Chapter 13

The tame and the wild valuation theory

13.1 Tame and purely wild extensions

Lemma 7.29 motivates the following definition. An algebraic extension $(L|K,v)$ of henselian fields is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $e(E|K,v)$ is prime to the characteristic exponent of $\overline{K}$.

(TE2) The residue field extension $\overline{E}|\overline{K}$ is separable.

(TE3) The extension $(E|K,v)$ is defectless.

Remark 13.1 This notion of "tame extension" does not coincide with the notion of "tamely ramified extension" as defined in the book of O. Endler [END8], page 180. The latter definition requires (TE1) and (TE2), but not necessarily (TE3). Our tame extensions are the defectless tamely ramified extensions in the sense of Endler's book. In particular, in our terminology, proper immediate algebraic extensions of henselian fields are not called tame, which is justified in view of the difficulties that they will give us later in this book.

It follows directly from this definition that every subextension of a tame extension is again a tame extension. If the residue characteristic of (K,v) is 0, then (TE1) and (TE2) are trivially satisfied for every finite extension $(E|K,v)$. By Theorem 11.23, also (TE3) is satisfied. Hence, every finite extension of henselian fields of residue characteristic 0 is tame. The following theorem characterizes the tame extensions of an arbitrary henselian field (K,v):

Theorem 13.2 *Let (K,v) be a henselian field. Then the absolute ramification field (K^r,v) is a tame extension of (K,v), and every other tame extension of (K,v) is contained in $(K,v)^r$. Hence, $(K,v)^r$ is* the *maximal tame extension of (K,v). In particular, the compositum of every two tame extensions of (K,v) is again a tame extension of (K,v).*

Proof: Let $(L|K,v)$ be a subextension of $(K^r|K,v)$ and $(E|K,v)$ a finite subextension of $(L|K,v)$. Then an application of Lemma 7.29 with $K_0 = K$ and $K_1 = E$ shows that $(E|K,v)$ satisfies (TE1), (TE2) and (TE3). Hence, $(L|K,v)$ is a tame extension.

Now let $(L|K,v)$ be a tame extension. To show that $L \subset K^r$, it suffices to prove that every finite subextension $(E|K,v)$ of $(L|K,v)$ is a subextension of $(K^r|K,v)$. By assumption, $(E|K,v)$ satisfies (TE1), (TE2) and (TE3). By Lemma 7.32 and (TE1), (TE2), $[L\cap K^r : K] = e(L|K,v)\cdot f(L|K,v)$. By (TE3), the latter is equal to $[L:K]$. Hence $L = L\cap K^r \subset K^r$. □

An extension $(L|K,v)$ of henselian fields is called **unramified** if the extension $vL|vK$ is trivial.

Theorem 13.3 *Let (K,v) be a henselian field. Every tame unramified extension of (K,v) is contained in the absolute inertia field $(K,v)^i$ of (K,v). Hence, $(K,v)^i$ is the maximal tame unramified extension of (K,v). In particular, the compositum of every two tame unramified extensions of (K,v) is again a tame extension of (K,v).*

The proof is similar to that of the foregoing theorem, taking K^i in the place of K^r.

The following consequence of Lemma 11.78 is one of the most important properties of tame extensions. It serves to reduce the study of the defect to more suitable situations (cf. Chapter 3 of [KU1] and [KU2]).

Lemma 13.4 *Let (K,v) be henselian, $(F|K,v)$ a tame extension and $(L|K,v)$ a finite extension. Then*

$$d(L|K,v) = d(L.F|F,v)\ .$$

Further, (L,v) is a defectless field if and only if (K,v) is.

A (not necessarily algebraic) extension $(L|K,v)$ of a henselian field (K,v) is called **purely wild** if it satisfies the following conditions:

(PW1) vL/vK is a p-group, where p is the characteristic exponent of $\overline{K}$.
(In particular, vL/vK is a torsion group.)

(PW2) The residue field extension $\overline{E}|\overline{K}$ is purely inseparable.
(In particular, $\overline{E}|\overline{K}$ is algebraic.)

We see: Every subextension of a purely wild extension is purely wild. Every purely inseparable extension is purely wild.

Since $K^r|K$ is normal, $L|K$ is linearly disjoint from $K^r|K$ if and only if $L\cap K^r = K$. By Lemma 7.32, the latter holds if and only if $e(L|K,v)$ is a power of the characteristic exponent of $\overline{K}$ and $\overline{L}|\overline{K}$ is purely inseparable. This proves:

Lemma 13.5 *An algebraic extension $(L|K,v)$ is purely wild if and only if $L|K$ is linearly disjoint from $K^r|K$, and this holds if and only if $L\cap K^r = K$.*

Let us consider a very special sort of purely wild extensions. Note that in the case of $\mathrm{char}\, K = 0$, we have by convention that $K^{1/p^\infty} = K$ (which is the perfect hull of K), even if $p = \mathrm{char}\, \overline{K}$.

Lemma 13.6 *Let (K,v) be a henselian field with $\mathrm{char}\, \overline{K} = p > 0$. Suppose that (K,v) admits no purely wild Artin-Schreier extensions. Then its value group vK is p-divisible and its residue field $\overline{K}$ is perfect, and (K,v) is dense in its perfect hull $(K^{1/p^\infty},v)$.*

Proof: To find that vK is p-divisible, employ Lemma 6.40 and note that the Artin-Schreier extensions given there are defectless and thus purely wild. Similarly, it follows from Lemma 6.41 that $\overline{K}$ is perfect. Now the proof for Theorem 11.74 shows that (K, v) is dense in its perfect hull $(K^{1/p^\infty}, v)$; just note that the immediate Artin-Schreier extensions provided by Lemma 11.73 are purely wild. $\square$

Observe that the hypothesis that (K, v) be henselian is only required since we have defined purely wild extensions only over henselian fields. The reader may try a more general definition but note that we do not want the henselization to be called a purely wild extension. The way out is to require that the extension in question is linearly disjoint from the henselization.

13.2 Valuation independence of Galois groups

In this section, we will introduce a valuation theoretical property that characterizes the Galois groups of tame Galois extensions. Take a (possibly infinite) Galois extension $(L|K, v)$ of henselian fields. Its Galois group $\mathrm{Gal}\, L|K$ will be called **valuation independent** if for every choice of elements $d_1, \ldots, d_n \in \tilde{L}$ and automorphisms $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}\, L|K$ there exists an element $d \in L$ such that (for the unique extension of the valuation v from L to $\tilde{L}$):

$$v\left(\sum_{i=1}^{n} \sigma_i(d) \cdot d_i\right) = \min_{1 \le i \le n} v(\sigma_i(d) \cdot d_i) . \tag{13.1}$$

Assume that (K, v) is henselian. Then $v\sigma(d) = vd$ for all $\sigma \in \mathrm{Gal}\, L|K$ and therefore, $v(\sigma_i(d) \cdot d_i) = vd + vd_i$. Suppose that $vd_1 = \min_i vd_i$; then (13.1) will hold if and only if

$$v\left(\sum_{i=1}^{n} \sigma_i(d) \cdot \frac{d_i}{d_1}\right) = vd .$$

So we see:

Lemma 13.7 *Assume that (K, v) is henselian. Then $\mathrm{Gal}\, L|K$ is valuation independent if and only if for every choice of elements $d_1, \ldots, d_n \in \tilde{L}$ such that $vd_i = 0$, $1 \le i \le n$, and automorphisms $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}\, L|K$ there exists an element $d \in L$ such that*

$$v\left(\sum_{i=1}^{n} \sigma_i(d) \cdot d_i\right) = vd . \tag{13.2}$$

Theorem 13.8 *A (possibly infinite) Galois extension of henselian fields is tame if and only if its Galois group is valuation independent.*

Proof: Take a Galois extension $(L|K, v)$ of henselian fields, elements $d_1, \ldots, d_n \in \tilde{L}$ such that $vd_i = 0$, $1 \le i \le n$, and automorphisms $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}\, L|K$. Equation 13.2 is equivalent to

$$v\left(\sum_{i} \frac{\sigma_i(d)}{d} \cdot d_i\right) = 0 . \tag{13.3}$$

Since (K,v) is henselian, $G^d(L|K,v) = \mathrm{Gal}\, L|K$. Hence, we can use the crossed homomorphism from $G^d(L|K,v)$ to the character group $\mathrm{Hom}(L^\times, \overline{L}^\times)$ introduced in (7.9) of Section 7.1 to rewrite (13.3) as

$$\sum_i \chi_{\sigma_i}(d)\cdot d_i v \neq 0\,.$$

The theorem of Artin on linear independence of characters (see [LANG3], VIII, §11, Theorem 18) tells us that if the χ_σ are mutually different, then the required element d will exist. This shows that G is valuation independent if the map $\sigma \mapsto \chi_\sigma$ is injective. The converse is also true: if $\sigma_1 \neq \sigma_2$ but $\chi_{\sigma_i 1} = \chi_{\sigma_2}$, then with $n=1$ and $d_1 = -d_2 = 1$, (13.2) does not hold for any d.

Since the kernel of (7.9) is the ramification group of $(L|K,v)$, we conclude that $\mathrm{Gal}\, L|K$ is valuation independent if and only if the ramification group is trivial. This is equivalent to $(L|K,v)$ being a tame extension. □

Note that we could give the above definition and the result of the theorem also for extensions which are not Galois, replacing automorphisms by embeddings; however, the normal hull of an algebraic extension $L|K$ of a henselian field K is a tame extension of K if and only if $L|K$ is a tame extension, so there is no loss of generality in restricting our scope to Galois extensions.

13.3 The diameter of the conjugates

Let (K,v) be a valued field and let v be extended to $\tilde{K}$. Further, take $a \in \tilde{K}$. Following J. Ax [AX2], we call the value

$$\mathrm{diam}(a,K) := \min\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\, K\}$$

the **diameter of the conjugates of** a (**over** K). For $a \in \tilde{K} \setminus K^{1/p^\infty}$, we set

$$\mathrm{kras}(a,K) := \max\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\, K \wedge \sigma a \neq a\}\,.$$

For $a \in K^{\mathrm{sep}}$, we know this counterpart of $\mathrm{diam}(a,K)$ already from Krasner's Lemma, which is property 7) of Theorem 9.1. Observe that $\mathrm{diam}(a,K) = \infty$ if $a \in K^{1/p^\infty}$. For this case, let us also define $\mathrm{kras}(a,K) := \infty$.

Krasner's Lemma tells us that if (K,v) is henselian and if we have $b \in K^{\mathrm{sep}}$ such that $v(a-b) > \mathrm{kras}(a,K)$, then $a \in K(b)$. In particular, we see that if (K,v) is henselian and if $a \in K^{\mathrm{sep}} \setminus K$, then there is no $b \in K$ such that $v(a-b) > \mathrm{kras}(a,K)$. But this result can be improved:

Theorem 13.9 *Let (K,v) be a valued field, and let v be extended to $\tilde{K}$. Then the following assertions are equivalent:*

1) *(K,v) is henselian,*

2) *For every $a \in \tilde{K}$ and every $b \in K$,* $\mathrm{diam}(a,K) \geq v(a-b)\,,$

3) *For every $a \in K^{\mathrm{sep}}$ and every $b \in K$,* $\mathrm{diam}(a,K) \geq v(a-b)\,,$

4) *For every $a \in K^{\mathrm{sep}}$ and every $b \in K$,* $\mathrm{kras}(a,K) \geq v(a-b)\,.$

Proof: 1)⇒2): Assume (K,v) to be henselian, $a \in \tilde{K}$ and $b \in K$. Then for every $\sigma \in \mathrm{Gal}\, K$, we have that $v(\sigma a - b) = v\sigma(a-b) = v(a-b)$ because $v\sigma = v$. Hence, $v(a - \sigma a) \geq \min\{v(a-b), v(\sigma a - b)\} = v(a-b)$. This proves that $\mathrm{diam}(a,K) \geq v(a-b)$.

2)⇒3) is trivial. 3)⇒4) holds since $\mathrm{kras}(a,K) \geq \mathrm{diam}(a,K)$. 4)⇒1): If 4) holds, then trivially, (K,v) satisfies Krasner's Lemma because its assumption is never true. By Theorem 9.1, it follows that (K,v) is henselian. □

Corollary 13.10 *If (K,v) is a henselian field and $a \in \tilde{K}$, then*

$$\mathrm{diam}(a,K) \geq \mathrm{dist}\,(a,K)\ .$$

The question arises whether $\mathrm{dist}\,(a,K)$ is always equal to $\mathrm{diam}(a,K)$. The following lemma, due to Ax [AX2], seems to lend credibility to our presumption:

Lemma 13.11 *Let $(K(a)|K,v)$ be an extension of degree not divisible by the residue characteristic of K. Then $v(a-b) \geq \mathrm{diam}(a,K)$ for $b := [K(a):K]^{-1}\mathrm{Tr}_{K(a)|K}(a)$.*

Proof: We know that $v[K(a):K] = 0$ since the residue characteristic does not divide $[K(a):K]$. We compute $v(a - [K(a):K]^{-1}\mathrm{Tr}_{K(a)|K}(a)) = v([K(a):K]\,a - \mathrm{Tr}_{K(a)|K}(a)) = v\sum a - \sigma a \geq \min v(a - \sigma a) = \mathrm{diam}(a,K)$. □

Corollary 13.12 *Let (K,v) be a valued field of residue characteristic 0. Then for all $a \in \tilde{K}$,*

$$\mathrm{diam}(a,K) \leq \mathrm{dist}\,(a,K)\ ,$$

and if (K,v) is henselian, then

$$\mathrm{diam}(a,K) = \mathrm{dist}\,(a,K)\ .$$

Unfortunately, the situation is not as nice for extensions of degree divisible by the residue characteristic. Here, Ax has shown:

If (K,v) is a henselian field with archimedean value group vK, and if $a \in \tilde{K}$, then

$$\mathrm{diam}(a,K) = \mathrm{dist}\,(a, K^{1/p^\infty})\ . \tag{13.4}$$

But this is not anymore true if the value group is non-archimedean, as we have shown in Example 11.61. If the characteristic of K is $p > 0$, then it may also happen that $\mathrm{diam}(a,K) > \mathrm{dist}\,(a,K)$. At present, it remains to draw the reader's attention to an interesting error in Ax' paper [AX2]. This error was found by S. K. Khanduja, who gave a counterexample in [KH1]. If we have the equality (13.4), does it follow that there is some $b \in K^{1/p^\infty}$ such that $v(a-b) = \mathrm{diam}(a,K)$? After having read Section 11.5, the reader should be able to answer "No!" (this is what we hope). Indeed, if $\mathrm{diam}(a,K) = \mathrm{dist}\,(a,K^{1/p^\infty})$ is the distance of a proper immediate extension of $(K^{1/p^\infty}, v)$, then we know that this distance can not be assumed. So let us reconsider Example **??**. Before we continue, let us note:

Lemma 13.13 *Let K be a field of characteristic $p > 0$ and a a root of the Artin-Schreier polynomial $X^p - X - c$ with $c \in K$. If $a \notin K$, then* $\mathrm{kras}(a,K) = \mathrm{diam}(a,K) = 0$.

Proof: According to Lemma 24.2, the roots of $X^p - X - c$ are $a, a+1, \dots, a+p-1$. If $a \notin K$, then $X^p - X - c$ is irreducible over K, and the conjugates of a are precisely these roots. Hence, $a - \sigma a = i$ for some $i \in \{0, \dots, p-1\}$. If $a \neq \sigma a$, then consequently, $v(a - \sigma a) = 0$. This proves our assertion. □

Example 13.14 Let $(K, v) = (\mathbb{F}_p((t)), v_t)$ and a a root of the Artin-Schreier polynomial $X^p - X - 1/t$. Then $\operatorname{dist}(a, K^{1/p^\infty}) = 0 = \operatorname{diam}(a, K^{1/p^\infty})$ by Corollary 14.12 and the foregoing lemma. But this distance is not assumed by any $b \in K^{1/p^\infty}$ since the extension $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$ is non-trivial and immediate. This is the example given by Khanduja.

To make the example a bit more drastical, one can replace $\mathbb{F}_p$ by $k := \tilde{\mathbb{F}}_p$. For $L := k((t))$, everything works as before, and we obtain an immediate Artin-Schreier extension $(L^{1/p^\infty}(a)|L^{1/p^\infty}, v)$. Now if there were some $b \in L^{1/p^\infty}$ such that $v(a - b) \geq 0$, then the polynomial $X^p - X - (1/t - b^p - b)$ would have integral coefficients. Since $\overline{L} = \tilde{\mathbb{F}}_p$ is algebraically closed, there would exist some $b_0 \in L^{1/p^\infty}$ such that $\overline{b_0}$ is a root of the reduction of $X^p - X - (1/t - b^p - b)$. That is, $v(1/t - b^p - b - b_0^p - b_0) > 0$, and the polynomial $X^p - X - (1/t - b^p - b - b_0^p - b_0)$ would have a root in L according to Example 9.3. This contradicts the fact that $L^{1/p^\infty}(a)|L^{1/p^\infty}$ is an Artin-Schreier extension. ♢

This example disproves Corollary 2 and case b) of Proposition 2′ of [AX2].

13.4 Artin-Schreier extensions of henselian fields

Recall that every Galois extension of degree p of a field K of characteristic $p > 0$ is an Artin-Schreier extension

$$K(\vartheta)|K\,,\ [K(\vartheta):K] = p\,,\ \vartheta^p - \vartheta \in K\,.$$

We have that $(\vartheta + c)^p - (\vartheta + c) = \vartheta^p + c^p - \vartheta - c = \vartheta^p - \vartheta + c^p - c$ (this is the additivity of the Artin-Schreier polynomial $\wp(X) = X^p - X$ in characteristic $p > 0$, cf. Section **??**). For $c \in K$, $K(\vartheta + c) = K(\vartheta)$. Hence if $a, b \in K$ such that $b - a = c^p - c \in \wp(K)$ and if ϑ is a root of $X^p - X - a$, then $\vartheta + c$ is a root of $X^p - X - b$, both generating the same extension of K. Since $i^p - i = 0$ for $i = 0, 1, \dots, p-1 \in K$, we also see that $\vartheta, \vartheta+1, \dots, \vartheta+p-1 \in K(\vartheta)$ are p distinct roots of $X^p - X - a$. Therefore, $X^p - X - a$ splits over $K(\vartheta)$, which shows that $K(\vartheta)|K$ is a Galois extension. We note:

Lemma 13.15 *If* $\operatorname{char} K = p > 0$ *and* $a \equiv b$ *modulo* $\wp(K)$*, i.e.,* $b - a \in \wp(K)$*, then the Artin-Schreier polynomials* $X^p - X - a$ *and* $X^p - X - b$ *generate the same Galois extension of* K*.*

We are interested in the structure of such extensions in case we have additional information about the field K. In particular, if (K, v) is henselian, we want to decide whether a given Artin-Schreier extension is tame or purely wild. This is done by considering the elements of $\vartheta + \wp(K) = \{\vartheta + c^p - c \mid c \in K\}$, which by the above Lemma all generate the same extension.

Throughout this section, let (K, v) be a henselian field with residue characteristic $p > 0$.

Consider an Artin-Schreier extension $K(\vartheta)|K$, and let $f = X^p - X - a$ be the minimal polynomial of ϑ over K. Since f is irreducible, we know from Example 9.3 that $va \leq 0$.

Assume that $va = 0$. We note that in every valued field,

$$v\vartheta > 0 \implies pv\vartheta > v\vartheta \text{ and } v(\vartheta^p - \vartheta) = v\vartheta > 0 \tag{13.5}$$

$$v\vartheta < 0 \implies pv\vartheta < v\vartheta \text{ and } v(\vartheta^p - \vartheta) = pv\vartheta < 0\,. \tag{13.6}$$

and thus,

$$v(\vartheta^p - \vartheta) < 0 \implies v\vartheta = \frac{v(\vartheta^p - \vartheta)}{p} < 0\,. \tag{13.7}$$

Consequently, $a = \vartheta^p - \vartheta$ together with $va = 0$ implies that $v\vartheta = 0$. Further, $\overline{a} \neq 0$ and we know from Example 9.3 that the polynomial $X^p - X - \overline{a}$ does not admit a root in $\overline{K}$. But then, it must be irreducible over $\overline{K}$ (since its splitting field is of degree p over $\overline{K}$ and thus admits no proper subextensions). Hence, it is the minimal polynomial of $\overline{\vartheta}$ over $\overline{K}$. Since it is separable, we find that $(K(\vartheta)|K, v)$ is a tame unramified extension, and $\overline{K(\vartheta)}|\overline{K}$ is an Artin-Schreier extension. Hence,

Lemma 13.16 *Every irreducible Artin-Schreier polynomial $f = X^p - X - a$ over a henselian field (K, v) with Artin-Schreier closed residue field $\overline{K}$ of characteristic p must satisfy $va < 0$.*

Assume that $(K(\vartheta)|K, v)$ is not a tame extension. Then $va < 0$. Moreover, it is then not a subextension of $K^r|K$. Hence, $K^r \cap K(\vartheta) = K$ because $K(\vartheta)|K$ being of prime degree, it admits no proper non-trivial subextension. Since $K^r|K$ is normal by virtue of Theorem 7.27, it follows from Lemma 24.34 that $K(\vartheta)|K$ is linearly disjoint from $K^r|K$. This shows that $(K(\vartheta)|K, v)$ is a purely wild extension.

By Lemma 9.4, $\mathcal{M}_{\mathbf{K}} \subset \wp(K)$, and if $\overline{K}$ is Artin-Schreier closed, then even $\mathcal{O}_{\mathbf{K}} \subset \wp(K)$. We thus obtain, by an application of Lemma 13.15:

Corollary 13.17 *Let (K, v) be a henselian field of characteristic p. Further, let $\vartheta, \eta \in \tilde{K}$ such that $\vartheta^p - \vartheta = a \in K$ and $\eta^p - \eta = b \in K$. If*

$$a \equiv b \pmod{\mathcal{M}_{\mathbf{K}}}\,,$$

then $K(\vartheta) = K(\eta)$. If in addition $\overline{K}$ is Artin-Schreier closed, then this holds already if $a \equiv b$ modulo $\mathcal{O}_{\mathbf{K}}$.

The following lemma gives a classification of Artin-Schreier extensions in positive characteristic. We set $v(a - \wp(K)) = \{v(a - c^p + c) \mid c \in K\}$.

Lemma 13.18 *Let (K, v) be a henselian field of characteristic $p > 0$ and $K(\vartheta)|K$ an Artin-Schreier extension with $\vartheta^p - \vartheta = a \in K$. Then $v(a - \wp(K)) \leq 0$ and there are the following cases:*

<u>case 1:</u> $v(a - \wp(K))$ has maximal element 0. Then $(K(\vartheta)|K, v)$ is a tame unramified extension and $\overline{K(\vartheta)}|\overline{K}$ is an Artin-Schreier extension.

<u>case 2:</u> $v(a - \wp(K))$ has a maximal element < 0. Then $(K(\vartheta)|K, v)$ is a defectless purely wild extension.

<u>case 3:</u> $v(a - \wp(K))$ has no maximal element. Then $(K(\vartheta)|K, v)$ is an immediate purely wild extension.

Case 2 will occur if and only if there is some $c \in K$ such that
case 2.1: $v(a - c^p + c) \notin pvK$, *in which case* $(vK(\vartheta) : vK) = p = [K(\vartheta) : K]$,
case 2.2: *there is some* $d \in K$ *such that* $vd^p = v(a - c^p + c)$ *but* $\overline{(a - c^p + c)/d^p} \notin \overline{K}^p$, *in which case* $[\overline{K(\vartheta)} : \overline{K}] = p = [K(\vartheta) : K]$.
In both cases, $v(a - c^p + c)$ *is the maximal value in* $v(a - \wp(K))$.

Proof: Suppose that $c \in K$ is such that $v(a - c^p + c) > 0$. But then by Example 9.3, the polynomial $X^p - X - (a - c^p + c)$ splits over K and consequently, its root $\vartheta - c$ lies in K, contrary to our assumption that $K(\vartheta)|K$ is an Artin-Schreier extension. Hence there is no such c, showing that $v(a - \wp(K)) \leq 0$.

Suppose that $c \in K$ is such that $v(a - c^p + c) = 0$. Then by what we have shown above, the root $\vartheta - c$ of $X^p - X - (a - c^p + c)$ generates a tame unramified extension $K(\vartheta - c) = K(\vartheta)$ of K, and the residue field $\overline{K(\vartheta)} = \overline{K}(\overline{\vartheta - c})$ is an Artin-Schreier extension.

Now let $c \in K$ be such that $vb < 0$ for $b := a - c^p + c$. By (13.6), applied to $\vartheta - c$ in the place of ϑ, we find that $v(\vartheta - c) = vb/p$. Hence if $vb \notin pvK$, then $v(\vartheta - c) \notin vK$. By the fundamental inequality $n = d \cdot e \cdot f$ it then follows that $(vK(\vartheta) : vK) = p = [K(\vartheta) : K]$, showing that $(K(\vartheta)|K, v)$ is a defectless purely wild extension. Now assume that $vb \in pvK$ and choose $d \in K$ such that $vd^p = vb$. We compute

$$v\left(\left(\frac{\vartheta - c}{d}\right)^p - \frac{b}{d^p}\right) = v\left((\vartheta - c)^p - b\right) - vd^p = v(\vartheta - c) - vb = -\frac{p-1}{p} vb > 0 \,.$$

This shows that $\overline{(\vartheta - c)/d} = \overline{b/d^p}^{1/p}$. Hence if $\overline{b/d^p} \notin \overline{K}^p$, then by the fundamental inequality, $[\overline{K(\vartheta)} : \overline{K}] = p = [K(\vartheta) : K]$, and $(K(\vartheta)|K, v)$ is a defectless purely wild extension. But if $\overline{b/d^p} = \overline{d_1}^p$ for some $d' \in K$, then $v(b/d^p - d_1^p) > 0$ and $0 > vdd_1 > v(dd_1)^p = vb$, which implies that

$$v(b - (dd_1)^p + dd_1) \geq v(b - (dd_1)^p) > vd^p = vb \,.$$

Then with $c_1 := c + dd_1$, we obtain

$$v(a - c_1^p + c_1) = v(a - (c + dd_1)^p + c + dd_1) = v(b - (dd_1)^p + dd_1) > vb = v(a - c^p + c) \,,$$

showing that $v(a - c^p + c)$ was not the maximal value in $v(a - \wp(K))$.

Assume that $a - c^p + c$ satisfies the assumptions of case 2.1 or 2.2. It remains to show that $v(a - c^p + c)$ is the maximal value in $v(a - \wp(K))$. Assume to the contrary that $c_1 \in K$ such that $v(a - c_1^p + c_1) > v(a - c^p + c)$. Then $v(a - c^p + c - (c_1 - c)^p + (c_1 - c)) > v(a - c^p + c)$ and $0 > v(c_1 - c) > v(c_1 - c)^p$. Consequently, $v(a - c^p + c - (c_1 - c)^p) > v(a - c^p + c)$, which yields that $v(a - c^p + c) = v(c_1 - c)^p) \in pvK$, showing that $a - c^p + c$ does not satisfy the assumption of case 2.1. So there must be some $d \in K$ such that $vd^p = v(a - c^p + c)$ and $\overline{(a - c^p + c)/d^p} \notin \overline{K}^p$. On the other hand, $v(a - c^p + c - (c_1 - c)^p) > v(a - c^p + c)$ yields that $v((a - c^p + c)/(c_1 - c)^p - 1) > 0$. Since $d^p/(c_1 - c)^p \in K^p$, we would obtain that $1 = \overline{(a - c^p + c)/(c_1 - c)^p} = \overline{(a - c^p + c)/d^p} \cdot \overline{d^p/(c_1 - c)^p} \notin \overline{K}^p$, a contradiction.

Finally, assume that $v(a - \wp(K))$ has no maximum. Then $v(a - \wp(K)) < 0$, and equation (13.7) shows that also $\Lambda^L(\vartheta, K) < 0$ has no maximum. This means that at (ϑ, K) is immediate (cf. **??**), and by Lemma 11.72 we find that $(K(\vartheta)|K, v)$ is immediate and thus a purely wild extension. □

13.5 Kummer extensions of henselian fields

If the field K has characteristic 0 and contains all p-th roots of unity, then every Galois extension of K of degree p is a Kummer extension. So we wish to carry through a similar analysis as we did for Artin-Schreier extensions in the last section. We note the following easy observation, which is the analogue to Lemma 13.15:

Lemma 13.19 *If $a \equiv b \pmod{(K^\times)^p}$ in the multiplicative sense, that is, if $b \in a(K^\times)^p$, then the elements $\vartheta, \eta \in \tilde{K}$ with $\vartheta^p = a \in K$ and $\eta^p = b \in K$ generate the same extension of K. (This extension is normal if and only if K contains the p-th roots of unity).*

As a first preparation, we note:

Lemma 13.20 *Let (K, v) an arbitrary valued field of residue characteristic $p > 0$ and $\vartheta \in \tilde{K}$ such that $\vartheta^p = b \in K$. Then there are the following cases:*

case 1: $vb \notin pvK$. Then $(vK(\vartheta) : vK) = p = [K(\vartheta) : K]$, and the extension $(K(\vartheta)|K, v)$ is defectless.

case 2: $vb \in pvF$. Then there exists an element $d \in K^\times$ such that $vd^pb = 0$ and $K(\vartheta) = K(d\vartheta)$.

case 2.1: $\overline{d^pb} \notin \overline{K}^p$. Then $[\overline{K(\vartheta)} : \overline{K}] = p = [K(\vartheta) : K]$, $\overline{K(\vartheta)}|\overline{K}$ is purely inseparable and the extension $(K(\vartheta)|K, v)$ is defectless.

case 2.2: $\overline{d^pb} \in \overline{K}^p$.
Then there exists an element $d_1 \in \mathcal{O}_K^\times$ such that $\overline{d_1^p d^p b} = 1$ and $K(\vartheta) = K(d_1 d\vartheta)$.

If (K, v) is henselian, then in case 1 and case 2.1, the extension $(K(\vartheta)|K, v)$ is purely wild. Only in case 2.2 we do not know its structure. For a further analysis of this case, we may replace b by $d_1^p d^p b$ and ϑ by $d_1 d\vartheta$. Then we have that $vb = 0$ and $\bar{b} = 1$, thus $b = 1 + a$ with $a \in \mathcal{M}_{\mathbf{K}}$. Note that this implies $v\vartheta = 0$. We write $v(a - K^p) = \{v(a - c^p) \mid c \in K\}$.

Lemma 13.21 *Let (K, v) an arbitrary valued field of residue characteristic $p > 0$ and $\vartheta \in \tilde{K}$ such that $\vartheta^p = 1 + a \in K$ with $va > 0$. Then there are the following cases:*

case 1: $v(a - K^p) \leq vp$.

case 1.1: $v(a - K^p)$ has a maximal element. Then $(K(\vartheta)|K, v)$ is a defectless purely wild extension.

case 1.2: $v(a - K^p)$ has no maximal element. Then $(K(\vartheta)|K, v)$ is an immediate purely wild extension.

case 2: There is some $c \in K$ such that $v(a - c^p) > vp$. Then $\vartheta^p(1 - c)^{-p} \equiv 1$ modulo $p\mathcal{M}_{\mathbf{K}}$.

Proof: If $v(a - K^p)$ has a maximal element $v(a - c^p) \leq vp$, then either $v(a - c^p) \notin pvK$, or there is some $d \in K$ such that $vd^p = v(a - c^p)$ and $\overline{(a - c^p)/d^p} \notin \overline{K}^p$. Indeed, otherwise there would exist some $c_1 \in K$ such that $v(a - c^p - c_1^p) > v(a - c^p)$. Note that $vc > 0$ and $vc_1 > 0$, so $v(c^p + c_1^p - (c + c_1)^p) > vp$. Hence, $v(a - (c + c_1)^p) > v(a - c^p)$, contradicting our maximality assumption.

Applying the transformation $X = Y + 1$ to the polynomial $X^p - 1$, we obtain the polynomial $f(Y) = Y^p + g(Y)$ with $g(Y) \in p\mathcal{O}_{\mathbf{K}}[Y]$. Hence if $c \in K$ and $b \in K(\vartheta)$ with $(\vartheta - c)^p = 1 + b$ and $vb < vp$, then $v((\vartheta - 1 - c)^p - b = vg(\vartheta - c) \geq vp > vb$. On the other hand, $b = (\vartheta - c)^p - 1 \equiv \vartheta^p - c^p - 1 = 1 + a - c^p - 1 = a - c^p$ modulo $\mathcal{M}_{\mathbf{K}(\vartheta)}$, showing that $v((\vartheta - c - 1)^p - (a - c^p)) > v(a - c^p)$. So if $v(a - c^p) \leq vp$ is the maximal element of $v(a - K^p)$, then either $v(\vartheta - 1 - c) \notin vK$ or there is some $d \in K$ such that $vd = v(\vartheta - 1 - c)$ and $\overline{(\vartheta - 1)/d} \notin \overline{K}^p$. In both cases, $(K(\vartheta)|K, v)$ is a defectless purely wild extension.

But if $v(a - K^p) \leq vp$ and $v(a - K^p)$ has no maximal element, then the above computation shows that $\Lambda^L(\vartheta - 1, K) \leq (vp)/p$ and that $\Lambda^L(\vartheta - 1, K)$ has no maximal element. Then by Lemma **??**, $(K(\vartheta)|K, v)$ is immediate and thus a purely wild extension.

Finally, assume that there is some $c \in K$ such that $v(a - c^p) > vp$. Then $1 + a \equiv 1 + c^p \equiv (1 + c)^p$ modulo $p\mathcal{M}_{\mathbf{K}}$. Since $vc = (va)/p > 0$, we have that $v(1 + c) = 0$. Hence the foregoing equivalence implies $(\vartheta^p(1 - c)^{-p} = (1 + a)(1 + c)^{-p} \equiv 1$ modulo $p\mathcal{M}_{\mathbf{K}}$. □

In case 2, we still do not know the structure of the extension $(K(\vartheta)|K, v)$. But we can replace ϑ by $\vartheta(1 - c)^{-1}$ and a by $\vartheta^p(1 - c)^{-p} - 1$. Hence, we may assume from now on that $va > p$. For the further analysis, We define

$$\tilde{\wp}(X) := X^p + pX$$

and set $v(a - \tilde{\wp}(K)) = \{v(a - c^p - pc) \mid c \in K\}$.

Lemma 13.22 *Let (K, v) an arbitrary valued field of residue characteristic $p > 0$ and $K(\vartheta)|K$ a Kummer extension such that $\vartheta^p = 1 + a \in K$ with $va > vp$. Then*

$$v(a - \tilde{\wp}(K)) \;\leq\; \frac{p}{p-1} vp\,,$$

and there are the following cases:

case 1: $v(a - \tilde{\wp}(K))$ has maximal element $\frac{p}{p-1}vp$. Then $(K(\vartheta)|K, v)$ is a tame unramified extension and $\overline{K(C, \vartheta)}|\overline{K(C)}$ is an Artin-Schreier extension.

case 2: $v(a - \tilde{\wp}(K))$ has a maximal element $< \frac{p}{p-1}vp$. Then $(K(\vartheta)|K, v)$ is a defectless purely wild extension.

case 3: $v(a - \tilde{\wp}(K))$ has no maximal element. Then $(K(\vartheta)|K, v)$ is an immediate purely wild extension.

Case 2 will occur if and only if there is some $c \in K$ such that

case 2.1: $v(a - c^p + pc) \notin pvK$, in which case $(vK(\vartheta) : vK) = p = [K(\vartheta) : K]$,

case 2.2: there is some $d \in K$ such that $vd^p = v(a - c^p + c)$ but $\overline{(a - c^p + pc)/d^p} \notin \overline{K}^p$, in which case $[\overline{K(\vartheta)} : \overline{K}] = p = [K(\vartheta) : K]$.

In both cases, $v(a - c^p + c)$ is the maximal value in $v(a - \tilde{\wp}(K))$.

Proof: Assume that there is some $c \in K$ such that $v(a - c^p - pc) \geq \frac{p}{p-1}vp$. If $v(a - c^p - pc) > \frac{p}{p-1}vp$, then by Lemma 9.36, $1 + a - c^p - pc \in (K(C)^\times)^p$ and thus, $1 + a \in (K(C)^\times)^p$. This would yield that $\vartheta \in K(C)$.

Assume now that there is some $c \in K$ such that $vb \geq \frac{p}{p-1}vp$ for $b := a - c^p - pc$. Then $vc^p > vp$ because otherwise, $vc^p < vpc$ and $v(a - c^p - pc) = \min\{va, vc^p, vpc\} = vc^p < \frac{p}{p-1}vp$. Hence part d) of Corollary 9.37 shows that $1 + a \in (1 + b)(K(C)^\times)^p$. Applying the transformation $X = CY + 1$ to the polynomial $X^p - 1 - b$ and dividing by $C^p = -pC$, we obtain the polynomial

$$f(Y) = Y^p + g(Y) - Y - bC^{-p}$$

with $g(Y)$ as in (9.15) in the proof of Lemma 9.35. By assumption, $vbC^{-p} \geq 0$. We find that $\overline{f(Y)} = Y^p - Y - \overline{bC^{-p}}$ is an Artin-Schreier polynomial. Assume that it is reducible over $\overline{K(C)}$. Then it splits over $\overline{K(C)}$ and by virtue of Hensel's Lemma, f splits over $K(C)$. This implies that $X^p - 1 - b$ splits over $K(C)$, which means that $\vartheta \in K(C)$. But $[K(C) : K] < p$ by Lemma 9.36, contradicting our assumption that $K(\vartheta)|K$ be a Kummer extension. This proves f must be irreducible over $K(C)$. Hence, $vbC^{-p} = 0$ and $v(a - \tilde{\wp}(K)) \leq \frac{p}{p-1}vp$. The former yields that $\overline{K(C,\vartheta)}|\overline{K(C)}$ is an Artin-Schreier extension. Consequently, $(K(C,\vartheta)|K(C), v)$ is a tame extension. Since also $(K(C)|K, v)$ is a tame extension, the same holds for $(K(C,\vartheta)|K, v)$ and its subextension $(K(\vartheta)|K, v)$.

From now on, assume that $v(a - \tilde{\wp}(K)) < \frac{p}{p-1}vp$. Applying the transformation $X = Y + 1$ to the polynomial $X^p - 1$, we obtain the polynomial

$$f(Y) = Y^p + \tilde{g}(Y) + pY$$

where $\tilde{g}(Y) = \sum_{i=2}^{p-1} \binom{p}{i} Y^i \in pY\mathcal{O}_{\mathbf{K}}[Y]$. We will show that this polynomial is "additive" in the following sense: for $\alpha := \frac{p}{p-1}vp$,

$$v\zeta, v\xi > \frac{vp}{p} \implies f(\zeta + \xi) \equiv f(\zeta) + f(\xi) \pmod{\mathcal{O}^\alpha}, \tag{13.8}$$

where $\mathcal{O}^\alpha = \{x \in K(\zeta,\xi) \mid vx \geq \alpha\}$. We first observe that $\frac{p+2}{p} \geq \frac{p}{p-1}$ for every prime p. Hence,

$$v\zeta > \frac{vp}{p} \implies v\tilde{g}(\zeta) \geq vp + 2v\zeta > \frac{p+2}{p}vp \geq \frac{p}{p-1}vp\,.$$

Therefore,

$$v\zeta > \frac{vp}{p} \implies f(\zeta) \equiv \zeta^p + p\zeta \pmod{\mathcal{O}^\alpha}.$$

On the other hand, the polynomial X^p is additive in the same sense:

$$v\zeta, v\xi > \frac{vp}{p} \implies (\zeta+\xi)^p = \zeta^p + \xi^p + p\zeta\xi \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \zeta^{i-1}\xi^{p-i-1} \equiv \zeta^p + \xi^p \pmod{\mathcal{O}^\alpha}$$

Consequently,

$$v\zeta, v\xi > \frac{vp}{p} \implies f(\zeta+\xi) \equiv (\zeta+\xi)^p - p(\zeta+\xi) \equiv \zeta^p + p\zeta + \xi^p + p\xi \equiv f(\zeta) + f(\xi) \pmod{\mathcal{O}^\alpha}$$

which proves (13.8). We find that for $c \in K$ with $v(\vartheta - 1 - c) > vp$,

$$f(\vartheta - 1 - c) \equiv f(\vartheta - 1) - f(c) = a - f(c) \equiv a - c^p - pc \pmod{\mathcal{O}^\alpha_{\mathbf{K}(\vartheta)}}.$$

Since $v(a - c^p - pc) < \frac{p}{p-1}vp$, it follows that $pv(\vartheta - 1 - c) < v(\vartheta - 1 - c) + vp$ and thus,

$$v\left((\vartheta - 1 - c)^p - (a - c^p - pc)\right) = v(\vartheta - 1 - c) > v(a - c^p - pc)\,.$$

Now the proof is similar to that of the foregoing lemma. □

13.6 Minimal purely wild extensions

Let (K, v) be a henselian field which is not tame and thus admits purely wild extensions. We will call a purely wild extension $(L|K, v)$ a **minimal purely wild extension** if it does not admit a proper non-trivial subextension. In this section, we will prove a result due to F. Pop (cf. [POP1]): every minimal purely wild extension is generated by the root ϑ of a polynomial $\mathcal{A}(X) - \mathcal{A}(\vartheta)$ where $\mathcal{A}(X)$ is an additive polynomial. This result will follow if we are able to show that the extension $L|K$ satisfies condition (12.9) which is the hypothesis of Lemma 12.28. As a natural candidate for an extension $K'|K$ which is Galois and linearly disjoint from $L|K$, we may take the maximal algebraic extension which is linearly disjoint from every purely wild extension, namely $K^r|K$ (cf. Lemma 13.5). By Lemma 7.28 we know that $L.K^r = L^r$. We set

$\mathcal{G}$:= $\mathrm{Gal}\, K$
$\mathcal{N}$:= $\mathrm{Gal}\, K^r$, which is a normal subgroup of $\mathcal{G}$ by Lemma 7.6 and a pro-p-group by Theorem 7.16
$\mathcal{H}$:= $\mathrm{Gal}\, L$, which is a maximal proper subgroup of $\mathcal{G}$ since $L|K$ is a minimal non-trivial extension, and which satisfies $\mathcal{N}.\mathcal{H} = \mathcal{G}$ since $K^r|K$ is linearly disjoint from $L|K$
$\mathcal{D}$:= $\mathcal{N} \cap \mathcal{H} = \mathrm{Gal}\, L.K^r = \mathrm{Gal}\, L^r$.

The next lemma examines this group theoretical situation.

Lemma 13.23 *Let $\mathcal{G}$ be a profinite group with maximal proper subgroup $\mathcal{H}$. Assume that the non-trivial pro-p-group $\mathcal{N}$ is a normal subgroup of $\mathcal{G}$ not contained in $\mathcal{H}$. Then $\mathcal{D} = \mathcal{N} \cap \mathcal{H}$ is a normal subgroup of $\mathcal{G}$ and the finite factor group $\mathcal{N}/\mathcal{D}$ is an elementary-abelian p-group. Further, $\mathcal{N}/\mathcal{D}$ is an irreducible right $\mathcal{G}/\mathcal{D}$-module.*

Proof: Since $\mathcal{N} \not\subset \mathcal{H}$, we have that $\mathcal{D}$ is a proper subgroup of $\mathcal{N}$. Since every maximal proper subgroup of a profinite group is of finite index, we have that $(\mathcal{N} : \mathcal{D}) = (\mathcal{G} : \mathcal{H})$ is finite. Observe that $\mathcal{D}$ is **$\mathcal{H}$-invariant** (which means that $\mathcal{D}^\sigma = \mathcal{D}$ for every $\sigma \in \mathcal{H}$). This is true since $\mathcal{N} \lhd \mathcal{G}$ and $\mathcal{H}$ are $\mathcal{H}$-invariant. Assume that $\mathcal{E}$ is an $\mathcal{H}$-invariant subgroup of $\mathcal{N}$ containing $\mathcal{D}$. Then $\mathcal{H}\mathcal{E}$ is a subgroup of $\mathcal{G}$ containing $\mathcal{H}$. From the maximality of $\mathcal{H}$ it follows that either $\mathcal{H}\mathcal{E} = \mathcal{H}$ or $\mathcal{H}\mathcal{E} = \mathcal{G}$, whence either $\mathcal{E} = \mathcal{D}$ or $\mathcal{E} = \mathcal{N}$ (this argument is as in the proof of Lemma 12.30). We have proved that $\mathcal{H}\mathcal{N} = \mathcal{G}$ and that $\mathcal{D}$ is a maximal $\mathcal{H}$-invariant subgroup of $\mathcal{N}$.

Now let $\Phi(\mathcal{N})$ denote the Frattini subgroup of $\mathcal{N}$. Since $\mathcal{D} \neq \mathcal{N}$, it follows that $\mathcal{D}.\Phi(\mathcal{N}) \neq \mathcal{N}$. Being a characteristic subgroup of $\mathcal{N}$, the Frattini subgroup $\Phi(\mathcal{N})$ is $\mathcal{H}$-invariant like $\mathcal{N}$. Consequently, also the group $\mathcal{D}.\Phi(\mathcal{N})$ is $\mathcal{H}$-invariant. From the maximality of $\mathcal{D}$ we deduce that $\mathcal{D}.\Phi(\mathcal{N}) = \mathcal{D}$, showing that

$$\Phi(\mathcal{N}) \subset \mathcal{D} .$$

On the other hand, for the p-group $\mathcal{N}$ we know by part a) of Corollary 24.55 that the factor group $\mathcal{N}/\Phi(\mathcal{N})$ is a (possibly infinite dimensional) $\mathbb{F}_p$-vector space. In view of $\Phi(\mathcal{N}) \subset \mathcal{D}$, this yields that also $\mathcal{D}$ is a normal subgroup of $\mathcal{N}$ and that also $\mathcal{N}/\mathcal{D}$ is an elementary-abelian p-group. Since $\mathcal{D}$ is $\mathcal{H}$-invariant, $\mathcal{D} \lhd \mathcal{N}$ implies that

$$\mathcal{D} \lhd \mathcal{H}\mathcal{N} = \mathcal{G} .$$

As a normal subgroup of $\mathcal{G}$, $\mathcal{N}$ is a $\mathcal{G}$-module, and in view of $\mathcal{D} \lhd \mathcal{G}$ it follows that $\mathcal{N}/\mathcal{D}$ is a $\mathcal{G}/\mathcal{D}$-module. If it were reducible then there would exist a proper subgroup $\mathcal{E}$ of $\mathcal{G}$ such that $\mathcal{E}/\mathcal{D}$ is a non-trivial $\mathcal{G}/\mathcal{D}$-module. But then, $\mathcal{E}$ must be a normal subgroup of $\mathcal{G}$ properly containing $\mathcal{D}$; in particular, $\mathcal{E}$ would be $\mathcal{H}$-invariant in contradiction to the maximality of $\mathcal{D}$. □

This lemma shows that $L^r|K$ is Galois and $L^r = L.K^r$ is a finite p-elementary extension of K^r. Hence $L|K$ satisfies (12.9) with $K' = K^r$. We may now apply Lemma 12.28 to obtain that there exist an additive polynomial $\mathcal{A}(X) \in K[X]$ and an element $\vartheta \in L$ such that $L = K(\vartheta)$ and $\mathcal{A}(\vartheta) \in K$. Since $L|K$ is a minimal purely wild extension by our assumption, it is in particular minimal with property (12.9) and thus satisfies the hypothesis of Lemma 12.30. Hence, $\mathcal{A}(X)$ may be chosen such that its coefficients lie in the ring $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H]$, where b is the generator of a normal basis of $K'|K$. Since vK is cofinal in $v\tilde{K} = \widetilde{vK}$, we may choose some $c \in K$ such that $vcb \geq 0$. Since (K,v) is henselian by assumption, it follows that $v\sigma(cb) = vcb \geq 0$ for all $\sigma \in \mathrm{Gal}\,K$. On the other hand, cb is still the generator of a normal basis of $K'|K$. So we may replace b by cb, which yields that $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H] \subset \mathcal{O}_{\mathbf{K}}$ and consequently, that $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{K}}[X]$.

Now assume in addition that k is a subfield of K such that the extension $(K|k,v)$ is immediate. Then we may infer from Lemma 7.30 that $K^r = k^r.K$. So the Galois extension K' of K is the compositum of K with a suitable Galois extension k' of k. In this case, b may be chosen to be already the generator of a normal basis of k' over k; it will then also be the generator of a normal basis of K' over K. With this choice of b, we obtain that the ring $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H]$ is contained in $K \cap k' = k$, yielding that $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{k}}[X]$. Let us summarize what we have proved:

Theorem 13.24 *Let (K,v) be a henselian field and $(L|K,v)$ a minimal purely wild extension. Then $L^r|K$ is a Galois extension and $L^r|K^r$ is a p-elementary extension. Hence, $L|K$ satisfies condition (12.9), and there exist an additive polynomial $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{K}}[X]$ and an element $\vartheta \in L$ such that $L = K(\vartheta)$ and $\mathcal{A}(\vartheta) \in K$. If k is a subfield of K such that the extension $(K|k,v)$ is immediate, then $\mathcal{A}(X)$ may already be chosen in $\mathcal{O}_{\mathbf{k}}[X]$.*

Let us conclude this section by discussing the following special case. Assume that the value group vK is divisible by all primes $q \neq p$. Then by Theorem 7.27, $K^r|K$ is an unramified extension. Consequently, $\mathrm{Gal}\,K'|K \cong \mathrm{Gal}\,\overline{K'}|\overline{K}$ and we may choose the element b such that $\bar{b}$ is the generator of a normal basis of $\overline{K'}|\overline{K}$. It follows that the residue map is injective on the ring $\mathbb{F}_p[\rho b \mid \rho \in H]$ and thus, also the map $\tau \mapsto \overline{\phi(\tau)}$ is injective. In this case, we obtain that

$$\overline{\mathcal{A}}(X) = \prod_{\tau\in N} (X - \overline{\phi(\tau)})$$

has no multiple roots and is thus separable.

13.7 Complements of normal subgroups

We will now look for criteria for a normal subgroup of a profinite group G to admit a group complement. In this case, we are naturally led to the question whether a given group complement is unique up to conjugation.

The following Theorem of Gaschütz gives a criterion for normal subgroups of finite groups to admit a group complement. We shall use the following generalization to profinite groups:

Theorem 13.25 *Let G be a profinite group, $\mathfrak{G}$ a closed subgroup and N a finite abelian subgroup of $\mathfrak{G}$. Assume further that N is normal in G. If $(G:\mathfrak{G})$ is relatively prime to $|N|$ and if N admits a closed group complement in $\mathfrak{G}$, then N admits a closed group complement in G.*

Proof: I) The case of G a finite group. Let $\mathfrak{H}$ be a complement of N in $\mathfrak{G}$. Further, let $\mathfrak{R}$ be a set of representatives for the right cosets of $\mathfrak{G}$ in G, that is, for every $g \in G$ there is a unique element of $\mathfrak{R}$ which lies in $\mathfrak{G}g$. For fixed $g \in G$, the equation $\mathfrak{G}rg = \mathfrak{G}r'$, $r, r' \in \mathfrak{R}$, defines a bijective map from $\mathfrak{R}$ onto itself. It follows that $rgr'^{-1} \in \mathfrak{G}$ for all $r \in \mathfrak{R}$. Since $\mathfrak{H}$ may be used as a set of representatives for the left cosets of N in $\mathfrak{G}$, there is a unique $\tilde{r} \in \mathfrak{H}$ for every $r \in \mathfrak{R}$ such that $rgr'^{-1}N = \tilde{r}N$. Using the normality of N, we find that this is equivalent to $rgNr'^{-1} = \tilde{r}N$, to $rgN = \tilde{r}Nr'$ and finally, to $rgN = \tilde{r}r'N$. In particular, $g^{-1}r^{-1}\tilde{r}r' \in N$. Hence,

$$\phi(g) := \prod_{r\in\mathfrak{R}} g^{-1}r^{-1}\tilde{r}r'$$

defines a map from G into N. Let $h \in G$ and $r'hN = \tilde{r'}r''N$, $r', r'' \in \mathfrak{R}$, $\tilde{r'} \in \mathfrak{H}$. Using again the normality of N, we compute

$$rghN = rgNh = \tilde{r}r'Nh = \tilde{r}r'hN = \tilde{r}\tilde{r'}r''N\ .$$

We may conclude that $r'' \in \mathfrak{R}$ and $\tilde{r}\tilde{r'} \in \mathfrak{H}$ are the unique representatives such that $\mathfrak{G}rgh = \mathfrak{G}r''$ and $rghr''^{-1}N = \tilde{r}\tilde{r'}N$; here, we have used our hypothesis that $\mathfrak{H}$ be a group. Further,

$$h^{-1}g^{-1}r^{-1}\tilde{r}\tilde{r'}r'' = h^{-1}g^{-1}r^{-1}\tilde{r}\,r'hh^{-1}r'^{-1}\,\tilde{r'}r'' = (g^{-1}r^{-1}\tilde{r}r')^h h^{-1}r'^{-1}\tilde{r'}r''\ .$$

Since N is abelian by assumption, it follows that $\phi(gh) = \phi(g)^h\phi(h)$, that is, ϕ is a twisted homomorphism from G into N. Let $H \subset G$ be its kernel. For $g \in N$ we have $rgN = rN$ which implies that $r = r'$ and $\tilde{r} = 1$ for all $r \in \mathfrak{R}$. Hence, $\phi(g) = g^{|\mathfrak{R}|} = g^{-(G:\mathfrak{G})}$ for all $g \in N$. Since we have assumed $(G:\mathfrak{G})$ to be relatively prime to $|N|$ it follows that ϕ is an automorphism on N. In particular, $H \cap N$ is trivial.

Now let $g \in G$ and $a := \phi(g) \in N$. Then we may choose $b \in N$ such that $\phi(b) = a$. Since N is assumed to be abelian, we find $\phi(gb^{-1}) = \phi(g)^{b^{-1}}\phi(b^{-1}) = \phi(g)a^{-1} = 1$. Hence $gb^{-1} \in H$, showing that $g \in HN$. We have thus shown that H is a group complement of N in G.

II) The case of G an arbitrary profinite group. Let $\mathfrak{H}$ be a closed complement of N in $\mathfrak{G}$. Since since $(\mathfrak{G}:\mathfrak{H}) = |N|$ is finite, $\mathfrak{H}$ is open in $\mathfrak{G}$. We may choose an open subset U of G such that $U \cap \mathfrak{G} = \mathfrak{H}$. Since U is an open neighborhood of 1 and since the open normal subgroups of G form a basis for these neighborhoods, there exists an open normal subgroup $\mathcal{N}$ of G such that $\mathcal{N} \cap \mathfrak{G} \subset \mathfrak{H}$. We show that $N\mathcal{N} \cap \mathfrak{H}\mathcal{N} = \mathcal{N}$. Let $g = nn_1 = hn_2$ with $n \in N$, $h \in \mathfrak{H}$ and $n_1, n_2 \in \mathcal{N}$. Then $h^{-1}n = n_2n_1^{-1} \in \mathfrak{G} \cap \mathcal{N} \subset \mathfrak{H}$ and thus, $n \in N \cap \mathfrak{H} = \{1\}$. Consequently, $g = n_1 \in \mathcal{N}$.

By the first part of our proof, there exists a group complement of $N\mathcal{N}/\mathcal{N}$ in the finite group $G/\mathcal{N}$. Let H be the foreimage in G of this complement under the canonical epimorphism. Being the foreimage of a finite group, H is an open and closed subgroup of G. Since H contains $\mathcal{N}$, we find $HN = \mathcal{N}HN = G$ and $N\mathcal{N} \cap H = \mathcal{N}$. It follows that $N \cap H \subset (N \cap N\mathcal{N}) \cap H \subset N \cap \mathcal{N} \subset N \cap \mathfrak{H} = \{1\}$, showing that H is a complement of N in G. □

The Theorem of Schur – Zassenhaus gives a criterion for normal subgroups of finite groups to admit a group complement which is unique up to conjugation. Using the foregoing theorem, we shall prove a special form and its generalization to profinite groups:

Theorem 13.26 *Let G be a profinite group and N a prosolvable normal subgroup of G. Suppose that the index $(G : N)$ is relatively prime to the order $|N|$. Then there exists a closed group complement of N in G. Moreover, all closed group complements of N are conjugate within G.*

Proof: I) The case of G a finite group. We shall first prove the assertion under the additional assumption that N be abelian. Afterwards, the general case will be proved by an induction argument.

For N abelian, the existence of H follows from the foregoing theorem, where we take $\mathfrak{G} = N$. It remains to prove the uniqueness of H up to conjugation. We use the notations of the foregoing proof, setting $\mathfrak{G} = N$ and consequently, $\mathfrak{H} = \{1\}$ and $\tilde{r} = 1 = \tilde{r'}$. First, we observe the following: If H_0 is an arbitrary group complement of N in G, then we may take H_0 as set of representatives for the right cosets of N in G, and we may consider the map ϕ defined with respect to this set of representatives. If $g \in H_0$ and $rgN = r'N$, then $r' = rg$ since H_0 is a group. It follows that $\phi(g) = 1$ for all $g \in H_0$. On the other hand, we know that $\ker\phi \cap N = \{1\}$. So we find that $H_0 = \ker\phi$. It thus suffices to show: If $\mathfrak{R}$, $\mathfrak{R}_0$ are two set of representatives for the right cosets of N in G and ϕ, ϕ_0 the maps defined with respect to them, then $\ker\phi$ and $\ker\phi_0$ are conjugate in G.

More precisely, we shall prove the existence of some $b \in N$ such that

$$\phi_0(g) = \phi(b^{-1}gb) \quad \text{for all } g \in G\ .$$

For $r \in \mathfrak{R}$ we let $r_0 \in \mathfrak{R}$ such that $rN = r_0N$, and $n_r \in N$ such that $r_0 = rn_r$. Now let $g \in G$ and $rgN = r'N$, $r_0gN = (r_0)'N$. Note that $(r_0)'N = r_0gN = r_0Ng = rNg = rgN = r'N$. It follows that $(r_0)' = (r')_0$ and thus, $(r_0)' = n_{r'}r'$. We compute

$$g^{-1}r_0^{-1}(r_0)' = g^{-1}n_r^{-1}r^{-1}r'n_{r'} = n_r^{-g}(g^{-1}r^{-1}r')n_{r'} = n_r^{-g}n_{r'}(g^{-1}r^{-1}r')\ .$$

Setting $a := \prod_{r\in\mathfrak{R}} n_r = \prod_{r\in\mathfrak{R}} n_{r'} \in N$ and choosing $b \in N$ such that $\phi(b) = a$, we deduce

$$\phi_0(g) = a^{-g}a\phi(g) = \phi(b^{-1}gb)\ .$$

For N not abelian, we use induction on $|N|$. If the commutator subgroup N' is trivial, then N is abelian and our assertion is proved. Assume that N' is not trivial. By what we have proved, the assertion holds for the group G/N' with its abelian normal subgroup N/N'. So let G_1 be a subgroup of G such that G_1/N is a group complement of N/N' in G/N'. Then the hypothesis of the theorem holds for G_1 in the place of G and N' in the place of N. By induction hypothesis, there exists a complement H of G_1 in G which is unique up to conjugation. Since also G_1 was unique up to conjugation, our assertion now follows.

II) The case of G a profinite group, but N finite. The existence is shown as in part II) of the foregoing proof, setting $\mathfrak{G} = N$ and $\mathfrak{H} = \{1\}$ and using part I) of the present proof. It remains to prove uniqueness. Let H' be a second closed group complement of N in G. Because of $(G : H) = |N| = (G : H_1)$, both H and H_1 are open subgroups of G, and so is their intersection. Hence, we may choose an open normal subgroup $\mathcal{N}$ of G such that $\mathcal{N} \subset H \cap H_1$. In view of $\mathcal{N} \subset H$ and $H \cap N = \{1\}$ we have $H \cap N\mathcal{N} = \mathcal{N}$. We find that $H/\mathcal{N}$ is a group complement of $N\mathcal{N}/\mathcal{N}$ in $G/\mathcal{N}$ and similarly, the same is shown for $H_1/\mathcal{N}$. By part I), both are conjugate. That is, there is some $g \in G$ such that $H^g = (H\mathcal{N})^g = (H_1\mathcal{N})^g = H_1^g$.

III) The case of G and N profinite groups. Let $N_i = N \cap G_i$ where the G_i, $i \in I$, run through all open normal subgroups of G. Then every N_i is an open subgroup of N and normal in G. By part II), the finite group N/N_i has a group complement H_i in G/N_i, and all such group complements are conjugate. Observe that there are only finitely many conjugates since $H_i^g = H_i^h$ if g and h are elements of the same right cosets of H_i in G/N_i and there are only finitely many since N/N_i is a set of representatives for them. If $N_i \subset N_j$ then the projection of H_i into G/N_j is a group complement of N/N_j. Hence, these group complements form an inverse system of finite sets. By Lemma 24.7, the inverse limit of this system is nonempty. Take an element $(H_i)_{i\in I}$ of it. Since the G_i satisfy the hypotheses of Lemma **??**, so do the N_i; hence, this lemma shows that we may represent G as the inverse limit $\varprojlim G/N_i$. Let H be the set of elements $g \in G$ which satisfy $\pi_i g \in H_i$ for all $i \in I$. Then H is a nonempty closed subgroup of G. Given $g \in G$, for all $i \in I$ we have that $\pi_i(N \cap gH) = N/N_i \cap \pi_i gH_i$ is nonempty since $H_i(N/N_i) = G/N_i$. Hence, $N \cap gH$ is nonempty, which shows that $HN = G$. On the other hand, $H_i \cap (N/N_i) = \{1\}$ for all $i \in I$, showing that $H \cap N = \{1\}$.

Now suppose that $\mathcal{H}$ is a second group complement of N in G. Then for all $i \in I$, $(\mathcal{H}/N_i)(N/N_i) = G/N_i$, and $(\mathcal{H}N_i/N_i)(N/N_i) = \{1\}$ because of $\mathcal{H}N_i \cap N_i = N_i$. This shows that $\mathcal{H}_i := \mathcal{H}N_i/N_i$ is a group complement of N/N_i in G/N_i. Consequently, there is some $g_i \in G/N_i$ such that $H_i^{g_i} = \mathcal{H}_i$. We may actually choose g_i in the finite group N/N_i. If $N_i \subset N_j$ then the projection of g_i into N/N_j conjugates H_j onto $\mathcal{H}_j$. Hence, all possible elements g_i form an inverse system of finite sets. As before, this is not empty and we may choose $g \in G$ such that $(\pi_i g)_{i\in I}$ is an element of its inverse limit. It follows that $H^g = \mathcal{H}$. □

The original Theorem of Schur – Zassenhaus is the finite case of the above theorem without the assumption of N being solvable. Its proof uses the deep Theorem of Feit – Thompson which states that every group of odd order is solvable. Since $(G : N)$ is assumed to be relatively prime to the order $|N|$, one of the groups G/N or N will thus be solvable. In view of the above theorem, only the case of G/N being solvable remains to be treated. For the proof, see e.g. [HUP].

Combining the solvability of p-groups with the Schur – Zassenhaus Theorem, we obtain:

Corollary 13.27 *Let N be a normal subgroup of the finite group G. Assume that N is a p-group and G/N is a p'-group. Then there exists a group complement of N in G, uniquely determined up to conjugation within G.*

We want to improve this result, using the Theorem of Gaschütz.

Theorem 13.28 *Let G be a profinite group and N a closed normal subgroup of G. Suppose that N is a pro-p-group and that all p-Sylow subgroups of G/N are p-free. Then there exists a complement of N in G.*

Proof: By **??** there is a p-Sylow subgroup P of G which contains N. Now P/N is a p-Sylow subgroup of G, and it is a free pro-p-group by assumption. Hence, the exact sequence $1 \to P \to N \to P/N$ splits which means that there is a complement of N in P. If N is a finite abelian group, then the assertion of our theorem now follows from Theorem **??**. If this is not the case, we need an additional argument.

We consider the set S of all closed subgroups $\mathcal{H}$ of G such that $\mathcal{H}N = G$ and $\mathcal{H} \cap N$ is a closed normal subgroup of G. Observe that S is nonempty since it contains G. Further, S is partially ordered by inclusion. We show that the intersection of every descending chain $\mathcal{H}_i$, $i \in I$, in S is again in S. Firstly, $\mathcal{H}_I := \bigcap_{i\in I} \mathcal{H}_i$ is a closed subgroup of G, and $\mathcal{H}_I \cap N = \bigcap_{i\in I}(\mathcal{H}_i \cap N)$ is a closed normal subgroup of G. Secondly, we have to prove that $\mathcal{H}_I N = G$, so let $g \in G$. For all $i \in I$, the set $N \cap g\mathcal{H}_i$ is nonempty since $G = \mathcal{H}_i N$. Therefore, the decreasing sequence $N \cap g\mathcal{H}_i$, $i \in I$, of closed sets has the finite intersection property. By compactness, the intersection $\bigcap_{i\in I}(N \cap g\mathcal{H}_i) = N \cap g\bigcap_{i\in I} \mathcal{H}_i = N \cap g\mathcal{H}_I$ is nonempty, showing that $g \in \mathcal{H}_I N$.

Now by Zorn's Lemma, there exists a maximal element $\mathcal{H}_0$ in S. We are done if we can show that $N_0 := N \cap \mathcal{H}_0 = \{1\}$. Assume that this is not true. Then N_0 is a non-trivial pro-p-group. Take any open normal subgroup U of G and let M be a maximal subgroup of N_0 containing $U \cap N_0$. Observe that $U \cap N_0$ is a subgroup of finite index in N_0 which is also normal in G. It follows that the normal subgroup $N_1 := \bigcap_{g\in G} M^g$ contains $U \cap N_0$ and is thus of finite index in N_0. On the other hand, N_1 contains the Frattini subgroup $\Phi(N_0)$, and we infer from **??** that $N_0/\Phi(N_0)$ is elementary abelian. The same holds for $\mathcal{N} := N_0/N_1$ because it is a quotient group of $N_0/\Phi(N_0)$. On the other hand, setting $\mathcal{H} = \mathcal{H}_0/N_1$, we have $\mathcal{H}/\mathcal{N} = \mathcal{H}_0/N_0 = \mathcal{H}_0/\mathcal{H}_0 \cap N \cong \mathcal{H}_0 N/N = G/N$ showing that all p-Sylow subgroups of $\mathcal{H}/\mathcal{N}$ are free pro-p-groups. According to the first part of our proof, it now follows that there is a group complement for $\mathcal{N}$ in $\mathcal{H}$. Let $\mathcal{H}_1 \subset \mathcal{H}_0$ be the foreimage in G of this complement. Then $\mathcal{H}_1$ is a closed subgroup of G and satisfies $N\mathcal{H}_1 = NN_0\mathcal{H}_1 = N\mathcal{H}_0 = G$ and $N \cap \mathcal{H}_1 = N_1$. This contradicts the maximality of $\mathcal{H}_0$ in S, and we have thus proved that $N \cap \mathcal{H}_0 = \{1\}$, as required. □

13.8 Field complements of the ramification field

In Lemma 13.5 we have characterized the purely wild extensions as being the extensions which are linearly disjoint from the absolute ramification field (which is the maximal tame extension). In this section, we will show that the maximal purely wild extensions are as large as possible, i.e., they are field complements to the ramification field in the algebraic closure. We will prove the following field theoretical assertion:

(F) *Let (K, v) be a henselian field. Then its ramification field K^r admits a field complement L in K^{sep} over K, that is,*

$$L \cap K^r = K \quad \text{and} \quad L.K^r = K^{\mathrm{sep}} .$$

Every separable algebraic extension K' of K which is linearly disjoint from K^r over K, is contained in some field complement L in K^{sep} over K.

Via Galois Theory (cf. Theorem 24.10), this assertion translates into the following group theoretical assertion:

(G) *The ramification subgroup $G^r = G^r(\tilde{K}|K,v)$ of the Galois group $G = \mathrm{Gal}\,K$ admits a group complement H in G. That is,*

$$H.G^r = G \qquad \textit{and} \qquad H \cap G^r = 1\,.$$

Every closed subgroup $G' \subset G$ which satisfies $G'.G^r = G$, contains some group complement H of G^r in G.

Both assertions are trivial if the residue characteristic of (K,v) is 0 because then, $K^r = \tilde{K}$ and $G^r = 1$. So we will from now on assume that $\mathrm{char}\,\overline{K} = p > 0$. To prove assertion (G) by the results of the last section, we need an additional lemma:

Lemma 13.29 *For a henselian field of residue characteristic $p > 0$, the p-Sylow subgroups of G/G^r are p-free.*

Proof: Since $K^r|K^i$ is a p'-extension by Theorem 7.27, the p-Sylow subgroups of G/G^r are isomorphic to those of G/G^i. These in turn are isomorphic to those of $\mathrm{Gal}\,\overline{K}$ because $G/G^i \cong \mathrm{Gal}\,\overline{K}$ by Theorem 7.27. It follows from Artin–Schreier Theory that the p-Sylow subgroups of the absolute Galois of a field of characteristic p are p-free (cf. [SER], page II–5, cor. 1). □

In view of this lemma, the first assertion of (G) follows from Theorem 13.28. The last assertion of (G) is seen as follows. Let $G' \subset G$ be a closed subgroup which satisfies $G'.G^r = G$. Since $G'/G' \cap G^r \cong G/G^r$, also the p-Sylow subgroups of $G'/G' \cap G^r$ are p-free. So we can apply Theorem 13.28 to G' in the place of G, with $N = G' \cap G^r$. We obtain a complement $H \subset G'$ of $G' \cap G^r$ in G'. That is,

$$H.(G' \cap G^r) = G' \qquad \text{and} \qquad H \cap (G' \cap G^r) = 1\,,$$

which implies that $H.G^r = G'.G^r = G$ and $H \cap G^r = H \cap (G' \cap G^r) = 1$.

Since (G) holds, also (F) holds. The field complements L of K^r in K^{sep} are purely wild extensions by Lemma 13.5. On the other hand, they are maximal separable purely wild extensions. Indeed, if $(L'|k,v)$ is a separable purely wild extension containing L, then $L'|K$ is linearly disjoint from $K^r|K$ and thus, $L'|L$ is linearly disjoint from $K^r.L|L$. But $L' \subset K^r.L$, showing that $L' = L$. Conversely, the last assertion of (F) shows that every separable purely wild extension of K can be embedded in a field complement of K^r. Hence every maximal separable purely wild extension is already such a field complement. So assertion (F) in fact shows that the maximal separable purely wild extensions coincide with the field complements of K^r in K^{sep}.

Every purely inseparable extension is purely wild. Hence if L is a maximal separable purely wild extension, then L^{1/p^∞} is also a purely wild extension of K. Every algebraic extension $L'|L^{1/p^\infty}$ is separable, and the maximal separable subextension L'_s of $L'|K$ contains L. If $L'|L^{1/p^\infty}$ is purely wild, then the same holds for $L'|K$ and $L'_s|K$. Since L is a maximal separable purely extension of K, it follows that $L'_s = L$, which yields that

$L' = (L'_s)^{1/p^\infty} = L^{1/p^\infty}$. We have shown that L^{1/p^∞} is a maximal purely wild extension of K.

Conversely, every maximal purely wild extension W of K must be perfect by our above argument, and its maximal separable subextension W_s is a maximal separable purely wild extension of K. The latter is true since a proper purely wild extension L of W_s would yield a proper purely wild extension L^{1/p^∞} of W, contradicting the maximality of W. We have proved:

Lemma 13.30 *The perfect hulls of the maximal separable purely wild extensions coincide with the maximal purely wild extensions.*

On the other hand, the4 separable extension L of K is a field complement of K^r in K^{sep} if and only if L^{1/p^∞} is a field complement of K^r in $\tilde{K}$. Indeed, $L^{1/p^\infty}|L$ is linearly disjoint from the separable extension $K^r.L|L$. Hence if $L|K$ is linearly disjoint from $K^r|K$, then also $L^{1/p^\infty}|K$ is linearly disjoint from $K^r|K$. If $K^r.L = K^{\mathrm{sep}}$, then $K^r.L^{1/p^\infty} = (K^{\mathrm{sep}})^{1/p^\infty} = \tilde{K}$. Hence if L is a field complement of K^r in K^{sep}, then L^{1/p^∞} is a field complement of K^r in $\tilde{K}$. For the converse, we only have to show that $K^r.L = K^{\mathrm{sep}}$ if $K^r.L^{1/p^\infty} = \tilde{K}$. The inclusion $K^r.L \subset K^{\mathrm{sep}}$ follows from our hypothesis that $L|K$ be separable. The other inclusion follows from the fact that $K^r.L^{1/p^\infty} = (K^r.L)^{1/p^\infty}$. We summarize what we have proved:

Theorem 13.31 *Let (K, v) be a henselian field with residue characteristic $p > 0$. There exist algebraic field complements W_s of K^{sep} over K, i.e. $K^r.W_s = K^{\mathrm{sep}}$ and W_s is linearly disjoint from K^{sep} over K. The perfect hull $W = W_s^{1/p^\infty}$ is an algebraic field complement of K^r over K, i.e. $K^r.W = \tilde{K}$ and W is linearly disjoint from K^r over K. The valued complements (W_s, v) can be characterized as the maximal separable algebraic purely wild extensions of (K, v), and the (W, v) are the maximal algebraic purely wild extensions of (K, v).*

WARNING: This theorem only states the existence of field complements for the *absolute* ramification field. It is not true that the ramification field of every Galois extension admits a field complement. Such an assertion can only be proved for suitable classes of valued fields, e.g. for the Kaplansky fields (see Section 13.11).

Let us also apply the uniqueness statement for the group complements as given in Theorem 13.26. By Theorem 7.16 we know that $K^{\mathrm{sep}}|K^r$ is a p-extension. Hence, G^r is a pro-p-group, and it is thus prosolvable (cf. Corollary 24.55). Let us assume that $\overline{K}$ does not admit finite separable extensions of degree divisible by p. Then we obtain that $\mathrm{Gal}\,\overline{K}$ and hence also G/G^r is a p'-group. In this case, the hypothesis of Theorem 13.26 is satisfied by $N = G/G^r$. It follows that all group complements of G^r are conjugates. Translated via Galois Theory (cf. Theorem 24.10), this means that all field complements of K^r in K^{sep} are isomorphic over K. That is, all maximal separable purely wild extensions are isomorphic over K. Since a field isomorphism extends to the perfect hulls of the fields, we find, in view of the last lemma, that also all maximal purely wild extensions are isomorphic over K. Since (K, v) is henselian, isomorphisms of algebraic extensions over K are valuation preserving (cf. Lemma 7.34). We have proved:

Theorem 13.32 *Let (K, v) be a henselian field whose residue field is of characteristic $p > 0$ and does not admit a finite separable extension of degree divisible by p. Then the maximal separable purely wild extensions and the maximal purely wild extensions of (K, v) are unique up to valuation preserving isomorphism over K.*

13.9 Tame fields

Let (K,v) be a henselian valued field and denote by p the characteristic exponent of its residue field $\overline{K}$. Then (K,v) is said to be a **tame field** if $(\tilde{K}|K,v)$ is a tame extension, and to be a **separably tame field** if $(K^{\rm sep}|K,v)$ is a tame extension. On the other extreme, we will call (K,v) a **purely wild field** if $(\tilde{K}|K,v)$ is a purely wild extension. For example, for every henselian field (K,v), its ramification field $(K,v)^r$ is a purely wild field.

Lemma 13.33 *a) Let (K,v) be a henselian field. Then (K,v) is a tame field if and only if $K^r=\tilde{K}$. Similarly, (K,v) is a separably tame field if and only if $K^r=K^{\rm sep}$. Further, (K,v) is a purely wild field if and only if $K^r=K$.*

b) Every algebraic extension of a tame (resp. separably tame, resp. purely wild) field is again a tame (resp. separably tame, resp. purely wild) field.

Proof: a): If (K,v) is a tame field, then by definition and Theorem 13.2, $\tilde{K}\subset K^r$, showing that $\tilde{K}=K^r$. The converse also follows from Theorem 13.2. For "separably tame", the same proof works with $K^{\rm sep}$ in the place of $\tilde{K}$ because by definition, K^r is a subfield of $K^{\rm sep}$.

By definition and Lemma 13.5, (K,v) is a purely wild field if and only if $\tilde{K}|K$ is linearly disjoint from $K^r|K$. This in turn is equivalent to $K^r=K$.

b): Every algebraic extension $L|K$ satisfies $\tilde{L}=\tilde{K}=L.\tilde{K}$, $L^{\rm sep}=L.K^{\rm sep}$ and by Lemma 7.28 $L^r=L.K^r$. In view of this, assertion b) follows directly from a). □

If (K,v) is a henselian field of residue characteristic 0, then every algebraic extension $(L|K,v)$ is tame, as we have seen in the last section. So we note:

Lemma 13.34 *Every algebraic extension of henselian fields of residue characteristic 0 is a tame extension. Every henselian field of residue characteristic 0 is a tame field.*

From the definition and the fact that every tame extension is defectless and separable, we obtain:

Lemma 13.35 *Every tame field is henselian defectless and perfect.*

In general, infinite extensions of defectless fields need not be defectless fields (cf. Theorem 11.45 and Theorem 11.57). But from the foregoing lemma and Lemma 13.33, we can deduce:

Corollary 13.36 *Every algebraic extension of a tame field is a defectless field.*

Using Theorem 13.31, we give some characterizations for tame fields. Recall that every algebraically maximal field is henselian (cf. Corollary 11.31).

Lemma 13.37 *The following assertions are equivalent:*
1) (K,v) is tame,
2) Every algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is algebraically maximal and closed under purely wild extensions by p-th roots,
4) (K,v) is algebraically maximal, vK is p-divisible and $\overline{K}$ is perfect.

Proof: Let (K,v) be a tame field, i.e., $K^r = \tilde{K}$. Then by Lemma 13.5, every algebraic purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Suppose that (K,v) has no algebraic purely wild extension. Then in particular, it has no purely wild extension by p-th roots. Since every immediate algebraic extension of a henselian field is purely wild by definition, we also obtain that (K,v) admits no proper immediate algebraic extension, i.e., (K,v) is algebraically maximal. This proves 2)⇒3).

Assume now that (K,v) is an algebraically maximal field closed under purely wild extensions by p-th roots. Let a be an arbitrary element of K. Assume that va is not divisible by p in vK; then the extension $K(b)|K$ generated by an element $b \in \tilde{K}$ with $b^p = a$ satisfies $(vK(b) : vK) = p = [K(b) : K]$ and thus admits a unique extension of v. With this extension it is purely wild, contrary to our assumption on (K,v). Assume that $va = 0$ and that $\overline{a}$ has no p-th root in $\overline{K}$; then the extension $K(b)|K$ generated as above satisfies $[\overline{K(b)} : \overline{K}] = p = [K(b) : K]$ and is again purely wild, contrary to our assumption. By this, we have shown that vK is p-divisible and $\overline{K}$ is perfect. This proves 3)⇒4).

Suppose that (K,v) is an algebraically maximal (and thus henselian) field such that vK is p-divisible and $\overline{K}$ is perfect. Choose a maximal purely wild extension (W,v) in accordance to Theorem 13.31. Our condition on the value group and the residue field yields that $(W|K,v)$ is immediate. But since (K,v) is assumed to be algebraically maximal, this extension must be trivial. This shows that $\tilde{K} = K^r$, i.e., (K,v) is a tame field. This proves 4)⇒1). □

If K has characteristic $p > 0$, then every extension by p-th roots is purely inseparable and thus purely wild. So the lemma yields:

Corollary 13.38 *A valued field (K,v) of characteristic $p > 0$ is tame if and only if it is algebraically maximal and perfect. Consequently, if (K,v) is an arbitrary valued field of characteristic $p > 0$, then every maximal immediate algebraic extension (W,v) of $(K^{1/p^\infty}, v)$ is a tame field having the p-divisible hull of vK as its value group and the perfect hull of $\overline{K}$ as its residue field.*

For perfect valued fields of positive characteristic, "algebraically maximal" and "henselian defectless" are equivalent.

The next corollary shows how to construct tame fields with suitable prescribed value group and residue field:

Corollary 13.39 *Let p be a prime number, Γ a p-divisible ordered abelian group and k a perfect field of characteristic p. Then there exists a tame field K of characteristic p having Γ as its value group and k as its residue field such that $K|\mathbb{F}_p$ admits a valuation transcendence basis and the cardinality of K is equal to the maximum of the cardinalities of Γ and k.*

Proof: According to Theorem 6.42, there is a valued field (K_0,v) of characteristic p with value group Γ and residue field k, and admitting a valuation transcendence basis over its prime field. Now take (K,v) to be a maximal immediate algebraic extension of (K_0,v). Then (K,v) is algebraically maximal, and Lemma 13.37 shows that it is a tame field. Since it is an algebraic extension of (K_0,v), it still admits a valuation transcendence basis over its prime field. Hence, it follows from Example 6.6 that $|K| = \max\{|\Gamma|, |k|\}$. (If v is non-trivial, then K is infinite. If v is trivial, then $\Gamma = \{0\}$ and $K = k$.) □

Now we will prove an important lemma on tame fields that we will need in several instances.

Lemma 13.40 *Let (L,v) be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $\overline{L}|\overline{K}$ is an algebraic extension, then K is also a tame field and moreover, vK is pure in vL and $\overline{K} = \overline{L}$.*

Proof: Since (L,v) is tame, it is henselian and perfect. Since K is relatively algebraically closed in L, it is henselian and perfect too. Assume that $(K_1|K,v)$ is a finite purely wild extension; in view of Lemma 13.37, we have to show that it is trivial. The degree $[K_1 : K]$ is a power of p, say p^m. Since K is perfect, $L|K$ and $K_1|K$ are separable extensions. Since K is relatively algebraically closed in L, we know that L and K_1 are linearly disjoint over K. Thus, K_1 is relatively algebraically closed in $K_1.L$, and

$$[K_1.L : L] = [K_1 : K] = p^m \;.$$

Since L is assumed to be a tame field, the extension $(K_1.L|L,v)$ must be tame. This implies that

$$\overline{K_1.L} \,|\, \overline{L}$$

is a separable extension of degree p^m. On the other hand, $\overline{K_1.L} \,|\, \overline{K_1}$ is an algebraic extension since by hypothesis, $\overline{L} \,|\, \overline{K}$ and thus also $\overline{K_1.L} \,|\, \overline{K}$ are algebraic extensions. Furthermore, $(K_1.L,v)$ being a henselian field and K_1 being relatively algebraically closed in $K_1.L$, Hensel's Lemma shows that

$$\overline{K_1.L} \,|\, \overline{K_1}$$

must be purely inseparable. This yields that

$$\begin{aligned} p^m &= [\overline{K_1.L} : \overline{L}]_{\rm sep} \leq [\overline{K_1.L} : \overline{K}]_{\rm sep} = [\overline{K_1.L} : \overline{K_1}]_{\rm sep} \cdot [\overline{K_1} : \overline{K}]_{\rm sep} \\ &= [\overline{K_1} : \overline{K}]_{\rm sep} \leq [\overline{K_1} : \overline{K}] \leq [K_1 : K] = p^m \;, \end{aligned}$$

showing that

$$\overline{K_1} \,|\, \overline{K}$$

is separable of degree p^m. Since $K_1|K$ was assumed to be purely wild, we have $p^m = 1$ and the extension $K_1|K$ is trivial.

We have now shown that K is a tame field; hence by Lemma 13.37, vK is p-divisible and $\overline{K}$ is perfect. Since $\overline{L}|\overline{K}$ is assumed to be algebraic, we obtain from Lemma 9.26 that $\overline{K} = \overline{L}$ and that the torsion subgroup of vL/vK is a p-group. But vK is p-divisible since K is perfect. Thus, vL/vK has no p-torsion, showing that vL/vK has no torsion at all. □

The same lemma holds for separably tame fields, as stated in Lemma 13.48 below. The following corollaries will show some nice properties of the class of tame fields. They also possess generalizations to separably tame fields, see Corollary 13.49 below.

Corollary 13.41 *For every valued function field F with given transcendence basis $\mathcal{T}$ over a tame field K, there exists a tame subfield K_0 of K of finite rank with $\overline{K_0} = \overline{K}$ and vK_0 pure in vK, and furthermore a function field F_0 with transcendence basis $\mathcal{T}$ over K_0 such that*

$$F = K.F_0 \tag{13.9}$$

and

$$[F_0 : K_0(\mathcal{T})] = [F : K(\mathcal{T})] \;. \tag{13.10}$$

Proof: Let $F = K(\mathcal{T})(a_1,\ldots,a_n)$. There exists a finitely generated subfield K_1 of K such that $a_1,\ldots,a_n$ are algebraic over $K(\mathcal{T})$ and $[F : K(\mathcal{T})] = [K_1(\mathcal{T})(a_1,\ldots,a_n) : K_1(\mathcal{T})]$. This will also hold for every extension field of K_1 in K. As a finitely generated field, (K_1,v) has finite rank. Now let y_j, $j\in J$, be a system of elements in K such that the residues $\overline{y_j}$, $j\in J$, form a transcendence basis of $\overline{K}$ over $\overline{K_1}$. According to Lemma 6.35, the field $K_1(y_j|j\in J)$ has residue field $\overline{K_1}(\overline{y_j}|j\in J)$ and the same value group as K_1, hence it is again a field of finite rank. Let K_0 be the relative closure of this field within K. Since by construction, $\overline{K}|\overline{K_1}(\overline{y_j}|j\in J)$ and thus also $\overline{K}|\overline{K_0}$ are algebraic, we can infer from the preceding lemma that K_0 is a tame field with $\overline{K_0}=\overline{K}$ and vK_0 pure in vK. As an algebraic extension of a field of finite rank it is itself of finite rank. Finally, the function field $F_0 = K_0(\mathcal{T})(a_1,\ldots,a_n)$ has transcendence basis $\mathcal{T}$ over K_0 and satisfies assertions (13.9) and (13.10). □

Corollary 13.42 *For every extension $(L|K,v)$ with (L,v) a tame field, there exists a tame intermediate field L_0 such that the extension $(L_0|K,v)$ admits a valuation transcendence basis and the extension $(L|L_0,v)$ is immediate.*

Proof: Take $\mathcal{T}$ to be a maximal algebraically valuation independent set in $(L|K,v)$. With this choice, $vL/vK(\mathcal{T})$ is a torsion group and $\overline{L}|\overline{K(\mathcal{T})}$ is algebraic. Let L_0 be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by Lemma 13.40, we have that (L_0,v) is a tame field, that $\overline{L}=\overline{L_0}$ and that vL_0 is pure in vL and thus $vL_0 = vL$. This shows that the extension $(L|L_0,v)$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $(L_0|K,v)$ by construction. □

13.10 Separably tame fields

Recall that a valued field (K,v) is called **separably defectless** if every finite separable extension is defectless, and that it is called **separable-algebraically maximal** if it does not admit proper immediate separable algebraic extensions. Since the henselization of (K,v) is an immediate separable-algebraic extension of (K,v), a separable-algebraically maximal field (K,v) will coincide with its henselization and thus be henselian. Note that “henselian separably defectless” implies “separable-algebraically maximal”.

Since every finite separable algebraic extension of a separably tame field is tame and thus defectless, a separably tame field is always henselian separably defectless. The converse is not true; it needs additional assumptions on the value group and the residue field. Under the assumptions that we are going to use frequently, the converse will even hold for “separable-algebraically maximal” in the place of “henselian separably defectless”. Before proving this, we need a lemma which makes essential use of Theorem 13.31.

Lemma 13.43 *A henselian field (K,v) is defectless if and only if every finite purely wild extension of (K,v) is defectless. Similarly, (K,v) is separably defectless if and only if every finite separable purely wild extension of (K,v) is defectless.*

Proof: By Theorem 13.31, there exists a field complement W of K^r over K in K^{sep}, and W^{1/p^∞} is a field complement of K^r in $\tilde{K}$. Consequently, given any finite extension (resp. finite separable extension) $(L|K,v)$, there is a finite subextension $N|K$ of $K^r|K$ and a finite (resp. finite separable) subextension $W_0|K$ of $W|K$ such that $L \subset N.W_0$. If $(N.W_0|K,v)$ is defectless, then so is $(L|K,v)$; hence (K,v) is defectless (resp. separably defectless), if and only if every such extension $(N.W_0|K,v)$ is defectless. Since $(N|K,v)$ is a tame extension (by virtue of Theorem 13.2), Lemma 13.4 shows that

$$d(N.W_0|N,v) = d(W_0|K,v) \; .$$

Hence, $(L|K,v)$ is defectless if and only if $(W_0|K,v)$ is defectless. This yields our assertion. □

Lemma 13.44 *The following assertions are equivalent:*
1) (K,v) is separably tame,
2) Every separable algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is separable-algebraically maximal and closed under purely wild Artin-Schreier extensions,
4) (K,v) is separable-algebraically maximal, vK is p-divisible and $\overline{K}$ is perfect.

Proof: Let (K,v) be a separably tame field, i.e., $K^r = K^{\mathrm{sep}}$. Then by Lemma 13.5, every separable algebraic purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Now suppose that every separable algebraic purely wild extension of (K,v) is trivial. Then in particular, (K,v) admits no purely wild Artin-Schreier extensions (because they are separable). Furthermore, (K,v) admits no proper separable algebraic immediate extension since they are also purely wild. Consequently, (K,v) is separable-algebraically maximal. This proves 2)⇒3).

If (K,v) is closed under purely wild Artin-Schreier extensions, then by Lemma 13.6, vK is p-divisible and $\overline{K}$ is perfect. This proves 3)⇒4).

Suppose that (K,v) is a separable-algebraically maximal field such that vK is p-divisible and $\overline{K}$ is perfect. By Lemma 11.30, (K,v) is henselian. Choose a maximal separable algebraic purely wild extension (W_s,v) in accordance to Theorem 13.31. Our condition on the value group and the residue field yields that $(W_s|K,v)$ is immediate. But since (K,v) is assumed to be separable-algebraically maximal, this extension must be trivial. This shows that $K^{\mathrm{sep}} = K^r$, i.e., (K,v) is a separably tame field. This proves 4)⇒1). □

Suppose that (K,v) separably tame. Choose (W_s,v) according to Theorem 13.31. Then by condition 2) of the lemma, the extension $(W_s|K,v)$ must be trivial. This yields that $(K^{1/p^\infty},v)$ is the unique maximal algebraic purely wild extension of $((K,v)$. Further, (K,v) also satisfies condition 3) of the lemma. From Lemma 13.6 it follows that (K,v) is dense in $(K^{1/p^\infty},v)$, i.e., K^{1/p^∞} lies in the completion of (K,v). This proves:

Corollary 13.45 *If (K,v) is separably tame, then the perfect hull K^{1/p^∞} of K is the unique maximal algebraic purely wild extension of $((K,v)$ and lies in the completion of (K,v). That is, every immediate algebraic extension of a separably tame field (K,v) is purely inseparable and included in the completion of (K,v), and every algebraic approximation type over (K,v) has distance ∞.*

Lemma 13.46 *(K,v) is a separably tame field if and only if $(K^{1/p^\infty},v)$ is a tame field. Consequently, if $(K^{1/p^\infty},v)$ is a tame field, then the extension (K,v) is dense in $(K^{1/p^\infty},v)$.*

Proof: Suppose that (K,v) is a separably tame field. Then $(K^{1/p^\infty},v)$ admits no purely wild algebraic extensions since otherwise, it would contain a proper separable algebraic purely wild subextension. Hence by Lemma 13.37, $(K^{1/p^\infty},v)$ is a tame field.

For the converse, suppose that $(K^{1/p^\infty},v)$ is a tame field. Observe that the extension $(K^{1/p^\infty}|K,v)$ is purely wild and contained in every maximal purely wild algebraic extension of (K,v). Consequently, if $(K^{1/p^\infty},v)$ admits no purely wild algebraic extension at all, then $(K^{1/p^\infty},v)$ is the unique maximal purely wild extension of (K,v). Then in view of Theorem 13.31, K^{1/p^∞} must be a field complement for K^r over K in $\tilde{K}$. This yields that $K^r = K^{\mathrm{sep}}$, i.e., $(K^{\mathrm{sep}}|K,v)$ is a tame extension by Theorem 13.2, showing that (K,v) is a separably tame field. By the foregoing corollary, it follows that (K,v) is dense in $(K^{1/p^\infty},v)$. □

The following lemma describes the behaviour of separably tame fields under a decomposition of their place.

Lemma 13.47 *Let (K,v) be a separably tame field and let P be the place associated with v. Assume $P = P_1P_2P_3$ where P_1 is a coarsening of P and P_2 is non-trivial. (P_3 may be trivial.) Then (KP_1,P_2) is a separably tame field. If also P_1 is non-trivial, then (KP_1,P_2) is a tame field.*

Proof: By Lemma 13.37, vK is p-divisible. The same is then true for $v_{P_2}KP_1$. We wish to show that the residue field KP_1P_2 is perfect. Indeed, assume that this were not the case. Then by Lemma 6.40 there is an Artin-Schreier extension of (K,P_1P_2) which adjoins a p-th root to the residue field KP_1P_2. Since already this residue field extension is purely inseparable, the induced extension of the residue field $\overline{K} = KP_1P_2P_3$ can not be separable of degree p. This shows that the constructed Artin-Schreier extension is a separable algebraic purely wild extension of (K,v), contrary to our assumption on (K,v).

By Lemma 13.44, (K,P) is separable-algebraically maximal. By Theorem **??**, this yields that the same is true for (K,P_1P_2). If P_1 is trivial (hence w.l.o.g. equal to the identity map), then $(KP_1,P_2) = (K,P_1P_2)$ is separable-algebraically maximal, and it follows from Lemma 13.44 that (KP_1,P_2) is a separably tame field. If P_1 is non-trivial, then again by Theorem **??**, we obtain that (KP_1,P_2) is an algebraically maximal field, and it follows from Lemma 13.37 that (KP_1,P_2) is a tame field. □

The following is an analogue of Lemma 13.40.

Lemma 13.48 *Let (K,v) be a separably tame field and $k \subset K$ a relatively algebraically closed subfield of K. If the residue field extension $\overline{K}|\overline{k}$ is algebraic, then (k,v) is also a separably tame field.*

Proof: Since k is relatively algebraically closed in K, it follows that also k^{1/p^∞} is relatively algebraically closed in K^{1/p^∞} (cf. Exercise 24.5). Since (K,v) is a separably tame field, $(K^{1/p^\infty},v)$ is a tame field by Lemma 13.46. From this lemma we also know that $\overline{K} == \overline{K^{1/p^\infty}}$ and $vK = vK^{1/p^\infty}$. Our assumption on $\overline{K}\,|\,\overline{k}$ yields that the extension $\overline{K^{1/p^\infty}}\,|\,\overline{k^{1/p^\infty}}$ is algebraic. From Lemma 13.40 we can now infer that $(k^{1/p^\infty},v)$ is a tame field with $\overline{k^{1/p^\infty}} = \overline{K^{1/p^\infty}} = \overline{K}$ and vk^{1/p^∞} pure in $vK^{1/p^\infty} = vK$. Again by Lemma 13.46, (k, v) is thus a separably tame field with $\overline{k} = \overline{k^{1/p^\infty}} = \overline{K}$ and $vk = vk^{1/p^\infty}$ pure in vK. □

Corollary 13.49 *Corollary 13.41 also holds for separably tame fields in the place of tame fields. More precisely, if $F|K$ is a separable extension, then F_0 and K_0 can be chosen such that $F_0|K_0$ (and thus also $F_0^h|K_0$) is a separable extension. Moreover, if vK is cofinal in vF then it can also be assumed that vK_0 is cofinal in vF_0.*

Proof: Since the proof of Corollary 13.41 only involves Lemma 13.40, it can be adapted by use of Lemma 13.48. The first additional assertion can be shown using the fact that the finitely generated separable extension $F|K$ is separably generated. The second additional assertion is seen as follows. If vF admits a biggest proper convex subgroup, then let K_0 contain a nonzero element whose value does not lie in this subgroup. If vF and thus also vK does not admit a biggest proper convex subgroup, then first choose F_0 and K_0 as in the (generalized) proof of Lemma 13.41; since F_0 has finite rank, there exists some element in K whose value does not lie in the convex hull of vF_0 in vF, and adding this element to K_0 and F_0 will make vK_0 cofinal in vF_0. □

With the same proof as for Corollary 13.42, but using Lemma 13.48 in the place of Lemma 13.40, one shows:

Corollary 13.50 *Corollary 13.42 also holds for separably tame fields in the place of tame fields.*

13.11 Kaplansky fields

A valued field (K, v) of residue characteristic $p = \operatorname{char} Kv$ is called a **Kaplansky field** if it satisfies

(KAP1) the value group is p-divisible if $p > 0$

(KAP2) the residue field is perfect

(KAP3) the residue field does not admit a finite separable extension of degree divisible by p.

It follows that every algebraic extension of a Kaplansky field is again a Kaplansky field. If a field (K, v) does not admit extensions of degree p, then it is a Kaplansky field.

Kaplansky fields have been introduced by Irving Kaplansky in the paper "Maximal fields with valuations" [KAP1]. In place of our above axioms, Kaplansky's definition in that paper was what he called

Hypothesis A: vK is p-divisible if $p > 0$, and Kv is p-closed.

A field K of characteristic $p > 0$ will be called **p-closed** if for every additive polynomial $\mathcal{A}(X)$ over K and every $c \in K$, the polynomial $\mathcal{A} - c$ admits a zero in K. Every p-closed field is perfect and Artin-Schreier-closed.

For valued fields (K, v) with $\operatorname{char} Kv = 0$, hypothesis A is empty. The condition of a field to be p-closed seemed obscure at the time of Kaplansky's paper. But we have learned to understand this condition better. The following theorem was first proved by Whaples in [WHA2], using the cohomology theory of additive polynomials. A more elementary proof was later given in [DEL1]. Then Kaplansky gave a short and elegant proof in his "Afterthought: Maximal Fields with Valuation" ([Ka2]).

Theorem 13.51 *A field K is additively closed if and only if it does not admit any finite extensions of degree divisible by p.*

Proof: "$\Leftarrow$": Assume that K does not admit any finite extensions of degree divisible by p. Take any p-polynomial $f \in K[X]$. Write $f = \mathcal{A} + c$ where $\mathcal{A} \in K[X]$ is an additive polynomial. Let h be an irreducible factor of f; by hypothesis, it has a degree d not divisible by p. Fix a root b of h in the algebraic closure $\tilde{K}$ of K. All roots of f are of the form $b + a_i$ where the a_is are roots of $\mathcal{A}$. By part a) of Corollary 12.3 the roots of $\mathcal{A}$ in $\tilde{K}$ form an additive group. The sum of the roots of h lies in K. This gives us $db + s \in K$, where s is a sum of a subset of the a_is and is therefore again a root of $\mathcal{A}$. Likewise, $d^{-1}s$ is a root of $\mathcal{A}$ (as d is not divisible by p, it is invertible in K). Then $b + d^{-1}s = d^{-1}(db + s)$ is a root of f, and it lies in K, as required.

"$\Rightarrow$": (This part of the proof is due to David Leep.) Assume that K is p-closed. Since K is perfect, it suffices to take a Galois extension $L|K$ of degree n and show that p does not divide n. By the normal basis theorem there is a basis $b_1, \ldots, b_n$ of $L|K$ where the b_is are the roots of some irreducible polynomial over K. Since they are linearly independent over K, their trace is non-zero. The elements

$$1, b_1, b_1^p, \ldots, b_1^{p^{n-1}}$$

are linearly dependent over K since $[L : K] = n$. So there exist elements $d_0, \ldots, d_{n-1}, e \in K$ such that the p-polynomial

$$f(X) \;=\; d_{n-1}X^{p^{n-1}} + \ldots + d_0X + e$$

has b_1 as a root. It follows that all the b_is are roots of f. Thus the elements $b_2 - b_1, \ldots, b_n - b_1$ are roots of the additive polynomial $f(X) - e$. Since these $n-1$ roots are linearly independent over K, they are also linearly independent over the prime field $\mathbb{F}_p$. This implies that the additive group G generated by the elements $b_2 - b_1, \ldots, b_n - b_1$ contains p^{n-1} distinct elements, which therefore must be precisely the roots of $f(X) - e$. So $G + b_1$ is the set of roots of f. By hypothesis, one of these roots lies in K; call it ϑ. There exist integers $m_2, \ldots, m_n$ such that

$$\vartheta \;=\; m_2(b_2 - b_1) + \ldots + m_n(b_n - b_1) + b_1\,.$$

In this equation take the trace from L to K. The elements $b_1, \ldots, b_n$ all have the same trace; hence the trace of every $m_i(b_i - b_1)$ is 0. It follows that the trace $n\vartheta$ of ϑ is equal to the trace of b_1; as we have remarked already, this trace is non-zero. Hence $n\vartheta \neq 0$, which shows that n is not divisible by p. □

The condition that Kv be p-closed can be split up into the two conditions (KAP2), applying to purely inseparable extensions, and (KAP3), applying to separable extensions. The reason for this separation of the two cases is to be seen in the role that (KAP3) played in Theorem 13.32 and the role that (KAP2) together with (KAP1) is going to play in Lemma 13.54 below.

The property of being p-closed can also be used to give a characterization of algebraically maximal Kaplansky fields:

Theorem 13.52 *A henselian valued field of characteristic $p > 0$ is p-closed if and only if it is an algebraically maximal Kaplansky field.*

Proof: We will use Theorem 13.51 throughout the proof without further mention. Assume first that (K, v) is henselian and that K is p-closed. Since every finite extension of the residue field Kv can be lifted to an extension of K of the same degree, it follows that Kv is p-closed. Likewise, if the value group vK were not p-divisible, then K would admit an extension of degree p; this shows that vK is p-divisible. We have thus proved that (K, v) is a Kaplansky field. Since the degree of every finite extension of K is prime to p, it follows that (K, v) is defectless, hence algebraically maximal.

For the converse, assume that (K, v) is an algebraically maximal Kaplansky field. Since the henselization is an immediate algebraic extension, it follows that (K, v) is henselian. By Theorem 13.31, there exists a field complement W of K^r in $\tilde{K}$. As vK is p-divisible and Kv is p-closed, hence perfect, the same theorem shows that W is an immediate extension of K. Hence $W = K$, which shows that $K^r = \tilde{K}$. So every finite extension $L|K$ is a subextension of $K^r|K$ and is therefore defectless; that is, $[L : K] = (vL : vK)[Lv : Kv]$. As the right hand side is not divisible by p, (K, v) being a Kaplansky field, we find that p does not divide $[L : K]$. By Theorem 13.51, this proves that K is p-closed. □

For a generalization of the notion "p-closed" and of this theorem to fields of characteristic 0 see [V], in particular Corollary 5.

Next, we shall give a characterization of algebraically maximal Kaplansky fields through tameness.

Theorem 13.53 *A Kaplansky field is algebraically maximal if and only if it is a tame field.*

Proof: Assume that (K, v) is a Kaplansky field. Then vK is p-divisible and Kv is perfect. Hence by the equivalence 1)⇔1) of Lemma 13.37, it is algebraically maximal if and only if it is a tame field. □

Exercise 13.1 *Take a henselian Kaplansky field (K, v) with residue characteristic p. Show the following. If $(L|K, v)$ is a finite tame extension, then its degree is not divisible by p. Further, every finite extension $(L|K, v)$ of degree a power of p is purely wild.*

13.12 Uniqueness of maximal immediate extensions

In this section, we will prove the uniqueness of maximal immediate extensions of Kaplansky fields. We will first consider maximal immediate algebraic extensions. Since such extensions are algebraically maximal, we see that the maximal immediate algebraic extensions of a Kaplansky field (K, v) are tame fields and contain the perfect hull of K.

Lemma 13.54 *If a henselian field (K, v) of residue characteristic $p = \mathrm{char}\, Kv > 0$ satisfies* (KAP1) *and* (KAP2)*, then the maximal immediate algebraic extensions coincide with the maximal purely wild extensions.*

Proof: Every immediate algebraic extension of a henselian field is purely wild and thus contained in a maximal purely wild extension. On the other hand, in view of the definition of purely wild extensions, conditions (KAP1) and (KAP2) yield that every purely wild extension of (K, v) is immediate. □

Theorem 13.32 has shown that the maximal purely wild extensions of a field of residue characteristic p are unique up to isomorphism if the field satisfies (KAP3). Hence, the foregoing lemma yields:

Corollary 13.55 *The maximal immediate algebraic extensions of a Kaplansky field (K, v) are unique up to valuation preserving isomorphism over K.*

For the step from algebraic to transcendental extensions, we need an improved version of Lemma 13.40 for Kaplansky fields:

Lemma 13.56 *Let (L, v) be an algebraically maximal Kaplansky field. If K is a relatively algebraically closed subfield of L, then (K, v) is again an algebraically maximal Kaplansky field, and $\overline{K}$ is relatively algebraically closed in $\overline{L}$.*

Proof: Let K be relatively algebraically closed in L. Then (K, v) is henselian and perfect like (L, v). By Corollary 13.53 it now suffices to show that K admits no extension of degree divisible by p. Since K is perfect, this extension is separable. Since K is relatively algebraically closed in L, we know that $F|K$ is linearly disjoint from $L|K$. Consequently, $[F.L : L] = [F : K]$. On the other hand, $[F.L : L]$ is not divisible by p because (L, v) is an algebraically maximal Kaplansky field. This proves our our first assertion.

By Lemma 9.26, $\overline{K}$ is relatively separable-algebraically closed in $\overline{L}$. Since K is perfect, so is $\overline{K}$, showing that $\overline{K}$ is relatively algebraically closed in $\overline{L}$. □

Corollary 13.57 *Let (L, v) and (F, v) be two algebraically maximal Kaplansky fields and (K, v) a common subfield of them. If the henselization of (K, v) in (L, v) and (F, v) does not admit any non-trivial tame algebraic extension inside of (L, v) or (F, v), then the relative algebraic closures of (K, v) in (L, v) and (F, v) are isomorphic over K.*

Proof: Since the respective henselizations of (K, v) in the henselian fields (L, v) and (F, v) are isomorphic by the uniqueness property of henselizations, we can assume from the start that (K, v) is henselian. By the foregoing lemma, the relative algebraic closures (L_0, v) and (F_0, v) of (K, v) in (L, v) and (F, v) are algebraically maximal Kaplansky fields, with $\overline{L}_0$ relatively algebraically closed in $\overline{L}$ and $\overline{F}_0$ relatively algebraically closed in $\overline{F}$. Since $(L_0|K, v)$ and $(F_0|K, v)$ do not contain proper tame subextensions by hypothesis, they are both maximal purely wild algebraic extensions of (K, v). Moreover, this shows that $\overline{K}$ does not admit any finite separable extension of degree divisible by p. Indeed, since this is true for $\overline{L}$, it also holds for $\overline{L}_0$. If it would not hold for $\overline{K}$, then $\overline{L}_0|\overline{K}$ would contain a proper separable subextension and $(L_0|K, v)$ would contain a proper tame subextension, contrary to our hypothesis. Now it follows from Corollary 13.55 that (L_0, v) and (F_0, v) are isomorphic over K. □

Now we are able to prove the following analogue to Theorem 11.32 for Kaplansky fields:

Theorem 13.58 *Let (K, v) be a Kaplansky field. Then the following assertions hold:*

a) The maximal immediate extension of (K, v) is unique up to valuation preserving isomorphism over K.

b) If (L, v) is a maximal Kaplansky field containing (K, v) such that vL/vK has no p-torsion and Kv is relatively separable-algebraically closed in Lv, then (L, v) contains a maximal immediate extension of (K, v).

Proof: a): The proof is the same as the proof of part a) of Lemma 11.32, with one additional argument. Suppose that we have a common henselian subfield (L, v) of the two maximal immediate extensions (L_1, v_1) and (L_2, v_2) of (K, v), such that there is no proper extension of (L, v) in (L_1, v_1) which can be embedded in (L_2, v_2) over L. By Lemma 13.56, the relative algebraic closures (L'_1, v_1) and (L'_2, v_2) in the immediate extensions (L_1, v_1) and (L_2, v_2) are algebraically maximal and thus, they are maximal immediate algebraic extensions of (L, v). By Corollary 13.55, they are isomorphic over L. Hence, $L'_1 = L$ by the maximality of (L, v), showing that (L, v) is algebraically maximal. Now we proceed as in the proof of Lemma 11.32.

b): Let $(K'|K, v)$ be the maximal immediate subextension of $(L|K, v)$. By Theorem 8.28 we know that the maximal field (L, v) is spherically complete. By Lemma 8.25, it follows that every immediate approximation type over (K', v) is algebraic. By the condition on the value groups and the residue fields, we find that the relative algebraic closure (K'', v) of (K', v) in (L, v) has the property that vK''/vK' is a p-group and $K''v|K'v$ is purely inseparable. But (K', v) being an immediate extension of (K, v), the value group vK' is p-divisible and the residue field $K'v$ is perfect. This yields that $(K''|K', v)$ is immediate, and thus trivial by virtue of the maximality of (K', v). So K' is relatively algebraically closed in L, and Lemma 13.56 now shows that (K', v) is algebraically maximal. Since it also admits no non-trivial immediate transcendental approximation types, it must be a maximal field. □

For henselian Kaplansky fields, we can directly use the Theorem of Schur – Zassenhaus (Theorem 13.26) to prove an assertion which is even stronger than that of Theorem 13.31 (since it works for the ramification field of *every* Galois extension):

Theorem 13.59 *Let (K, v) be a henselian Kaplansky field and $(L|K, v)$ a Galois extension. Then the ramification field $(L|K, v)^r$ of this extension admits a field complement W in L:*

$$(L|K, v)^r.W = L \quad \text{and} \quad (L|K, v)^r \cap W = K \,.$$

These field complements coincide with the maximal immediate algebraic extensions of (K, v) within L. They are unique up to valuation preserving isomorphism over K.

We leave it as an exercise to the reader to give a proof of this theorem along the lines of Section 13.8.

13.13 amc-structures

Value group and residue field are invariants of a valued field, and one main goal of valuation theory is to determine the structure of valued fields with respect to these invariants. In the model theory of valued fields, one studies the question under which additional conditions the elementary theory of valued fields is determined by that of their value groups and residue fields. However, it turns out that these invariants some do not carry enough information. In this section, we will introduce a stronger structure, which will play an important role when we discuss quantifier elimination for valued fields. This structure encodes the additive and multiplicative congruences which hold in a valued field, and the relation between them.

For every initial segment δ of vK, let $\mathcal{M}_{\mathbf{K}}^{\delta}$ be the ideal $\{a \in \mathcal{M}_{\mathbf{K}} : va > \delta\}$ of $\mathcal{O}_{\mathbf{K}}$. In particular, $\mathcal{M}_{\mathbf{K}}^{0} = \mathcal{M}_{\mathbf{K}}$ is the maximal ideal of the valuation ring $\mathcal{O}_{\mathbf{K}}$. Note that $\mathcal{M}_{\mathbf{K}}^{\delta} = \mathcal{M}_{\mathbf{K}}^{0}$ for every $\delta \leq 0$. Further, $\mathcal{O}_{\mathbf{K}}^{\delta}$ will denote the factor ring $\mathcal{O}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$; this is a local ring with maximal ideal $\mathcal{M}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}^{\delta}$. In particular, $\mathcal{O}_{\mathbf{K}}^{0} = \overline{K}$. We write π_δ for the canonical projection $\mathcal{O}_{\mathbf{K}} \longrightarrow \mathcal{O}_{\mathbf{K}}^{\delta}$. Note that for $a \in \mathcal{O}_{\mathbf{K}}$, the projection $\pi_\delta a$ is an invertible element of $\mathcal{O}_{\mathbf{K}}^{\delta}$ if and only if $va = 0$.

On the other hand, consider the multiplicative groups $G_{\mathbf{K}}^{\delta} = K^\times/1{+}\mathcal{M}_{\mathbf{K}}^{\delta}$. In particular,

$$G_{\mathbf{K}} := G_{\mathbf{K}}^{0} = K^\times/1 + \mathcal{M}_{\mathbf{K}} \,.$$

We write π_δ^* for the canonical projection $K^\times \longrightarrow G_{\mathbf{K}}^{\delta}$. Note that $G_{\mathbf{K}}^{\delta}$ is the group of multiplicative congruence classes modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$ in the sense of Hasse. The group $G_{\mathbf{K}}$ reminds of the power predicates P_n. Indeed, if $\mathbf{K}$ is henselian and n is not divisible by the residue characteristic of $\mathbf{K}$ then Hensel's Lemma shows that $a \in \mathbf{K}$ admits an n-th root in $\mathbf{K}$ if and only if $\pi_0^* a$ admits an n-th root in $G_{\mathbf{K}}$. If n is divisible by the residue characteristic then this does not work. But if in this case, the characteristic of $\mathbf{K}$ itself is 0, then the groups $G_{\mathbf{K}}^{\delta}$ for $\delta > 0$ may be used to overcome this difficulty, as we will see below.

The local ring $\mathcal{O}_{\mathbf{K}}^{\delta}$ and the group $G_{\mathbf{K}}^{\delta}$ are related through a relation given by

$$\forall x \in \mathcal{O}_{\mathbf{K}}^{\delta}\, \forall y \in G_{\mathbf{K}}^{\delta} : \quad \Theta_\delta(x,y) \Leftrightarrow \exists z \in \mathcal{O}_{\mathbf{K}} : \pi_\delta z = x \wedge \pi_\delta^* z = y \,.$$

For elements of value 0, additive congruence modulo $\mathcal{M}_{\mathbf{K}}^{\delta}$ implies multiplicative congruence modulo $1 + \mathcal{M}_{\mathbf{K}}^{\delta}$. Hence Θ_δ induces a group homomorphism from ${\mathcal{O}_{\mathbf{K}}^{\delta}}^\times$ into $G_{\mathbf{K}}^{\delta}$ given by

$$\vartheta_\delta :\ a + \mathcal{M}_{\mathbf{K}}^{\delta} \mapsto a(1 + \mathcal{M}_{\mathbf{K}}^{\delta}) \ \text{ for all } a \in \mathcal{O}_{\mathbf{K}}^\times \,.$$

We have

$$\pi_\delta^* a = \vartheta_\delta \pi_\delta a \ \text{ for all } a \in \mathcal{O}_{\mathbf{K}}^\times \,. \tag{13.11}$$

For every initial segment δ of vK, we consider the system

$$\mathbf{K}_\delta = (\mathcal{O}_{\mathbf{K}}^{\delta},\ G_{\mathbf{K}}^{\delta},\ \Theta_\delta)$$

and call it the **structure of additive and multiplicative congruences of level δ in** K, or shorter: the **amc-structure of level** δ. In particular, $\mathbf{K}_0$ is the pair $(\overline{K},\ G_{\mathbf{K}})$ together with the embedding

$$\vartheta_0 : \overline{K}^\times \longrightarrow G_{\mathbf{K}}$$

whose cokernel is just the value group of $\mathbf{K}$:

$$vK \cong G_{\mathbf{K}}/\vartheta_0\overline{K}^{\times}\;, \tag{13.12}$$

and together with a unary predicate

$$\mathrm{Pos}(x) :\equiv \Theta_0(0,x)$$

on $G_{\mathbf{K}}$ whose range is exactly $\pi_0^*\mathcal{M}_{\mathbf{K}}$ and which maps modulo $\vartheta_0\overline{K}^{\times}$ onto the subset of positive elements in vK. More generally, on $G_{\mathbf{K}}^{\delta}$ we define

$$\mathrm{Pos}_{\delta}(x) :\equiv \Theta_{\delta}(0,x)$$

whose range is exactly $\pi_{\delta}^*\mathcal{M}_{\mathbf{K}}^{\delta}$. For an arbitrary valued field $\mathbf{K}$,

$$vK \cong G_{\mathbf{K}}/\{g \in G_{\mathbf{K}} \mid \neg\mathrm{Pos}(g) \wedge \neg\mathrm{Pos}(g^{-1})\} \tag{13.13}$$

and the order on vK (more precisely, the subset of all elements > 0) is just the image of the predicate Pos.

13.14 An Isomorphism Theorem for tame algebraic extensions

We will first describe the structure of a finite tame extension $\mathbf{L}|\mathbf{K}$ of henselian fields.

The residue field extension $\overline{L}|\overline{K}$ is finite and separable, hence simple. Let $\overline{c}$ be a generator of it. We choose some monic polynomial $f \in K[X]$ whose reduction modulo v is the irreducible polynomial of $\overline{c}$ over $\overline{K}$. Since the latter is separable, we may use Hensel's Lemma to find a root $c \in L$ of f with residue $\overline{c}$. From the fundamental inequality it follows that the extension $K(c)|K$ is of the same degree as $\overline{K}(\overline{c})|\overline{K}$ and that $\overline{K(c)} = \overline{K}(\overline{c}) = \overline{L}$. Let us mention that $(K(c),v)$ is the inertia field of our extension $\mathbf{L}|\mathbf{K}$.

Now we have to treat the case of $vL \neq vK$. Let $\alpha \in vL \setminus vK$ and assume that $n \neq 0$ is the minimal natural number such that $n\alpha \in vK$. If $a \in L$ with $va = \alpha$, then $va^n \in vK$ and thus there is some $b \in K$ with $v(ba^n) = 0$. Then the v-residue $\overline{ba^n} \in Lv = K(c)v$ is not zero, hence there is some $h \in K[X]$ with $\overline{a^{-n}b^{-1}h(c)} = 1$. By the minimality of n and condition 2) for tame extensions, n is prime to p if $\mathrm{char}(Kv) = p > 0$. Hence in the henselian field $\mathbf{L}$, we may use Hensel's Lemma to deduce the existence of some element $a_0 \in L$ which satisfies $a_0^n = a^{-n}b^{-1}h(c)$; putting $d = aa_0 \in L$ we get $bd^n = h(c)$. Note that we may choose h with v-integral coefficients since it only has to satisfy $\overline{h}(\overline{c}) = \overline{a^n b} \in \overline{K}(\overline{c})$ where $\overline{h}$ denotes the reduction of h modulo v.

Since $L|K$ is a finite extension, the group vL/vK is a finite torsion group, say

$$vL/vK = \mathbb{Z}\cdot(\alpha_1 + vK) \times \ldots \times \mathbb{Z}\cdot(\alpha_r + vK) \tag{13.14}$$

where every α_i has finite order, say n_i. Using the above procedure, for $1 \leq i \leq r$ we choose elements

- $d_i \in L$ with $b_i d_i^{n_i} = h_i(c)$, where
- $b_i \in K$ with $v(b_i) = -n_i\alpha_i$

- $h_i \in \mathcal{O}_{\mathbf{K}}[X]$ with $vh_i(c) = 0$.

Then $vL = vK(c, d_1, \ldots, d_r)$, and since $K(c) \subset K(c, d_1, \ldots, d_r) \subset L$ and $K(c)v = Lv$, we also have $Lv = K(c, d_1, \ldots, d_r)v$. From condition 3) on tame extensions it follows that $L = K(c, d_1, \ldots, d_r)$. On the other hand,

$$\begin{aligned} [L : K(c)] &\geq (vL : vK(c)) = (vL : vK) = n_1 \cdot \ldots \cdot n_r \\ &\geq [K(c, d_1, \ldots, d_r) : K(c)] = [L : K(c)] , \end{aligned}$$

whence $[L : K(c)] = n_1 \cdot \ldots \cdot n_r$ which shows that the extensions

$$K(c, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_r)|K(c) \quad \text{and} \quad K(c, d_i)|K(c)$$

are linearly disjoint for every i, $1 \leq i \leq r$. Hence, if $\mathbf{K} \subset \mathbf{F}$ and $z, t_1, \ldots, t_r \in F$ such that $f(z) = 0$ and $b_i t_i^{n_i} = h_i(z)$, then

$$(c, d_1, \ldots, d_r) \mapsto (z, t_1, \ldots, t_r)$$

induces an embedding of L in F over K. Since $\mathbf{K}$ is henselian, this is valuation preserving, i.e. an embedding of $\mathbf{L}$ in $\mathbf{F}$ over $\mathbf{K}$.

Let us note the following special cases:
– if $\overline{L} = \overline{K}$, then we may take $\overline{c}$, c and all $h_i(c)$ to be equal to 1,
– if $\mathbf{L}|\mathbf{K}$ is unramified, then we may set $r = 0$.

Using the "normal form" for finite tame extensions that we have now introduced, we will prove the main embedding lemma for tame algebraic extensions:

Lemma 13.60 *Let* $\mathbf{K}$ *be an arbitrary valued field,* $\mathbf{L}$ *a tame algebraic extension of some henselization of* $\mathbf{K}$ *and* $\mathbf{F}$ *an arbitrary henselian extension of* $\mathbf{K}$*. If* $\mathbf{L}$ *is embeddable in* $\mathbf{F}$ *over* $\mathbf{K}$*, then* $\mathbf{L}_0$ *is embeddable in* $\mathbf{F}_0$ *over* $\mathbf{K}_0$*. Conversely, every embedding* τ *of* $\mathbf{L}_0$ *in* $\mathbf{F}_0$ *over* $\mathbf{K}_0$ *can be pulled back to an embedding of* $\mathbf{L}$ *in* $\mathbf{F}$ *over* $\mathbf{K}$ *which induces* τ*.*

If in addition, $\mathbf{L}|\mathbf{K}$ *is unramified, then the same works for every embedding of* $\overline{L}$ *in* $\overline{F}$ *over* $\overline{K}$*. If on the other hand* $\overline{L} = \overline{K}$*, then the same works for every embedding of* $G_{\mathbf{L}}$ *in* $G_{\mathbf{F}}$ *over* $G_{\mathbf{K}}$*.*

Proof: The proof of the first statement is straightforward and thus left to the reader. Let now be given an embedding τ of $\mathbf{L}_0$ in $\mathbf{F}_0$ over $\mathbf{K}_0$.

Since both $\mathbf{L}$ and $\mathbf{F}$ are assumed to be henselian, they both contain henselizations of $\mathbf{K}$. By the uniqueness property of henselizations, these are isomorphic over $\mathbf{K}$ and we may identify them. This henselization has the same amc-structure of level 0 as $\mathbf{K}$: for every a in a henselization of $\mathbf{K}$ there is some $a' \in K$ such that $v(a - a') > va$, so a and a' have the same images under π_0 and π_0^*. Hence it suffices to prove our lemma under the additional hypothesis that $\mathbf{K}$ be henselian.

In view of the Compactness Principle for Algebraic Extensions (Theorem 24.5), it suffices to prove our lemma only in the case of $\mathbf{L}|\mathbf{K}$ a finite extension of henselian fields. Let $\mathbf{L}|\mathbf{K}$ be given as described above. By the remark preceding our lemma, it suffices to find an image in $\mathbf{F}$ for the tuple $(c, d_1, \ldots, d_r)$ in order to obtain an embedding of $\mathbf{L}$ in $\mathbf{F}$ over $\mathbf{K}$. This tuple satisfies

$$\overline{f}(\overline{c}) = 0 \quad \text{and} \quad \bigwedge_{1 \leq i \leq r} \Theta_0(\overline{h_i}(\overline{c}) \, , \, \tilde{b}_i \tilde{d}_i^{\,n_i}) \wedge \overline{h_i}(\overline{c}) \neq 0$$

where $\tilde{b}_i = \pi_0^* b_i$ and $\tilde{d}_i = \pi_0^* d_i$. Now τ sends $\bar{c}$ to some element $x \in \overline{F}$ and every $\tilde{d}_i$ to some $y_i \in G_{\mathbf{F}}$ which satisfy

$$\overline{f}(x) = 0 \quad \text{and} \quad \bigwedge_{1\le i\le r} \Theta_0(\overline{h_i}(x)\,,\, \tilde{b}_i y_i^{n_i}) \wedge \overline{h_i}(x) \neq 0$$

The polynomial $\overline{f}$ being irreducible and separable over $\overline{K}$, the zero x is simple and thus gives rise to a zero $z \in F$ of f with residue x by virtue of Hensel's Lemma.

Now let $i \in \{1, \ldots, r\}$. We choose $\eta_i \in F$ such that $\pi_0^* \eta_i = y_i$. Since $\overline{h_i}(x) \neq 0$, the relation $\Theta_0(\overline{h_i}(x), \tilde{b}_i y_i^{n_i})$ is equivalent to $\vartheta_0 \overline{h_i}(x) = \tilde{b}_i y_i^{n_i}$ which in turn gives

$$\pi_0^* h_i(z) = \vartheta_0 \pi_0 h_i(z) = \vartheta_0 \overline{h_i}(x) = \tilde{b}_i y_i^{n_i} = \pi_0^* b_i \eta_i^{n_i}\ ,$$

that is, $h_i(z) b_i^{-1} \eta_i^{-n_i} \equiv 1 \bmod \mathcal{M}_{\mathbf{F}}$. So the polynomial

$$X^{n_i} - h_i(z) b_i^{-1} \eta_i^{-n_i} \in \mathcal{O}_F[X] \tag{13.15}$$

reduces modulo v to the polynomial $X^{n_i} - 1$ which admits 1 as a simple root since n_i is not divisible by the characteristic of $\overline{K}$. By virtue of Hensel's Lemma, the polynomial (13.15) admits a root η_i' in the henselian field $\mathbf{F}$. Putting $t_i := \eta_i' \eta_i \in F$, we obtain $b_i t_i^{n_i} = h_i(z)$. Consequently, the assignment $(c, d_1, \ldots, d_r) \mapsto (z, t_1, \ldots, t_r)$ induces an embedding of $\mathbf{L}$ in $\mathbf{F}$ over $\mathbf{K}$.

We still have to show that it is a lifting of τ. But this will follow if we are able to show that the assignment $(\bar{c}, \tilde{d}_1, \ldots, \tilde{d}_r) \mapsto (x, y_1, \ldots, y_r)$ determines the embedding of $\mathbf{L}_0$ in $\mathbf{F}_0$ over $\mathbf{K}_0$ uniquely. Since $\bar{c}$ generates $\overline{L}$ over $\overline{K}$, it just remains to show that the elements $\tilde{d}_1, \ldots, \tilde{d}_r$ generate $G_{\mathbf{L}}$ over the group compositum $G_{\mathbf{K}}.\vartheta_0 \overline{L}$. Given an element $a \in L$, our choice of the d_i implies that there exist integers $m_1, \ldots, m_r$, an element $d' \in K$ and an element $g(c) \in \mathcal{O}_K[c]$ of value 0 such that the value of $a^{-1} d_1^{m_1} \cdot \ldots \cdot d_r^{m_r} d' g(c)$ is 0 and its residue is 1. Hence

$$\pi_0^* a = \tilde{d}_1^{m_1} \cdot \ldots \cdot \tilde{d}_r^{m_r} \cdot \pi_0^* d' \cdot \vartheta_0 \overline{g}(\bar{c})$$

with $\pi_0^* d' \in G_{\mathbf{K}}$ and $\vartheta_0 \overline{g}(\bar{c}) \in \vartheta_0 \overline{L}$. This concludes our proof. (The special cases mentioned in the lemma are shown by straightforward modifications of this proof.) □

From this proof, we extract one more interesting case, namely the case where the relation Θ_0 can be omitted. We see from the proof that it is indeed superfluous if all $h_i(c)$ can be chosen to be an element of K. But this means that in $\mathbf{L}$ there exists a subfield $\mathbf{C}$ which has the same value group as $\mathbf{L}$, the same residue field as $\mathbf{K}$ and is a field complement of the inertia field $\mathbf{L}^i$ of $\mathbf{L}|\mathbf{K}$ over the henselization $\mathbf{K}^h$ of $\mathbf{K}$ in $\mathbf{L}$. This means, $\mathbf{C}$ is linearly disjoint from $\mathbf{L}^i$ over $\mathbf{K}^h$ and the compositum $\mathbf{L}^i.\mathbf{C}$ equals $\mathbf{L}$. Conversely, one can show that every field complement $\mathbf{C}$ of the inertia field $\mathbf{L}^i$ in $\mathbf{L}$ over $\mathbf{K}^h$ has the property $vC = vL$ and $\overline{C} = \overline{K}$. Since $\mathbf{L}|\mathbf{K}^h$ is supposed to be a tame algebraic extension, the same is true for the subextension $\mathbf{C}|\mathbf{K}^h$.

Lemma 13.61 *Let $\mathbf{K}$ be an arbitrary valued field, $\mathbf{C}$ a tame algebraic extension of some henselization $\mathbf{K}^h$ of $\mathbf{K}$ such that $\overline{C} = \overline{K}$. Then $\mathbf{C}$ is generated over $\mathbf{K}^h$ by its subset*

$$R = \bigcup_{n\,\in\,\mathbb{N}} \{x \in C \mid x^n \in K\}\ .$$

of radicals over K, i.e. $\mathbf{C} = (K^h(R), v)$ (which is equal to the henselization of $(K(R), v)$ inside of $\mathbf{C}$).

Proof: Let $\alpha \in vC \setminus vK$ and $n \in \mathbb{N} \setminus \{0\}$ minimal with $n\alpha \in vK$. Choose $a \in C$ with $va = \alpha$ and $c \in K$ such that $v(a^n - c) > 0$ (which is possible since $va^n \in vK$ and $\overline{C} = \overline{K}$). By our hypothesis it follows that $\mathbf{C}|\mathbf{K}^h$ is a tame extension, so n (being minimal with $n\alpha \in vK$) is not divisible by the residue characteristic. Hence, by virtue of Hensel's Lemma there exists an element $a_0 \in C$ of value 0 such that $a_0^n = a^{-n}c$. Replacing a by aa_0, we obtain an element $a \in C$ of value α which satisfies $a^n \in K$.

Let R be the collection of all radicals a obtained in this way for all $\alpha \in vC \setminus vK$. Then $(K^h(R), v)$ has the same value group as $\mathbf{C}$. Since $\overline{C} = \overline{K}$, it also has the same residue field as $\mathbf{C}$. As a part of a tame extension, $\mathbf{C}|(K^h(R), v)$ is tame. Since it is immediate, it must be trivial: $C = K^h(R)$. □

From the proof of Lemma 13.60, we can now deduce the following lemma:

Lemma 13.62 *Let the hypothesis be as in Lemma 13.60 and assume in addition that there exists a field complement* $\mathbf{C}$ *of the inertia field* $\mathbf{L}^i$ *in* $\mathbf{L}$ *over* $\mathbf{K}^h$*.*

a) For all embeddings ρ *of* $\overline{L}$ *in* $\overline{F}$ *over* $\overline{K}$ *and* σ *of* $G_{\mathbf{L}}$ *in* $G_{\mathbf{F}}$ *over* $G_{\mathbf{K}}$*, there exists an embedding of* $\mathbf{L}$ *in* $\mathbf{F}$ *over* $\mathbf{K}$ *which induces* ρ *and* σ*.*

b) If $\forall n \in \mathbb{N} : K \cap L^n \subset K \cap F^n$*, then for every embedding* ρ *of* $\overline{L}$ *in* $\overline{F}$ *over* $\overline{K}$ *there exists an embedding of* $\mathbf{L}$ *in* $\mathbf{F}$ *over* $\mathbf{K}$ *which induces* ρ*.*

Note: if also $\mathbf{F}$ is a tame algebraic extension of some henselization of $\mathbf{K}$ which admits a field complement of its inertia field, then the embedding σ in a) may be replaced by an embedding of vL in vF over vK. The proof is left to the reader.

Two algebraic extensions of $\mathbf{K}$ are isomorphic over $\mathbf{K}$ if they can be embedded in each other over $\mathbf{K}$. Hence we get the following theorem as an immediate corollary to Lemma 13.60 and Lemma 13.62. It may be seen as a classification of tame algebraic extensions relative to their amc-structures of level 0.

Theorem 13.63 *Let* $\mathbf{K}$ *be an arbitrary valued field and* $\mathbf{L}$*,* $\mathbf{F}$ *tame algebraic extensions of some henselizations of* $\mathbf{K}$*. Then* $\mathbf{L}$ *and* $\mathbf{F}$ *are isomorphic over* $\mathbf{K}$ *if and only if their amc-structures of level 0 are isomorphic over* $\mathbf{K}_0$*. Under the additional hypothesis of Lemma 13.62, the isomorphism will follow already from*

1) an isomorphism of the amc-structures without the Θ_0*-relation, or*

2) an isomorphism $\overline{L} \cong \overline{F}$ *over* $\overline{K}$ *together with the condition*
$\forall n \in \mathbb{N} : K \cap L^n = K \cap F^n$*.*

Moreover, if $\mathbf{L}|\mathbf{K}$ *and* $\mathbf{F}|\mathbf{K}$ *are unramified, then the isomorphism follows already if* $\overline{L}$ *and* $\overline{F}$ *are isomorphic over* $\overline{K}$*. If on the other hand,* $\mathbf{L}$*,* $\mathbf{F}$ *and* $\mathbf{K}$ *have the same residue field, then the isomorphism follows already if* $G_{\mathbf{L}}$ *and* $G_{\mathbf{F}}$ *are isomorphic over* $G_{\mathbf{K}}$*, or if* $\forall n \in \mathbb{N} : K \cap L^n = K \cap F^n$*.*

In all preceding conditions, the isomorphism $G_{\mathbf{L}} \cong G_{\mathbf{F}}$ *over* $G_{\mathbf{K}}$ *may be replaced by an isomorphism* $vL \cong vF$ *over* vK*, if the hypothesis of Lemma 13.62 applies to both* $\mathbf{L}$ *and* $\mathbf{F}$*.*

For the case of $\overline{L}, \overline{K}, \overline{F}$ not all being equal, we need a generalization of Lemma 13.60. A general not necessarily algebraic extension $(k_1, w)|(k, w)$ will be called **pretame**, if the following holds:

1) the residue field extension $k_1w|kw$ is separable,

2) if $p = \mathrm{char}(kw) > 0$, then the order of every torsion element of vk_1/vk is prime to p.

Note that every extension of a tame field is pretame, and that every algebraic pretame extension of a defectless field is tame.

Suppose now that $\mathbf{K}$ is a defectless field, $\mathbf{F}$ a henselian extension field of $\mathbf{K}$ and $\mathbf{L}|\mathbf{K}$ a pretame extension which admits a valuation transcendence basis $\mathcal{T}$ of the form

$$\left.\begin{array}{l} \mathcal{T} = \{x_i, y_j \mid i \in I, j \in J\} \text{ such that} \\ \text{the values } vx_i,\ i \in I \text{ form a maximal system of values} \\ \text{in } vL \text{ which are rationally independent over } vK, \text{ and} \\ \text{the residues } \overline{y_j},\ j \in J \text{ form a transcendence basis of } \overline{L}|\overline{K}. \end{array}\right\} \tag{13.16}$$

Suppose that τ is an embedding of $\mathbf{L}_0$ in $\mathbf{F}_0$ over $\mathbf{K}_0$. Then $\overline{L}$ is embeddable in $\overline{F}$ over $\overline{K}$. But also vL is embeddable in vF over vK since $vL \cong G_{\mathbf{L}}/\vartheta_0\overline{L}^\times$ and the order relation on vK is induced by Pos. Denote these embeddings by σ and ρ and choose a set $\mathcal{T}' = \{x'_i, y'_j \mid i \in I, j \in J\} \subset F$ such that $vx'_i = \rho v x_i$ for all $i \in I$, and $\overline{y'_j} = \sigma\overline{y_j}$ for all $j \in J$. Then $\mathcal{T}'$ is a valuation transcendence basis of the subextension $(K(\mathcal{T}), v)|\mathbf{K}$ of $\mathbf{F}|\mathbf{K}$, and the assignment $x_i \mapsto x'_i$, $i \in I$, $y_j \mapsto y'_j$, $j \in J$, induces a valuation preserving isomorphism from $(K(\mathcal{T}), v)$ onto $(K(\mathcal{T}'), v)$ over $\mathbf{K}$ (more precisely, this isomorphism induces the embedding ρ on the value groups and the embedding σ on the residue fields). This isomorphism is even a lifting of the isomorphism of the respective amc-structures of level 0 which is a restriction of τ. Similarly as in the proof of Lemma 13.60, this is shown by proving that the residues $\overline{y}_j$, $j \in J$, generate $\overline{K(\mathcal{T})}$ over $\overline{K}$ and that the elements $\pi_0^* x_i$, $i \in I$, generate $G_{(K(\mathcal{T}),v)}$ over the compositum $G_{\mathbf{K}} \,.\, \vartheta_0\overline{K(\mathcal{T})}^\times$.

Hence we may identify $K(\mathcal{T})$ and $K(\mathcal{T}')$ as a common valued subfield of $\mathbf{L}$ and $\mathbf{F}$. We may now apply Lemma 13.60 to get:

Lemma 13.64 *Let $\mathbf{K}$ be a common subfield of the henselian fields $\mathbf{L}$ and $\mathbf{F}$. Assume that $\mathbf{L}$ admits a valuation transcendence basis $\mathcal{T}$ such that $\mathbf{L}$ itself is a tame extension of some henselization $(K(\mathcal{T}), v)^h$. Then for every embedding τ of $\mathbf{L}_0$ in $\mathbf{F}_0$ over $\mathbf{K}_0$ there is an embedding of $\mathbf{L}$ in $\mathbf{F}$ over $\mathbf{K}$ which induces τ.*

The special cases mentioned in Lemma 13.60 go through as follows. If $\mathbf{L}|\mathbf{K}$ is unramified then again, an embedding of $\overline{L}$ in $\overline{F}$ over $\overline{K}$ will suffice. On the other hand, if $\overline{L} = \overline{K}$ then a simple embedding of $G_{\mathbf{L}}$ in $G_{\mathbf{F}}$ over $G_{\mathbf{K}}$ may not suffice. We have seen above that in the ramified case, an embedding ρ of vL in vF over vK is needed. But in view of (13.13), this will be induced by an embedding of $(G_{\mathbf{L}}, \mathrm{Pos})$ in $(G_{\mathbf{F}}, \mathrm{Pos})$ over $(G_{\mathbf{K}}, \mathrm{Pos})$; hence in the case $\overline{L} = \overline{K}$, such an embedding will suffice.

13.15 An Isomorphism Theorem in the mixed characteristic case

Theorem 13.65 *Let $\mathbf{K} = (K, v)$ be a valued field, $\mathbf{L} = (L, v)$ an algebraic extension of $\mathbf{K}$ and $\mathbf{F} = (F, v)$ an arbitrary henselian extension of $\mathbf{K}$. If Δ is a convex subgroup of vK such that (L, v_Δ) is a tame extension of some henselization of (K, v_Δ), then the next statements are equivalent:*

i) $\mathbf{L}$ *is embeddable in* $\mathbf{F}$ *over* K,

ii) $\mathbf{L}_\delta$ *is embeddable in* $\mathbf{F}_\delta$ *over* $\mathbf{K}_\delta$ *for each* $\delta \in \Delta^{\geq 0}$.

Note that the existence of such a convex subgroup Δ of vK implies that $L|K$ is separable; conversely, if $L|K$ is separable, then there is always such a Δ, namely $\Delta = vK$. With this choice, v_Δ is the trivial valuation, and $\overline{v}_\Delta$ may be identified with v.

In view of our remark preceding to Theorem 13.63, the following isomorphism theorem is an immediate consequence of Theorem 13.65.

Theorem 13.66 *Let* $\mathbf{K} = (K, v)$ *be a valued field and* $\mathbf{L} = (L, v)$, $\mathbf{F} = (F, v)$ *two henselian algebraic extensions of* $\mathbf{K}$. *If* Δ *is a convex subgroup of* vK *such that both* (L, v_Δ) *and* (F, v_Δ) *are tame extensions of some henselizations of* (K, v_Δ), *then the next statements are equivalent:*

i) $\mathbf{L}$ *and* $\mathbf{F}$ *are isomorphic over* K,

ii) $\mathbf{L}_\delta$ *and* $\mathbf{F}_\delta$ *are isomorphic over* $\mathbf{K}_\delta$ *for each* $\delta \in \Delta^{\geq 0}$.

As a special case, we want to consider fields of characteristic 0. Assume $\mathrm{char} K = 0$ and let p be the *characteristic exponent* of the residue field $\overline{K}$, i.e. $p = \mathrm{char}(\overline{K}) > 0$ or $p = 1$ if $\mathrm{char}(\overline{K}) = 0$. The **canonical decomposition** of the valuation v is defined as follows. Denote by $\Delta_{\mathbf{K}}$ the smallest convex subgroup of vK containing the value vp; note that the value set $\{n \cdot vp \mid n \in \mathbb{N}\}$ is cofinal in Δ. We write $\dot{v} := v_{\Delta_{\mathbf{K}}}$; this is called the **coarse valuation** assigned to v. Denote by $\dot{\mathbf{K}}$ the valued field $(K, \dot{v})$. The valuation ring $\mathcal{O}_{\dot{\mathbf{K}}}$ of $\dot{\mathbf{K}}$ is characterized as the smallest overring of $\mathcal{O}_{\mathbf{K}}$ in which p becomes a unit, i.e. $\mathcal{O}_{\dot{\mathbf{K}}}$ is the ring of fractions of $\mathcal{O}_{\mathbf{K}}$ with respect to the multiplicatively closed set $\{p^n \mid n \in \mathbb{N}\}$; consequently, the residue field $K\dot{v}$ is of characteristic 0. Note that $\dot{v} = v$ iff $p = 1$, and $\dot{v}$ is trivial if and only if $\Delta_{\mathbf{K}} = vK$. For $n \in \mathbb{N}$ we write $\mathbf{K}_n$ instead of $\mathbf{K}_{n \cdot vp}$.

Since $(K, \dot{v})$ has residue characteristic $\mathrm{char} K\dot{v} = 0$, every algebraic extension of a henselization of $(K, \dot{v})$ is tame. This is immediately seen from the second characterization of tame extensions given above; note that in this case, condition 3) is a consequence of the Lemma of Ostrowski. Hence, with $\Delta = \Delta_{\mathbf{K}}$, we obtain from Theorems 13.65 and 13.66 the following corollary:

Corollary 13.67 *Let* $\mathbf{K} = (K, v)$ *be a valued field of characteristic zero,* $\mathbf{L} = (L, v)$ *a henselian algebraic extension of* $\mathbf{K}$ *and* $\mathbf{F} = (F, v)$ *an arbitrary henselian extension of* $\mathbf{K}$. *The next statements are equivalent:*

i) $\mathbf{L}$ *is embeddable in* $\mathbf{F}$ *over* K,

ii) $\mathbf{L}_n$ *is embeddable in* $\mathbf{F}_n$ *over* $\mathbf{K}_n$ *for each* $n \in \mathbb{N}$.

If also $F|K$ *is algebraic, then the next statements are equivalent:*

i) $\mathbf{L}$ *and* $\mathbf{F}$ *are isomorphic over* K.

ii) $\mathbf{L}_n$ *and* $\mathbf{F}_n$ *are isomorphic over* $\mathbf{K}_n$ *for each* $n \in \mathbb{N}$.

Theorem 13.68 *Let* $\mathbf{K} = (K, v)$ *be a* p*-valued field and* $\mathbf{L} = (L, v)$, $\mathbf{F} = (F, v)$ *be two henselian* p*-valued algebraic extensions of the same* p*-rank as* $\mathbf{K}$. *Then* $\mathbf{L}$ *and* $\mathbf{F}$ *are isomorphic over* K *if and only if* $K \cap L^n = K \cap F^n$ *for each* $n \in \mathbb{N}$.