Two longstanding open problems in positive characteristic and their relation to valuation theory
Part I: Local uniformization

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Two open problems

Longstanding open problems in positive characteristic:

• Resolution of Singularities and its local form, Local Uniformization, over fields of positive characteristic;

• Decidability of Laurent Series Fields over finite fields (see Part II).

In 1940, Zariski [Z 1940] proved Local Uniformization over base fields of characteristic 0;
in 1964, Hironaka [H 1964] proved Resolution of Singularities over base fields of characteristic 0.

Partial solutions to the first problem

Closest approximations to the first problem to date:

Abhyankar [A 1998], Cossart & Piltant [CP 2008], [CP 2009]: resolution up to dimension 3 (see also [C 2009]);

de Jong [dJ 1996]: resolution by alteration;

Knaf & K [KK 2005]: local uniformization for Abhyankar places;

Knaf & K [KK 2009]: local uniformization by separable alteration;

Temkin [T 2013]: local uniformization by purely inseparable alteration.

See also [K 2000].

Basic idea of local uniformization

While resolution seeks to associate to a given variety a birationally equivalent variety without singularities, local uniformization only seeks to "get rid of just one singularity at a time".

Over base fields of positive characteristic, this problem is open in general for dimensions > 3.

For local uniformization we can just consider affine varieties V, given by polynomials

$$f_1, \dots, f_n \in K[X_1, \dots, X_\ell] .$$

Naive concept: points of V are common zeros $(a_1, \dots, a_\ell)$ in some extension field L of K. This means that the kernel of the evaluation homomorphism

$$K[X_1, \dots, X_\ell] \longrightarrow L \qquad X_i \mapsto a_i$$

contains the ideal $(f_1, \dots, f_n)$, so it induces a homomorphism η from the coordinate ring

$$K[V] \ := \ K[X_1, \dots, X_\ell]/(f_1, \dots, f_n)$$

into L over K ("over K" means that it leaves the elements of K fixed).

The modern concept, but...

Modern concept: a point is given by the kernel of η.

However, the homomorphism η is more important for us.

The function field of an affine algebraic variety

We may write

$$K[V] \;=\; K[x_1,\dots,x_\ell] \quad \text{with } x_i = X_i + (f_1,\dots,f_n)\,,$$

Then its quotient field

$$K(V) \;:=\; K(x_1,\dots,x_\ell)$$

is finitely generated over K. Every finitely generated extension of a field K is called (algebraic) function field over K.

Every function field over an arbitrary field K is in fact the function field of a suitable variety defined over K.

Correspondence of points

Problem of local uniformization: given a variety V with a singular point, find a variety V' on which the point "becomes" non-singular. V' should be related to V through a proper birational morphism; this implies that $K(V) = K(V')$. But what is the corresponding point on V'?

Why not extend η to a homomorphism on $K(V) = K(V')$ and then restrict it to $K[V']$? The problem is that a homomorphism on $K[V]$ can, and usually does, send nonzero elements to zero; so on $K(V)$ their inverses have to be sent to ∞. Zariski's idea was to allow this by using places on $K(V)$.

Places

A place P on a field F is a homomorphism on a subring $\mathcal{O}_P$ of F whose quotient field is F that sends every element outside of $\mathcal{O}_P$ to ∞. $\mathcal{O}_P$ is called the valuation ring of P.

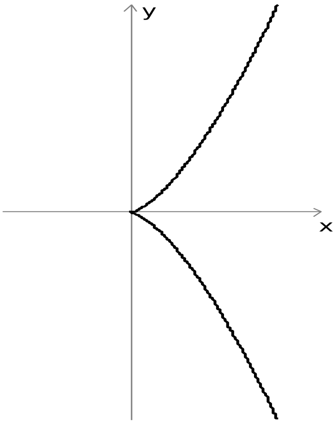
To solve the local uniformization problem for P, we need to find V' such that $\mathcal{O}_P$ contains $K[V']$, so that the restriction of P to $K[V']$ is a homomorphism and thus identifies a point of V'; this point should be non-singular.

We can forget about the variety V and the homomorphism η and consider the function field $F = K(V)$ together with a place P instead.

In order to make the problem of valuation theoretical nature, we ask for a bit more: we ask for the new point to be smooth, meaning that the Implicit Function Theorem is satisfied in this point. We then talk of Smooth Local Uniformization.

In the following example, the point at the origin is not smooth; the Implicit Function Theorem does not hold here because when we increase x, the point can move into the upper or the lower branch.

Example: the Neil curve



$$y^2 = x^3$$

The topology

The topology needed for the Implicit Function Theorem is provided by the valuation v_P that is associated to the place P:

$$v_P : F^\times \longrightarrow F^\times / \mathcal{O}_P^\times .$$

The quotient carries in a natural way the structure of an ordered abelian group, denoted by $v_P F$. We set $v_P(0) = \infty$, an element larger than all elements of $v_P F$.

Valuations

In general, a valuation of F is a function from F to an ordered abelian group together with ∞ such that
$v(x) = \infty \Leftrightarrow x = 0$,
$v(xy) = v(x) + v(y)$,
$v(x+y) \geq \min\{v(x), v(y)\}$.

The value group of v is $vF := v(F^\times)$,
and its valuation ring is $\mathcal{O}_v := \{x \in F \mid v(x) \geq 0\}$.
The unique maximal ideal of $\mathcal{O}_v$ is $\mathcal{M}_v := \{x \in F \mid v(x) > 0\}$.
The field $Fv := \mathcal{O}_v / \mathcal{M}_v$ is called the residue field of v.
The residue map of v is the canonical epimorphism $\mathcal{O}_v \to Fv$; extending it to all of F by sending all elements outside of $\mathcal{O}_v$ to ∞ produces the place associated to v.

Basic examples of valuations will be given in Part II.

Problem of Smooth Local Uniformization

We can formulate the task of Smooth Local Uniformization as follows:

Given a function field F with a place P, we need to find a coordinate ring $K[V'] \subseteq \mathcal{O}_P$ having quotient field F and such that the homomorphism induced by P on $K[V']$ is a smooth point. In other words, we are looking for generators $x_1, \ldots, x_m \in \mathcal{O}_P$ of $F|K$, such that the homomorphism induced by P on $K[x_1, \ldots, x_m]$ is a smooth point.

The question is: when can we find such generators? The answer is provided by ramification theory: if $F|K$ admits a transcendence basis T such that F lies in the absolute inertia field of $(K(T), P)$. This is a problem in the structure theory of valued function fields.

Disclaimer

At this point, a word of warning is in place. Local Uniformization also requires that the new variety V' is connected with V by a proper birational morphism. This amounts to an extra condition, one may see as "Local Uniformization for $(K(T), P)$", which we will ignore for now.

Finding suitable generators of $F|K$ is known as Elimination of Ramification.

The defect

In positive characteristic the main obstruction in the structure theory of valued function fields is the defect (see [K 2011]). Take a finite extension $(E|F,v)$ where the extension of v from F to E is unique. If char $Fv = 0$, then

$$[E:F] \;=\; (vE:vF)\,[Ev:Kv]\,.$$

But if char $Fv = p > 0$, then by the Lemma of Ostrowski,

$$[E:F] \;=\; p^{\nu}(vE:vF)\,[Ev:Kv]$$

with $\nu \geq 0$. The factor p^{ν} is called the defect of $(E|F,v)$. We either have to work with settings where no nontrivial defect appears, or work around the defect.

Defectless fields

A valued field (F, v) is henselian if the extension of v to every algebraic extension field is unique. A henselian field (F, v) is defectless if no finite extension of (K, v) has nontrivial defect. Every henselian field (K, v) with char $Kv = 0$ is defectless. The field $\mathbb{Q}_p$ of p-adic numbers is defectless.

Examples for valued fields that are not defectless:

1) certain infinite algebraic extensions of $\mathbb{Q}_p$, such as $\mathbb{Q}_p^{\mathrm{ab}}$,

2) if (K, v) is a nontrivially valued field that is not perfect, then its separable-algebraic closure is henselian, but not defectless,

3) the perfect hull of the field of Laurent series over $\mathbb{F}_p$ (see Part II).

The Abhyankar Inequality

If Γ is any abelian group, then the rational rank of Γ is $\mathrm{rr}\,\Gamma := \dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma$. This is the maximal number of rationally independent elements in Γ.
If $(F|K, v)$ is an arbitrary valued field extension of finite transcendence degree, then we have the Abhyankar inequality:

$$\mathrm{trdeg}\, F|K \;\geq\; \mathrm{rr}\; vF/vK \;+\; \mathrm{trdeg}\, Fv|Kv\,. \tag{1}$$

We call v an Abhyankar valuation and its associated place P an Abhyankar place if equality holds in (1).

Note: if v is trivial on K, then $vK = 0$ and $Kv \simeq K$.

The Generalized Stability Theorem

Theorem (K, thesis 1989; K 2010)

(Generalized Stability Theorem)
Assume that v is an Abhyankar valuation on the function field $F|K$, not necessarily trivial on K. If (K, v) is a defectless field, then (F, v) is a defectless field.

This theorem is crucial for the proof of the next result, which shows that local uniformization is possible for all Abhyankar places.

Local uniformization for Abhyankar places

If P is a place on F with associated valuation v, then the residue field Fv is also denoted by FP.

Theorem (Knaf–K, KK 2005)

Let P be an Abhyankar place of the function field $F|K$, trivial on K, and assume that $FP|K$ is separable. Then P admits smooth local uniformization.

Local uniformization by alteration

Theorem (Knaf–K, KK 2009)

Let P be an arbitrary place of the function field $F|K$, trivial on K. Then there is a finite separable extension $\mathcal{F}|F$ and an extension of P to $\mathcal{F}$ which admits smooth local uniformization.

If K is perfect of characteristic $p > 0$, then the extension $\mathcal{F}|F$ can be chosen to be
- *either Galois,*
- *or such that $v_P\mathcal{F}/v_PF$ is a p-group and $\mathcal{F}P|FP$ is purely inseparable.*

The proof uses the Generalized Stability Theorem and a second theorem which will be presented in Part II.

Inseparable local uniformization

While the last theorem shows that local uniformization is always possible after a finite separable extension of the function field, there is also a complementary result:

Theorem (Temkin, T 2013)

Let P be an arbitrary place of the function field $F|K$, trivial on K. Then there is a finite purely inseparable extension $\mathcal{F}|F$ and an extension of P to $\mathcal{F}$ which admits smooth local uniformization.

Unfortunately, so far nobody knows how to derive the least common denominator of these two theorems: local uniformization without alteration.

References: Resolution of Singularities

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More detailed information

This presentation can be found on the web page

https://math.usask.ca/fvk/Fvkslides.html,

and a lecture series on valued function fields and the defect can be found on the web page

https://math.usask.ca/fvk/Fvkls.html.

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.