Valuation bases for generalized algebraic series fields*

Franz-Viktor Kuhlmann, Salma Kuhlmann and Jonathan W. Lee†

January 28, 2008

Abstract

We investigate valued fields which admit a valuation basis. Given a countable ordered abelian group G and a real closed or algebraically closed field F with subfield K, we give a sufficient condition for a valued subfield of the field of generalized power series $F((G))$ to admit a K-valuation basis. We show that the field of rational functions $F(G)$ and the field $F(G)^\sim$ of power series in $F((G))$ algebraic over $F(G)$ satisfy this condition. It follows that for archimedean F and divisible G the real closed field $F(G)^\sim$ admits a restricted exponential function.

1 Introduction

Before describing the motivation for this research, and stating the main results obtained, we need to briefly remind the reader of some terminology and background on valued and ordered fields (see [KS1] for more details).

Definition 1. Let K be a field and V be a K-vector space. Let Γ be a totally ordered set, and ∞ be an element larger than any element of Γ. A surjective map $v : V \to \Gamma \cup \{\infty\}$ is a *valuation* on V if for all $x, y \in V$ and $r \in K$, the following holds: (i) $v(x) = \infty$ if and only if $x = 0$, (ii) $v(rx) = v(x)$ if $r \neq 0$, (iii) $v(x - y) \geq \min\{v(x), v(y)\}$.

*2000 Mathematics Subject Classification: Primary: 12J10, 12J15, 12L12, 13A18; Secondary: 03C60, 12F05, 12F10, 12F20.

†Research of the first two authors was supported by NSERC Discovery Grants; of the third by an NSERC Undergraduate Student Research Award.

An important example arises from any ordered abelian group G. Set $|g| := \max\{g, -g\}$ for $g \in G$; for non-zero $g_1, g_2 \in G$, say g_1 is archimedean equivalent to g_2 if there exists an integer r such that $r\,|g_1| \geq |g_2|$ and $r\,|g_2| \geq |g_1|$. Denote by $[g]$ the equivalence class of $g \neq 0$, and by v the *natural valuation* on G, that is, $v(g) := [g]$ for $g \neq 0$, and $v(0) := \infty$. If G is divisible, then G is a valued $\mathbb{Q}$-vector space.

Definition 2. We say that $\{b_i : i \in I\} \subseteq V$ is *K-valuation independent* if whenever $r_i \in K$ such that $r_i \neq 0$ for only finitely many $i \in I$,

$$v\left(\sum_{i\in I} r_i b_i\right) = \min_{\{i \in I\,:\, r_i \neq 0\}} v(b_i)\ .$$

A *K-valuation basis* is a K-basis which is K-valuation independent.

We now recall some facts about valued fields (see [Ri] for more details).

Definition 3. Let K be a field, G an ordered abelian group and ∞ an element greater than every element of G.

A surjective map $w : K \to G \cup \{\infty\}$ is a *valuation* on K if for all $a, b \in K$
(i) $w(a) = \infty$ if and only if $a = 0$, (ii) $w(ab) = w(a) + w(b)$, (iii) $w(a - b) \geq \min\{w(a), w(b)\}$.

We say that (K, w) is a *valued field.* The *value group* of (K, w) is $wK := G$. The *valuation ring* of w is $\mathcal{O}_K := \{a : a \in K \text{ and } w(a) \geq 0\}$ and the *valuation ideal* is $\mathcal{I}(K) := \{a : a \in K \text{ and } w(a) > 0\}$. We denote by $\mathcal{U}(K)$ the multiplicative group $1 + \mathcal{I}(K)$ (the group of 1-units); it is a subgroup of the group of units (invertible elements) of $\mathcal{O}_K$. If $\mathcal{U}(K)$ is divisible, that is, closed under n-th roots for all integers $n > 1$, it is a valued $\mathbb{Q}$-vector space under the valuation $w_{\mathcal{U}}$ defined by $w_{\mathcal{U}}(a) = w(1 - a)$.

We denote by P the place associated to a valuation w; we denote the residue field by $KP = \mathcal{O}_K/\mathcal{I}(K)$. (We shall omit the K from the above notations whenever it is clear from the context.) For $b \in \mathcal{O}_K$, bP or bw is its image under the residue map. For a subfield E of K, we say that P is *E-rational* if P restricts to the identity on E and $KP = E$.

A valued field (K, w) is *henselian* if given a polynomial $p(x) \in \mathcal{O}[x]$, and $a \in Kw$ a simple root of the reduced polynomial $p(x)w \in Kw[x]$, we can find a root $b \in K$ of $p(x)$ such that $bw = a$.

There are important examples of valued fields. If $(K,+,\times,0,1,<)$ is an ordered field, we denote by v its natural valuation, that is, the *natural valuation* v on the ordered abelian group $(K,+,0,<)$. (The set of archimedean classes becomes an ordered abelian group by setting $[x]+[y]:=[xy]$.) Note that the residue field in this case is an archimedean ordered field, and that v is *compatible* with the order, that is, has a convex valuation ring.

Given an ordered abelian group G and a field F, denote by $F((G))$ the (generalized) *power series field* with coefficients in F and exponents in G; elements of $F((G))$ take the form $f=\sum_{g\in G} a_g t^g$ with $a_g \in F$ and well-ordered *support* $\{g\in G : a_g \neq 0\}$. We define $g(f)=a_g$ (the coefficient of f corresponding to the exponent g), $\mathrm{coeffs}(f)=\{a_g : g\in G\}$, $\mathrm{expons}(f)=\{g\in G : a_g\neq 0\}$, and the *minimal support valuation* to be $v_{\min}(f)=\min\mathrm{support}(f)$. By convention, $v_{\min}(0)=\infty$.

Definition 4. Let E be a field and G an ordered abelian group. Given P a place on E, we define the ring homomorphism:

$$\varphi_P : \mathcal{O}_E((G)) \to (EP)((G)); \quad \sum_g a_g t^g \mapsto \sum_g (a_g P)\, t^g\,.$$

1.1 Motivation

Brown in [B] proved that a valued vector space of countable dimension admits a valuation basis. This result was applied in [KS1] to show that every countable ordered field K, henselian with respect to its natural valuation, admits a restricted exponential function, that is, an order preserving isomorphism from the ideal of infinitesimals $(\mathcal{I}(K),+,0)$ onto the group of 1-units $(\mathcal{U}(K),\times,1)$. We address the following question: *does every ordered field K, which is henselian with respect to its natural valuation, admit a restricted exponential function?* Let us consider the following illustrative example.

Example 5. **Puiseux series fields:** Let F be a real closed field. Then the function field $F(t)$ becomes an ordered field when we set $0<t<a$ for all $a\in F$. Define the real closed field of (generalized) Puiseux series over F to be

$$\mathrm{PSF}(F)=\bigcup_{n\in\mathbb{N}} F((t^{\frac{1}{n}}))\,,$$

and let $F(t)^{\sim}$ denote the real closure of $F(t)$. We then have the following containments of ordered fields:

$$F(t)\subset F(t)^{\sim}\subset \mathrm{PSF}(F)\subset F((\mathbb{Q}))\,.$$

(Note that throughout this paper, when we write "$\subset$", we mean "$\subsetneq$".) Now, since F has characteristic 0, the power series field $K = F((\mathbb{Q}))$ admits a restricted exponential exp with inverse log. These are defined by

$$\exp(\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \quad \text{and} \quad \log(1+\varepsilon) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\varepsilon^i}{i} \quad \text{where} \quad \varepsilon \in \mathcal{I}(K)\,.$$

(See [A].) The same argument shows that each term $F((t^{\frac{1}{n}}))$ in $\mathrm{PSF}(F)$ admits a restricted exponential. Therefore, so does $\mathrm{PSF}(F)$ itself. We now turn to the question of whether $F(t)^{\sim}$ admits a restricted exponential. Note that one could not just take the restriction of the exponential map exp defined above to the subfield $F(t)^{\sim} \subseteq F((\mathbb{Q}))$. Indeed, it can be shown that the map exp sends algebraic power series to transcendental power series, so the restriction of the exponential map exp to $F(t)^{\sim}$ is not even a well-defined map.

Following the strategy outlined at the beginning of this section, we shall instead investigate whether the multiplicative group of 1-units and the valuation ideal of $F(t)^{\sim}$ admit valuation bases.

It turns out that this question is interesting to ask for any valued field (not only for ordered valued fields):

Definition 6. Given a valued field (L, w), define a *w-restricted exponential* exp to be an isomorphism from $(\mathcal{I}(K), +, 0)$ onto $(\mathcal{U}(K), \times, 1)$ which is *w-compatible*, that is,

$$wa = w(1 - \exp(a))\,.$$

Since $\mathcal{U}(K)$ is endowed with the valuation $w_{\mathcal{U}}$ given by $w_{\mathcal{U}}(b) = w(1-b)$, this means that exp is valuation preserving.

Note that the same definitions as above render a $v_{\min}$-restricted exponential exp with inverse log on every power series field $F((G))$, for all fields F of characteristic 0 and all ordered abelian groups G.

The main results are Theorem 2.1 and Theorem 2.2 (see Section 2). We consider valued subfields L of a field of power series $F((G))$, where F is algebraically (or real) closed, and G is a countable ordered abelian group, which satisfy the *transcendence degree reduction property* (**TDRP**) over a countable ground field K (see Definitions 7 and 9; Section 2). We prove that the additive group of L admits a valuation basis as a K-valued vector space.

In particular, the valuation ideal of L admits a valuation basis as a K-valued vector space. If the group of 1-units of L is divisible, we show that it admits a valuation basis over the rationals. We exhibit some interesting intermediate fields $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K. For instance, the field of rational functions $F(G)$ and the field $F(G)^{\sim}$ of power series in $F((G))$ algebraic over $F(G)$ satisfy it (see Theorem 3.12 and Theorem 3.13). We show that the class of fields satisfying the TDRP over K is closed under adjunction of countably many elements of $K((G))$ — if L satisfies the TDRP over K, then so does $L(f_1, f_2, \ldots)$ (see Theorem 3.15).

In particular, if F is an archimedean ordered real closed field, and G is a countable divisible ordered abelian group, then the real closed field $F(G)^{\sim}$ admits a restricted exponential function. This gives a partial answer to the original question posed.

It is interesting to note that similar arguments are used in Section 11, p. 35 of [A-D] to show that certain ordered fields admit a derivation function.

The paper is organized as follows. In Section 2, we give a detailed statement of the main results. In Section 3, we work out several technical valuation theoretic results, needed for the proofs of the main results. In Section 3.2, we develop interesting tests to decide whether a generalized power series is rational, or algebraic over the field of rational functions. In Section 3.3, we discuss the TDRP in detail and prove Theorems 3.12, 3.13 and 3.15. Section 4 is devoted to the proofs of Theorems 2.1 and 2.2. Finally, in Section 5, we apply the results to ordered fields and to the complements of their valuation rings, and we provide counterexamples (see Remark 16) to a theorem of Banaschewski ([BAN], Satz, p. 435).

It turns out that by assuming $|F| \leq \aleph_1$, one can provide elementary proofs of Theorems 2.1 and 2.2 not requiring the technical machinery developed in Sections 3 and 4. We provide details in Appendix A (Theorems A.1 and A.2).

2 Main Results

In this paper, we will be particularly interested in subfields of $F((G))$ satisfying a certain closure property. We first provide a definition in the case where F is algebraically closed.

Definition 7 (TDRP — algebraic)**.** Let F be an algebraically closed field, K a countably infinite subfield of F and G a countable ordered abelian group. We say that an intermediate field L, for

$$F(G) \subseteq L \subseteq F((G)),$$

satisfies the *transcendence degree reduction property* (or TDRP) over K if:

1. whenever the intermediate field E, for $K \subseteq E \subseteq F$, is countable, then $E((G)) \cap L$ is countable; moreover, L is the union of the fields $E((G)) \cap L$ taken over such E;

2. whenever $K \subseteq E \subset E' \subseteq F$ for algebraically closed intermediate fields E, E' and E'/E is a field extension of transcendence degree 1, then for finitely many power series $s_1, \ldots, s_n$ in $E'((G)) \cap L$, there exists an E-rational place P of E' such that $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i) \in E((G)) \cap L$ for all i;

3. for E, E', P as above, if $\{\alpha\}$ is a fixed transcendence basis of E'/E, we may assume that P sends α, α^{-1} to K.

Remark 8. The key point of the third axiom is that if P restricts to the identity on some intermediate field $K \subseteq K' \subseteq E'$ and is finite on some element c algebraic over $K'(\alpha)$, then cP is algebraic over K'. Indeed, if c is algebraic over $K'(\alpha)$, then $[K'(\alpha, c) : K'(\alpha)] < \infty$ and hence $[K'(\alpha, c)P : K'(\alpha)P] < \infty$, which shows that cP is algebraic over $K'(\alpha)P = K'(\alpha P) = K'$.

It turns out that many results for the real closed case are implied by those for the algebraically closed case; hence, we make the following analogous definition.

Definition 9 (TDRP — real algebraic)**.** Let F be a real closed field, K a countably infinite subfield of F, and G a countable ordered abelian group. We say that an intermediate field L, for

$$F(G) \subseteq L \subseteq F((G))$$

satisfies the *transcendence degree reduction property* over K if the intermediate field

$$F^a(G) \subseteq L(\sqrt{-1}) \subseteq F^a((G))$$

does, where $F^a = F(\sqrt{-1})$ denotes the algebraic closure of F.

Note that $F^a(G) = F(G)(\sqrt{-1})$ by part b) of Lemma 3.1 below.

Consider an algebraically or real closed field F and a countable ordered abelian group G. We will exhibit later some interesting intermediate fields $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K. For instance, the field of rational functions $F(G)$ and the field $F(G)^\sim$ of power series in $F((G))$ algebraic over $F(G)$ satisfy it. Moreover, the class of fields satisfying the TDRP over K is closed under adjunction of countably many elements of $K((G))$ — if L satisfies the TDRP over K, then so does $L(f_1, f_2, \ldots)$.

Remark 10. Note that $L(f_1, f_2, \ldots)$ doesn't necessarily have countable dimension over L, so we cannot resort to any generalization of Brown's theorem ([B]) in this situation.

Our primary objective of this paper is to prove the following result.

Theorem 2.1 (Additive). *Let F be an algebraically or real closed field, K a countably infinite subfield of F and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K, then the valued K-vector spaces $(L, +)$ and $(\mathcal{I}(L), +)$ admit valuation bases.*

Note that this theorem refers to the valuation $v_{\min}$. It induces a valuation $w_{\mathcal{U}}$ on the group $(\mathcal{U}(L), \times)$ given by $w_{\mathcal{U}}(a) = v_{\min}(1-a)$. In the case of F being a real closed field, this group is ordered, and $w_{\mathcal{U}}$ coincides (up to equivalence) with its natural valuation (see [KS1], Corollary 1.13). With respect to this valuation $w_{\mathcal{U}}$, we also prove the following multiplicative analogue to the last theorem.

Theorem 2.2 (Multiplicative). *Let F be an algebraically or real closed field of characteristic zero, and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over $\mathbb{Q}$ and the group $(\mathcal{U}(L), \times)$ is divisible, then $(\mathcal{U}(L), \times)$ is a valued $\mathbb{Q}$-vector space and admits a $\mathbb{Q}$-valuation basis.*

Note that these results are trivial whenever F is assumed to be countable; by the TDRP axioms, L would be countable, and we could apply Brown's theorem ([B]). So, suppose F is uncountable. Our strategy then involves expressing uncountable objects, such as F, as the colimits of countable objects. In particular, suppose we express F as the colimit of countable subfields, say K_λ for indices λ in a directed set. (This is always possible; how we do it will depend whether we may assume $\operatorname{trdeg} F \leq \aleph_1$.) From this, it will follow that, in the additive situation, the group $\mathcal{I}(L)$ is the colimit of the countable groups $\mathcal{I}(K_\lambda((G)) \cap L)$; in the multiplicative situation, the group $\mathcal{U}(L)$ is the colimit of the countable groups $\mathcal{U}(K_\lambda((G)) \cap L)$.

We now restrict ourselves to the additive case; analogous remarks apply to the multiplicative case. Since each $\mathcal{I}(K_\lambda((G)) \cap L)$ is countable, we can find a valuation basis for it by Brown's theorem ([B]), say B_λ. If we are fortunate enough that these valuation bases are consistent in the sense that $B_{\lambda'}$ extends B_λ whenever $\lambda < \lambda'$, then we may take the colimit of the B_λ, which will be our desired valuation basis of $\mathcal{I}(K_\lambda((G)) \cap L)$. How are we to choose the B_λ consistently? The answer lies in a generalization of Brown's theorem (Proposition 2.3 below), which follows from Corollary 3.6 in [KS2].

Definition 11. Let $W \subseteq V$ be an extension of valued k-vector spaces with valuation w. For $a \in V$, we say that a has an *optimal approximation* in W if there exists $a' \in W$ such that for all $b \in W$, $w(a' - a) \geq w(b - a)$. We say that W has the *optimal approximation property* in V if every $a \in V$ has an optimal approximation in W.

The following proposition follows from Corollary 3.6 in [KS2]. (There, the term "nice" is used for the optimal approximation property.)

Proposition 2.3. *Let $W \subseteq V$ be an extension of valued k-vector spaces. If W has the optimal approximation property in V and $\dim_k V/W$ is countable, then any k-valuation basis of W may be extended to one of V.*

We are then left to show that $\mathcal{I}(K_\lambda((G)) \cap L)$ has the optimal approximation property in $\mathcal{I}(K_{\lambda'}((G)) \cap L)$ whenever $\lambda < \lambda'$; this will occupy the bulk of our arguments. Once we establish this, we are able to construct our desired valuation bases inductively.

We conclude with two remarks concerning the two main theorems.

Remark 12. Note that the assumption that $\operatorname{char} F = 0$ is necessary in Theorem 2.2. If $\operatorname{char} F = p$, then for any non-trivial element $f \in \mathcal{U}(L)$, we have $v_{\min}(1 - f^p) = p \cdot v_{\min}(1 - f) \neq v_{\min}(1 - f)$. Hence, $(\mathcal{U}(L), \times)$ does not admit a valued $\mathbb{Q}$-vector space structure, even if it is divisible.

Remark 13. Note that it can make a difference over which subfield we wish to take a valuation basis. By the results of this paper, we know that $\mathbb{R}(t)$ and $\mathbb{R}(t)^\sim$ both admit $\mathbb{Q}$-valuation bases. We claim they do not admit $\mathbb{R}$-valuation bases. Indeed, since $\mathbb{R}(t)$ and $\mathbb{R}(t)^\sim$ have residue field $\mathbb{R}$, if $\mathcal{B}$ is an

$\mathbb{R}$-valuation independent subset, then the elements of $\mathcal{B}$ have pairwise distinct values. Therefore, $|\mathcal{B}| \leq |\mathbb{Q}| = \aleph_0$. On the other hand, the dimension of $\mathbb{R}(t)$, as a vector space over $\mathbb{R}$ is uncountable (e.g., the subset $\{(1-xt)^{-1}\}_{x\in\mathbb{R}}$ is $\mathbb{R}$-linearly independent).

Concerning the choice of the ground field, we also record the following observation (which is of independent interest). The proof is straightforward, and we omit it.

Proposition 2.4. *Let V be a valued K-vector space and k be a subfield of K. If B denotes a K-valuation basis of V and B' denotes a k-vector space basis of K, then $BB' = \{bb' : b \in B, b' \in B'\}$ is a k-valuation basis of V.*

3 Technical results and key examples

We isolate here some results common to the proofs of our main theorems; note that the proofs of these results hold in every characteristic unless noted otherwise. As an application, we then give examples of fields satisfying the TDRP.

We start with a useful lemma. Its easy proof is similar to the well known special case of rational function fields, so we leave it to the reader.

Lemma 3.1. *Take an ordered abelian group G and an algebraic field extension $L|K$.*
a) Suppose that $L|K$ is normal. To every automorphism $\sigma \in \mathrm{Gal}(L|K)$ define an automorphism σ_G of $L(G)|K(G)$ by letting σ act on the coefficients of the polynomials in $L[G]$. Then $L(G)|K(G)$ is a normal algebraic extension, and $\sigma \mapsto \sigma_G$ induces an isomorphism $\mathrm{Gal}(L|K) \to \mathrm{Gal}(L(G)|K(G))$.
b) Suppose that $L|K$ is finite. Then also $L(G)|K(G)$ is finite, $[L(G) : K(G)] = [L : K]$, and every basis of $L|K$ is also a basis of $L(G)|K(G)$.
Corresponding statements hold for $L((G))|K((G))$, provided that $L|K$ is finite.

3.1 Constructing places

A basic tool in this paper will be the existence of certain places; these will often be used to decrease transcendence degrees.

Proposition 3.2. *Consider a tower of fields*

$$K \subseteq E \subseteq E'$$

where K is infinite and E'/E is an extension of algebraically closed fields with transcendence basis $\{\alpha\}$. Suppose R is a subring of E' that is finitely generated over E. Then there exists an E-rational place P of E' such that the elements α and α^{-1} are sent to K and the place P is finite on R.

Proof. We assume without loss of generality that $\alpha, \alpha^{-1} \in R$; if not, simply adjoin them. We first exhibit a place of $\operatorname{Quot} R$ satisfying the stated conditions.

There are infinitely many E-rational places P of $\operatorname{Quot} R$ sending α and α^{-1} to K. Indeed, for each $q \in K$, we obtain the $(\alpha - q)$-adic place P_q on $E[\alpha]$ and therefore on $\operatorname{Quot} R$ by Chevalley's place extension theorem. Note that for $q \neq q'$, we necessarily have $P_q \neq P_{q'}$.

Moreover, we may select some q such that P_q is finite on R. For suppose $R = E[c_1, \ldots, c_n]$. Since the P_q are trivial on E, they are necessarily finite on any c_i algebraic over E. On the other hand, for any c_i transcendental over E, the $(1/c_i)$-adic place on $E(c_i)$ is the only one not finite on c_i; by extension, there are at most $[\operatorname{Quot}(R) : E(c_i)] < \infty$ places on $\operatorname{Quot}(R)$ not finite on c_i. Since of the infinitely many places P_q only finitely many map c_i to ∞ for some i, we may fix a q such that P_q is finite on all c_i and thus finite on R.

Henceforth, write P to denote this place. By Chevalley's place extension theorem again, P extends from $\operatorname{Quot} R$ to a place on E' having the desired properties. □

Intuitively, the place P given by Proposition 3.2 is used to replace a field subextension of K in F of transcendence degree d by one of transcendence $d-1$. We may also make use of this tool for power series via the induced ring homomorphism φ_P. We now present a finiteness condition that enables us to apply this previous result. For its proof we will need a lemma and two definitions.

Lemma 3.3. *Let (M, v) be a henselian valued field with divisible value group and algebraically closed residue field.*
a) If $\operatorname{char}(Mv) = 0$, then M is algebraically closed.
b) If $\operatorname{char}(M) = p > 0$, M is perfect and closed under Artin-Schreier extensions (i.e., every polynomial $X^p - X - a$ with $a \in M$ has a root in M), then M is algebraically closed.

Proof. Take any henselian valued field (M,v) with divisible value group and algebraically closed residue field and satisfying a) or b). Extend v to the algebraic closure M^a and denote this extension again by v. If $\mathrm{char}(Mv)=0$, then $\mathrm{char}(M)=0$, so under all of our assumptions, M is perfect. We consider the ramification theory of the normal extension $M^a|M$; for the basic facts of general ramification theory, we refer the reader to [E]. We denote by M^r the ramification field of the extension $M^a|M$ with respect to the chosen extension of v. Suppose that $M^r\neq M$ and choose a non-trivial finite subextension $M'|M$ of $M^r|M$. By (22.2) of [E], the Fundamental Equality

$$[M':M]=(vM':vM)[M'v:Mv] \tag{1}$$

holds. But by our assumptions on value group and residue field, $(vM':vM)[M'v:Mv]=1$, that is, $M'|M$ must be trivial. This contradiction shows that $M^r=M$.

If $\mathrm{char}(Mv)=0$, then $M^r=M^a$ by Theorem (20.18) of [E], showing that M is algebraically closed. Now assume that b) holds. As $M=M^r$ and M is assumed to be perfect, it follows from Theorem (20.18) of [E] that $M^a|M$ is a p-extension. Suppose it is not trivial, and pick a non-trivial finite normal subextension $M'|M$. It follows from the general theory of p-groups (cf. [H], Chapter III, §7, Satz 7.2 and the following remark) via Galois correspondence that $M'|M$ is a tower of Galois extensions of degree p. But every Galois extension of degree p of a field of characteristic p is an Artin-Schreier extension (cf. Theorem 6.4 of [L]). But by assumption, M does not have such extensions. Hence, $M'=M$, and this contradiction shows that $M=M^a$, i.e., M is algebraically closed. □

Definition 14. Let (M,v) be a valued field. A *contraction* Φ on a subset S of M is a map $S\to S$ such that

$$v(\Phi a-\Phi b)>v(a-b) \text{ for all } a,b\in S \text{ such that } a\neq b\,.$$

By a *finitely generated ring* we mean a ring that is a finitely generated ring extension of its prime ring.

Theorem 3.4. *Take an algebraically closed field $\mathcal{K}$ and a divisible ordered abelian group H. Set*

$$\begin{aligned} M \;=\; & \bigcup\{\mathcal{R}((\Delta)) : \mathcal{R} \textit{ a finitely generated subring of } \mathcal{K} \\ & \quad \textit{and } \Delta \textit{ a finitely generated subgroup of } H\} \end{aligned}$$

if $\mathrm{char}(K) = 0$*; for* $\mathrm{char}(K) = p > 0$*, we replace* $\mathcal{R}((\Delta))$ *by* $\mathcal{R}^{1/p^\infty}((\frac{1}{p^\infty}\Delta))$*, where* $\mathcal{R}^{1/p^\infty}$ *denotes the closure of* $\mathcal{R}$ *under* p*-th roots, and* $\frac{1}{p^\infty}\Delta$ *denotes the* p*-divisible hull of* Δ*. Then* M *is an algebraically closed subfield of* $\mathcal{K}((H))$*.*

Proof. Let v denote the minimal support valuation $v_{\min}$ on $\mathcal{K}((H))$, as well as its restriction to M. We first establish that (M, v) is a henselian subfield. It is easily verified that M is in fact a field. Denote by A the prime ring of $\mathcal{K}$. If the coefficients of $r, r' \in M$ are contained in finitely generated subrings $\mathcal{R}, \mathcal{R}' \subset \mathcal{K}$, respectively, and the exponents of r, r' are contained in finitely generated subgroups $\Delta, \Delta' \subset H$, respectively, then the coefficients of $r - r'$ belong to the finitely generated ring $A[\mathcal{R}, \mathcal{R}'] \subset \mathcal{K}$, and the exponents of $r - r'$ belong to the finitely generated group $\Delta + \Delta' \subset H$. If $r' \neq 0$, then the coefficients of r/r' belong to the finitely generated ring $A[\mathcal{R}, \mathcal{R}', 1/c]$, where c is the leading coefficient of r', and the exponents of r/r' belong to the finitely generated group $\Delta + \Delta' \subset H$.

Being the union of power series rings, M is henselian. For the convenience of the reader, we include a short proof. Take a monic polynomial $Q \in \mathcal{O}_M[t]$ and an element $r \in \mathcal{O}_M$ such that $vQ(r) > 0$ and $vQ'(r) = 0$. Write $Q(t) = a_0 + a_1 t + \cdots + a_n t^n$, and let c be the leading coefficient of $Q'(r)$. We claim that r can be refined to a root f with coefficients in the ring S generated by $1/c$ and the coefficients of the a_i and of r. By the Newton Approximation Method, we obtain a contraction:

$$\begin{aligned} \Phi : r + \mathcal{I}(\,S((G))\,) &\to r + \mathcal{I}(\,S((G))\,) \\ x &\mapsto x - Q(x)/Q'(r)\,. \end{aligned}$$

Since $\mathcal{I}(\,S((G))\,)$ is spherically complete, Φ has a fixed point, which is a root of Q in $r + \mathcal{I}(\,S((G))\,)$. Thus, M is henselian.

The value group $vM = H$ is divisible and the residue field $Mv = \mathcal{K}$ is algebraically closed. Hence if $\mathrm{char}(\mathcal{K}) = 0$, then M is algebraically closed by part a) of Lemma 3.3. Now assume that $\mathrm{char}(\mathcal{K}) = p > 0$. Since $\mathcal{R}^{1/p^\infty}((\frac{1}{p^\infty}\Delta))$ is closed under p-th roots for any subring $\mathcal{R}$ of $\mathcal{K}$ and any subgroup Δ of H, we find that M is perfect. Take an element in any power series ring $\mathcal{R}^{1/p^\infty}((\frac{1}{p^\infty}\Delta))$, where $\mathcal{R}$ is a finitely generated subring of $\mathcal{K}$ and Δ a finitely generated subgroup of H. Write it as $a + r + b$ where a is a power series with only negative exponents, r is an element of $\mathcal{R}^{1/p^\infty}$, and b is a power series with only positive exponents. Since $vb > 0$, $X^p - X - b$ has a root β in the henselian field M. Further, take ρ to be a root of $X^p - X - r$ in the algebraically closed field $\mathcal{K}$; since $\mathcal{R}$ is a finitely generated subring of $\mathcal{K}$, so is $\mathcal{R}[\rho]$, and thus $\rho \in M$. Finally, the sum

$$\alpha \;=\; \sum_{i=1}^{\infty} a^{1/p^i}$$

is again an element of $\mathcal{R}^{1/p^\infty}((\frac{1}{p^\infty}\Delta))$, and it is a root of $X^p - X - a$. So we have that $\alpha+\rho+\beta \in M$, and $(\alpha+\rho+\beta)^p-(\alpha+\rho+\beta) = \alpha^p-\alpha+\rho^p-\rho+\beta^p-\beta = a+r+b$. This proves that M is closed under Artin-Schreier extensions. Now it follows from part b) of Lemma 3.3 that M is algebraically closed. □

In order to obtain our desired auxiliary result from this theorem, we need another lemma.

Lemma 3.5. *Take an algebraic field extension $\mathcal{K}|K$ and a finitely generated ring $\mathcal{R} \subseteq \mathcal{K}$ such that $K(\mathcal{R})|K$ is separable. Then there exists a finitely generated ring $R \subseteq K$ which contains $\mathcal{R} \cap K$.*

Proof. By our assumptions, $K(\mathcal{R})|K$ is a finite separable extension. Hence there is a primitive element a such that $K(\mathcal{R}) = K(a)$. Choose a finitely generated subring R of K such that the generators of $\mathcal{R}$ and the coefficients of the minimal polynomial of a over K are contained in R. Then $\mathcal{R} \subseteq R[a]$. Every element in $R[a]$ can be written in a unique way as a polynomial in a with coefficients in R and degree less than $[K(a) : K]$. It is an element of K only if it is a constant polynomial in a, i.e., equal to an element in R. This proves that $\mathcal{R} \cap K \subseteq R[a] \cap K = R$. □

Corollary 3.6. *For K a field and G an ordered abelian group, let $f \in K((G))$ be algebraic over $K(G)$. Then there exists a finitely generated subring $R \subseteq K$ and a finitely generated subgroup $\Delta \subseteq G$ such that* $\operatorname{coeffs} f \subseteq R$ *and* $\operatorname{expons} f \subseteq \Delta$ *if* $\operatorname{char}(K) = 0$*, and* $\operatorname{coeffs} f \subseteq R^{1/p^\infty}$ *and* $\operatorname{expons} f \subseteq \frac{1}{p^\infty}\Delta$ *if* $\operatorname{char}(K) = p > 0$.

Proof. Let $\mathcal{K}$ be the algebraic closure of K and H be the divisible hull of G. Since f is algebraic over $K(G)$, it is also algebraic over $\mathcal{K}(H)$. Hence by Theorem 3.4, there is a finitely generated subring $\mathcal{R} \subseteq \mathcal{K}$ and a finitely generated subgroup $\Gamma \subseteq H$ such that $f \in \mathcal{R}((\Gamma))$ if $\operatorname{char}(K) = 0$, or $f \in \mathcal{R}^{1/p^\infty}((\frac{1}{p^\infty}\Gamma))$ if $\operatorname{char}(K) = p > 0$. If $\operatorname{char}(K) = p > 0$ then we can find some integer $\mu \geq 0$ such that $K(\mathcal{R}^{p^\mu})|K$ is separable; then we set $\mathcal{R}_1 = \mathcal{R}^{p^\mu}$, and

$\mathcal{R}_1 = \mathcal{R}$ if $\mathrm{char}(K) = 0$. By the foregoing lemma there is a finitely generated subring $R \subseteq K$ such that $\mathcal{R}_1 \cap K \subseteq R$. As a subgroup of a finitely generated abelian group, also $\Delta = \Gamma \cap G$ is a finitely generated group.

For $\mathrm{char}(K) = 0$ it follows immediately that $\mathrm{coeffs}\, f \subseteq R$ and $\mathrm{expons}\, f \subseteq \Delta$. If $\mathrm{char}(K) = p > 0$ and $c \in \mathrm{coeffs}\, f \subseteq \mathcal{R}^{1/p^\infty} \cap K = \mathcal{R}_1^{1/p^\infty} \cap K$, then $c^{p^\nu} \in \mathcal{R}_1 \cap K \subseteq R$ for some integer $\nu \geq 0$, and thus $c \in R^{1/p^\infty}$. It is clear that $\mathrm{expons}\, f \subseteq \frac{1}{p^\infty}\Delta$. This proves our assertion. □

Note that in positive characteristic, the statement that $\mathrm{coeffs}\, f \subseteq R^{1/p^\infty}$ cannot be strengthened to $\mathrm{coeffs}\, f \subseteq R$. Indeed, let $K = \mathbb{F}_p(y)$ and $G = \mathbb{Q}$. Then the power series

$$f(t) = \sum_{i \geq 1} y^{1/p^i} t^{-1/p^i}$$

satisfies the relation $f^p - f - yt^{-1}$ and is therefore algebraic over $K(\mathbb{Q})$; on the other hand, the coefficient set of $f(t)$ is $\{y^{1/p^i} : i \geq 1\}$, which is clearly not contained in any ring finitely generated over $K = \mathbb{F}_p(y)$.

We now apply our previous results to rational and algebraic series.

Proposition 3.7. *Let E'/E be an extension of algebraically closed fields with transcendence basis $\{\alpha\}$ and take an infinite subfield K of E.*
a) Given finitely many power series $s_1, \ldots, s_n \in E'(G) \subseteq E'((G))$, there exists an E-rational place P of E' sending α, α^{-1} to K such that $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i) \in E(G) \subseteq E((G))$ for each i.
b) We have that $E((G)) \cap E'(G) = E(G)$.

Proof. a): For each i, take $f_i, g_i \in E'[G]$ such that $s_i = f_i/g_i$; without loss of generality, assume that the g_i are monic. Observe that $\mathrm{coeffs}(s_i, f_i, g_i)$ is contained in the ring R generated by the finitely many coefficients of the f_i and g_i. Hence by Proposition 3.2, there exists an E-rational place P of E' sending α, α^{-1} to K that is finite on R. Since each g_i is monic, the $\varphi_P(g_i)$ are non-zero; hence, $\varphi_P(s_i) = \varphi_P(f_i)/\varphi_P(g_i)$.
b): The inclusion "$\supseteq$" is clear. Now take some $s \in E((G)) \cap E'(G)$ and apply part a) to find a place P such that $\varphi_P(s) \in E(G)$. But $s \in E((G))$ and φ_P is trivial on $E((G))$, hence $s = \varphi_P(s) \in E(G)$. □

Proposition 3.8. *Let E'/E be an extension of algebraically closed fields with transcendence basis $\{\alpha\}$ and take an infinite subfield K of E.*

a) Given finitely many power series $s_1, \dots, s_n \in E'((G))$ that are algebraic over $E'(G)$, there exists an E-rational place P of E' sending α, α^{-1} to K such that for each i, $s_i \in \mathcal{O}_P((G))$ and $\varphi_P(s_i)$ lies in the relative algebraic closure of $E(G)$ in $E((G))$.
b) We have that $E((G)) \cap E'(G)^{\sim} = E(G)^{\sim}$, where $E'(G)^{\sim}$ denotes relative algebraic closure in $E'((G))$ and $E(G)^{\sim}$ and denotes relative algebraic closure in $E((G))$.

Proof. a): By Corollary 3.6, there exists a subring R of E', finitely generated over E, such that $\operatorname{coeffs} s_i \subseteq R$ if $\operatorname{char} E = 0$, and $\operatorname{coeffs} s_i \subseteq R^{1/p^\infty}$ if $\operatorname{char} E = p$, for each i. By Proposition 3.2, we may take an E-rational place P of E' that is finite on R and sends α, α^{-1} to K.

Take s to be any of the s_i. As s is algebraic, suppose it is a root of the non-trivial (not necessarily monic!) polynomial $Q \in E[\alpha, t^g : g \in G][y]$. Notice that in the polynomial ring $E[\alpha]$, the kernel of P is the prime ideal $(\alpha - \alpha P)$. Since $E[\alpha]$ is a unique factorization domain, we may divide out coefficients of Q if necessary in order to assume that the polynomial $\varphi_P Q$ is non-zero. (In a slight abuse of notation, we extend φ_P to the polynomial ring over $\mathcal{O}_P((G))$.) As $\varphi_P s$ is a root of $\varphi_P Q \neq 0$, it is algebraic over $E(G)$. Since it also lies in the image $E((G))$ of φ_P, it lies in the relative algebraic closure of $E(G)$ in $E((G))$, as desired.
b): The inclusion "$\supseteq$" is clear. Now take some $s \in E((G)) \cap E'(G)^{\sim}$ and apply part a) to find a place P such that $\varphi_P(s) \in E(G)^{\sim}$. As in the previous proposition, we get $s = \varphi_P(s) \in E(G)^{\sim}$. □

Let us also prove a consequence of the TDRP similar to the parts b) of the previous two propositions.

Lemma 3.9. *Let the setting be as in the formulation of the TDRP. Take any $h \in E((G))$. Then $E((G)) \cap (E'((G)) \cap L)(h) = (E((G)) \cap L)(h)$.*

Proof. We show the "$\subseteq$" direction; the other is clear. Take $s \in E((G)) \cap (E'((G)) \cap L)(h)$ and write

$$s = \frac{f_0 + f_1 h + \cdots + f_n h^n}{g_0 + g_1 h + \cdots + g_m h^m}$$

with $f_i, g_i \in E'((G)) \cap L$. If h is algebraic over L, we may assume the denominator above is 1; otherwise, we may assume that $g_0 = 1$. Apply condition 2 of the TDRP to find a place P such that $\varphi_P(f_i), \varphi_P(g_i) \in E((G)) \cap L$. Since $s \in E((G))$ and φ_P is trivial on $E((G))$, we have that $h = \varphi_P(h)$ and

$$s = \varphi_P(s) = \frac{\varphi_P(f_0) + \varphi_P(f_1)h + \cdots + \varphi_P(f_n)h^n}{\varphi_P(g_0) + \varphi_P(g_1)h + \cdots + \varphi_P(g_m)h^m} \in (E((G)) \cap L)(h) \ .$$

Note that our assumption on the denominator implies that it does not vanish. □

3.2 Coefficient tests for rational and algebraic power series

Using the results developed in the previous section, we can develop a simple coefficient test; in this section, G will denote an arbitrary ordered abelian group with no restrictions on its cardinality. For now, we make no assumptions about characteristic.

Proposition 3.10. *Let E/K be an extension of fields. Then,*

$$K((G)) \cap E(G) = K(G) \,.$$

Proof. The inclusion "$\supseteq$" is clear. Now take $s \in K((G)) \cap E(G)$ and write $s = f/g$ with $f, g \in E[G]$. Replacing E by a subfield generated by the finitely many coefficients of f, g over K, we may assume that $n = \operatorname{trdeg} E/K$ is finite. Take a filtration

$$K^a = E_0 \subset E_1 \subset \cdots \subset E_n = E \,,$$

where $\operatorname{trdeg} E_{i+1}/E_i = 1$ for all i and K^a denotes the algebraic closure of K. We apply part b) of Proposition 3.7 n times to see that $h \in K^a(G)$.

We now show that $h \in K^s(G)$, where K^s denotes the separable closure of K. We may suppose $\operatorname{char} K = p$. We may take some $m \geq 0$ such that $f_0^{p^m} \in K^s[G]$; then we have $s = f_0/g_0 = f_1/g_1$, where

$$f_1 = f_0^{p^m} \quad \text{and} \quad g_1 = g \cdot f_0^{p^m - 1} \,.$$

Since $f_1 \in K^s[G]$ and $s \in K((G))$, it follows that $g_1 \in K^s((G)) \cap K^a[G] = K^s[G]$, as desired.

Finally, we show that $h \in K(G)$. Take a finite Galois extension F/K such that $f_1, g_1 \in F[G]$. Similar to before, we have $s = f_1/g_1 = f_2/g_2$, where

$$f_2 = \prod_{\sigma \in \mathrm{Gal}(F/K)} \sigma(f_1) \in K(G) \quad \text{and} \quad g_2 = g_1 \cdot \prod_{\sigma \neq \mathrm{id}} \sigma(f_1)\,,$$

where we identify $\mathrm{Gal}(F/K)$ with $\mathrm{Gal}(F(G)/K(G))$ (see Lemma 3.1). Since $s \in K((G))$, it follows that $g_2 \in K[G]$, as desired. □

We have an algebraic power series analogue corresponding to Proposition 3.10.

Proposition 3.11. *Let E/K be an extension of fields. If E and K are both real closed or both algebraically closed, then*

$$K((G)) \cap E(G)^{\sim} = K(G)^{\sim}\,,$$

where $\cdot^{\sim}$ denotes relative algebraic closure in $E((G))$.

Proof. Since K is relatively algebraically closed in E, it follows that $K((G))$ is relatively algebraically closed in $E((G))$. This is so because every finite extension M' of the henselian field $M = K((G))$ satisfies the fundamental equality (1), hence if it is a proper extension, then it has a value group larger than G or a residue field larger than K. Thus, $K(G)^{\sim} \subseteq K((G))$ and therefore, $K(G)^{\sim} \subseteq K((G)) \cap E(G)^{\sim}$.

To see the "$\subseteq$" inclusion, first assume that E, K are algebraically closed. Take some $s \in K((G)) \cap E(G)^{\sim}$. Since s satisfies a polynomial relation in $E(G)$, we may assume that $\operatorname{trdeg} E/K$ is finite, after replacing E by a suitable subfield if necessary. Taking a filtration

$$K = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

where $\operatorname{trdeg} E_{i+1}/E_i = 1$ for all i, we apply part b) of Proposition 3.8 n times to see that $s \in K(G)^{\sim}$, as desired.

If E, K are real closed, the above procedure shows that $s \in K^a(G)^{\sim}$, the relative algebraic closure of $K^a(G)$ in $K^a((G))$. Hence s is algebraic over $K^a(G)$ and therefore also over $K(G)$, since $K^a(G)|K(G)$ is algebraic by Lemma 3.1. As $s \in K((G))$, it follows that s lies in $K(G)^{\sim}$, the relative algebraic closure of $K(G)$ in $K((G))$. □

3.3 TDRP for rational and algebraic power series

Fix an algebraically or real closed field F, a countably infinite subfield K and a countable ordered abelian group G. In this section, we exhibit some intermediate fields $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K.

Theorem 3.12. *The field $F(G)$ satisfies the TDRP over K.*

Proof. Suppose that F is real closed. Since $F(G)(\sqrt{-1}) = F(\sqrt{-1})(G) = F^a(G)$, the TDRP in this case will follow if we can show it in the case of F algebraically closed. In the latter case, the first condition of the TDRP is obvious: if $E \subseteq F$ is a field extension of K and E is countable, then $E((G)) \cap F(G) = E(G)$ (with equality from Proposition 3.10) is countable. The second and third conditions are simply the statement of Proposition 3.7. □

Theorem 3.13. *The relative algebraic closure $F(G)^\sim$ of $F(G)$ in $F((G))$ satisfies the TDRP over K.*

Proof. As above, we may assume that F is algebraically closed after verifying that $F(G)^\sim(\sqrt{-1}) = F^a(G)^\sim$, where $F(G)^\sim$ denotes relative algebraic closure in $F((G))$ and $F^a(G)^\sim$ denotes relative algebraic closure in $F^a((G))$. The inclusion "$\subseteq$" is clear. The converse follows from the well known facts that $F^a((G)) = F(\sqrt{-1})((G)) = F((G))(\sqrt{-1})$ and that $a + b\sqrt{-1}$ is algebraic over $F(\sqrt{-1})(G) = F(G)(\sqrt{-1})$ only if a and b are algebraic over $F(G)$.

If $E \subseteq F$ is a field extension of K and E is countable, then by Proposition 3.11, $E^a((G)) \cap F^a(G)^\sim = E^a(G)$, which is countable. Hence, $E^a((G)) \cap F^a(G)^\sim \subseteq E^a((G)) \cap F^a(G)^\sim = E^a(G)$ is also countable. For the second and third conditions of the TDRP, use Proposition 3.8 instead of 3.7. □

We now show that the class of fields satisfying the TDRP over K is closed under the adjunction of countably many power series in $K((G))$.

Lemma 3.14. *Suppose that the intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfies the TDRP over K, where F is algebraically closed. Consider an algebraically closed and countable subextension $K \subseteq E \subseteq F$. Then, for any power series $h \in K((G))$, we have*

$$E((G)) \cap L(h) = (E((G)) \cap L)(h)\,.$$

Proof. We show the "$\subseteq$" direction; the other is clear. Suppose that $s \in E((G)) \cap L(h)$. Write s as a rational function in h and choose a countable algebraically closed field E' that contains E and the necessary coefficients from L. Thus, $s \in E((G)) \cap (E'((G)) \cap L)(h)$. Then take a chain $E = E_0 \subset E_1 \subset \cdots \subset E_n = E'$ of algebraically closed intermediate fields E_i such that each E_{i+1}/E_i is an extension of transcendence degree 1. By applying Lemma 3.9 n times, we obtain that $s \in (E((G)) \cap L)(h)$. □

Theorem 3.15. *Suppose that the intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfies the TDRP over K. Then if $\{h_i\}_{i\geq 1}$ are power series in $K((G))$, the field $L(h_i : i \geq 1)$ also satisfies the TDRP over K.*

Proof. As usual, it suffices to prove the result when F is algebraically closed, because $L(h_i : i \geq 1)(\sqrt{-1}) = L(\sqrt{-1})(h_i : i \geq 1)$.

Henceforth, suppose F is algebraically closed. It suffices to verify the second condition of the TDRP, the rest being trivial. Furthermore, it suffices to show that if L satisfies the TDRP over K, then so does $L(h)$: given finitely many power series $s_1, \ldots, s_n$ in

$$E((G)) \cap L(h_i : i \geq 1) = (E((G)) \cap L)(h_i : i \geq 1)$$

(with equality from a repeated application of Lemma 3.14), we may select finitely many $h_1, \ldots, h_m$ such that $s_1, \ldots, s_n \in L(h_1, \ldots, h_m)$ and proceed by induction.

Let E, E' be algebraically closed fields and E'/E an extension of transcendence degree 1. Given $s_1, \ldots, s_n$ in $E'((G)) \cap L(h)$, Lemma 3.14 allows us to write $s_i = S_i(h)/Q_i(h)$ with polynomials $S_i(x), Q_i(x)$ in $L[x]$. Moreover, if h is algebraic, we may assume that each Q_i is constant; otherwise, we may assume that each Q_i is monic.

Using that L satisfies the TDRP over K, we pick an E-rational place P of E' such that $\text{coeffs}(S_i, Q_i) \subseteq \mathcal{O}_P((G))$ and $\varphi_P(\text{coeffs}(S_i, Q_i)) \subseteq E((G)) \cap L$. Since $h \in K((G))$, $\varphi_P h = h$ and consequently, $\varphi_P(S_i(h)) = \varphi_P(S_i)(h)$ and $\varphi_P(Q_i(h)) = \varphi_P(Q_i)(h)$, where $\varphi_P(S_i)(x), \varphi_P(Q_i)(x)$ are the polynomials in $(E((G)) \cap L)[x]$ obtained from $S_i(x), Q_i(x)$ through an application of φ_P to their coefficients. By our assumptions on the denominators, all $\varphi_P(Q_i)(h)$ are non-zero and therefore, $\varphi_P(s_i) \in (E((G)) \cap L)(h) = E((G)) \cap L(h)$ for all i, as desired. □

Remark 15. The hypothesis and assertion of Theorem 3.15 can be simultaneously weakened. Indeed, if the power series h_i are only assumed to be in

$F((G))$ (instead of in $K((G))$), one cannot immediately apply Theorem 3.15. However, the countability of G permits one to take a countable extension field K' of K containing coeffs(h_i). Since L satisfies the TDRP over K, it does so over K'; applying Theorem 3.15 with K' in place of K, one concludes that $L(h_i : i \geq 1)$ satisfies the TDRP over K'.

4 Constructing valuation bases via TDRP

In this section, we seek out to prove Theorems 2.1 and 2.2. In what follows, F denotes an algebraically or real closed field, and we consider a countable subfield $K \subseteq F$.

Our strategy is to express F as the union of countable subfields of finite transcendence degree over K. More precisely, fix a transcendence basis $\{\alpha_\lambda\}_{\lambda \in I}$ of F over K. Notice that the family of finite subsets of I forms a directed set under inclusion — for each such finite subset $X \subseteq I$, define the subfield

$$K_X = K(\alpha_\lambda : \lambda \in X)^\sim \subseteq F\,,$$

where $\cdot^\sim$ denotes relative algebraic closure in F. Observe that just as $\varinjlim X = I$, $\varinjlim K_X = F$. Moreover, by the first TDRP axiom,

$$\begin{aligned} \varinjlim K_X((G)) \cap L &= L\,, \\ \varinjlim \mathcal{I}(K_X((G)) \cap L) &= \mathcal{I}(L) \quad \text{and} \\ \varinjlim \mathcal{U}(K_X((G)) \cap L) &= \mathcal{U}(L)\,. \end{aligned}$$

Given any finite subset X of I, we will need the optimal approximation property for the valued vector space extensions

$$\langle \mathcal{I}(K_Y((G)) \cap L) : Y \subset X \rangle \subseteq \mathcal{I}(K_X((G)) \cap L)\,.$$

Consequently, we will fix X throughout this section. For notational convenience, label the elements of X to be $x_1, x_2, \ldots, x_N$, so that

$$X = \{x_1, x_2, \ldots, x_N\}\,.$$

For $1 \leq i \leq N$, set $Y_i = X \setminus \{x_i\}$ and $Y_{i,j} = X \setminus \{x_i, x_j\}$.

Our desired results in the case that F is real closed will follow from the corresponding results when F is algebraically closed. Hence, we will assume that F is algebraically closed for now.

4.1 Complements of valuation rings in characteristic 0

The results of this subsection will not be needed later in this paper; they are provided for the sake of independent interest and perspective. Instead, we will need the weaker Lemma 4.3 that will be established in the next subsection. Throughout this section, we need to assume that char $F = 0$.

Suppose that we have a K_{Y_N}-rational place P of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Consider a sum

$$a = a_1 + a_2 + \cdots + a_N \text{ with } a_i \in K_{Y_i}\,.$$

We would like to show that whenever aP is finite, we may assume that we also have a representation of the form

$$a = b_1 + b_2 + \cdots + b_N \text{ with } b_i \in K_{Y_i}\,,$$

where each b_iP is finite.

Note that since we consider F to be algebraically closed, we have that the residue field $\overline{K_X}$ of K_X under P is equal to K_{Y_N}. Because of our assumption that P sends $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K, we have that $\overline{K_{Y_i}} = K_{Y_{i,N}}$. We consider $\mathcal{O}_{K_X}$, the valuation ring of P on K_X, as a K_{Y_N}-vector space. There exists a K_{Y_N}-vector space complement C of $\mathcal{O}_{K_X}$ in K_X; that is, $K_X = C \oplus \mathcal{O}_{K_X}$. Observe that for $1 \le i \le k$,

$$(K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X}) \subseteq K_{Y_i}\,.$$

Assuming equality held in the equation above, we could uniquely write $a_i = b_i + c_i$ for $b_i \in \mathcal{O}_{K_{Y_i}}$ and $c_i \in C$ — note that $\mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$. Our immediate aim is therefore to construct such a complement C where equality in fact holds.

Lemma 4.1. *Suppose that F is algebraically closed and P is a K_{Y_N}-rational place of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Then there exists a K_{Y_N}-vector space complement C of $\mathcal{O}_{K_X}$ in K_X such that for $1 \le i \le N$, $C \cap K_{Y_i}$ is a $K_{Y_{i,N}}$-vector space complement of $\mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$ in K_{Y_i}, that is,*

$$K_{Y_i} = (K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X})\,. \tag{2}$$

Proof. Let $\mathrm{PSF}(\overline{K_X})$ denote the field of Puiseux series over $\overline{K_X}$; that is,

$$\mathrm{PSF}(\overline{K_X}) = \bigcup_{n=1}^{\infty} \overline{K_X}((t^{1/n}))\,.$$

We consider $\mathrm{PSF}(\overline{K_X})$ to be a valued field with the minimal support valuation $v_{\min}$. Since the residue field $\overline{K_X}$ is algebraically closed and of characteristic 0, it is well-known that $\mathrm{PSF}(\overline{K_X})$ is algebraically closed.

As $\alpha_{x_N}P, \alpha_{x_N}^{-1}P \in K$ by construction, we see that the element $\beta = \alpha_{x_N} - \alpha_{x_N}P \in K_{\{x_N\}}$ is transcendental over K_{Y_N}; note that $\beta P = 0$. We thus define the embedding

$$\iota : K_{Y_N}(\beta) \to \mathrm{PSF}(\overline{K_X})$$

such that ι restricts to the identity on K_{Y_N} and sends β to t. Since $\beta P = 0$, we have that ι preserves the valuation v_P on $K_{Y_N}[\beta]$; it follows that it does so on $K_{Y_N}(\beta)$ as well.

Since $\mathrm{PSF}(\overline{K_X})$ is algebraically closed and K_X is an algebraic field extension of $K_{Y_N}(\beta)$, ι extends to an embedding:

$$\iota : K_X \to \mathrm{PSF}(\overline{K_X})\,.$$

Note that this induces a valuation $w = v_{\min} \circ \iota$ on K_X. We may assume without loss of generality that $w = v_P$; for K_X is algebraic over $K_{Y_N}(\beta)$, and therefore there exists $\sigma \in \mathrm{Gal}(K_X/K_{Y_N}(\beta))$ such that $w \circ \sigma = v_P$. Thus, if we consider instead the embedding $\iota' = \iota \circ \sigma$, we have that ι' preserves valuations; that is, $v_P = v_{\min} \circ \iota'$.

Moreover, for each $1 \leq i \leq N$, we have that

$$\iota(K_{Y_i}) \subseteq \mathrm{PSF}(\overline{K_{Y_i}})\,.$$

Note that this is immediate for $i = N$, as ι restricts to the identity on K_{Y_N}. For $i \neq N$, notice that K_{Y_i} is algebraic over $K_{Y_{i,N}}(\alpha_{x_N})$; moreover, ι restricted to $K_{Y_{i,N}} \subset K_{Y_N}$ is the identity. Consequently, $\iota(K_{Y_i})$ is algebraic over $K_{Y_{i,N}}(\iota(\alpha_{x_N}))$. Since $\alpha_{x_N} = \beta + \alpha_{x_N}P$, we have $\iota(\alpha_{x_N}) = t + \alpha_{x_N}P$; this implies that $\iota(K_{Y_{i,N}}(\alpha_{x_N}))$ and, by algebraicity, $\iota(K_{Y_i})$ are contained in $\mathrm{PSF}(K_{Y_{i,N}}) = \mathrm{PSF}(\overline{K_{Y_i}})$.

We are ready to construct our complement of C with the stated properties. Note first that

$$C_P = \{\sum_{q\in S} c_q t^q : c_q \in \overline{K_X} \text{ and } S \text{ a finite negative subset of } \mathbb{Q}\}$$

is a complement to the valuation ring of $\mathrm{PSF}(\overline{K_X})$. Moreover, since it is contained in the image of ι and ι preserves the valuation, we deduce that

$\iota^{-1}(C_P)$ is a complement of $\mathcal{O}_{K_X}$. That is, if

$$C = \iota^{-1}(C_P) = \{\sum_{q\in S} c_q \beta^q : c_q \in \overline{K_X} \text{ and } S \text{ a finite negative subset of } \mathbb{Q}\}\,,$$

then

$$K_X = C \oplus \mathcal{O}_{K_X}\,.$$

It remains to verify that "$\subseteq$" holds in (2). Note that for $i = N$, this follows immediately, as $K_{Y_N} \subseteq \mathcal{O}_{K_X}$. For other i, the fact that $\iota(K_{Y_i}) \subseteq \mathrm{PSF}(\overline{K_{Y_i}})$, together with $\mathcal{O}_{K_{Y_i}} = K_{Y_{i,N}}$, shows that

$$\iota(K_{Y_i}) \cap C_P = \{\sum_{q\in S} c_q t^q : c_q \in \overline{K_{Y_i}} \text{ and } S \text{ a finite negative subset of } \mathbb{Q}\}\,,$$

which is a complement of the valuation ring $\mathcal{O}_{\iota(K_{Y_i})}$ in $\iota(K_{Y_i})$. Pulling back by the valuation-preserving embedding ι, it follows that

$$\begin{aligned} K_{Y_i} \cap C &= \iota^{-1}(\iota(K_{Y_i}) \cap C_P) \\ &= \{\sum_{q\in S} c_q \beta^q : c_q \in K_{Y_{i,N}} \text{ and } S \text{ a finite negative subset of } \mathbb{Q}\} \end{aligned}$$

is a complement to $\mathcal{O}_{K_{Y_i}}$ in K_{Y_i}; that is,

$$K_{Y_i} = (K_{Y_i} \cap C) \oplus \mathcal{O}_{K_{Y_i}} = (K_{Y_i} \cap C) \oplus (K_{Y_i} \cap \mathcal{O}_{K_X})\,.$$

□

It is still possible to prove the previous result in the case that F is real closed; however, significantly more work is needed to eliminate negative parts of the power series given by ι in the proof above. We do not provide details here, as it suffices to consider the case that F is algebraically closed for now.

We can now construct complements as suggested at the beginning of this section.

Lemma 4.2. *Let $\langle K_Y : Y \subset X\rangle$ denote the additive subgroup of K_X generated by the subgroups K_Y and suppose that P is a K_{Y_N}-rational place of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Then, with respect to the place P,*

$$\mathcal{O}_{K_X} \cap \langle K_Y : Y \subset X\rangle = \langle \mathcal{O}_{K_Y} : Y \subset X\rangle\,.$$

More precisely,

$$\mathcal{O}_{K_X} \cap \langle K_Y : N \in Y \subset X\rangle = \langle \mathcal{O}_{K_Y} : Y \subset X\rangle\,.$$

Proof. It suffices to show the "$\subseteq$" direction; the other is immediate. Take $a \in \mathcal{O}_{K_X} \cap \langle K_Y : Y \subset X \rangle$. We may write $a = a_1 + a_2 + \cdots + a_N$ with $a_i \in K_{Y_i}$.

By Lemma 4.1, we may take a decomposition $K_X = C \oplus \mathcal{O}_{K_X}$ such that (2) holds for $1 \leq i \leq N$. Accordingly, we write $a_i = c_i + d_i$ with $c_i \in C$ and $d_i \in \mathcal{O}_{K_{Y_i}} = \mathcal{O}_{K_X} \cap K_{Y_i}$. Since $a = \sum c_i + \sum d_i$ is in $\mathcal{O}_{K_X}$, it follows that $\sum c_i = 0$; that is, $a = d_1 + d_2 + \cdots + d_N$. Since P is trivial on K_{Y_N}, we have $c_N = 0$ and therefore $a_N = d_N$. Both claims now follow. $\square$

Note that in positive characteristic, we can no longer assume that there exist complements as given in Lemma 4.1; the proof fails as we can no longer assume that the negative part of the support of an algebraic power series, and particularly of an element in the image of ι, is finite. (For example, see the remarks following Corollary 3.6.) In the next section we will prove a weakened version of Lemma 4.2 that holds independently of char F. In the case of char $F = 0$, it follows as an immediate corollary from the proof of the previous lemma.

4.2 A weaker result for arbitrary characteristic

Our later combinatorial arguments will depend on a cancellation property of a ring homomorphism φ_P implied by the result here.

Lemma 4.3. *Let P be a K_{Y_N}-rational place of K_X sending $\alpha_{x_N}, \alpha_{x_N}^{-1}$ to K. Suppose that $a \in \mathcal{O}_{K_X} \cap \langle K_Y : Y \subset X \rangle$; that is, $a = a_1 + a_2 + \cdots + a_N$ with $a_i \in K_{Y_i}$. Then $aP = b_1 + b_2 + \cdots + b_N \in \langle K_Y : Y \subset X \rangle$ with $b_i \in K_{Y_{i,N}}$ for $1 \leq i \leq N-1$ and $b_N = a_N$. Further, we may assume that $b_i = 0$ if $a_i = 0$.*

Proof. Since in positive characteristic, the Puiseux series field is not algebraically closed, we replace the embedding ι used in the proof of Lemma 4.1 by an embedding in $\overline{K_X}((\mathbb{Q}))$. Now we have that $\iota(K_{Y_i}) \subseteq K_{Y_{i,N}}((\mathbb{Q}))$ for $1 \leq i \leq N-1$, and ι is the identity on K_{Y_N}. Define $b_i = 0(\iota(a_i))$, the constant term of $\iota(a_i)$. Then $aP = 0(\iota(a)) = 0(\iota(a_1)) + \cdots + 0(\iota(a_N)) = b_1 + \cdots + b_N$, and the b_i have all of the required properties. $\square$

4.3 The optimal approximation property

Consider an intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K. We give a combinatorial formula for an optimal approximation h from a particular subspace to a given power series $f \in L$ in terms of ring homomorphisms φ_P as in the second axiom of the TDRP. Since it will follow that $h \in L$ as well, this conceptually means that the field L is "closed under taking optimal approximations."

Theorem 4.4. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is algebraically closed. Take $\langle \cdot \rangle$ in the context of additive groups. If $f \in K_X((G)) \cap L$, then there exists for each $1 \leq i \leq N$ a K_{Y_i}-rational place P_i of K_X such that*

$$h = f - (\mathrm{id} - \varphi_{P_1}) \circ \cdots \circ (\mathrm{id} - \varphi_{P_N}) f$$

is an element of $\langle K_Y((G)) \cap L : Y \subset X \rangle$ and an optimal approximation to f in $\langle K_Y((G)) : Y \subset X \rangle$. The respective statement also holds for $\mathcal{I}(K_X((G)))$, $\mathcal{I}(K_Y((G)))$ in the place of $K_X((G))$, $K_Y((G))$.

Proof. It suffices to prove the first statement, the second being an easy consequence.

We define the places P_k by decreasing induction on k from N to 1. For notational ease, whenever the places P_i have been defined for all $k < i \leq N$, we define

$$f_k = (\mathrm{id} - \varphi_{P_{k+1}}) \circ \cdots \circ (\mathrm{id} - \varphi_{P_N})\, f\,;$$

moreover, for any tuple $\sigma = (e_k, e_{k+1}, \ldots, e_N)$ of elements in $\{0, 1\}$ we define

$$f_\sigma = \psi^\sigma(f)\,, \quad \text{where } \psi^\sigma = (-\varphi_{P_k})^{e_k} \circ \cdots \circ (-\varphi_{P_N})^{e_N}\,.$$

(For any function φ, we let $\varphi^1 = \varphi$ and $\varphi^0 = \mathrm{id}$.)

Suppose that for some k, P_i has been defined for all $k < i \leq N$ and that f_k is a power series in $K_X((G)) \cap L$. Then by the second TDRP axiom, we may take a K_{Y_k}-rational place P_k of K_X such that for all $(N-k+1)$-tuples σ of elements in $\{0, 1\}$, P is finite on coeffs(f_k, f_σ) and $\varphi_P(f_k)$, $\varphi_P(f_\sigma)$ are power series in $K_{Y_k}((G)) \cap L$. Note that we then have $f_{k-1} \in K_X((G)) \cap L$, as desired.

Having defined our places, we check that they have the desired properties. Let $\sigma = (e_1, \ldots, e_N)$ denote a non-zero tuple. If i denotes the least index such that $e_i = 1$, then $f_\sigma \in K_{Y_i}((G)) \cap L$. Since

$$-h = \sum f_\sigma\,,$$

the sum over non-zero N-tuples σ, we see that $h \in \langle K_Y((G)) \cap L : Y \subset X \rangle$, as desired.

To see that h is an optimal approximation to f in $\langle K_Y((G)) : Y \subset X\rangle$, it suffices to show that if $g(f) \in \langle K_Y : Y \subset X\rangle$ for some exponent $g \in G$, then $g(h) = g(f)$. Indeed, for such a g, write

$$g(f) = a_1 + a_2 + \cdots + a_N \quad \text{where } a_i \in K_{Y_i} .$$

By decreasing induction on k, we show that

$$g(f_k) = b_1 + b_2 + \cdots + b_k \quad \text{where } b_i \in K_{Y_i}$$

for $0 \le k \le N$. This holds for $k = N$ with $b_i = a_i$, as $f_N = f$. If it holds for $k > 0$, then $g(f_{k-1}) = g((\mathrm{id} - \varphi_{P_{k+1}})(f_k)) = g(f_k) - g(\varphi_{P_{k+1}}(f_k))$. By Lemma 4.3, where we replace N by k, the latter is of the form $b_1 + \cdots + b_k - (b'_1 + \cdots + b'_k)$ with $b'_i \in K_{Y_i}$ and $b_k = b'_k$. So $g(f_{k-1}) = b_1 - b'_1 + \cdots + b_{k-1} - b'_{k-1}$ with $b_i - b'_i \in K_{Y_i}$. Hence, $g(f_0) = 0$ and $g(h) = g(f - f_0) = g(f)$, as desired. □

We have a multiplicative analogue.

Theorem 4.5. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is algebraically closed and of characteristic 0. Take $\langle\cdot\rangle$ in the context of multiplicative groups. If $f \in \mathcal{U}(K_X((G)) \cap L)$, then there exists for each $1 \le i \le k$ a K_{Y_i}-rational place P_i of K_X such that*

$$h = f \,/\, (\mathrm{id} \,/\, \varphi_{P_1}) \circ \cdots \circ (\mathrm{id} \,/\, \varphi_{P_k})\, f$$

is an element of $\langle \mathcal{U}(K_Y((G)) \cap L) : Y \subset X\rangle$ and an optimal approximation to f in $\langle \mathcal{U}(K_Y((G))) : Y \subset X\rangle$.

Proof. The construction of the places P_k proceeds as in Theorem 4.4. Verification of the stated properties is a straightforward modification from before, after recalling that the map exp introduced by the formulas in Example 5 is a valuation preserving group isomorphism from $(\mathcal{I}(F((G))), +)$ onto $(\mathcal{U}(F((G))), \times)$ with inverse log. Moreover, note that the maps exp and log commute with the ring homomorphism φ_P. Thus, if we denote by f^a_σ the elements f_σ introduced in the previous proof for the additive case, and by f^m_σ their multiplicative counterparts, we have that

$$\exp(\log(f)^a_\sigma) = f^m_\sigma \ .$$

This shows that each f_σ, for σ a non-zero tuple, is contained in some $K_Y((G))$, and that

$$h^m = 1/\prod f_\sigma^m$$

is an optimal approximation to f in $\langle \mathcal{U}(K_Y((G))) : Y \subset X\rangle$ because $h^a = -\sum \log(f)_\sigma^a$ is an optimal one to $\log(f)$ in $\langle \mathcal{I}(K_Y((G))) : Y \subset X\rangle$, and $\exp(h^a) = h^m$. $\square$

Our optimal approximation results that we will use to extend valuation bases now follow immediately. We also consider the case when F is real closed.

Theorem 4.6. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field. Take $\langle\cdot\rangle$ in the context of additive groups. Then,*

$$\langle \mathcal{I}(K_Y((G)) \cap L) : Y \subset X\rangle$$

has the optimal approximation property in $\mathcal{I}(K_X((G)) \cap L)$.

Proof. If F is algebraically closed, then this is deduced immediately from Theorem 4.4. Otherwise, if F is real closed, we reduce to the algebraically closed case. In particular, given an element $f \in \mathcal{I}(K_X((G)) \cap L)$, we may regard f as an element of $K_X^a((G)) \cap L(\sqrt{-1})$. As $L(\sqrt{-1})$ is a subfield of the algebraically closed field F^a and satisfies the TDRP over K, we can apply the statement of our theorem for the algebraically closed case. We obtain an optimal approximation h to f in

$$\left\langle K_Y^a((G)) \cap L(\sqrt{-1}) : Y \subset X \right\rangle .$$

Take σ to be the conjugation of $F(\sqrt{-1})|F$ and extend it to $F^a(G) = F(\sqrt{-1})((G))$ as described in Lemma 3.1. Then we can define the real part of h as $(h + \sigma(h))/2$. It is easy to see that because h is an optimal approximation to f, the real part of h is an optimal approximation in $\langle K_Y((G)) \cap L : Y \subset X\rangle$ to the real part of f, which is f itself. (This is because $\{1, \sqrt{-1}\}$ is a valuation basis of $F(\sqrt{-1})((G))|F((G))$.) $\square$

The following theorem is proved analogously, again using the valuation preserving isomorphism exp as in the proof of Theorem 4.5.

Theorem 4.7. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field of characteristic 0. Take $\langle\cdot\rangle$ in the context of multiplicative groups. Then,*

$$\langle \mathcal{U}(\, K_Y((G)) \cap L\,) : Y \subset X \rangle$$

has the optimal approximation property in $\mathcal{U}(\, K_X((G)) \cap L\,)$.

4.4 Extending valuation bases

Using our optimal approximation results, we can now exhibit valuation bases for $\mathcal{I}(\,L, +\,)$ and $\mathcal{U}(\,L, \times\,)$. (Note that when $\operatorname{char} K > 0$, only the additive case applies.) Recall that we have chosen a transcendence basis $\{\alpha_\lambda\}_{\lambda \in I}$ of F over K, and for each finite subset X of I, we set $K_X = K(\alpha_\lambda : \lambda \in X)^{\sim}$.

For each X, let V_X denote the valued K-vector space $\mathcal{I}(\, K_X((G)) \cap L\,)$. If $\mathcal{U}(\, K_X((G)) \cap L\,)$ is a divisible group, let W_X denote the valued $\mathbb{Q}$-vector space $\mathcal{U}(\, K_X((G)) \cap L\,)$.

For successively larger n, our aim is to define a valuation basis B_X for each valued vector space V_X (or W_X in the multiplicative case) where $|X| = n$, extending the valuation bases B_X for $|X| < n$. We first give a lemma assuring that the valuation bases B_X for $|X| = n$ can be chosen arbitrarily, as long as they extend the valuation bases B_Y for $Y \subset X$.

Lemma 4.8. *Suppose $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field. Let $\langle\cdot\rangle$ denote K-vector space span. For a finite subset $X \subseteq I$,*

$$K_X \cap \langle K_Z : Z \text{ finite and } X \not\subseteq Z \rangle = \langle K_Y : Y \subset X \rangle \; .$$

Proof. We first assume that F is algebraically closed. It suffices to show the inclusion "$\subseteq$", the other direction being clear. Let $Z_1, \ldots, Z_k$ be finite subsets of the index set I not containing the subset X, take any $y_j \in K_{Z_j}$, $1 \leq j \leq k$, and suppose that $y = y_1 + \cdots + y_k \in K_X$. We show that $y \in \langle K_Y : Y \subset X \rangle$.

Writing $Z = X \cup Z_1 \cup \cdots \cup Z_k$, we see that K_Z has finite transcendence degree over K_X. Labelling the elements of $Z \setminus X$ as $\alpha_1, \ldots, \alpha_n$, we may take a chain of algebraically closed fields

$$K_X = E_0 \subset E_1 \subset \cdots \subset E_n = K_Z$$

where each field extension E_i/E_{i-1} has transcendence basis $\{\alpha_i\}$. By repeated application of Proposition 3.2, for decreasing values of i from n to 1, we can take an E_{i-1}-rational place P_i of E_i that is finite on the $y_jP_{i+1}P_{i+2}\cdots P_n$ and sends the transcendence basis $\{\alpha_i\}$ of E_i over E_{i-1} to K. Observe that by Remark 8, we must have that

$$y_jP_iP_{i+1}\cdots P_n \in K_{Z_j\setminus\{\alpha_i,\alpha_{i+1},\ldots,\alpha_n\}}\,.$$

In particular, $y_jP \in K_{X\cap Z_j}$ for all $1\le j\le k$, where we write P to denote the composition of places $P_0P_1\cdots P_{n-1}$. Since $X\cap Z_j$ is a proper subset of X, we have

$$y = yP = y_1P+\cdots+y_kP \in \langle K_{X\cap Z_1},\ldots,K_{X\cap Z_k}\rangle \subseteq \langle K_Y : Y\subset X\rangle\ .$$

Now if F is real closed, then taking algebraic closures and applying the result in the algebraically closed case, we see that

$$K_X^a \cap \langle K_Z^a : Z \text{ finite and } X\nsubseteq Z\rangle = \langle K_Y^a : Y\subset X\rangle\ ,$$

from which the desired result follows by considering the real parts of the involved elements. □

Theorem 4.9. *Suppose $F(G)\subseteq L\subseteq F((G))$ is an intermediate field satisfying the TDRP over K and F is a real closed or algebraically closed field. Let $n\ge 0$, and suppose that for each subset X of I of cardinality at most n, we have a valuation basis B_X of the K-vector space*

$$V_X = \mathcal{I}(\,K_X((G))\cap L\,)\,.$$

Suppose that

$$\mathcal{B}_n = \bigcup\{B_X : X\subseteq I, |X|\le n\}$$

is valuation independent. Then, for each subset X of I of cardinality $n+1$, we may define a valuation basis B_X of V_X such that

$$\mathcal{B}_{n+1} = \bigcup\{B_X : X\subseteq I, |X|\le n+1\}$$

is valuation independent.

Proof. Observe that since $\mathcal{B}_n$ is valuation independent, the mapping $X\mapsto B_X$ must be inclusion-preserving. Indeed, suppose $X'\subseteq X$ of cardinality at most n. By assumption, $B_{X'}\cup B_X$ is valuation independent and therefore a valuation basis of V_X. Since B_X is a valuation basis of V_X and therefore maximally valuation independent, we must have $B_{X'} \cup B_X = B_X$ and $B_{X'} \subseteq B_X$.

We now define a valuation basis B_X of V_X for each subset X of I of cardinality $n+1$. For such a subset X, observe that $\bigcup\{B_Y : Y \subset X\}$ is a valuation basis of $\langle V_Y : Y \subset X\rangle$. By Theorem 4.6, the subspace $\langle V_Y : Y \subset X\rangle$ has the optimal approximation property in V_X; moreover, since V_X is countable, it has countable dimension over $\langle V_Y : Y \subset X\rangle$. Therefore, Proposition 2.3 allows us to extend $\bigcup\{B_Y : Y \subset X\}$ to a valuation basis B_X of V_X.

It remains to show that $\mathcal{B}_{n+1}$ is valuation independent. Consider a finite sum

$$a = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k$$

for non-zero scalars $c_i \in K$ and distinct elements $b_i \in \mathcal{B}_{n+1}$ such that $q = v_{\min}(b_1) = v_{\min}(b_2) = \cdots = v_{\min}(b_k)$. Since we know that $\mathcal{B}_n$ is valuation independent, we may assume that $b_i \notin \mathcal{B}_n$ for some i. So there exists some subset $X \subseteq I$ of cardinality $n+1$ and (after reindexing if necessary) an index j such that $1 \leq j \leq k$ and

$$b_i \in B_X \setminus \bigcup\{B_Y : Y \subset X\} \quad \text{if and only if} \quad i \leq j\ .$$

Since B_X is valuation independent, the coefficient $q(c_1b_1 + c_2b_2 + \cdots + c_jb_j)$ is in $K_X \setminus \langle K_Y : Y \subset X\rangle$. As the coefficient $q(c_{j+1}b_{j+1} + c_{j+2}b_{j+2} + \cdots + c_kb_k)$ is clearly in $\langle K_Z : Z \neq X, |Z| \leq n+1\rangle$, Lemma 4.8 implies that $q(a) \neq 0$. Hence, $v_{\min}(a) = q$, and $\mathcal{B}_{n+1}$ is valuation independent. □

In the case char $F = 0$, we have a multiplicative analogue.

Theorem 4.10. *In addition to the assumptions of Theorem 4.9, assume that* char $F = 0$. *Let $n \geq 0$, and suppose that for each subset X of I of cardinality at most n, we have a valuation basis B_X of $\mathcal{U}(K_X((G)) \cap L)$. Suppose that*

$$\mathcal{B}_n = \bigcup\{B_X : X \subseteq I, |X| \leq n\}$$

is valuation independent. Then, for each subset X of I of cardinality $n+1$, we may define a valuation basis B_X of $\mathcal{U}(K_X((G)) \cap L)$ such that

$$\mathcal{B}_{n+1} = \bigcup\{B_X : X \subseteq I, |X| \leq n+1\}$$

is valuation independent.

Proof. As above, using Theorem 4.7 instead of 4.6, and again employing the valuation preserving isomorphism exp as in the proof of Theorem 4.5. When checking valuation independence, it is essential to note that for scalars $c_i \in \mathbb{Q}$ and elements $b_i \in \mathcal{U}(F((G)))$ such that $q = v_{\min}(1 - b_i)$ for all i, then we have equality

$$q(b_1^{c_1} \cdot b_2^{c_2} \cdots b_k^{c_k}) = c_1 q(b_1) + c_2 q(b_2) + \cdots + c_k q(b_k)\,.$$

□

We can now prove Theorem 2.1. We first show that $(\mathcal{I}(L), +)$ admits a valuation basis.

Proof. By Theorem 4.9, we may take a valuation basis B_X of each valued K-vector space $\mathcal{I}(K_X((G)) \cap L)$ such that whenever $X' \subseteq X$, then $B_{X'} \subseteq B_X$. It follows that the union of the B_X, taken over all finite subsets X of I, is a valuation basis for $\mathcal{I}(L)$. □

The arguments to establish that $(L, +)$ admits a valuation basis are similar. Indeed, the statement and proof of Theorem 4.9 remains valid if we replace $V_X = \mathcal{I}(K_X((G)) \cap L)$ by $V_X = K_X((G)) \cap L$.

The proof of Theorem 2.2 is analogous, using Theorem 4.10 instead of 4.9.

5 Applications

Now, suppose that F is real closed. Applying Theorems 2.1, 2.2, and 3.13 we immediately obtain:

Corollary 5.1. *Assume that F is a real closed field, and G a countable divisible ordered abelian group. There exist $\mathbb{Q}$-valuation bases of $(F(G)^\sim, +)$ with respect to the minimal support valuation $v_{\min}$, and of $(\mathcal{U}(F(G)^\sim), \times)$ with respect to the derived valuation $v_{\min}(1 - \cdot)$.*

If F is archimedean, then the $v_{\min}$ valuation coincides with the natural valuation on $F((G))$; we obtain

Corollary 5.2. *Let F be an archimedean real closed field, and G a countable divisible ordered abelian group. Then $(F(G)^\sim, +)$ and $(\mathcal{U}(F(G)^\sim), \times)$ admit $\mathbb{Q}$-valuation bases with respect to their natural valuations.*

We can now obtain a partial answer to the original question posed in the introduction. Define the *skeleton* of a valued K-vector space V with value set Γ to be the ordered system of K-vector spaces $S(V) := [\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$, where the *component* $B(\gamma)$ is the K-vector space

$$B(\gamma) = \{x \in V : v(x) \geq \gamma\}/\{x \in V : v(x) > \gamma\} .$$

Now, given an ordered system of K-vector spaces $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$, the Hahn product $\prod_{\gamma\in\Gamma} B(\gamma)$ is the $v_{\min}(s)$-valued K-vector space consisting of sequences with well-ordered support (where support(s) and $v_{\min}(s)$ are defined as for fields of power series.) The Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ is the subspace of elements with finite support; its skeleton is precisely the given system $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$. By considering "leading coefficients", one sees that if V has skeleton $[\Gamma, \{B(\gamma)\}_{\gamma\in\Gamma}]$ and admits a valuation basis, then $V \simeq \coprod_{\gamma\in\Gamma} B(\gamma)$.

Corollary 5.3. *Let F be an archimedean real closed field, and G a countable divisible ordered abelian group. Then the real closed field $F(G)^\sim$ admits a restricted exponential.*

Proof. Since $\mathcal{I}(F(G)^\sim)$ and $\mathcal{U}(F(G)^\sim)$ both admit valuation bases, they are both isomorphic as ordered abelian groups to the Hahn sums over their skeleta, which are themselves isomorphic (as proven in Theorem 1.15 in [KS1]). □

Our final application is to the structure of complements to valuation rings in fields of algebraic series. Observe that for the field $F((G))$, an additive complement to the valuation ring is given by $F((G^{<0}))$, where $F((G^{<0}))$ is the (non-unital) ring of power series with negative support. It follows easily (see [B-K-K]) that for the subfield $L = F(G)^\sim$ of $F((G))$, an additive complement to the valuation ring is given by $\mathrm{Neg}(L)$, where $\mathrm{Neg}(K) := F((G^{<0})) \cap L$. We shall call $\mathrm{Neg}(L)$ the *canonical complement* to the valuation ring of L. Note that $F[G^{<0}] \subseteq \mathrm{Neg}(L)$, where $F[G^{<0}]$ is the *semigroup ring* consisting of power series with negative and finite support. Observe that the additive group of $F[G^{<0}]$ is just the Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ with $\Gamma = G^{<0}$ and $B(\gamma) = F$ for each γ. We are interested in understanding under which conditions $F[G^{<0}] = \mathrm{Neg}(L)$. In Proposition 2.4 of [B-K-K], we proved the following

Proposition 5.4. *Assume that G is archimedean and divisible, and that F is a real closed field. Then* $\mathrm{Neg}(L) = F[G^{<0}]$.

On the other hand, in Remark 2.5 of [B-K-K], we observed that if G is not archimedean, then $F[G^{<0}] \neq \mathrm{Neg}(L)$. The results of this paper imply that:

Proposition 5.5. *Let $L = F(G)^{\sim}$, where F is a real closed field and G is a countable divisible ordered abelian group. Then $\mathrm{Neg}(L) \simeq F[G^{<0}]$ as ordered groups under addition.*

Proof. We know that $L = \mathrm{Neg}(L) \oplus \mathcal{O}_L$, and this is a lexicographic decomposition. Now the lexicographic sum of valued vector spaces admits a valuation basis if and only if each summand admits a valuation basis (see [KS1]). It follows that $\mathrm{Neg}(L)$ admits a valuation basis. Clearly, $F[G^{<0}]$ also admits a valuation basis, since it is just a Hahn sum. Since $\mathrm{Neg}(L)$ and $F[G^{<0}]$ both have skeleton $[G^{<0}, \{F\}_{\gamma \in G^{<0}}]$, it follows that they are isomorphic as valued vector spaces, and in particular, as ordered groups under addition. □

Remark 16. This proposition shows that $\mathrm{Neg}(L)$ (which contains series of infinite support if G is not archimedean as shown in Remark 2.5 of [B-K-K]) is nevertheless *isomorphic* to the Hahn sum $F[G^{<0}]$ as ordered group under addition. This shows a theorem of Banaschewski ([BAN], Satz, p. 435) to be false. This theorem characterizes the Hahn sums to be those valued vector spaces in which every element a admits a *convex decomposition* (that is, $a = a_1 + \cdots + a_n$ for distinct $v(a_1), \ldots, v(a_n)$) of maximal length. (Note that Banaschewski uses the name "schwache Hahnsche Summe" for the Hahn sum and "Hahnsche Summe" for the Hahn product.) It is *not* true that all elements in a Hahn sum admit convex decompositions of maximal length. For if the value set admits a chain $v(a) = \beta_1 < \beta_2 < \beta_3 < \cdots$, take elements b_i such that $v(b_i) = \beta_i$ and $b_1 = a$. Then for arbitrary $n \in \mathbb{N}$, a convex decomposition of a of length n is given by $(b_1 - b_2) + (b_2 - b_3) + \cdots + b_n$. In his purported proof, Banaschewski erroneously claims (writing "wie man sofort sieht") that a tuple of the Hahn product with only one non-zero entry admits only convex decompositions of length 1.

Most remarkably, Fleischer [FL] cites the false theorem of Banaschewski in order to disprove a theorem of Hill and Mott ([H-M], Theorem 5.1). Their theorem states that a countable ordered abelian group G, whose archimedean components are each isomorphic to $\mathbb{Z}$, can be embedded in the ordered Hahn sum $\coprod_{\gamma \in v(G)} \mathbb{Z}$. Fleischer gives an interesting example which, because of Banaschewski's error, does not show what he wants, but rather lends credibility to the theorem of Hill and Mott.

Consider the Hahn product $\prod_{n\in\mathbb{N}} \mathbb{Z}$ and its subgroup H of all cofinitely constant tuples, generated over the Hahn sum $\coprod_{n\in\mathbb{N}} \mathbb{Z}$ by the constant tuple $(1, 1, 1, \ldots)$. According to Banaschewski's theorem, H would not be isomorphic to the Hahn sum over its skeleton; hence, it would not admit a $\mathbb{Z}$-valuation basis. As a counterexample, consider the basis consisting of tuples of the form $(0, 0, 0, \ldots, 0, 1, 1, 1, \ldots)$.

Acknowledgements: We wish to thank Antongiulio Fornasiero for contributing to many insightful discussions and suggesting a proof of Corollary 3.6.

A Appendix

We prove alternative versions of our main theorems, weakened by the assumption that the residue field of our power series field has transcendence degree at most $\aleph_1$. That is, we take F to be an algebraically or real closed field and assume that $\operatorname{trdeg} F \leq \aleph_1$; as in the body of the paper, G denotes a countable ordered abelian group.

In the body of the paper, we write F as the union of countable subfields of finite transcendence degree over K; the new assumption $\operatorname{trdeg} F \leq \aleph_1$ enables us to additionally assume this is a linear colimit over countable fields. The linearity renders the prior combinatorial arguments (and supporting technical results) unnecessary, as now we need only verify the optimal approximation property for valued vector space extensions of the form (in the additive case):

$$\mathcal{I}(\, K_\lambda((G)) \cap L \,) \subseteq \mathcal{I}(\, K_{\lambda+1}((G)) \cap L \,)\,.$$

In particular, we may fix a transcendence basis $\{\alpha_\lambda\}_{\lambda<\aleph_1}$ of F over K. Notice that the $\lambda < \aleph_1$ form a directed set. For each $\lambda \leq \aleph_1$, define the field

$$K_\lambda = K(\alpha_\gamma : \gamma < \lambda)^{\sim} \subseteq F$$

where $\cdot^{\sim}$ denotes relative algebraic closure in F. Observe that we have the following colimits of countable objects:

$$\varinjlim \lambda = \aleph_1 \quad \varinjlim K_\lambda = F\,.$$

Moreover, given an intermediate field $F(G) \subseteq L \subseteq F((G))$ satisfying the TDRP over K, the first axiom implies

$$\varinjlim L_\lambda = L \quad \varinjlim \mathcal{I}(L_\lambda) = \mathcal{I}(L) \quad \varinjlim \mathcal{U}(L_\lambda) = \mathcal{U}(L)\,.$$

where $L_\lambda = K_\lambda((G)) \cap L$.

Theorem A.1 (Bounded Additive)**.** *Let F be an algebraically or real closed field such that* $\operatorname{trdeg} F \leq \aleph_1$*, K a countably infinite subfield of F and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over K, then the valued K-vector spaces $(L, +)$ and therefore $(\mathcal{I}(L), +)$ admit valuation bases.*

Proof. For each λ, define the K-vector space $V_\lambda = (L_\lambda, +)$. We wish to define a valuation basis B_λ for each countable vector space V_λ such that $B_{\lambda'}$ extends B_λ whenever $\lambda < \lambda'$.

First, we verify that V_λ has the optimal approximation property in $V_{\lambda+1}$. Indeed, suppose that $f \in V_{\lambda+1} \setminus V_\lambda$; by definition of V_λ, there exists a minimal $q \in \operatorname{support} f$ such the power series coefficient $q(f)$ lies in $K_{\lambda+1} \setminus K_\lambda$. Thus, if h is any approximation to f in V_λ, we necessarily have $v_{\min}(f - h) \leq q$.

Assume that F is algebraically closed. By the second TDRP property, we may take a K_λ-rational place P of $K_{\lambda+1}$ such that $\varphi_P(f) \in V_\lambda$. Then $\varphi_P(f)$ is our desired optimal approximation because for each exponent g such that $g(f) \in K_\lambda$, one has $g(\varphi_P(f)) = g(f)$.

If F is real closed, we reduce to the previous case: if f has an optimal approximation h in $V_\lambda \oplus \sqrt{-1}V_\lambda$, then as in the proof of Theorem 4.6, the real part of h is an optimal approximation to f in V_λ.

Having established the optimal approximation property, we are in a position to define the B_λ via transfinite induction. For $\lambda = 0$, simply select an arbitrary valuation basis B_0 of V_0. For any successor ordinal $\lambda + 1$, note that $V_{\lambda+1}$ is countable and thus has countable dimension over V_λ; hence, by Proposition 2.3, the valuation basis B_λ of V_λ extends to one $B_{\lambda+1}$ of $V_{\lambda+1}$. Now for a limit ordinal λ, we see that V_λ is the colimit of the V_ρ for $\rho < \lambda$; hence, we may simply define B_λ to be the colimit of the B_ρ for $\rho < \lambda$. Then $B_{\aleph_1}$ is the desired valuation basis for $V_{\aleph_1} = L$. □

The proof of a multiplicative version is analogous — simply define $V_\lambda = (\mathcal{U}(L_\lambda), \times)$ and replace the valuation $v_{\min}$ by $v_{\min}(1 - \cdot)$ in the above proof. We thus have

Theorem A.2 (Bounded Multiplicative)**.** *Let F be an algebraically or real closed field of characteristic zero such that* $\operatorname{trdeg} F \leq \aleph_1$, *and G a countable ordered abelian group. If $F(G) \subseteq L \subseteq F((G))$ is an intermediate field satisfying the TDRP over $\mathbb{Q}$ and the group $(\mathcal{U}(L), \times)$ is divisible, then $(\mathcal{U}(L), \times)$ is a valued $\mathbb{Q}$-vector space and admits a $\mathbb{Q}$-valuation basis.*

Remark 17. A subfield L of $F((G))$ is *truncation closed* if for any element $s = \sum_{g \in G} a_g t^g$ in L and any $q \in G$, the restriction $s_{<q} = \sum_{g<q} a_g t^g$ of s to the initial segment $G^{<q}$ of G also belongs to L. For example, the fields $F(G)$ of rational series and $F(G)^\sim$ of algebraic series are both truncation closed (see [F]).

We note that Theorems A.1 and A.2 remain true if we assume that L is a truncation closed subfield that satisfies only the first axiom of the TDRP. Indeed, as in the proof of Theorem A.1 above, we let $h := f_{<q}$ be the truncation of f at $q \in G$ where q is the least exponent for which $f(q) \notin K_\lambda$. Then, $h \in V_\lambda$ is an optimal approximation to f.

References

[A] Alling, Norman: *On exponentially closed fields,* Proc. Amer. Math. Soc. **13** (1962), 706–711

[A-D] Aschenbrenner, Matthias and van den Dries, Lou: *Liouville closed H-fields,* J. Pure Appl. Algebra **197** (2005), 83-139

[BAN] Banaschewski, Bernhard: *Totalgeordnete Moduln,* Arch. Math. (Basel) **7** (1956), 430-440, **MR 19 #385**

[B] Brown, Ron: *Valued vector spaces of countable dimension*, Publ. Math. Debrecen **18** (1971), 149–151

[B-K-K] Biljakovic, Darko - Kotchetov, Mikhail - Kuhlmann, Salma: *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields,* in: Logic, Algebra and Arithmetic, Lecture Notes in Logic, **Vol 26**, Association for Symbolic Logic, AK Peters (2006), 42–65

[E] Endler, Otto: *Valuation theory*, Springer, Berlin (1972)

[FL] Fleischer, Isidore: *On Hill and Mott, Trans. Amer. Math. Soc.* ***175*** *(1973), 283-297,* ***MR 47 #102,*** Trans. Amer. Math. Soc. **231** (1977), 273-274

[F] Fornasiero, Antongiulio: *Embedding Henselian fields into power series,* Journal of Algebra **304** (2006), 112-156

[H] Huppert, Bertram: *Endliche Gruppen I,* Springer, Berlin (1967)

[H-M] Hill, Paul and Mott, Joe L.: *Embedding theorems and generalized discrete ordered abelian groups,* Trans. Amer. Math. Soc. **175** (1973), 283-297, **MR 47 #102**

[KS1] Kuhlmann, Salma: *Ordered Exponential Fields,* The Fields Institute Monograph Series, **Volume 12**, Amer. Math. Soc. (2000)

[KS2] Kuhlmann, Salma: *Valuation bases for extensions of valued vector spaces,* Forum Math. **8** (1996), 723–735

[L] Lang, Serge: *Algebra,* Springer-Verlag (2002)

[P] Prestel, Alexander: *Lectures on Formally Real Fields (Lecture Notes in Mathematics, 1093),* Springer-Verlag (1984)

[Ri] Ribenboim, Paulo: *Théorie des valuations,* Les Presses de l'Université de Montréal, Montréal (1968)

F.-V. Kuhlmann:
Research Center for Algebra, Logic and Computation,
University of Saskatchewan, S7N 5E6, Canada.
Email: fvk@math.usask.ca

S. Kuhlmann:
Research Center for Algebra, Logic and Computation,
University of Saskatchewan, S7N 5E6, Canada.
Email: skuhlman@math.usask.ca

J. W. Lee:
Department of Mathematics, Stanford University, California, 94305-2125, USA.
Email: jlee@math.stanford.edu