening w of v corresponding to the convex subgroup $(v_t t)\mathbb{Z}$ of vL, also (L, w) is not extremal. Its value group $wL = vL/(v_t t)\mathbb{Z}$ is divisible and its residue field $Lw = \mathbb{F}_p((t))$ is large, but not perfect. Note that (L, v) and (L, w) are of equal characteristic.

In order to prove the remaining existence statements of Proposition 1.10, we consider compositions of valuations. Unfortunately, contrary to our assertion that the proof of Lemma 5.2 of [1] is easy (and thus left to the reader), we are unable to prove it in the cases that are not covered by Proposition 1.6. (However, we also do not know of any counterexample.) In fact, a slightly different version can easily be proved: *If (K, v) is $\mathcal{O}_v$-extremal, then also (K, w) is $\mathcal{O}_v$-extremal.* We do not know whether the latter impies that (K, w) is $\mathcal{O}_w$-extremal. Proposition 3.2 is of no help here because $\mathcal{O}_v$ is in general not a ball of the form $B_\alpha(a)$ in (K, w).

It appears, though, that we actually had in mind the following result, which is indeed easy to prove:

Lemma 4.1. *If (K, v) is extremal and $v = w \circ \overline{w}$, then $(Kw, \overline{w})$ is extremal.*

Proof. Assume that (K, v) is extremal with $v = w \circ \overline{w}$; note that for any $a, b \in \mathcal{O}_w$, $\overline{w}(aw) > \overline{w}(aw)$ implies $va > vb$.

Assume further that $g \in Kw[X_1, \ldots, X_n]$. Then choose $f \in \mathcal{O}_w$ such that $fw = g$. By assumption, there are $b_1, \ldots, b_n \in \mathcal{O}_v$ such that

$$vf(b_1, \ldots, b_n) \;=\; \max\{vf(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathcal{O}_v\}\,.$$

Since $b_1, \ldots, b_n \in \mathcal{O}_v \subseteq \mathcal{O}_w$ we have that

$$f(b_1, \ldots, b_n)w \;=\; fw(b_1 w, \ldots, b_n w) \;=\; g(b_1 w, \ldots, b_n w)\,.$$

We claim that

$$\overline{w}g(b_1 w, \ldots, b_n w) \;=\; \max\{\overline{w}g(\overline{a}_1, \ldots, \overline{a}_n) \mid \overline{a}_1, \ldots, \overline{a}_n \in \mathcal{O}_{\overline{w}}\}\,.$$

Indeed, if there were $\overline{a}_1, \ldots, \overline{a}_n \in \mathcal{O}_{\overline{w}}$ with $\overline{w}g(\overline{a}_1, \ldots, \overline{a}_n) > \overline{w}g(b_1 w, \ldots, b_n w)$, then for any choice of $a_1, \ldots, a_n \in \mathcal{O}_w$ with $a_i w = \overline{a}_i$ for $1 \leq i \leq n$ we would obtain that $vf(a_1, \ldots, a_n) > vf(b_1, \ldots, b_n)$, a contradiction. □

It remains to prove the existence of the non-extremal fields in mixed characteristic as claimed in Proposition 1.10. We consider again the two non-extremal fields (L, v) and (L, w) mentioned above. By Theorem 2.14 of [6] there is an extension

(K_0,v_0) of $(\mathbb{Q},v_p)$ with divisible value group and L as its residue field. We replace (K_0,v_0) by a maximal immediate extension (M,v_0). Then (M,v_0) is algebraically complete, and so are $(M,v_0\circ v)$ $(M,v_0\circ w)$. The value group of $(M,v_0\circ v)$ is a $\mathbb{Z}$-group, and $(M,v_0\circ w)$ has divisible value group and nonperfect large residue field. But by Lemma 4.1, both fields are non-extremal. This completes the proof of Proposition 1.10.

For the conclusion of this paper, let us discuss how the property of extremality behaves in a valued field extension $(L|K,v)$ where (K,v) is existentially closed in (L,v). In this case, it is known that $L|K$ and $Lv|Kv$ are regular extensions and that vL/vK is torsion free. (An extension $L|K$ of fields is called **regular** if it is separable and K is relatively algebraically closed in L.)

Proposition 4.2. *Take a valued field extension $(L|K,v)$ such that (K,v) is existentially closed in (L,v), a subset S_K of K that is definable with parameters in K, and a polynomial f in n variables over K. Denote by S_L the subset of L defined by the sentence that defines S_K in K. Then the following assertions hold.*
a) If (K,v) is S_K-extremal w.r.t. f, then (L,v) is S_L-extremal w.r.t. f and $\max vf(S_L^n)=\max vf(S_K^n)$. In particular, if (K,v) is extremal, then (L,v) is extremal w.r.t. all polynomials with coefficients in K.
b) Assume in addition that $vL=vK$. If (L,v) is S_L-extremal w.r.t. f, then (K,v) is S_K-extremal w.r.t. f and $\max vf(S_L^n)=\max vf(S_K^n)$. In particular, if (L,v) is extremal, then so is (K,v).

Proof. a): Assume that $a\in S_K^n$ such that $vf(a)=\max vf(S_K^n)$. Then the assertion that there exists an element b in S_L^n such that $vf(b)>vf(a)$ is an elementary existential sentence with parameters in K. Hence if it held in L, then there would be an element b' in S_K^n such that $vf(b')>vf(a)$, which is a contradiction to the choice of a. It follows that $\max vf(S_L^n)\leq \max vf(S_K^n)$. Since $S_K\subseteq S_L$, we obtain that $\max vf(S_L^n)=\max vf(S_K^n)$.
b): Take $b\in S_L^n$ such that $vf(b)=\max vf(S_L^n)$. Since $vL=vK$ by assumption, there is $c\in K$ such that $vc=vf(b)$. Now the assertion that there exists an element b in S_L^n such that $vf(b)=vc$ is an elementary existential sentence with parameters in K. Hence there is $a\in S_K^n$ such that $vf(a)=vc=\max vf(S_L^n)$. Since $vf(a)\in vf(S_K^n)\subseteq vf(S_L^n)$, we obtain that $vf(a)=\max vf(S_K^n)$. □

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Department of Mathematics and Statistics, University of Saskatchewan,
106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6
email: fvk@math.usask.ca