The theory of the defect and its application to the problem of local uniformization, III

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Valued function fields

As we have seen in the previous lectures, we are interested in the structure theory of valued function fields $(F|K, v)$. Here, the restriction of v to K may or may not be trivial. In applications to the model theory of valued fields or to arithmetic algebraic geometry, it is usually not trivial.

In contrast, for our problem of local uniformization we start from the situation where the restriction of v to K is trivial. However, we have to deal with intermediate fields of the extension $F|K$ (e.g. those of lower transcendence degree over K), on which the restriction of v will be non-trivial.

Our goals

We would like to understand the variety of all valuations on a function field F that extend a given (trivial or non-trivial) valuation of K. Roughly, this problem can be broken down in two parts:
1) describe all possible extensions from K to $K(T)$, where T is a transcendence basis of $F|K$,
2) describe all possible extensions from $K(T)$ to the algebraic extension F.

The second, the algebraic part of the problem, is more or less taken care of by ramification theory.
One may believe that the first part of the problem can be attacked by induction on the transcendence degree of $F|K$, but that is *very wrong*. Nevertheless, there is extensive literature on the transcendence degree 1 case, while very little has been published on the case of higher transcendence degree.

Approaches to the transcendence degree 1 case

There are three different aspects of our problem:
• construction of extensions to a given function field,
• analysis of a given extension,
• description of all possible extensions.

The third task is relatively easy for algebraic extensions: all of them are conjugate. In the transcendental case, one of the points to start with is by actually clarifying what we mean by "description". It can mean to descibe all possible constructions (at least by their basic properties). It can also mean to look at the entirety of all possible extensions as a set with a certain structure (e.g. topology). In this spirit, we will later introduce the Zariski spaces of all valuations or places of a given function field.

The case of transcendence degree 1

Although a rational function field $K(x)|K$ of transcendence degree 1 is a very simple object, the task of describing all possible extensions of a valuation v from K to $K(x)$ is surprisingly complex. That also depends on how much information one wants to obtain. Mark Spivakovsky for instance sees the description of such extensions as a main key to local uniformization – this explains the name "key polynomials" (just joking).

Approaches to the transcendence degree 1 case

There are several different approaches and tools for the transcendence degree 1 case:

• key polynomials, introduced by MacLane,
• limits of residue-transcendental extensions,
• minimal pairs,
• pseudo Cauchy sequences (defined later), introduced by Ostrowski and Kaplanski,
• approximation types,
• power series,
• generating sequences (often used by algebraic geometers).

These are more or less strongly interrelated, but in certain cases passing from one to the other is technically challenging (example: passing between pseudo Cauchy sequences and generating sequences).

The headache of algebraic geometers

Take a valued function field $(K(x)|K, v)$. It is an important task to compute the value of every element of $K(x)$. It suffices to compute the value of all polynomials in $K[x]$. We know the values of all elements in K. If in addition we know the value vx, then everything would be easy if for every polynomial

$$f(x) = \sum_{i=0}^{n} c_i x^i \in K[x] \tag{1}$$

the following equation would hold:

$$vf(x) = \min_{0 \le i \le n} vc_i + ivx \,. \tag{2}$$

Indeed, we can define valuations on $K(x)$ in this way, choosing vx to be any element in some ordered abelian group which contains vK. If we choose $vx = 0$, we obtain the Gauß valuation.

By the ultrametric triangle law, (2) always holds when the monomials in the polynomial (1) have pairwise distinct values. Unfortunately, this is rarely the case.

If we choose vx in some ordered abelian group containing vK such that it is non-torsion over vK, then all $vx^i = ivx$ lie in distinct cosets modulo vK and therefore, the values vc_ix^i will be pairwise distinct. In this case our problem is solved.

Valuations for which the value of a polynomial (written in a suitable form) is always equal to the minimum of the values of its monomials are called monomial valuations.

The Abhyankar Inequality

Our next goal is to give a basic classification of all possible extensions in the transcendence degree 1 case. We employ a powerful tool that we will also need later in the case of higher transcendence degree.

If Γ is any abelian group, then we denote by $\tilde{\Gamma}$ its divisible hull (which can be represented as $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$). The rational rank of Γ is $\operatorname{rr}\Gamma := \dim_{\mathbb{Q}} \tilde{\Gamma}$. This is the maximal number of rationally independent elements in Γ.

If $(L|K, v)$ is an arbitrary valued field extension of finite transcendence degree, then we have the Abhyankar inequality:

$$\operatorname{trdeg} L|K \;\geq\; \operatorname{trdeg} Lv|Kv \,+\, \operatorname{rr}\,(vL/vK)\,. \qquad (3)$$

The famous Abhyankar inequality is in fact a version of this inequality for the case of noetherian local rings.

Abhyankar valuations and Abhyankar places

We call v an Abhyankar valuation and its associated place P an Abhyankar place if equality holds in (3). In this case we will also say that $(F|K,v)$ is a valued function field without transcendence defect.

As we will see later, all Abhyankar valuations are monomial valuations, and they are very important for our purposes.

Classification in the transcendence degree 1 case

For valued rational function fields $(K(x)|K,v)$, the Abhyankar Inequality reads as follows:

$$1 \;=\; \operatorname{trdeg} K(x)|K \;\geq\; \operatorname{rr} vK(x)/vK \;+\; \operatorname{trdeg} K(x)v|Kv\,. \qquad (4)$$

Thus there are the following mutually exclusive cases:

- $(K(x)|K,v)$ is valuation-algebraic:
$vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic,
- $(K(x)|K,v)$ is value-transcendental:
$vK(x)/vK$ has rational rank 1, but $K(x)v|Kv$ is algebraic,
- $(K(x)|K,v)$ is residue-transcendental:
$K(x)v|Kv$ has transcendence degree 1, but $vK(x)/vK$ is a torsion group.

Classification in the transcendence degree 1 case

We combine the value-transcendental case and the residue-transcendental case by saying that

- $(K(x)|K,v)$ is valuation-transcendental:

$vK(x)/vK$ has rational rank 1 or $K(x)v|Kv$ has transcendence degree 1.

A special case of the valuation-algebraic case is the following:

- $(K(x)|K,v)$ is **immediate**:

$vK(x) = vK$ and $K(x)v = Kv$.

The case of algebraically closed K

If K is algebraically closed, then the residue field Kv is algebraically closed, and the value group vK is divisible. So we see that for an extension $(K(x)|K,v)$ with algebraically closed K, there are only the following mutually exclusive cases:

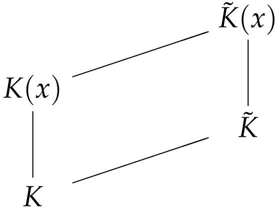
$(K(x)|K,v)$ is immediate: $vK(x) = vK$ and $K(x)v = Kv$,

$(K(x)|K,v)$ is value-transcendental: $\mathrm{rr}\, vK(x)/vK = 1$, but $K(x)v = Kv$,

$(K(x)|K,v)$ is residue-transcendental: $\mathrm{trdeg}\, K(x)v|Kv = 1$, but $vK(x) = vK$.

Passing to the algebraic closure of K

The appearance of non-trivial algebraic residue field extensions or of non-trivial torsion extensions of the value groups (e.g. when the extension is valuation-algebraic but not immediate) poses particular problems.
It is useful to pass to the algebraic closure $\tilde{K}$ of K:



Passing to the algebraic closure of K

If $vK(x)/vK$ is a torsion group, then $v\tilde{K} \subseteq v\tilde{K}(x) \subseteq v\widetilde{K(x)} = v\tilde{K}$, so equality holds.

If $K(x)v/Kv$ is algebraic, then $\tilde{K}v \subseteq \tilde{K}(x)v \subseteq \widetilde{K(x)}v = \tilde{K}v$, so equality holds.

Hence we obtain:

• $(K(x)|K,v)$ is valuation-algebraic if and only if $(\tilde{K}(x)|\tilde{K},v)$ is immediate.

• $(K(x)|K,v)$ is value-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is value-transcendental. In this case, $\tilde{K}(x)v = \tilde{K}v$.

• $(K(x)|K,v)$ is residue-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is residue-transcendental. In this case, $v\tilde{K}(x) = v\tilde{K}$.

Immediate extensions

Immediate extensions are much easier to analyze than valuation-algebraic extensions that are not immediate. Recall that if $(K(x)|K,v)$ is immediate, then the set $v(x-K)$ has no largest element. We are going to unravel the information that is contained in this set.

A sequence $(a_\nu)_{\nu<\lambda}$ of elements in K (where λ is a limit ordinal) is called a pseudo Cauchy sequence if for all $\rho<\sigma<\tau<\lambda$ we have that

$$v(a_\rho - a_\sigma) \;<\; v(a_\sigma - a_\tau)\,.$$

An element a in some valued field extension (L,v) of (K,v) is called a (pseudo) limit of $(a_\nu)_{\nu<\lambda}$ if for all $\rho<\lambda$, we have that

$$v(a-a_\rho) \;=\; v(a_{\rho+1}-a_\rho)\,.$$

Since the values $v(a_\rho - a_\sigma)$ may be bounded from above in vK, limits of pseudo Cauchy sequences are in general not unique.

Remark: Different pseudo Cauchy sequences can have the same limits. Like for Cauchy sequences, one then calls them equivalent. Approximation types correspond to such equivalence classes, and can concretely be thought of as filters consisting of ultrametric balls.

Using the fact that the set $v(x-K)$ has no largest element, one can construct a pseudo Cauchy sequence in K which does not have a limit in K but admits x as a limit. It makes visible the information contained in $v(x-K)$.

Pseudo Cauchy sequences can also be used to construct immediate extensions.

Pseudo Cauchy sequences

A pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ is said to be of transcendental type if for every polynomial f over K, the value $vf(a_\nu)$ is fixed for all large enough ν. Otherwise, $(a_\nu)_{\nu<\lambda}$ is said to be of algebraic type.

Kaplansky shows:

• If $(a_\nu)_{\nu<\lambda}$ is of transcendental type and x is transcendental over K, then there is an extension of v from K to $K(x)$ such that $(K(x)|K, v)$ is immediate and x is a limit of $(a_\nu)_{\nu<\lambda}$.

• If $(a_\nu)_{\nu<\lambda}$ is of algebraic type without a limit in K, f is a polynomial of minimal degree whose value is not ultimately fixed, and a is a root of f, then there is an extension of v from K to $K(a)$ such that $(K(a)|K, v)$ is immediate and a is a limit of $(a_\nu)_{\nu<\lambda}$.

Maximality

Using these facts, one can prove:

• A valued field (K, v) is maximal (i.e., does not admit non-trivial immediate extensions) if and only if every pseudo Cauchy sequence in K has a limit in K.

• A valued field (K, v) is algebraically maximal (i.e., does not admit non-trivial immediate algebraic extensions) if and only if every pseudo Cauchy sequence of algebraic type in K has a limit in K.

Since the algebraically closed field $(\tilde{K}, v)$ is algebraically maximal, every pseudo Cauchy sequence in $\tilde{K}$ without a limit in $\tilde{K}$ must be of transcendental type. In contrast, (K, v) itself may not be algebraically maximal, and x, though being transcendental, may be the limit of a pseudo Cauchy sequence of algebraic type.

Construction of extensions to $K(x)$

We would like to construct extensions of v from K to $K(x)$ for each of the three types with prescribed extensions of value groups and residue fields. First we have to find out about possible constraints.

Theorem

Take any extension of v from K to $K(x)$. Then $vK(x)/vK$ is countable, and $K(x)v|Kv$ is countably generated.

The following is J. Ohm's Ruled Residue Theorem:

Theorem

If $K(x)v|Kv$ is transcendental, then $K(x)v$ is a rational function field in one variable over a finite extension of Kv.

If $y \in K(x)$ is such that vy is non-torsion over vK, then

$$vK(y) \;=\; vK \oplus \mathbb{Z}vy \quad \text{and} \quad K(y)v \;=\; Kv\,.$$

If $y \in K(x)$ is such that $vy = 0$ and yv is transcendental over Kv, then

$$K(y)v \;=\; Kv(yv) \quad \text{and} \quad vK(y) \;=\; vK\,.$$

In both cases, y is transcendental over K. These facts follow from a much more general theorem that we will see later.

Construction of extensions to $K(x)$

As $K(x)|K(y)$ is finite, the Fundamental Inequality shows that the extensions of value group and residue field from $K(y)$ to $K(x)$ can only be finite. This proves:

Theorem

If v is a value-transcendental extension from K to $K(x)$, then the torsion group of $vK(x)/vK$ and the extension $K(x)v|Kv$ are finite.

If v is a residue-transcendental extension from K to $K(x)$, then $vK(x)/vK$ is finite and the relative algebraic closure of Kv in $K(x)v$ is a finite extension.

The implicit constant field

Take any valued rational function field $(K(x)|K,v)$. What is the relative algebraic closure of K in the henselization $K(x)^h$? You may guess it is the henselization K^h of K. *Wrong*, in general it can be larger. We call it the implicit constant field of $(K(x)|K,v)$ and denote it by

$$\mathrm{IC}\,(K(x)|K,v)\,.$$

The implicit constant field

Theorem

Let $(K_1|K,v)$ be a countable separable-algebraic extension of non-trivially valued fields. Then there is an extension of v from K_1 to the algebraic closure $\widetilde{K_1(x)} = \widetilde{K(x)}$ such that, upon taking henselizations in $(\widetilde{K(x)}, v)$,

$$\mathrm{IC}\,(K(x)|K,v) \;=\; K_1^h\,. \tag{5}$$

This is essentially proved by constructing suitable extensions of v from K_1 to $K_1(x)$.

This theorem can be used to prove the following solution of our construction problem.

Construction of extensions to $K(x)$

Theorem

Take any valued field (K,v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0 .

Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0 .

Now assume that v is non-trivial on K and that Γ_0/vK and $k_0|Kv$ are countably generated. Suppose that at least one of them is infinite or that (K,v) admits an immediate transcendental extension. Then there is an extension of v from K to $K(x)$ which has value group Γ_0 and residue field k_0.

Higher transcendence degree

The case of higher transcendence degree is very complex. Although rational function fields are finite extensions, for certain extensions of the valuation the corresponding value group and residue field extensions are not finitely generated. This was already shown by MacLane and Schilling and later by Zariski and Samuel:

Higher transcendence degree

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and k to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation whose restriction to K is trivial, whose value group is Γ and whose residue field is k.

In particular, every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of K can be realized as value group and residue field of a valuation of the rational function field $K(x, y)|K$ whose restriction to K is trivial.

This is a special case of a much more comprehensive theorem that almost completely describes all value groups and residue fields that can be realized by extensions of a valuation to rational function fields of higher transcendence degree.

It can get even worse:

Theorem

Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial, such that $(K(x,y),v)$ admits an infinite tower of immediate Galois extensions of degree p and defect p.

Bourbaki helps

We will now present a result that is very important for us. For the technical, but easy, proof, see Bourbaki, *Commutative Algebra*, Chapter VI, §10.3, Theorem 1.

Let $(L|K, v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.

Bourbaki helps

Moreover, if we write

$$f = \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]$$

in such a way that for every $k \neq \ell$ there is some i such that $\mu_{k,i} \neq \mu_{\ell,i}$ or some j such that $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf = \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} = \min_k vc_k + \sum_{i\in I} \mu_{k,i} vx_i\,. \qquad (6)$$

That is, the value of the polynomial f is equal to the least of the values of its monomials.

In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i \in I} \mathbb{Z} v x_i \\ K(x_i, y_j \mid i \in I, j \in J)v &= Kv\,(y_j v \mid j \in J)\,. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues $y_j v$.

Conversely, if (K, v) is any valued field and we assign to the vx_i any values in an ordered group extension of vK which are rationally independent, then (6) defines a valuation on L, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv.

An important consequence

From this result, together with the Fundamental Inequality, one deduces:

Corollary

Let $(L|K, v)$ be an extension of finite transcendence degree of valued fields. Then the Abhyankar Inequality holds:

$$\operatorname{trdeg} L|K \;\geq\; \operatorname{trdeg} Lv|Kv \,+\, \operatorname{rr}\,(vL/vK)\,. \tag{7}$$

If in addition $L|K$ is a function field and if equality holds in (7), then the extensions $vL|vK$ and $Lv|Kv$ are finitely generated.

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