SUBFIELDS OF ALGEBRAICALLY MAXIMAL KAPLANSKY FIELDS

F.-V. KUHLMANN

Abstract. Using the ramification theory of tame and Kaplansky fields, we show that maximal Kaplansky fields contain maximal immediate extensions of each of their subfields. Likewise, algebraically maximal Kaplansky fields contain maximal immediate algebraic extensions of each of their subfields. This study is inspired by problems that appear in henselian valued fields of rank higher than 1 when a Hensel root of a polynomial is approximated by the elements generated by a (transfinite) Newton algorithm. The main result of this article has been applied in the theory of separated (also called vector space defectless) extensions of valued fields.

1. Introduction

For real functions, the Newton Algorithm is a nice tool to approximate their zeros. An analogue works in valued fields, such as the fields of p-adic numbers. Take a complete discretely valued field K with valuation ring $\mathcal{O}$ and a polynomial f in one variable with coefficients in $\mathcal{O}$. If $b \in \mathcal{O}$ satisfies $vf(b) > 2vf'(b)$, then set $x_0 := b$ and

$$x_{i+1} := x_i - \frac{f(x_i)}{f'(x_i)}$$

for $i \geq 0$. It can easily be shown, using the Taylor expansion of f, that this sequence is a Cauchy sequence, and if a limit exists, then it is a root of f. This fact can be used to prove Hensel's Lemma in the

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complete fields of p-adic numbers, which states the existence of p-adic roots of suitable polynomials.

Hensel's Lemma also holds in other valued fields (such as power series fields with arbitrary ordered abelian groups of exponents), which can be much larger than the fields of p-adic numbers. Still, it can be proved using the Newton algorithm, but if the value group of the field is not archimedean ordered (i.e., if it is not an ordered subgroup of $\mathbb{R}$), then the algorithm may not deliver the root after the first ω many iterations, and transfinite induction is needed. For those who do not like the technicalities of transfinite induction, S. Prieß-Crampe presented in 1990 an elegant alternative (see [11]). The setting of the Newton algorithm gives rise to a contracting function

$$\Phi(x) \;:=\; x \;-\; \frac{f(x)}{f'(x)}$$

defined on a suitable subset of the field, whose unique fixed point, if it exists, is a root of the polynomial. The existence is guaranteed if f satisfies the assumptions of Hensel's Lemma and the valued field from which its coefficients are taken is spherically complete (which is a property shared by all power series fields). While it is not the original definition, it suffices here to say that a valued field is spherically complete if and only if every pseudo Cauchy sequence in the sense of [4] has a (pseudo) limit.

The members of the sequence obtained from the Newton algorithm are approximations to a root of f. If the value group of the field in which we are working is not archimedean ordered, then the problem can appear that the sequence is not Cauchy, but only a pseudo Cauchy sequence. The limits of pseudo Cauchy sequences which are not Cauchy are not uniquely determined. In a recent paper [12], Prieß-Crampe considers the question whether this situation can be repaired, for a given polynomial f, by passing to a suitable subfield of K in which the sequence becomes Cauchy and has a limit. She proves that this can indeed be done when K is a spherically complete Kaplansky field. Let us provide the necessary definitions and background.

We will work with (Krull) valuations v and write them in the classical additive way, that is, the value group of v on a field K, denoted by vK, is an additively written ordered abelian group, and the ultrametric triangle law reads as $v(a+b) \geq \min\{va, vb\}$. We denote by Kv the residue field of v on K, by va the value of an element $a \in K$, and by av its residue.

A polynomial f over a field of characteristic $p > 0$ is called **additive** if $f(a+b) = f(a) + f(b)$ for all elements a, b in any extension of the field. This holds if and only if f is of the form $\sum_{0\leq i\leq n} c_i X^{p^i}$. Following Kaplansky, we call a polynomial g a **p-polynomial** if $g = f + c$ where f is an additive polynomial and c is a constant.

A valued field (K, v) is called a **Kaplansky field** if $\mathrm{char}\, Kv = 0$ or if it satisfies **Kaplansky's hypothesis A**: for $\mathrm{char}\, Kv = p > 0$,

(A1) every p-polynomial with coefficients in Kv has a zero in Kv,

(A2) vK is p-divisible.

Prieß-Crampe proves her main result in [12] by a direct application of this formulation of hypothesis A. While this proof is not without interest, a more modern approach can lead to more insight, as we wish to show in the present paper. A lot of important work has been done over time by several authors in order to better understand Kaplansky's hypothesis A, e.g. by G. Whaples in [14], by F. Delon in her thesis, and in the paper [10]. The final touch to this development was given by Kaplansky himself, in cooperation with D. Leep (see [5, Section 9]). We will use the following characterization. By Theorem 1 of [14] as well as the other cited sources, hypothesis A is equivalent to the conjunction of the following three conditions, where p denotes the characteristic of the residue field:

(K1) if $p > 0$, then the value group is p-divisible,

(K2) the residue field is perfect,

(K3) the residue field admits no finite separable extension of degree divisible by p.

A valued field is **henselian** if it satisfies Hensel's Lemma, or equivalently, admits a unique extension of its valuation to its algebraic closure. Note that every algebraic extension of a henselian field is again henselian, with respect to the unique extension of the valuation. A **henselization** of a valued field (K, v) is an algebraic extension which is henselian and minimal in the sense that it can be embedded over K in every other henselian extension field of (K, v). Henselizations exist for every valued field (K, v), in fact, every henselian extension field of (K, v) contains a henselization of (K, v). Henselizations are unique up to valuation preserving isomorphism over K. Therefore, we will speak of "the henselization of (K, v)" and denote it by (K^h, v).

An extension $(L|K, v)$ of valued fields is called **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto, or in other words, value group and residue field remain unchanged under the extension. A valued field is called **maximal** if it does not admit any nontrivial immediate extensions; by [4, Theorem 4], this holds if and only if it is spherically complete. A valued field is called **algebraically**

maximal if it does not admit any nontrivial immediate algebraic extension. Since henselizations are immediate algebraic extensions, every algebraically maximal field is henselian.

The notion of "purely wild extension" (an algebraic extension of a henselian field that is linearly disjoint from all tame extensions) was introduced in [10]. We will give the precise definitions for both types of extensions in the Preliminaries Section.

The following is our main theorem:

Theorem 1.1. *Take a valued field extension $(L|K,v)$, with (L,v) an algebraically maximal Kaplansky field. Then L contains a maximal immediate algebraic extension of (K,v), as well as a maximal purely wild extension of the henselization of (K,v) inside of (L,v).*

If in addition (L,v) is maximal, then it also contains a maximal immediate extension of (K,v).

This theorem is used to prove Theorem 4.9 in [1], which states:

Assume that (K,v) is a Kaplansky field and that (L,v) is an algebraic extension of (K,v). If $L|K$ is linearly disjoint from every immediate extension $M|K$ (in every common field extension), then the extension $(L|K,v)$ is vector space defectless, i.e., every finitely generated K-vector subspace of L admits a basis $b_1,\ldots,b_n$ such that for any choice of $c_i\in K$,

$$v\sum_{i=1}^{n} c_ib_i \;=\; \min_{1\le i\le n} v(c_ib_i)\,.$$

Modulo the fact that a valued field is maximal if and only if it is spherically complete, the last assertion of Theorem 1.1 is proved in [12, Theorem 3.5]. As already indicated, we present a different proof, which will be based on the ramification theory of Kaplansky fields and tame fields. A crucial tool in this proof is the following analogue of Lemma 3.7 of [9], which deals with the case of tame fields. Interestingly, in the case of algebraically maximal Kaplansky fields we do not need the assumption of that lemma that the residue field extension $Lv|Kv$ be algebraic. Note that this assumption also guarantees that vK is pure in vL, but without it, this cannot even be achieved when $\mathrm{char}\,Kv=0$ (see [6, Example 3.9]).

Proposition 1.2. *Take an algebraically maximal Kaplansky field (L,v) and let K be a relatively algebraically closed subfield of L. Then also (K,v) is an algebraically maximal Kaplansky field, with its residue field is relatively algebraically closed in that of L.*

These results are a nice complement to the theory of tame fields as developed in [9]. The proofs of the above results will be given in Section 3, along with some more facts about algebraically maximal Kaplansky fields.

2. Preliminaries

We start with the following well known fact:

Lemma 2.1. *If (L, v) is henselian and K a relatively algebraically closed subfield of L, then also (K, v) is henselian.*

Proof. Assume that f is a polynomial with coefficients in the valuation ring of (K, v) which satisfies the conditions of Hensel's Lemma. Since the valuation ring of (K, v) is contained in that of (L, v) and (L, v) is henselian, f admits a root a in L which satisfies the assertions of Hensel's Lemma. Being a root of f, a is algebraic over K and since K is relatively algebraically closed in L, we have that $a \in K$. □

Take a finite extension $(L|K, v)$ of valued fields. The **Lemma of Ostrowski** says that whenever the extension of v from K to L is unique, then

$$[L : K] \; = \; p^\nu \cdot (vL : vK) \cdot [Lv : Kv] \quad \text{with } \nu \geq 0 \,, \tag{1}$$

where p is the **characteristic exponent** of Lv, that is, $p = \operatorname{char} Lv$ if this is positive, and $p = 1$ otherwise. For the proof, see [13, Théorème 2, p. 236]) or [15, Corollary to Theorem 25, p. 78]). If $p^\nu = 1$, then we say that the extension $(L|K, v)$ is **defectless**. Note that $(L|K, v)$ is always defectless if $\operatorname{char} Kv = 0$. We call a henselian field a **defectless field** if all of its finite extensions are defectless. Each valued field of residue characteristic 0 is a defectless field. The Lemma of Ostrowski shows that every henselian defectless field is algebraically maximal; however, the converse does not hold (see [7, Theorem 3.26]).

An algebraic extension $(L|K, v)$ of henselian fields is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) the ramification index $(vE : vK)$ is not divisible by $\operatorname{char} Kv$,

(TE2) the residue field extension $Ev|Kv$ is separable,

(TE3) the extension $(E|K, v)$ is defectless.

A **tame valued field** (in short, **tame field**) is a henselian field for which all algebraic extensions are tame. From the definition of a tame extension it follows that (K, v) is a tame field if and only if it is a defectless field satisfying conditions (K1) and (K2). Indeed, it is an easy observation that every extension of a defectless field (K, v) satisfying conditions (K1) and (K2) must be tame and so (K, v) must be a tame field. The converse can be derived from the arguments in the second part of the proof of Proposition 3.1 below.

By (K1) and (K2), the perfect hull of a tame field is an immediate extension, and by (TE3), this extension must be trivial. This shows that every tame field is perfect.

If char $Kv = 0$, then conditions (TE1) and (TE2) are void, and every finite extension of (K, v) is defectless. Hence every algebraic extension of a henselian field of residue characteristic 0 is a tame extension, and every henselian field of residue characteristic 0 is a tame field.

While "algebraically maximal" does in general not imply "defectless" as we have remarked above, Theorem 3.2 of [9] shows that a valued field is tame if and only if it is algebraically maximal and satisfies conditions (K1) and (K2). Together with the facts about tame fields that we have mentioned above, this implies:

Lemma 2.2. *Every algebraically maximal valued field satisfying conditions (K1) and (K2), and in particular every algebraically maximal Kaplansky field, is a tame field and hence a perfect and defectless field.*

The converse of this lemma does not hold: as noted above, an algebraically maximal field that satisfies conditions (K1) and (K2) (such as the power series field $\mathbb{F}_p((\mathbb{Q}))$) is a tame field, but its residue field may have finite separable extensions of degree divisible by p.

Take a valued field (K, v), fix an extension of v to the separable-algebraic closure K^{sep} of K and call it again v. The fixed field of the closed subgroup

$$G^r := \{\sigma \in \mathrm{Gal}\,(K^{\mathrm{sep}}|K) \mid v(\sigma a - a) > va \text{ for all } a \in \mathcal{O}_{K^{\mathrm{sep}}} \setminus \{0\}\}$$

of $\mathrm{Gal}(K^{\mathrm{sep}}|K)$ (cf. [2, Corollary (20.6)]) is called the **absolute ramification field of** (K, v). For the following fact, see [2, (20.15 b)].

Lemma 2.3. *Take a henselian field (K, v) and denote by Z the absolute ramification field of (K, v). If (K', v) is an extension of (K, v) inside of Z, then Z is also the absolute ramification field of (K', v).*

The next result follows from [2, Theorem (22.7)] (see also [10, Proposition 4.1]).

Proposition 2.4. *The absolute ramification field Z of a henselian field (K, v) is the maximal tame extension of (K, v), that is, every tame extension of (K, v) lies in Z. Hence (K, v) is tame if and only if Z is algebraically closed.*

An algebraic extension of a henselian field is called **purely wild** if it is linearly disjoint from the absolute ramification field and thus from every tame extension. If (K,v) is tame, then it does not admit any nontrivial purely wild extension. In [10] it was shown that maximal purely wild extensions are field complements to the absolute ramification field. In the case of Kaplansky fields, they are at the same time maximal immediate algebraic extensions, and they are unique up to isomorphism. See the cited paper for details.

Lemma 2.5. *Every immediate algebraic extension of a henselian field is purely wild.*

Proof. By [9, Lemma 2.6], every immediate algebraic extension of a henselian field is linearly disjoint from every tame extension, as every finite subextension of the latter is defectless. □

Finally, we will need the following results about the absolute ramification field.

Lemma 2.6. *Take an algebraic extension $(L|K,v)$ of henselian fields. Denote by Z the absolute ramification field of (K,v). Then $(Z\cap L,v)$ is a tame extension of (K,v) maximal with respect to being contained in L, and (L,v) is a purely wild extension of $(Z\cap L,v)$.*

Proof. Write $Z_0 := Z\cap L$. By Proposition 2.4, (Z_0,v) is a tame extension of (K,v). Assume that (E,v) is a tame extension of (K,v) inside of (L,v). Applying Proposition 2.4 shows that E lies in Z and hence in $Z\cap L = Z_0$. This proves that (Z_0,v) is a tame extension of (K,v) maximal with respect to being contained in L.

Now we wish to prove that $(L|Z_0,v)$ is a purely wild extension. Lemma 2.3 shows that Z is also the absolute ramification field of (Z_0,v). Hence it suffices to show that $L|Z_0$ is linearly disjoint from $Z|Z_0$. Since $Z_0 = Z\cap L$, this follows from the fact that according to [3, Theorem 5.3.3 (2)], $Z|Z_0$ is a Galois extension. □

3. Results on algebraically maximal Kaplansky fields

We start with the following characterization of algebraically maximal Kaplansky fields:

Proposition 3.1. *A valued field (K,v) is an algebraically maximal Kaplansky field if and only if it is henselian and does not admit any finite extension of degree divisible by* $\mathrm{char}\,Kv$.

Proof. If $\mathrm{char}\,Kv = 0$, then the assertion is trivial since in this case, (K,v) is an algebraically maximal Kaplansky field if and only if it is henselian, and 0 does not divide the degree of any finite extension.

Now we consider the case of $\operatorname{char} Kv = p > 0$. Assume first that (K,v) is an algebraically maximal Kaplansky field. Then in particular, it is henselian. By Lemma 2.2, it is a defectless field, that is, every finite extension $(K'|K,v)$ satisfies

$$[K':K] \;=\; (vK':vK)\cdot[K'v:Kv]\,.$$

Since (K,v) satisfies (K1), $(vK':vK)$ is not divisible by p. Because of (K2) and (K3), the same holds for $[K'v:Kv]$. Hence also $[K':K]$ is not divisible by p.

Now assume that (K,v) is henselian and does not admit any finite extension of degree divisible by p. Pick any finite extension $(K'|K,v)$. By the Lemma of Ostrowski,

$$[K':K] \;=\; p^{\nu}\cdot(vK':vK)\cdot[K'v:Kv]$$

with $\nu\geq 0$. Since $[K':K]$ is not divisible by p, it follows that $\nu=0$. This implies that (K,v) is a defectless field and thus also an algebraically maximal field. Further, if there was an element $\alpha\in vK$ not divisible by p, then adjoining to K the p-th root of any element of K having value α would generate an extension of K of degree p. This shows that vK must be p-divisible, i.e., (K,v) satisfies (K1). Finally, if there was a finite extension of Kv with a degree divisible by p, then it could be lifted to an extension of K of the same degree, which should not exist. This shows that Kv does not admit any finite extension of degree divisible by p, so (K,v) satisfies (K2) and (K3). □

Proof of Proposition 1.2. Since (L,v) is algebraically maximal, it is henselian, and by Lemma 2.1, also (K,v) is henselian. Hence if $\operatorname{char} Kv=0$, then Proposition 3.1 proves the first assertion.

Let us now assume that $\operatorname{char} Kv = p > 0$. Since L is perfect by Lemma 2.2, the same holds for its relatively algebraically closed subfield K. Take any finite extension $K'|K$. Since K is perfect and relatively algebraically closed in L, $K'|K$ is linearly disjoint from $L|K$. Therefore, $[K':K]=[L.K':L]$, which according to Proposition 3.1 is not divisible by p. Using Proposition 3.1 again, we conclude that (K,v) is also an algebraically maximal Kaplansky field.

By what we have shown so far, K is perfect, hence so is Kv. Suppose that Kv is not relatively algebraically closed in Lv. Then there exists some $\bar{a}\in Lv\setminus Kv$ separable-algebraic over Kv. Lift its minimal polynomial $\bar{f}$ over Kv up to a polynomial f over K of the same degree. Since $\bar{a}$ is a simple root of $\bar{f}$, Hensel's Lemma yields the existence of a root $a\in L$ of f with $av=\bar{a}$. Then a is algebraic over, but not in

K, which contradicts our assumption that K is relatively algebraically closed in L. Hence Kv is relatively algebraically closed in Lv. □

Proposition 3.2. *Take an algebraically maximal Kaplansky field* (L, v) *and a henselian subfield* K *such that* $L|K$ *is algebraic. Denote the absolute ramification field of* (K, v) *by* Z *and set* $Z_0 := Z \cap L$. *Then the following assertions hold:*

a) (Z_0, v) *is a tame extension of* (K, v) *maximal with respect to being contained in* L*,*
b) (L, v) *is a maximal purely wild extension of* (Z_0, v)*,*
c) (K3) *also holds for* (Z_0, v)*.*

Proof. The assertions are trivial if $\operatorname{char} Kv = 0$, in which case Z is algebraically closed, $Z_0 = L$, and there are no nontrivial purely wild extensions; so let us assume that $\operatorname{char} Kv = p > 0$.

It follows from Lemma 2.6 that (Z_0, v) is a tame extension of (K, v) maximal with respect to being contained in L, and that $(L|Z_0, v)$ is purely wild. In fact, (L, v) is a maximal purely wild extension of (Z_0, v) since by Lemma 2.2 it is a tame field. Since $(L|Z_0, v)$ is purely wild, the extension of the respective residue fields is purely inseparable. As L satisfies (K3), the same is consequently true for Z_0. □

Proposition 3.3. *Take an algebraically maximal Kaplansky field* (L, v) *and a subfield* K *such that* $L|K$ *is algebraic. Then* (L, v) *contains:*

a) a maximal immediate algebraic extension of (K, v)*,*
b) a henselization (K^h, v) *of* (K, v) *and a maximal purely wild extension of* (K^h, v)*.*

Proof. The algebraically maximal field (L, v) is henselian, hence it contains a henselization K^h of (K, v), and this is an immediate algebraic extension of (K, v). If $\operatorname{char} Kv = 0$, then it is the maximal immediate algebraic extension of (K, v) and the assertions of our proposition are trivial. Therefore, let us assume that $\operatorname{char} Kv = p > 0$; we may also assume that (K, v) itself is henselian.

For the proof of part a), we take (K', v) to be an immediate extension of (K, v), maximal with respect to being contained in L. Then (K', v) is henselian since it is an algebraic extension of the henselian field (K, v). We set $Z_0 := Z \cap L$, where Z is the absolute ramification field of (K', v). Suppose that (K', v) is not algebraically maximal. Then there exists a nontrivial immediate algebraic extension $(K'(a)|K', v)$; let f be the minimal polynomial of a over K'. By Lemma 2.5 the extension is purely wild, that is, it is linearly disjoint from the tame extensions $Z|K'$ and $Z_0|K'$. It follows that on the one hand, f remains the minimal polynomial of a over Z_0, and on the other hand, $Z_0(a)|Z_0$ is linearly disjoint from $Z|Z_0$. By Lemma 2.3, Z is also the absolute ramification field of Z_0, so this shows that $(Z_0(a)|Z_0,v)$ is purely wild. It can be extended to a maximal purely wild extension of (Z_0,v). Since by Proposition 3.2, Z_0 satisfies (K3), all maximal purely wild extensions of (Z_0,v) are isomorphic over Z_0 by [10, Proposition 3.2 and Theorem 4.3]. As (L,v) is a maximal purely wild extension of (Z_0,v) by Proposition 3.2, it follows that $Z_0(a)$ admits an embedding in L over Z_0. Hence there is some $a' \in L$ such that $Z_0(a)$ and $Z_0(a')$ are isomorphic over Z_0, which means that a' is also a root of f. Therefore, the isomorphism induces an isomorphism $K'(a) \simeq K'(a')$ over K'. Since (K',v) is henselian, this isomorphism preserves the valuation, so $(K'(a')|K',v)$ is a nontrivial immediate algebraic extension contained in L. As this contradicts the maximality of (K',v), we find that (K',v) must be algebraically maximal. This proves part a) of our proposition.

In order to prove part b), take (K',v) to be a purely wild extension of K, maximal with respect to being contained in L. Again, (K',v) is henselian. By the same proof as before, just replacing "immediate" by "purely wild", one shows that (K',v) is a maximal purely wild extension of (K,v). Note that in this case, the extension $K'(a')|K'$ obtained in the proof is linearly disjoint from $Z|K'$, and hence purely wild, because a' and a have the same minimal polynomial over K'. □

Proof of Theorem 1.1. In order to prove the first part of the theorem, we use Proposition 1.2 to replace L by the relative algebraic closure of K in L. Then the assertions follow from Proposition 3.3.

For the proof of the second part of the theorem, assume that (L,v) is maximal, and take an immediate extension (K',v) of (K,v), maximal with respect to being contained in L. By the first part of our theorem, the relative algebraic closure of K' in L contains a maximal immediate algebraic extension of (K',v); by the maximality of K', it must be equal to K'. Hence (K',v) does not admit any nontrivial immediate algebraic extensions.

Suppose that (K',v) is not maximal. Then by [4, Theorem 4], (K',v) admits a pseudo Cauchy sequence without a limit in K'. This must be of transcendental type, because if it were of algebraic type, then by [4, Theorem 3], there would exist a nontrivial immediate algebraic extension of (K',v), contrary to what we have shown. Since the pseudo Cauchy sequence we took in K' is also a pseudo Cauchy sequence in (L,v) and (L,v) is maximal, [4, Theorem 4] shows the existence of a limit a in L. It follows from [4, Theorem 2] that $(K'(a)|K',v)$ is an immediate transcendental extension. As it is contained in L, this contradicts the maximality of (K', v). We have now proved that (K', v) is a maximal immediate extension of (K, v). □

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Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland

Email address: fvk@usz.edu.pl