The theory of the defect and its application to the problem of local uniformization, VI

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Valued function fields: the general case

Assume that $(F|K, v)$ is a valued function field. We are trying to prove that it admits Elimination of Ramification. Thanks to the Generalized Stability Theorem, we succeeded when $(F|K, v)$ is without transcendence defect, i.e., v is an Abhyankar valuation. This has been used to show Local Uniformization for all function fields with Abhyankar places. We now wish to consider the general case. This means we now assume that

$$\operatorname{trdeg} F|K \; > \; \operatorname{rr} vF/vK \, + \, \operatorname{trdeg} Fv|Kv \, .$$

Abhyankar sub-function fields

As before, we set

$$\rho \;:=\; \mathrm{rr}\, vF/vK \quad \text{and} \quad \tau \;:=\; \mathrm{trdeg}\, Fv|Kv\,.$$

Then we choose a set

$$\mathcal{T}_0 \;=\; \{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\} \;\subset\; F$$

such that

• the values $vx_1\,,\ldots,vx_\rho$ are rationally independent over vK,

• the residues $y_1v,\ldots,y_\tau v$ are algebraically independent over Kv.

Abhyankar sub-function fields

We set

$$F_0 \ := \ K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)\,.$$

Then $F_0|K$ is a rational function field of transcendence degree $\rho+\tau$ and v is an Abhyankar valuation on $F_0|K$. Further, $F|F_0$ is a transcendental extension and by our choice of $\mathcal{T}_0$,

- vF/vF_0 is a torsion group,
- $Fv|F_0v$ is algebraic.

Both vF/vF_0 and $Fv|F_0v$ may be non-trivial, even if we replace F_0 by its relative algebraic closure in F; so this idea does not really help.

Reduction to immediate extensions

To the present day, we still do not have the tools to deal with the case where the extension $(F|F_0, v)$ is not immediate. Hence we employ the same approach as in the case of transcendence degree 1 that we discussed earlier: we use the fact that when we extend v to the compositum $F.\widetilde{F_0}$, we find that the extension

$$(F.\widetilde{F_0}|\widetilde{F_0}, v)$$

is immediate. Roughly speaking, the idea is now to use this extension to prove Elimination of Ramification and then collect the finitely many elements from $\widetilde{F_0}$ that are needed to write down the generators that give us the inertial generation of the function field. (This is again the "field of definition" approach.) This means that, if we are successful, we arrive at Elimination of Ramification (and based on it, Local Uniformization) for a finite extension of the function field (alteration).

How much alteration is needed?

Two questions arise:

(a) Do we really have to go up to the algebraic closure which has divisible value group and algebraically closed residue field? Or can we control the alteration in such a way that the corresponding extensions of value group and residue field are confined to, in some sense, the minimal damage we would expect in positive characteristic?

For the residue field Fv, this would mean that the extensions we are willing to accept are the purely inseparable ones. Analogously, for the value group vF the extensions we are willing to accept are those generated by dividing elements by powers of p.

How much alteration is needed?

(b) In the case where the immediate extension $(F.\widetilde{F_0}|\widetilde{F_0}, v)$ has transcendence degree greater than 1, what further alteration will we possibly need if we proceed by induction on the transcendence degree? The tools that have been developed to the present day for dealing with immediate extensions, in essence work only for simple extensions that are algebraic or of transcendence degree 1.

If $t \in F$ is transcendental over F_0 and F is still transcendental over $F_0(t)$, will we then have to pass to the extension $F.\widetilde{F_0(t)}|\widetilde{F_0(t)}$ or can we do better?

The role of separably tame fields

The best answer that has been found so far is that instead of algebraic closures we can use separably tame fields. Recall that a henselian valued field is called a tame field if its absolute ramification field is algebraically closed, and separably tame field if its absolute ramification field is separable-algebraically closed. The following lemma shows that working with these fields can help limit the alteration to the minimal damage we have discussed:

Lemma

Take a valued field (L, v) of residue characteristic $p > 0$. Then there is an algebraic extension (or separable-algebraic extension) (L', v) which is a tame field (or separably tame field, respectively) such that vL'/vL is a p-group and $L'v|Lv$ is purely inseparable.

Another crucial lemma

Assume that we have, using the previous lemma, replaced (F, v) by a tame or separably tame extension field (F', v). What about corresponding extensions of (F_0, v), $(F_0(t), v)$, and other intermediate fields of $(F|F_0, v)$? An answer is provided by the following lemma:

Lemma

Let (L, v) be a separably tame field and $K \subset L$ a relatively algebraically closed subfield of L. If the residue field extension $Lv|Kv$ is algebraic, then (K, v) is also a separably tame field and moreover, vL/vK is torsion free and $Kv = Lv$. The same holds for "tame" in place of "separably tame".

Note that the condition that $Lv|Kv$ is algebraic is necessary.

Another crucial lemma

Since $Fv|F_0v$ is algebraic and $F'|F$ is an algebraic extension, also $F'v|F_0v$ is algebraic. Hence for every intermediate field E of $F'|F_0$, also $F'v|Ev$ is algebraic, and consequently, the relative algebraic closure E' of E in F' will be a tame field (or separably tame field, respectively) with $E'v = F'v$. Moreover, as vF/vF_0 and hence also vF'/vF_0 are already torsion groups, we will have that $vF' = vE'$.

Induction on the transcendence degree

Using the extension to tame or separably tame fields, we have achieved the reduction to immediate extensions. However, as we now have to deal with alteration, the reduction to rank 1 is more intricate than in the proof of the GST. Our induction will have to proceed from the top to the bottom.

Assume that $\{t_1, \ldots t_k\}$ is a transcendence basis of $F|F_0$. For $1 \le i \le k-1$, we take F'_i to be the relative algebraic closure of $F_i := F_0(t_1, \ldots, t_i)$ in F'. Then all (F'_i, v) are tame fields (or separably tame fields, respectively).

Induction on the transcendence degree

Let us assume that we can achieve Elimination of Ramification for all immediate valued function fields of transcendence degree 1 over tame (or separably tame) ground fields. Then we first apply this to the function field

$$(F.F'_{k-1}|F'_{k-1},v)\,.$$

We collect the finitely many elements in F'_{k-1} that are needed for this elimination and also for generating F over $F_0(t_1,\ldots,t_k)$. They lead to a finite extension F''_{k-1} of F_{k-1} .

Induction on the transcendence degree

Now we repeat the procedure with the function field

$$(F''_{k-1}.F'_{k-2}|F'_{k-2}, v) .$$

In this way we work our way downward through the transcendence basis until we arrive at an algebraic extension of F_0 as the ground field. Adjoining all necessary elements to F_0, we then obtain a function field over K with an Abhyankar valuation. At that point we use the GST to finish the procedure.

The case of transcendence degree 1

The remaining case of transcendence degree 1 is covered by the Henselian Rationality Theorem (HRT). Recall that a function field $(F|K,v)$ is henselized rational if F admits a transcendence basis $\mathcal{T}$ such that $F \subset K(\mathcal{T})^h$. Recall also that if $(F|K,v)$ is immediate and a henselized inertially generated function field, then it is already henselized rational since there is no non-trivial residue field extension.

Theorem (Henselian Rationality)

Let (K,v) be a tame field and $(F|K,v)$ an immediate function field. If its transcendence degree is 1, then $(F|K,v)$ is henselized rational.

In the general case of transcendence degree ≥ 1, given any immediate extension (N,v) of (F,v) which is a tame field, there is a finite immediate extension (F_1,v) of (F,v) within (N,v) such that $(F_1|K,v)$ is henselized rational.

Separably tame ground fields

If our ground field is "only" a separably tame field, then we have to require that the function field $F|K$ is separable. This is automatic in the case of a tame ground field, since every tame field is perfect.

Theorem

Let (K, v) be a separably tame field and $(F|K, v)$ an immediate function field, with $F|K$ a separable extension. If its transcendence degree is 1, then $(F|K, v)$ is henselized rational.

In the general case of transcendence degree ≥ 1, given any immediate separable extension (N, v) of (F, v) which is a separably tame field, there is a finite immediate separable extension (F_1, v) of (F, v) within (N, v) such that $(F_1|K, v)$ is henselized rational.

Applications

Together with the GST, the HRT has found two major applications:

1) It has been shown that Elimination of Ramification and Local Uniformization by alteration can be achieved for arbitrary valued function fields. There is also a version of Local Uniformization by alteration in arithmetic algebraic geometry, which uses the versions of both theorems for mixed characteristic.

2) Model theoretic results have been proven for tame and for separably tame fields: Ax–Kochen–Ershov principles (also known as relative completeness and relative model completeness), relative decidability. However, relative quantifier elimination for tame fields is still an open problem.

Separable Local Uniformization

Local Uniformization by alteration actually follows from J. de Jong's Resolution of Singularities by alteration. However, we have provided a purely valuation theoretical proof that also allows for a more detailed description of the alteration. Thanks to the use of separably tame fields, it can be shown that the alteration, that is, the finite extension of the function field, can be taken to be separable and

- either Galois,
- or purely wild, so that the minimal damage conditions on value group and residue field extensions are met.

M. Temkin has achieved “Inseparable Local Uniformization”, i.e., Local Uniformization by purely inseparable alteration. This is, so to say, linearly disjoint from Local Uniformization by separable alteration. Does this mean that one can deduce from this the “common denominator”: no alteration at all? The answer is *no*; in both cases, the alteration stows away the defect.

Nevertheless, Temkin’s result appears to reveal an interesting fact. Only dependent defect, the one that is connected with purely inseparable defect extensions, can be killed by purely inseparable alteration. Hence Temkin’s result indicates that independent defect is more harmless and can be dealt with. Unfortunately, we have not succeeded to read off from Temkin’s paper how this can be done with our purely valuation theoretical methods.

A possible direction for future research

One possible direction for future research is to try to refine the valuation theoretical approach by replacing the use of separably tame fields by that of a larger class of valued fields that admit only independent defect in its algebraic extensions. One such class are the deeply ramified fields whose valuation theory has recently been studied in detail. Note that all perfect valued fields of positive characteristic are deeply ramified fields.

One important question is whether the model theoretic results on tame and separably tame fields can be generalized to deeply ramified fields. It has been conjectured that the perfect hull of $\mathbb{F}_p((t))$ has a decidable elementary theory, but so far no proof has been given. As this field is perfect of characteristic $p > 0$, it is a deeply ramified field.

Two special cases

Let us discuss two special cases in which the proof of the HRT is easy. A finitely ramified field is a valued field of mixed characteristic whose value group has a smallest element α and there is a natural number e such that $e\alpha = vp$.

Theorem

Let (K, v) be a valued field of residue characteristic 0 or a finitely ramified field. Then every immediate function field over (K, v) is henselized rational.

Two special cases

For the proof, take an immediate function field $(F|K,v)$ and an arbitrary transcendence basis $\mathcal{T}$ of $F|K$. Then also $(F|K(\mathcal{T}),v)$ is immediate. Hence

$$(F^h|K(\mathcal{T})^h,v) \tag{1}$$

is an immediate algebraic extension.
If the residue characteristic of (K,v) is 0 or if (K,v) is finitely ramified, then the same holds for every immediate extension; in particular, it holds for $(K(\mathcal{T})^h,v)$. We know that every valued field of residue characteristic 0 is a defectless field, and the same is known for finitely ramified fields. Consequently, the immediate unibranched extension (1) must be trivial, which shows that $F \subset F^h = K(\mathcal{T})^h$.

Two special cases

For the next theorem, note that the completion K^c of a henselian field (K, v) is again henselian (see Warner, *Topological fields*, Theorem 32.10). Hence henselizations of any subfields of the completion can be taken inside of it.

Theorem

Let (K, v) be a henselian field of arbitrary characteristic. If the valued function field $(F|K, v)$ is a separable subextension of the extension $(K^c|K, v)$, then $(F|K, v)$ is henselized rational. More precisely, $F \subset K(\mathcal{T})^h$ for every separating transcendence basis $\mathcal{T}$ of $F|K$.

For the proof, let $\mathcal{T}$ be a separating transcendence basis of $F|K$. Then $F.K(\mathcal{T})^h|K(\mathcal{T})^h$ is a separable-algebraic subextension of $K^c|K(\mathcal{T})^h$. This extension must be trivial since a henselian field is relatively separable-algebraically closed in its completion (see Warner, *Topological fields*, Theorem 32.19). We found that $F.K(\mathcal{T})^h = K(\mathcal{T})^h$, which shows that $F \subset K(\mathcal{T})^h$.

We will now sketch parts of the proof of the transcendence degree 1 case of these two theorems. We have to work with immediate function fields $(F|K,v)$ of transcendence degree 1 over a tame or separably tame ground field (K,v). Pick an element $x \in F \setminus K$. Recall that then x is a (pseudo) limit of a pseudo Cauchy seqence $(a_\nu)_{\nu<\lambda}$ in K that does not have a limit in K. Here, λ is a limit ordinal and by definition,

$$\rho < \sigma < \tau < \lambda \quad \text{implies that} \quad v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)\,.$$

Recall that $(a_\nu)_{\nu<\lambda}$ is of transcendental type if for every polynomial $f \in K[X]$, the value $vf(a_\nu)$ is ultimately fixed. This property is crucial for our proof.

The case of tame ground fields

All tame fields are henselian and defectless, hence algebraically maximal. This implies that every pseudo Cauchy sequence in K without a limit in K is of transcendental type. This is because a pseudo Cauchy sequence in K without a limit in K of algebraic type gives rise to a non-trivial immediate algebraic extension (Kaplansky, *Maximal Fields with valuations*, Theorem 3).

The case of separably tame ground fields

Separably tame fields in general are not algebraically maximal. All of their separable-algebraic extensions are defectless, so they are separable-algebraically maximal. But they may have purely inseparable defect extensions.

Suppose that (K, v) is a separably tame field and that x is limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in K without a limit in K that is of algebraic type. Then there is an immediate algebraic extension $(K(a)|K, v)$ with a being a limit of $(a_\nu)_{\nu<\lambda}$. Any non-trivial separable subextension of $(K(a)|K, v)$ would be defectless, so that the extension could not be immediate. Therefore, $K(a)|K$ must be purely inseparable.

We wish to show that $(a_\nu)_{\nu<\lambda}$ must be a Cauchy sequence. Suppose that this is not the case. Then a does not lie in the completion of (K,v) and it is easy to derive from $(K(a)|K,v)$ a purely inseparable defect extension of degree $p = \mathrm{char}\, K$ that does not lie in the completion of (K,v). Recall that from such an extension we can construct a separable defect extension of degree p; this is done by replacing a minimal polynomial $X^p - c$ by $X^p - dX - c$ with vd large enough. However, as (K,v) does not admit non-trivial immediate separable-algebraic extensions, this leads to a contradiction. We have thus shown that $(a_\nu)_{\nu<\lambda}$ must be a Cauchy sequence.

The case of separably tame ground fields

Now we know that both x and a are limits of the same Cauchy sequence. A naive argument would say: *so they must be equal*, which then yields a contradiction since x is transcendental and a is algebraic over K. However, we have to be very careful with such an argument. In order to compare the two elements they must both be elements in some valued field extension of (K, v). Let us extend v to a valuation $\tilde{v}$ of the algebraic closure of $K(x)$. This contains a, and as $K(a)|K$ is purely inseparable, the unique extension of v from K to $K(a)$ must coincide with the restriction of $\tilde{v}$.

Now we are able to derive information about the value $v(x-a)$. Since both x and a are limits of $(a_\nu)_{\nu<\lambda}$, we have that for all $\nu < \lambda$,

$$v(x-a_\nu) \;=\; v(a_{\nu+1}-a_\nu) \;=\; v(a-a_\nu)$$

and therefore,

$$v(x-a) \;\geq\; v(a_{\nu+1}-a_\nu)\,.$$

Since $(a_\nu)_{\nu<\lambda}$ is a Cauchy sequence, the sequence $(v(a_{\nu+1}-a_\nu))_{\nu<\lambda}$ is cofinal in vK, we find that $v(x-a) > vK$. This implies that $vK(x,a)/vK$ is not a torsion group, and since $K(x,a)|K(x)$ is algebraic, it follows that also $vK(x)/vK$ is not a torsion group. This contradicts our assumption that the extension $(K(x)|K,v)$ is immediate. We thus find that every pseudo Cauchy sequence in K without a limit in K having x as a limit is of transcendental type.

Characteristic blind Taylor expansion

We will need a Taylor expansion of polynomials that also works in positive characteristic. That is, denominators with natural numbers that become zero in positive characteristic have to be avoided.

For every $j \in \mathbb{N}$, we have:

$$(X+Y)^j = \sum_{i=0}^{n} \binom{j}{i} X^{j-i} Y^i .$$

This is also true in fields of characteristic $p > 0$ since the binomial coefficients are natural numbers which then will just be taken modulo p. For an arbitrary polynomial $f(X) = c_n X^n + c_{n-1} X^{n-1} + \ldots + c_0$, summation now gives

$$f(X+Y) = \sum_{j=0}^{n} \sum_{i=0}^{j} c_j \binom{j}{i} X^{j-i} Y^i = \sum_{i=0}^{n} \sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i} Y^i .$$

Characteristic blind Taylor expansion

We set

$$\partial_i f(X) := \sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i} = \sum_{j=0}^{n-i} c_{j+i} \binom{j+i}{i} X^j , \qquad (2)$$

which we will call the *i*-th Hasse-Schmidt derivative of f (also called the *i*-th formal derivative of f). We obtain the Taylor expansion

$$f(X+Y) = \sum_{i=0}^{n} \partial_i f(X) Y^i . \qquad (3)$$

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