Local Uniformization and the defect

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Two events in the mid 1960's

In the mid 1960's two important but seemingly unrelated results appeared:

• in 1964, H. Hironaka proved resolution of singularities for arbitrary algebraic varieties over ground fields of characteristic 0;

• in 1965, J. Ax and S. Kochen, and independently Yu. Ershov, proved that the elementary theory of $\mathbb{Q}_p$ is decidable.

In fact, Ax and Kochen first proved a corrected version of Artin's Conjecture by means of model theory.

Two deep open problems in positive characteristic

Since then, the positive characteristic analogues of these results have been much studied, but have yet remained unproven:

• resolution of singularities for arbitrary algebraic varieties over ground fields of positive characteristic;

• decidability of formal Laurent series fields over finite fields.

The field $\mathbb{F}_p((t))$, where $\mathbb{F}_p$ is the field with p elements, is very similar to $\mathbb{Q}_p$ (it "just" has positive characteristic), but we still do not understand its model theoretic properties.

Some progress on the first problem has been made:

• S. Abhyankar proved resolution in positive characteristic for dimension 2 and, with restrictions, for dimension 3.

• J. de Jong proved resolution in positive characteristic for all dimensions by alteration, i.e., taking into account a finite extension of the function field.

• V. Cossart and O. Piltant proved resolution in positive characteristic for dimension 3 without restrictions.

If we cannot solve a problem globally, we try to solve it locally. Naively speaking, we choose a singularity and try to get rid of it. As for resolution, starting with a singular variety V, we are looking for a "better" variety V'. But V' cannot be arbitrary, we require that it is connected with V by a proper birational morphism.

As we consider just one singular point x_0 on V (and are not interested in other singular points), we can restrict our attention to an affine neighborhood of x_0 . So we may assume from the start that V is an affine variety.

Starting with the point x_0 on V, we need a correspondence which singles out a point on the new variety V', so that we can require that this corresponding point be non-singular.

Zariski's idea

If V and V' are birationally equivalent, then they have the same function field F. Adopting the view that a point of V is given by a homomorphism of the coordinate ring $K[V]$, one may have the idea to extend this homomorphism to F and then restrict it to the coordinate ring $K[V']$. The problem is that if the homomorphism is not an isomorphism (which it usually isn't), then its extension to F cannot be a homomorphism; some elements will have to be sent to ∞. So Zariski's idea was to extend the homomorphism to a place P of F, which is a homomorphism on a large enough subring of F (the valuation ring $\mathcal{O}_P$ of P) and sends all elements outside of $\mathcal{O}_P$ to ∞.

Basic idea of local uniformization

Hence, given the function field $F|K$ and a place P on F that is trivial (i.e., identity) on K, we are looking for a variety V' such that $K[V'] \subset \mathcal{O}_P$ and that the point represented by the homomorphism that P induces on $K[V']$ is non-singular. Every place has an associated valuation $v = v_P$, so we are talking about a problem concerning a valued function field.

In order to make local uniformization an entirely valuation theoretical problem, we will require that the new point on V' be smooth, which means that the implicit function theorem holds in this point. The topology needed to formulate the implicit function theorem is induced by the valuation v_P .

Implicit function theorem and Hensel's Lemma

The implicit function theorem is closely related to the multi-dimensional Hensel's Lemma.

Let $f = (f_1, \ldots, f_n)$ be a system of polynomials in the variables $X = (X_1, \ldots, X_n)$ and with coefficients in $\mathcal{O}_v$. Consider the Jacobian matrix

$$J_f(X) := \left(\frac{\partial f_i}{\partial X_j}(X) \right)_{i,j} .$$

Assume that there exists a tuple $b = (b_1, \ldots, b_n) \in \mathcal{O}_v^n$ such that

$$vf_i(b) > 0 \text{ for } 1 \le i \le n \quad \text{and} \quad v \det J_f(b) = 0 .$$

Then there exists a unique tuple $a = (a_1, \ldots, a_n) \in \mathcal{O}_v^n$ such that $f_i(a) = 0$ and that $v(a_i - b_i) > 0$ for all i.

It is the uniqueness, not the existence, that matters here.

Henselian fields

Hensel's Lemma was originally a lemma proven by Hensel for $\mathbb{Q}_p$. Today we often see it as a property of valued fields: we call a valued field henselian if it satisfies Hensel's Lemma (the one-dimensional or, equivalently, the multi-dimensional version).

The henselization of a valued field is a minimal algebraic extension which is henselian. It is unique up to valuation preserving isomorphism, and it is an immediate extension, i.e., value group and residue field of the valuation do not change.

Smooth local uniformization

We consider a valued function field $(F|K, v)$ where $v = v_P$ for a given place P. Our problem now is to find a transcendence basis $\mathcal{T} = \{t_1, \ldots, t_n\} \subset \mathcal{O}_v$ of $F|K$ and an element $a \in \mathcal{O}_v$ algebraic over $K(\mathcal{T})$ such that $F = K(\mathcal{T}, a)$ and the point $(t_1 P, \ldots, t_n P, aP)$ is smooth. Note that in order to admit local uniformization, the function field must be separably generated, i.e., admit a transcendence basis $\mathcal{T}$ such that $F|K(\mathcal{T})$ is separable and consequently, simple. The smoothness means that a satisfies the assumptions of the one-dimensional Hensel's Lemma:

if f is the minimal polynomial of a over $K(\mathcal{T})$, then $vf(a) > 0$ (which is automatic as $f(a) = 0$) and $vf'(a) = 0$.

Disclaimer

At this point, a word of warning is in place. Local uniformization also requires that the new variety V' is connected with the original one by a proper birational morphism. This amounts to an extra condition, which we will address later.

The extension $(K(\mathcal{T},a)|K(\mathcal{T}))$

How do we know that the extension $(K(\mathcal{T},a)|K(\mathcal{T}),v)$ satisfies the condition on a we have just derived? Does our condition on a mean that a lies in the henselization of $K(\mathcal{T})$? The answer is no. We will need ramification theory to give the correct answer.

Ramification theory

We consider a normal algebraic extension $(N|L, v)$ and set $G := \mathrm{Aut}\,(N|L)$. One defines three subgroups G^d, G^i and G^r of G. Their fixed fields in the maximal separable extension field of L within N are called decomposition field, inertia field and ramification field of $(N|L, v)$, respectively.

Remark: In contrast to the classical definition used by other authors, we take the fixed fields not in N, but *in the maximal separable subextension*. The reason for this will become clear in a moment.

We speak of absolute ramification theory if N is taken to be the algebraic closure $\tilde{L}$ of L. Then the fixed fields are taken in the separable-algebraic closure K^{sep} of K.

Absolute ramification theory

Galois group		field			value group		residue field	
		$\tilde{L}$			$\widetilde{vL}$		$\widetilde{Lv}$	
		$\vert$	purely inseparable		$\Vert$		$\Vert$	
1		L^{sep}		separable-algebraic closure	$\widetilde{vL}$		$\widetilde{Lv}$	
$\vert$		$\vert$	Galois p-extension		$\vert$	division by p	$\vert$	purely inseparable
G^r		L^r		absolute ramification field	$\frac{1}{p'^{\infty}} vL$		$(Lv)^{\mathrm{sep}}$	
$\vert$	Char	$\vert$	abelian Galois p'-extension		$\vert$	division prime to p	$\Vert$	
G^i		L^i		absolute inertia field	vL		$(Lv)^{\mathrm{sep}}$	
$\vert$	$\mathrm{Gal}\, Lv$	$\vert$	Galois		$\Vert$		$\vert$	Galois
G^d		$L^d = L^h$		absolute decomposition field = henselization	vL		Lv	
$\vert$		$\vert$			$\Vert$		$\Vert$	
$\mathrm{Gal}\, L$		L			vL		Lv	

Elimination of ramification

Ramification is the valuation theoretical expression of the failure of the implicit function theorem. So we wish to eliminate ramification in a given valued function field $(F|K,v)$. However, ramification means more than just the change of the value groups. Already in classical algebraic number theory one calls an extension ramified also if the residue field extension is not separable. Even more general, for us ramification is everything that happens above the absolute inertia field. Then elimination of ramification means to find a transcendence basis $\mathcal{T}$ such that F lies in the absolute inertia field (also called strict henselization) of $(K(\mathcal{T}),v)$. In other words, the element a we talked about does not have to lie in $K(\mathcal{T})^h$, but should lie in $K(\mathcal{T})^i$. We then also say that the extension $(F|K,v)$ is inertially generated.

Elimination of ramification

Hence the task of elimination of ramification for a given valued function field $(F|K,v)$ is to show that it is inertially generated.

If the residue field Lv of L has characteristic 0, then L^r is already algebraically closed and there is no wild ramification. Zariski proved in 1940 that local uniformization holds for all places of algebraic function fields over fields of characteristic 0. He "only" had to eliminate the tame ramification.

Elimination of ramification

Elimination of ramification is also a crucial tool for the proof of model theoretic results about valued fields, such as Ax–Kochen–Ershov principles (also known as relative completeness and relative model completeness), and decidability or relative decidability. One approach for the proof of such results are embedding theorems: If $(F|K,v)$ is a valued function field and (K^*,v^*) is a henselian extension of (K,v) (with some additional properties), then we are looking for a valuation preserving embedding of F in K^* over K that lifts given embeddings $vF \hookrightarrow v^*K^*$ and $Fv \hookrightarrow K^*v^*$. The only way we can do this (so far) is using Abhyankar sub-function fields, Kaplansky's results on immediate extensions, and Hensel's Lemma. Therefore, we need the function field to be inertially generated.

Defect is the common enemy

Hence it turns out that the defect is the common hurdle for both, local uniformization and the model theory of valued fields in positive charactersitic.

It is therefore important to find ways to avoid the defect or to work around it.

The Abhyankar Inequality

If Γ is any abelian group, then the rational rank of Γ is $\mathrm{rr}\,\Gamma := \dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma$. This is the maximal number of rationally independent elements in Γ.
If $(L|K, v)$ is an arbitrary valued field extension of finite transcendence degree, then we have the Abhyankar inequality:

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{rr}\,(vL/vK) \,+\, \mathrm{trdeg}\, Lv|Kv\,. \tag{1}$$

We call v an Abhyankar valuation and its associated place P an Abhyankar place if equality holds in (1).

We set

$$\rho \;:=\; \mathrm{rr}\, vF/vK \quad \text{and} \quad \tau \;:=\; \mathrm{trdeg}\, Fv|Kv\,.$$

Then we choose a set

$$\mathcal{T}_0 \;=\; \{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\} \;\subset\; F$$

such that

• the values $vx_1\,,\ldots,vx_\rho$ are rationally independent over vK,

• the residues $y_1v,\ldots,y_\tau v$ are algebraically independent over Kv.

Abhyankar sub-function fields

Now $K(\mathcal{T}_0)|K$ is a rational function field of transcendence degree $\rho + \tau$ and v is an Abhyankar valuation on $K(\mathcal{T}_0)|K$. By our choice of $\mathcal{T}_0$,

- $vF/vK(\mathcal{T}_0)$ is a torsion group,
- $Fv|K(\mathcal{T}_0)v$ is algebraic.

Function fields with Abhyankar valuations

Let us first assume that $\mathcal{T}_0$ is already a transcendence basis of $F|K$, so that v on F is an Abhyankar valuation. Then the elements x_i can be chosen such that $vF = vK(\mathcal{T}_0)$. Things are a bit more complicated for the residue fields (which makes it necessary to work with absolute inertia fields), but let us assume for simplicity that $Fv = K(\mathcal{T}_0)v$. Does all this imply that F lies in $K(\mathcal{T}_0)^i$ (and then, in this simplified case, already in $K(\mathcal{T}_0)^h$)?

We have to answer the question whether the extension

$$F^h|K(\mathcal{T}_0)^h$$

of henselian fields must be trivial, which then implies that

$$F \subset K(\mathcal{T}_0)^h \subset K(\mathcal{T}_0)^i .$$

The Lemma of Ostrowski

If $(F|E,v)$ is a finite extension of valued fields and the valuation v of E admits a unique extension to F, then by the Lemma of Ostrowski,

$$[F:E] \;=\; p^{\nu}\cdot(vF:vE)[Fv:Ev]\,,$$

where $\nu \geq 0$ is an integer and p is the characteristic exponent of Ev, that is, $p = \operatorname{char} Ev$ if it is positive and $p = 1$ otherwise. The factor p^{ν} is called the defect of the extension $(L|K,v)$.

Since passing to henselizations does not change value groups and residue fields, we have that $vF^h = vF = vK(\mathcal{T}_0) = vK(\mathcal{T}_0)^h$ and $F^h v = Fv = K(\mathcal{T}_0)v = K(\mathcal{T}_0)^h v$. Hence if the extension $F^h|K(\mathcal{T}_0)^h$ has no (non-trivial) defect, it must be trivial.

We give two examples of extensions with non-trivial defect.

The example of F. K. Schmidt

We consider the formal Laurent series field $\mathbb{F}_p((t))$ with its canonical t-adic valuation $v = v_t$. (This is the unique valuation that satisfies $vt = 1$, and $(\mathbb{F}_p((t)), v)$ is the completion of $(\mathbb{F}_p(t), v)$.) Since $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose some element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. We have that

$$v\mathbb{F}_p((t)) = \mathbb{Z} = v\mathbb{F}_p(t) \text{ and } \mathbb{F}_p((t))v = \mathbb{F}_p = \mathbb{F}_p(t)v\,,$$

showing that $(\mathbb{F}_p((t))|\mathbb{F}_p(t), v)$ is an immediate extension. The same also holds for

$$(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p), v)\,.$$

This extension is purely inseparable of degree p, and there is only one extension of the valuation v from $\mathbb{F}_p(t, s^p)$ to $\mathbb{F}_p(t,s)$. Hence, the defect of this extension is equal to its degree p.

Abhyankar's example

In 1956 S. Abhyankar gave the following example (without talking about the defect at all). We consider

$$K := \mathbb{F}_p((t))^{1/p^\infty}$$

with the t-adic valuation. We take ϑ to be a root of the polynomial

$$X^p - X - \frac{1}{t}\,.$$

It can then be proven that the separable extension $(K(\vartheta)|K, v)$ has defect p.

Abhyankar's example

In the field $\mathbb{F}_p((t^{\mathbb{Q}}))$ of all power series with coefficients in $\mathbb{F}_p$ and exponents in $\mathbb{Q}$, the element ϑ can be represented as a power series

$$\vartheta = \sum_{i=1}^{\infty} t^{-\frac{1}{p^i}} .$$

Although ϑ is algebraic over $\mathbb{F}_p((t))$, the exponents do not have a common denominator. This phenomenon is only possible in positive characteristic, and its observation is what made Abhyankar's example famous.

The Generalized Stability Theorem

We call (K, v) a defectless field if no finite extension of its henselization has non-trivial defect.

Theorem

(Generalized Stability Theorem)
Assume that v is an Abhyankar valuation on the function field $F|K$, not necessarily trivial on K. If (K, v) is a defectless field, then (F, v) is a defectless field.

This theorem is crucial for the proof of the next result.

Local uniformization for Abhyankar places

Theorem (Knaf–K, 2005)

Let P be an Abhyankar place of the function field $F|K$, trivial on K, and assume that $FP|K$ is separable. Take any finite set $Z \subset \mathcal{O}_P$. Then there is model X for F such that P is centered in a smooth point $x \in X$, Z is a subset of the local ring $\mathcal{O}_{X,x}$ at x, and $\dim \mathcal{O}_{X,x} = \mathrm{rr}\, v_P F$. Moreover, X can be chosen such that all elements of Z are $\mathcal{O}_{X,x}$-monomials in $\{a_1, \ldots, a_d\}$ for some regular parameter system $(a_1, \ldots, a_d)$ of $\mathcal{O}_{X,x}$.

The assertion on the finite sets Z is what complements the elimination of ramification; it is proved by a Perron algorithm. It ensures that the original variety and the new variety are connected by a proper birational morphism. Indeed, as mentioned in the beginning, we may assume from the start that V is an affine variety, i.e., $V = \mathrm{Spec}\,(A)$, where A is a finitely generated K-algebra with quotient field F. If Z is chosen to be a set of generators of A, then $Z \subset \mathcal{O}_{X,x}$ implies that $A \subset \mathcal{O}_{X,x}$, which in turn implies the existence of a morphism $V \to X$.

Beyond Abhyankar valuations

Let us now consider the case where $F|K(\mathcal{T}_0)$ is transcendental, i.e., v is *not* an Abhyankar valuation of $F|K$. Recall that $vF/vK(\mathcal{T}_0)$ is a torsion group and $Fv|K(\mathcal{T}_0)v$ is algebraic. However, we are unable to treat the case where any one of them is non-trivial. To force the extension to become immediate, we have to replace $K(\mathcal{T}_0)$ by some algebraic extension. This gives rise to an alteration. Even more extensions are needed in order to proceed by induction on the transcendence degree of $F|K(\mathcal{T}_0)$.

When dealing with immediate extensions, we cannot avoid the defect, but we can get around it.

The Henselian Rationality Theorem

A valued field (K, v) is called a separably tame field if it is henselian and its absolute ramification field is separable-algebraically closed (so all wild ramification over (K, v) is coming from purely inseparable extensions, which moreover all can be shown to lie in its completion).

Theorem

Let (K, v) be a separably tame field and $(F|K, v)$ an immediate function field, with $F|K$ a separable extension. If its transcendence degree is 1, then there is $x \in F$ such that $F \subset K(x)^h$.

This shows that we can eliminate ramification also in this special case. This result, together with the previous theorem, is crucial for the proof of the next theorem.

Local uniformization by alteration

Theorem (Knaf–K, 2009)

Let P be an arbitrary place of the function field $F|K$, trivial on K. Then there is a finite extension $\mathcal{F}|F$ and an extension of P to $\mathcal{F}$ which admits local uniformization, with the smoothness and the assertions on the finite sets Z satisfied as before.

If K is perfect of characteristic $p > 0$, then the extension $\mathcal{F}|F$ can be chosen to be

- *either Galois*
- *or such that $v_P\mathcal{F}/v_PF$ is a p-group and $\mathcal{F}P|FP$ is purely inseparable.*

This time, the Perron algorithm is augmented by an adaptation of methods that I. Kaplansky developed for dealing with immediate extensions.

The arithmetic case

For both theorems on local uniformization there are arithmetic versions where P is not trivial on K, but other conditions have to be satisfied.

Inseparable local uniformization

M. Temkin has achieved inseparable local uniformization, i.e., local uniformization by purely inseparable alteration. This is, so to say, linearly disjoint from our local uniformization by separable alteration. Does this mean that one can deduce from this the "common denominator": no alteration at all? The answer is *no*; in both cases, the alteration stows away the defect.

Nevertheless, Temkin's result appears to reveal an interesting fact. Only one type of defect, the one that is connected with purely inseparable defect extensions, can be killed by purely inseparable alteration. Hence Temkin's result indicates that the remaining type of defect (the one appearing in Abhyankar's example) is more harmless and can be dealt with.
Unfortunately, we have not succeeded to read off from Temkin's paper how this can be done with our purely valuation theoretical methods.

A possible direction for future research

One possible direction for future research is to try to refine the valuation theoretical approach by replacing the use of separably tame fields by that of a larger class of valued fields which admit only the more harmless defect in its finite extensions. One such class are the deeply ramified fields whose valuation theory has recently been studied in detail.

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