The theory of the defect and its application to the problem of local uniformization, II

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Elimination of Ramification

Our problem:
Given a (separable) function field $F|K$ with a place P and associated valuation v, find a transcendence basis $T \subset \mathcal{O}_P$ such that $F \subseteq K(T)^i$.

A simple example

Assume that the function field satisfies $FP = K$ and has value group $vF = \bigoplus_{1 \le i \le n} \mathbb{Z}$ endowed with an arbitrary ordering. Then we can choose a set $T_0 = \{t_i \mid 1 \le i \le n\} \subset \mathcal{O}_P$ such that $vF = \bigoplus_{1 \le i \le n} \mathbb{Z} vt_i$. We will see later that the elements t_i must be algebraically independent and that $vK(T_0) = vF$ and $K(T_0)P = K = FP$. Hence $(F|K(T_0), v)$ is an immediate extension.

Now there are two cases:

1) $n = \operatorname{trdeg} F|K$, so that $T := T_0$ is a transcendence basis of $F|K$.

2) $n < \operatorname{trdeg} F|K = n'$. In this case we can add $n' - n$ elements $t_{n+1}, \ldots, t_{n'} \in \mathcal{O}_P$ to T_0 to obtain a transcendence basis T.

In both cases, the finite extension $(F|K(T), v)$ will be immediate.

A simple example

We ask: does this imply that F lies in $K(T)^i$? If so, then because $(F|K(T),v)$ is immediate, F will already lie in $K(T)^h$. This is equivalent to $F^h = K(T)^h$. Recall that henselizations are immediate extensions, hence because $(F|K(T),v)$ is immediate, the same holds for the extension

$$(F^h|K(T)^h,v)\,.$$

This is again a finite extension, because by ramification theory, F^h is equal to the compositum $F.K(T)^h$ (which is the smallest subfield of $\tilde{F}$ containing F and $K(T)^h$). Now the question is: does this imply that the extension is trivial? We will see: the answer is YES in case 1), but in general NO in case 2), unless char $Kv = 0$.

The Fundamental Inequality

We will say that a valued field extension $(L|K,v)$ is unibranched if the extension of v from K to L is unique. Now assume that $L|K$ is finite. Then the Fundamental Inequality holds:

$$[L:K] \;\geq\; (vL:vK)\cdot[Lv:Kv]\,.$$

What is the missing factor?

The Lemma of Ostrowski

Set $p = \mathrm{char}\, Kv$ if this is positive, and $p = 1$ otherwise (then p is called the characteristic exponent of Kv). The Lemma of Ostrowski states that

$$[L:K] \;=\; p^{\nu} \cdot (vL:vK) \cdot [Lv:Kv]\,, \tag{1}$$

where ν is a nonnegative integer. The factor

$$\mathrm{d}(L|K,v) \;:=\; p^{\nu}$$

is called the defect of the extension $(L|K,v)$.

The Fundamental Equality

We set
$\mathrm{e}(L|K,v) := (vL : vK)$; this is called the ramification index of the extension,
$\mathrm{f}(L|K,v) := [Lv : Kv]$; this is called the inertia degree.

Then equation (1) can be rewritten as follows:

$$[L:K] \;=\; \mathrm{d}(L|K,v)\cdot \mathrm{e}(L|K,v)\cdot \mathrm{f}(L|K,v)\,. \tag{2}$$

This is called the Fundamental Equality (for unibranched extensions).

Multiplicativity of d, e and f

The defect, ramification index and inertia degree, like the degree of field extensions, are multiplicative: if $(L|K,v)$ and $(N|L,v)$ are finite extensions, then

$$\begin{aligned} \mathrm{d}(N|K,v) &= \mathrm{d}(N|L,v)\cdot\mathrm{d}(L|K,v)\,, \\ \mathrm{e}(N|K,v) &= \mathrm{e}(N|L,v)\cdot\mathrm{e}(L|K,v)\,, \\ \mathrm{f}(N|K,v) &= \mathrm{f}(N|L,v)\cdot\mathrm{f}(L|K,v)\,. \end{aligned}$$

The defect

If $\mathrm{d}(L|K,v) > 1$, then $(L|K,v)$ is called a defect extension.
If $\mathrm{d}(L|K,v) = 1$, then we call $(L|K,v)$ a defectless extension; in this case, equality holds in the Fundamental Inequality.
By the Lemma of Ostrowski, $(L|K,v)$ is always defectless if $\mathrm{char}\, Kv = 0$.

If each extension of the henselization of (K,v) is defectless, then we call it a defectless field (this definition does not depend on the choice of the henselization). Every valued field with residue characteristic 0 is a defectless field.

Note: when we speak of a defect extension of prime degree, we always tacitly assume that it is unibranched (which is actually a consequence in this situation.)

Back to our example

In our example, the extension $(F^h|K(T)^h, v)$ is unibranched, because $(K(T)^h, v)$ is henselian. We have that $\mathrm{e}(F^h|K(T)^h, v) = 1$ and $\mathrm{f}(F^h|K(T)^h, v) = 1$ because the extension is immediate. Hence by the Fundamental Equality,

$$\mathrm{d}(F^h|K(T)^h, v) \;=\; [F^h : K(T)^h]\,.$$

If the extension is not trivial, then it is a defect extension. Consequently, if $\mathrm{char}\, Kv = 0$, then the extension is trivial, and our problem of Elimination of Ramification is solved.

When is $K(T)^h$ a defectless field?

If $K(T)^h$ is a defectless field, then the extension $(F^h|K(T)^h, v)$ is trivial. If char $Kv \neq 0$, when do we still know that $(K(T)^h, v)$ is a defectless field?
The Generalized Stability Theorem which we will discuss in this lecture series tells us that $(K(T)^h, v)$ is a defectless field in case 1), where $n = \operatorname{trdeg} F|K$.
But in case 2), where $n < \operatorname{trdeg} F|K$, the field $(K(T)^h, v)$ will in general not be a defectless field.

Examples of defect extensions

We will now present examples of defect extensions. Throughout, we will deal with valued fields of residue characteristic $p > 0$. We look for defect extensions of degree p. Then by the Fundamental Equality, $\mathrm{e}(F^h|K(T)^h, v) = 1$ and $\mathrm{f}(F^h|K(T)^h, v) = 1$, that is, the extension is immediate.

The example of F. K. Schmidt

We consider the formal Laurent series field $\mathbb{F}_p((t))$ with its canonical t-adic valuation $v = v_t$. (This is the unique valuation that satisfies $vt = 1$, and $(\mathbb{F}_p((t)), v)$ is the completion of $(\mathbb{F}_p(t), v)$.) Since $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose some element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. We have that

$$v\mathbb{F}_p((t)) = \mathbb{Z} = v\mathbb{F}_p(t) \text{ and } \mathbb{F}_p((t))v = \mathbb{F}_p = \mathbb{F}_p(t)v\,,$$

showing that $(\mathbb{F}_p((t))|\mathbb{F}_p(t), v)$ is an immediate extension. The same holds for

$$(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p), v)\,.$$

This extension is purely inseparable of degree p. As all extensions of the valuation v from $\mathbb{F}_p(t,s^p)$ to $\mathbb{F}_p(t,s)$ are conjugate, there is only one extension. Consequently, the defect of this extension is equal to its degree p.

Separable-algebraic closures

Take a nontrivially valued field (K, v) which is not perfect. Then the extension $\tilde{K}|K^{\text{sep}}$ is not trivial. As it is purely inseparable, again the extension of v from K^{sep} to $\tilde{K}$ is unique. We have already seen in our picture of absolute ramification theory that the extension $(\tilde{K}|K^{\text{sep}}, v)$ is immediate. Thus we find that the defect of every finite subextension is equal to its degree. This shows that (K^{sep}, v) is not a defectless field.

It is interesting that it is not hard to show that $(\tilde{K}, v)$ lies in the completion of (K^{sep}, v) (even for arbitrary rank).

Is defect always inseparable?

From our first examples one might conclude that defect extensions are always inseparable. But this is false. If (K,v) admits a purely inseparable defect extension of degree p that does not lie in its completion, then one can transform it into a separable defect extension as follows. Assume that $(K(\eta)|K,v)$ with $\eta^p \in K$ is this defect extension. The minimal polynomial of η is $X^p - \eta^p$. We can make this polynomial separable by adding a summand dX with $0 \neq d \in K$. If η does not lie in the completion of (K,v), then by choosing d with large enough value, we can obtain that for each root ϑ of the new polynomial

$$X^p - dX - \eta^p ,$$

the extension $(K(\vartheta)|K,v)$ is a defect extension such that

$$v(\vartheta - c) \;=\; v(\eta - c) \quad \text{for all } c \in K .$$

Dependent defect

We say that the defect in a separable defect extension of degree p is dependent if it can be derived from a purely inseparable defect extension in the way we have just described.

One may now be tempted to conclude that all defect is dependent, at least for valued fields of positive characteristic. Wrong again. There are perfect fields of positive characteristic that admit defect extensions, as we will see now.

Before we start, we introduce an important class of field extensions in characteristic $p > 0$.

Artin–Schreier extensions

If char $K = p > 0$, then for every $a \in K$ and each root ϑ of the polynomial

$$X^p - X - a\,,$$

the extension $K(\vartheta)|K$ is called an Artin–Schreier extension if it is non-trivial. If so, then it is a Galois extension of degree p because all other roots of the polynomial are $\vartheta + i$ for $i \in \mathbb{F}_p$. Conversely, every Galois extension of degree p of a field of characteristic p is an Artin–Schreier extension.

Independent defect

Let (K_0, v) be a valued field of characteristic $p > 0$ whose value group is not p-divisible. Take $a \in K_0$ such that va is negative and not divisible by p. Let ϑ be a root of the polynomial $X^p - X - a$. Then $v\vartheta = va/p$ and $[K_0(\vartheta) : K_0] = p = (vK_0(\vartheta) : vK_0)$. The Fundamental Inequality shows that $K_0(\vartheta)v = K_0v$ and that the extension of v from K_0 to $K_0(\vartheta)$ is unique. Also the further extension of v to the perfect hull $K_0(\vartheta)^{1/p^\infty} = K_0^{1/p^\infty}(\vartheta)$ of $K_0(\vartheta)$ is unique, as it is a purely inseparable extension. It follows that the extension of v from $K := K_0^{1/p^\infty}$ to $K(\vartheta)$ is unique. On the other hand, $[K(\vartheta) : K] = p$ since the separable extension $K_0(\vartheta)|K_0$ is linearly disjoint from the purely inseparable extension $K|K_0$.

Independent defect

We have already noted that $K(\vartheta)$ is the perfect hull of $K_0(\vartheta)$. Thus the value group $vK(\vartheta)$ is the p-divisible hull of $vK_0(\vartheta) = vK_0 + \mathbb{Z}v\vartheta$. Since $pv\vartheta = va \in vK_0$, this is the same as the p-divisible hull of vK_0, which in turn is equal to vK. The residue field of $K(\vartheta)$ is the perfect hull of $K_0(\vartheta)v = K_0v$. Hence it is equal to the residue field of K. It follows that the Artin–Schreier extension

$$(K(\vartheta)|K, v)$$

is immediate and that its defect is p, equal to its degree.

The field K is perfect, so the extension cannot originate from a purely inseparable defect extension. Here, we speak of independent defect.

Abhyankar's example

In 1956 S. Abhyankar gave the above example in the special case of

$$K = \mathbb{F}_p((t))^{1/p^\infty} \quad \text{and} \quad a = \frac{1}{t}$$

(without talking about the defect at all). For ϑ a root of the polynomial

$$X^p - X - \frac{1}{t},$$

the field $K(\vartheta)$ is a subfield of the field $\mathbb{F}_p((t^{\mathbb{Q}}))$ of all power series with coefficients in $\mathbb{F}_p$ and exponents in $\mathbb{Q}$ (this power series field is also written as $\mathbb{F}_p((\mathbb{Q}))$).

Abhyankar's example

The element ϑ can be represented as a power series

$$\vartheta := \sum_{i=1}^{\infty} t^{-1/p^i} \in \mathbb{F}_p((\mathbb{Q}))\,.$$

Indeed, this is a root of the Artin-Schreier polynomial $X^p - X - \frac{1}{t}$ because

$$\begin{aligned} \vartheta^p - \vartheta - \frac{1}{t} &= \sum_{i=1}^{\infty} t^{-1/p^{i-1}} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1} \\ &= \sum_{i=0}^{\infty} t^{-1/p^i} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1} = 0\,. \end{aligned}$$

Abhyankar's example

This power series expansion for ϑ was presented by Abhyankar. It became famous since it shows that there are elements algebraic over $\mathbb{F}_p(t)$ with a power series expansion in which the exponents do not have a common denominator. This phenomenon does not occur in the case of residue characteristic 0.

(With $p = 2$, the example was also used by I. Kaplansky to show that if his "hypothesis A" is violated, then the maximal immediate extension of a valued field may not be unique up to isomorphism.)

The set $v(\zeta - K)$

If ζ is an element in an arbitrary valued field extenson of (K, v), then we set

$$v(\zeta - K) \;:=\; \{v(\zeta - c) \mid c \in K\}\,.$$

This set has the following properties:

1) $v(\zeta - K) \cap vK$ is an initial segment of vK,

2) if $v(\zeta - K)$ has no maximal element, then $v(\zeta - K) \subseteq vK$,

3) if the extension $(K(\zeta)|K, v)$ is immediate, then $v(\zeta - K)$ has no maximal element.

The set $v(\zeta - K)$

Let us prove assertion 3), which is Theorem 1 in Kaplansky's famous paper "Maximal fields with valuations". Consider $\zeta - c$ for an arbitrary $c \in K$. Since $vK(\zeta) = vK$ by assumption, there is $d_1 \in K$ such that $vd_1 = v(\zeta - c)$. Hence $vd_1^{-1}(\zeta - c) = 0$, and since $K(\zeta)v = Kv$ by assumption, there is $d_2 \in K$ such that $vd_2 = 0$ and $d_2 v = d_1^{-1}(\zeta - c)v$. It follows that $d_2^{-1}d_1^{-1}(\zeta - c)v = 1$. Consequently,

$$v(d_2^{-1}d_1^{-1}(\zeta - c) - 1) \;>\; 0\,,$$

whence

$$v(\zeta - c - d_2 d_1) \;>\; vd_2 d_1 \;=\; v(\zeta - c)\,.$$

After setting $c' := c + d_2 d_1$, we obtain that $v(\zeta - c') > v(\zeta - c)$.

The set $v(\vartheta - K)$

We consider the field $K = \mathbb{F}_p((t))^{1/p^\infty}$ and the element ϑ as in Abhyankar's example. From the power series expansion of ϑ, together with assertion 1), we see that $vK^{<0} = \{\alpha \in vK \mid \alpha < 0\}$ is contained in $v(\vartheta - K)$. We wish to show that equality holds.

We will discuss a more general case.

The set $v(\vartheta - K)$

We take a henselian field (K, v) of characteristic $p > 0$, an element $a \in K$, and a root ϑ of the polynomial $X^p - X - a$.

Suppose that there is some $c \in K$ such that $v(\vartheta - c) \geq 0$. Then also $v(\vartheta - c)^p \geq 0$ and therefore, $v((\vartheta - c)^p - (\vartheta - c)) \geq 0$. On the other hand,

$$b := (\vartheta - c)^p - (\vartheta - c) = \vartheta^p - \vartheta - c^p + c = a - c^p + c \in K .$$

As $vb \geq 0$, we can consider the polynomial $X^p - X - bv$ over Kv. If this does not have a root in Kv, it follows that the extension $(K(\vartheta)|K, v)$ is not immediate. If $X^p - X - bv$ has a root in Kv, then Hensel's Lemma shows that the polynomial $X^p - X - b$ has a root in K; in this case, the extension $(K(\vartheta)|K, v)$ is trivial.

The set $v(\vartheta - K)$

We see that for $(K(\vartheta)|K,v)$ to be a defect extension, it is necessary that

$$v(\vartheta - K) \; \subseteq \; vK^{<0} \, .$$

In particular, we must have that $va < 0$.

A classification of defects

It has been shown that in the case of rank 1, the independent defects are characterized by the equality $v(\vartheta - K) = vK^{<0}$. In arbitrary rank, where vK may have proper nontrivial convex subgroups, independent defects are characterized by the equality

$$v(\vartheta - K) \; = \; vK^{<0} \setminus H$$

for some proper convex subgroup H.

However, all this only works when K has positive characteristic. When (K, v) has mixed characteristic, that is, $\mathrm{char}\, K = 0$ while $\mathrm{char}\, Kv > 0$, our original definition of "dependent defect" does not make sense, as there are no non-trivial inseparable extensions. Nevertheless, recently the classification has been generalized to the case of mixed characteristic.

What about complete valued fields?

Are complete valued fields of rank 1 defectless fields? The answer is YES for complete discrete valued fields, but NO in general.

Take the defect extension $(K(\vartheta)|K,v)$ of Abhyankar's example. Consider the completion (K^c,v) of (K,v). Since every finite extension of a complete valued field is again complete, $(K^c(\vartheta),v)$ is the completion of $(K(\vartheta),v)$; note that the completion of a henselian field is again henselian, so the extension of the valuation v from (K^c,v) to $K^c(\vartheta)$ is unique. Completions are immediate extensions, hence the extension $(K^c(\vartheta)|K(\vartheta),v)$ is immediate. Since $(K(\vartheta)|K,v)$ is immediate and the property "immediate" is transitive, also the extension $(K^c(\vartheta)|K,v)$ is immediate. It follows that $(K^c(\vartheta)|K^c,v)$ is immediate. On the other hand, this extension is non-trivial since $v(\vartheta - K) = vK^{<0}$ shows that $\vartheta \notin K^c$. Therefore, this extension has defect p.

Defects in mixed characteristic

The field $\mathbb{Q}_p$ of p-adic numbers with its p-adic valuation v_p is a defectless field. Nevertheless, there are infinite algebraic extensions of $\mathbb{Q}_p$, making the value group $v_p\mathbb{Q}_p = \mathbb{Z}$ p-divisible while keeping the residue field $\mathbb{F}_p$ unchanged, that are not defectless fields.

Examples of defectless fields

• fields with residue characteristic 0,
• algebraically closed valued fields,
• complete discrete valued fields,
• power series fields $k((\Gamma))$, where k is any field and Γ is any ordered abelian group,
• maximal immediate extensions of arbitrary valued fields.

The latter are called maximal fields. All power series fields are maximal fields, and so is $(\mathbb{Q}_p, v_p)$. Finite extensions of maximal fields are again maximal. By definition, a valued field is maximal if and only if it does not admit non-trivial immediate extensions.

Algebraically maximal fields

A valued field is called algebraically maximal if it does not admit non-trivial immediate *algebraic* extensions. Since henselizations are immediate algebraic extensions, all algebraically maximal fields are henselian. The converse is not true, as several of our examples have shown. All henselian defectless fields are algebraically maximal, but also here, the converse does not hold. (The known counterexamples are quite complicated.) However, for valued fields of residue characteristic 0, these three properties are equivalent.

Characterization of defectless fields

Using the classification of defects, one can prove a useful characterization of defectless fields in positive characteristic:

Theorem (K)

A valued field of positive characteristic is defectless if it is algebraically maximal and every finite purely inseparable extension is defectless.

Note that algebraically maximal fields are easy to construct using Zorn's Lemma, and there is an easy criterion for the second condition to hold:

Characterization of defectless fields

Proposition

If (K, v) is a valued field of characteristic $p > 0$ such that $[K : K^p] < \infty$, then each finite purely inseparable extension of (K, v) is defectless if and only if

$$[K : K^p] \;=\; (vK : pvK) \cdot [Kv : (Kv)^p] \,.$$

This makes it possible to construct examples of defectless fields that are not as big as power series fields or maximal fields.

Tame extensions

A unibranched extension $(L|K, v)$ is called a tame extension if every finite subextension $E|K$ satisfies the following conditions:

(TE1) the ramification index $(vE : vK)$ is not divisible by char Kv,

(TE2) the residue field extension $Ev|Kv$ is separable,

(TE3) the extension $(E|K, v)$ is defectless.

Proposition

For a henselian field, its absolute ramification field is the maximal tame extension.

Tame fields

A henselian valued field is called a tame field if all of its algebraic extensions are tame extensions, or equivalently, its absolute ramification field is algebraically closed. All tame fields are perfect and defectless fields.

Lifting defect extensions

The following fact is very important for the investigation of defect extensions:

Proposition

Take a henselian field (K,v) and a tame extension (N,v) of (K,v). Then for any finite extension $(L|K,v)$,

$$\mathrm{d}(L|K,v) \;=\; \mathrm{d}(L.N|N,v)\,.$$

In particular, (K,v) is a defectless field if and only if (N,v) is.

If we want to investigate a defect extension $(L|K,v)$, then we can consider the extension $(L.K^r|K^r,v)$ which has the same defect. The fact that G^r is a p-group, i.e., $K^{\mathrm{sep}}|K^r$ is a p-extension, implies that $L.K^r|K^r$ is a tower of Galois extensions of degree p and purely inseparable extensions of degree p.

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