Perfectoid and deeply ramified fields: algebra and model theory

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Absolute Galois groups

The absolute Galois group of a field K is the Galois group of the extension $K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K. Absolute Galois groups are of great interest in algebraic number theory. For instance, it is said that the absolute Galois group of $\mathbb{Q}$ knows everything that number theorists want to know. Unfortunately, we do not know everything about that Galois group. The situation becomes better when we pass to $\mathbb{Q}_p$, and from there to certain extensions of $\mathbb{Q}_p$, for which Coates and Greenberg in 1996 introduced the name "deeply ramified".

The Fontaine-Wintenberger Theorem

The field $\mathbb{Q}_p(p^{1/p^\infty}) = \mathbb{Q}_p(p^{1/p^n} \mid n \in \mathbb{N})$ is such a deeply ramified extension of $\mathbb{Q}_p$. The celebrated Fontaine-Wintenberger Theorem states that the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))(t^{1/p^\infty})$ have isomorphic absolute Galois groups. Note that since these fields are henselian, passing to their (p-adic or t-adic, respectively) completions does not change the absolute Galois group.

Peter Scholze (Fields Medal winner 2018) vastly generalized this result. For valued fields, he introduced the notion of "perfectoid field" and a construction, called "tilting", that associates to every perfectoid field of mixed characteristic a perfectoid field of equal positive characteristic. He showed that tilting preserves absolute Galois groups.

Perfectoid fields and their tilts

A perfectoid field is a non-discrete complete real-valued field K of residue characteristic $p > 0$ with valuation ring $\mathcal{O}_K$ such that $\mathcal{O}_K/p\mathcal{O}_K$ is semi-perfect, i.e., the Frobenius map

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K$$

is surjective. The tilt $K^\flat$ of a perfectoid field K can essentially be presented as $\varprojlim_{x \mapsto x^p} = \{(x_n)_{n\in\mathbb{N}} \mid x_{n+1}^p = x_n\}$.

Examples of perfectoid fields and their tilts

The p-adic completions of the fields $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty}) = \mathbb{Q}_p(\zeta_{p^n} \mid n \in \mathbb{N})$ where ζ_{p^n} denotes a p^n-th root of unity, and $\mathbb{Q}_p^{\mathrm{ab}}$ are perfectoid fields.

The tilts of the first two are isomorphic to the t-adic completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$, and the tilt of the latter is isomorphic to the t-adic completion of $\mathbb{F}_p^{\mathrm{ac}}((t))(t^{1/p^\infty})$, where $\mathbb{F}_p^{\mathrm{ac}}$ denotes the algebraic closure of $\mathbb{F}_p$.

Two main tasks

In a natural way, two main tasks arise in relation to Scholze's work:

1) develop the model theory of the tilting construction (Jahnke and Kartas),

2) develop the valuation theory for a suitable elementary class of valued fields containing the perfectoid fields (Cutkosky, K, Rzepka).

Note that the notion "perfectoid field" is not elementary (in the language L_{val} of valued rings), as "complete" is not elementary, and neither is "real-valued" (= value group is archimedean = valuation is of rank 1).

Deeply ramified fields

The class of perfectoid fields is contained in the elementary class of deeply ramified fields. A valued field (K, v) of residue characteristic $p > 0$ is deeply ramified if it satisfies the following conditions:

(DRvg) if $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) $\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}}$ is semi-perfect, where $\hat{K}$ denotes the completion of (K, v).

Note that "completion" (w.r.t. the topology induced by the valuation) also makes sense for valuations of arbitrary rank.

In mixed characteristic, (DRvr) can be replaced by saying that $\mathcal{O}_K/p\mathcal{O}_K$ is semi-perfect. In positive characteristic, a valued field is deeply ramified if and only if it lies dense in its perfect hull.

Limitations to the model theory of deeply ramified fields

Ax-Kochen/Ershov principles like

$$(K,v) \subseteq (L,v) \wedge vK \prec vL \wedge Kv \prec Lv \Rightarrow (K,v) \prec (L,v)$$

where vK, vL denote the respective value groups and Kv, Lv the respective residue fields cannot hold generally in the class of all henselian deeply ramified fields. There are nontrivial finite extensions $(L|K,v)$ of henselian deeply ramified fields that are immediate, i.e., $vK = vL$ and $Kv = Lv$, but for them $(K,v) \prec (L,v)$ cannot be true.

Abhyankar's Example

Consider the henselian field $K := \mathbb{F}_p((t))^{1/p^\infty}$ with its t-adic valuation v_t . It is a deeply ramified field as it is perfect. Take a root ϑ of the Artin-Schreier polynomial

$$X^p - X - \frac{1}{t}\,.$$

The extension $(K(\vartheta)|K, v_t)$ can be proven to be immediate. The same still holds if we replace K by its completion, which is a perfectoid field.
Here we are confronted with an extension $(L|K, v)$ of henselian fields where $[L : K] > (vL : vK)[Lv : Kv]$, in which case we speak of a defect extension. While Abhyankar never explicitly talked of "defect" and constructed his example for a different purpose, this is an example for a defect of a particular type that we are very interested in.

Analoguous examples can be given in mixed characteristic; for this and an abundance of other examples for defect extensions, see

K: *Defect*, in: Commutative Algebra - Noetherian and non-Noetherian perspectives, Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I. (Eds.), Springer-Verlag, New York, 2011.

Example 3.20 in that paper presents the p-adic analogue of Abhyankar's Example:
Take $K := \mathbb{Q}_p(p^{1/p^\infty})$ and ϑ to be a root of $X^p - X - 1/p$. Then $(K(\vartheta)|K, v)$ is immediate and hence a defect extension.

Model theory for perfectoid fields?

The classical Ax-Kochen/Ershov principles can only work for defectless fields, that is, henselian valued fields (K, v) not admitting finite defect extensions, i.e., every finite extension $(L|K, v)$ satisfies $[L : K] = (vL : vK)[Lv : Kv]$. (Note that this definition can be generalized to valued fields that are not henselian.) So we see that we cannot expect classical Ax-Kochen/Ershov principles to hold for perfectoid fields.

In their recent paper

Jahnke, F. – Kartas, K.: *Beyond the Fontaine-Wintenberger theorem*, arXiv:2304.05881 (2023),

Franziska and Konstantinos get around this problem by a method which they informally call "taming perfectoid fields". This refers to the elementary class of tame fields, which conveniently are henselian defectless fields.

Tame extensions

An algebraic extension $(L|K, v)$ of henselian valued fields is called tame if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by char Kv,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ has no defect.

Tame fields

A henselian valued field (K, v) is called a tame field if $(K^{\mathrm{ac}}|K, v)$ is a tame extension. All tame fields are perfect and are deeply ramified fields (but not vice versa).

Theorem (K)

Tame fields (K, v) satisfy model completeness and decidability relative to the elementary theories of their value groups vK and their residue fields Kv. If $\mathrm{char}\, K = \mathrm{char}\, Kv$, then also relative completeness holds.

$\mathbb{F}_p((t))^{1/p^\infty}$ is perfect, but does not satisfy (T3).

A reference

For the theory of tame fields, see

K: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43.

Decomposition of the valuation in mixed characteristic

Take a valued field (K, v) of characteristic 0 with residue characteristic $p > 0$. Decompose $v = v_0 \circ v_p \circ \overline{v}$, where v_0 is the finest coarsening of v that has residue characteristic 0, v_p is a rank 1 valuation on Kv_0 , and $\overline{v}$ is the valuation induced by v on the residue field of v_p (which is of characteristic $p > 0$). The valuations v_0 and $\overline{v}$ may be trivial. Further, following Johnson, we call the value group vK roughly p-divisible if $v_p \circ \overline{v}\,(Kv_0)$ (the value group of $v_p \circ \overline{v}$ on Kv_0) is p-divisible.
Note that (K, v_0) is always defectless since it has residue characteristic 0, and therefore, (K, v) is defectless if and only if the core field $(Kv_0, v_p \circ \overline{v})$ is.

Because of this, we define a henselian valued field (K, v) to be a roughly tame field if it is either a tame field of equal characteristic, or it is of mixed characteristic and $(Kv_0, v_p \circ \overline{v})$ is a tame field. Similarly, we define a valued field (K, v) with residue characteristic $p > 0$ to be a roughly deeply ramified field if it is either a deeply ramified field of positive characteristic or it is of characteristic 0, $\mathcal{O}_K / p\mathcal{O}_K$ is semi-perfect and vp is not the smallest positive element of vK; in this case, vK is roughly p-divisible.

In their paper, Franziska and Konstantinos generalize my model theoretic results for tame fields to the case of roughly tame fields. They crucially apply these new results after "taming" perfectoid fields.

A useful elementary class of valued fields

We have seen that Ax-Kochen/Ershov principles will in general fail for perfectoid fields. Therefore, Franziska and Konstantinos have developed a method to make the model theory of roughly tame fields available for their purposes. They consider the class $\mathcal{C}$ of henselian valued fields (K, v) of residue characteristic $p > 0$ with $\mathcal{O}_K / p\mathcal{O}_K$ semi-perfect, together with a distinguished element ϖ in the maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$ such that $\mathcal{O}_K[\varpi^{-1}]$ is algebraically maximal (which we will define on the next slide). This class contains the perfectoid fields. Moreover, Franziska and Konstantinos prove that $\mathcal{C}$ is an elementary class in the language L_{val} of valued rings enriched by a constant symbol for ϖ.

Note that $\mathcal{O}_K[\varpi^{-1}]$ properly contains $\mathcal{O}_K$, so it is the valuation ring of a proper coarsening v_ϖ of v. The requirement on $\mathcal{O}_K[\varpi^{-1}]$ on the previous slide says that (K, v_ϖ) shall be algebraically maximal, i.e, does not admit any proper immediate algebraic extensions. Franziska and Konstantinos prove that if $(K, v, \varpi) \in \mathcal{C}$, then (K, v_ϖ) is a defectless field. This is remarkable, since in general “algebraically maximal” does not imply “defectless” and it is often a tricky task to prove that a given valued field is a defectless field.

Taming perfectoid fields

Franziska and Konstantinos prove even more:

Theorem (Jahnke and Kartas 2023)

Take $(K_0, v_0, \varpi) \in \mathcal{C}$ *and let* (K, v) *be an* $\aleph_1$*-saturated elementary extension of* (K_0, v_0, ϖ) *and* w *be the coarsest coarsening of* v *with* $w\varpi > 0$*. Then* (K, w) *is a roughly tame field, and if in addition* (K_0, v_0) *is perfectoid, then* (K, w) *is a tame field.*

Hence they can apply the model theory of (roughly) tame fields to (K, w).

Ax-Kochen/Ershov principles for $\mathcal{C}$

This leads to Ax-Kochen/Ershov principles for the members of the class $\mathcal{C}$, such as:

Theorem (Jahnke and Kartas 2023)

Let $(K, v, \varpi) \subseteq (K', v', \varpi)$ be an extension of members of $\mathcal{C}$. Then $(K, v) \prec (K', v')$ in L_{val} if and only if $\mathcal{O}_K/\varpi\mathcal{O}_K \prec \mathcal{O}_{K'}/\varpi\mathcal{O}_{K'}$ in the language of rings and $vK \prec v'K'$ in the language of ordered abelian groups.

This theorem is used to prove:

Theorem (Jahnke and Kartas 2023)

Let (K, v) be a perfectoid field and $0 \neq \varpi \in \mathcal{M}_K$. Let U be a non-principal ultrafilter on $\mathbb{N}$ and (K_U, v_U) be the corresponding ultrapower. Let w be the coarsest coarsening of v_U such that $w\varpi > 0$. Then the tilt $(K^\flat, v^\flat)$ embeds elementarily in $(K_U w, \bar{v})$, where $\bar{v}$ is the induced valuation of v_U on $K_U w$.

Further applications

Theorem (Jahnke and Kartas 2023)

Suppose that (K_1, v_1) *and* (K_2, v_2) *are two perfectoid fields with a common perfectoid subfield* (K_0, v_0). *Then*

$$(K_1, v_1) \equiv_{(K_0,v_0)} (K_2, v_2) \quad \textit{in } L_{\mathrm{val}}$$

if and only if

$$(K_1^\flat, v_1^\flat) \equiv_{(K_0^\flat,v_0^\flat)} (K_2^\flat, v_2^\flat) \quad \textit{in } L_{\mathrm{val}}\,.$$

This implies that tilting preserves the relations "elementary substructure", "elementarily equivalent" and "existentially closed" between perfectoid fields.

Further applications

Corollary (Jahnke and Kartas 2023)

For a perfectoid field K, the L_{val}-theory of $K^\flat$ is decidable relative to the one of K.

And finally another striking application. It is still not known whether the henselization $\mathbb{F}_p(t)^h$ of $\mathbb{F}_p(t)$ is an elementary substructure of $\mathbb{F}_p((t))$. However,

Corollary (Jahnke and Kartas 2023)

The perfect hull of $F_p(t)^h$ is an elementary substructure of the perfect hull of $\mathbb{F}_p((t))$.

Let us explore the reason why things work so well for perfectoid fields though they admit defect extensions. Since the class of deeply ramified fields is elementary and contains all perfectoid fields, all $\aleph_1$-saturated elementary extensions of perfectoid fields are deeply ramified fields - but not perfectoid, as they are neither complete nor of rank 1. As shown in the paper

K and Rzepka, A.: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738,

the defects appearing in finite extensions of deeply ramified fields belong to one of two types of defects that are "more harmless" than those of the other type.

Classification of defects

Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. For every σ in its Galois group $\mathrm{Gal}\,(L|K)$ with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\} .$$

This set is a final segment of vK and independent of the choice of σ; we denote it by $\Sigma_{\mathcal{E}}$. We say that $\mathcal{E}$ has independent defect if $\Sigma_{\mathcal{E}}$ is equal to $\{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ for some proper convex subgroup $H_{\mathcal{E}}$ of vK such that $vK/H_{\mathcal{E}}$ has no smallest positive element; otherwise we will say that $\mathcal{E}$ has dependent defect. If (K, v) has rank 1, then the only proper convex subgroup of vK is $\{0\}$ and the condition just means that $\Sigma_{\mathcal{E}}$ consists of all positive elements in vK.
The name "independent defect" comes from the fact that in positive characteristic these defects cannot by obtained from "twisting" purely inseparable defect extensions.

The following results were proven in the above cited paper with Rzepka:

1) all defects appearing in Galois extensions of prime degree of a deeply ramified field are independent,

2) every algebraic extension of a deeply ramified field is again a deeply ramified field,

3) if $(L|K,v)$ is a finite extension and (L,v) is a deeply ramified field, then so is (K,v).

The same holds for roughly deeply ramified fields, and for them there is also infinite descent through tame extensions.

Note that 2) also holds if "deeply ramified" is replaced by "tame", whereas infinite extensions of defectless fields may not again be defectless fields.

The Gabber-Ramero Theorem

In the book

Gabber, O. – Ramero, L.: *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003,

results 2) and 3) are deduced from the following characterization of deeply ramified fields:

Theorem (Gabber and Ramero 2003)

Take a valued field (K, v) with valuation ring $\mathcal{O}_K$. Choose any extension of v to K^{sep}. Then (K, v) is a deeply ramified field if and only if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0 ,$$

where $\Omega_{B|A}$ denotes the module of relative differentials (Kähler differentials) when A is a ring and B is an A-algebra. This definition does not depend on the chosen extension of v from K to K^{sep}.

Gabber and Ramero used heavy machinery to prove this theorem. A much more down-to-earth proof is given in

Cutkosky, S.D. – K: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, arXiv:2306.04967 (2023).

This paper is a continuation of

Cutkosky, S.D. – K – Rzepka, A.: *Characterizations of Galois extensions with independent defect*, arXiv:2305.10022 (2023).

Because of the role independent defect plays for deeply ramified fields and the experience that in some way we can handle them, it is important to have at hand various characterizations of Galois extensions (of prime degree) with independent defect.

Two of these characterizations are:

Theorem (Cutkosky – K – Rzepka 2023)

Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree $p = \operatorname{char} Kv$. Then the following assertions are equivalent:

a) $\mathcal{E}$ has independent defect,

b) $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$,

c) $\operatorname{Tr}_{L|K}(\mathcal{M}_L) = \mathcal{M}_{v_H} \cap K$ for some convex subgroup H of vK such that vK/H has no smallest positive element.

If assertion c) holds, then $H = H_{\mathcal{E}}$. The value group of the corresponding coarsening of v does not have a smallest positive element.

Surprise, surprise

Consider the Abhyankar Example which we introduced earlier. There, $K = \mathbb{F}_p((t))^{1/p^\infty}$ is a deeply ramified field. Hence with $L = K(\vartheta)$, the Galois extension $(L|K, v)$ has independent defect. Therefore, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. This remains true if K is replaced by its t-adic completion. However, in the scarce sources from almost mathematics we have found so far, it is only said that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is almost zero, that is, annihilated by the maximal ideal $\mathcal{M}_L$ of $\mathcal{O}_L$.

In Example 1.6.1, Franziska and Konstantinos reproduce an example given by Scholze. Fix a prime $p \neq 2$, let K be the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$ and $L = K(p^{1/2})$. Then the Galois extension $(L|K, v)$ is a tame extension and thus has no defect. In my paper with Cutkosky cited above it is shown that also in this case, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. However, in the paper of Franziska and Konstantinos it is only stated that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is almost zero.

THE END

Thank you for your attention!

More detailed information

This presentation will be posted on the web page

https://www.fvkuhlmann.de/Fvkslides.html

and a lecture series on valued function fields and the defect can be found on the web page

https://www.fvkuhlmann.de/Fvkls.html.

Preprints and further information:

https://www.valth.eu/Valth.html.