# Chapter 14

# Extremality, maximality, and defectless fields

## 14.1 Extremality

Take a valued field $(K,v)$ with valuation ring $\mathcal{O}$. If $f$ is a polynomial in $n$ variables with coefficients in $K$, then we will say that $(K,v)$ is **$K$-extremal with respect to $f$** if the set

$$v\,\mathrm{im}_K(f) \;:=\; \{vf(a_1,\ldots,a_n) \mid a_1,\ldots,a_n \in K\} \;\subseteq vK \cup \{\infty\} \tag{14.1}$$

has a maximum, and we we will say that $(K,v)$ is **$\mathcal{O}$-extremal with respect to $f$** if the set

$$v\,\mathrm{im}_{\mathcal{O}}(f) \;:=\; \{vf(a_1,\ldots,a_n) \mid a_1,\ldots,a_n \in \mathcal{O}\} \;\subseteq vK \cup \{\infty\} \tag{14.2}$$

has a maximum. The former means that

$$\exists Y_1,\ldots,Y_n \forall X_1,\ldots,X_n :\; vf(X_1,\ldots,X_n) \leq vf(Y_1,\ldots,Y_n)$$

holds in $(K,v)$. For the latter, one has to build into the sentence the condition that the $X_i$ and $Y_j$ only run over elements of $\mathcal{O}$. It follows that being $K$-extremal or $\mathcal{O}$-extremal with respect to $f$ is an elementary property in the language of valued fields with parameters from $K$. Note that in the first case the maximum is $\infty$ if and only if $f$ admits a zero in $K^n$; in the second case, this zero has to lie in $\mathcal{O}^n$. A valued field $(K,v)$ is called **extremal** if for all $n \in \mathbb{N}$, it is $\mathcal{O}$-extremal with respect to every polynomial $f$ in $n$ variables with coefficients in $K$. This property can be expressed by a countable scheme of elementary sentences (quantifying over the coefficients of all possible polynomials of degree at most $n$ in at most $n$ variables). Hence, it is elementary in the language of valued fields.

If we would have chosen $K$-extremality for the definition of "extremal valued field" (as Yu. Ershov in [Er2]), then we would have obtained precisely the class of algebraically closed valued fields. Using $\mathcal{O}$-extremality instead yields a much more interesting class of valued fields. See [A–Ku–Pop] for details.

The properties "algebraically maximal", "separable-algebraically maximal" and "inseparably defectless" are each equivalent to $K$- or $\mathcal{O}$-extremality restricted to certain (elementarily definable) classes of polynomials, as we will see in the following sections.

## 14.2 Inseparably defectless fields

In this section, we shall give a characterization of inseparably defectless fields. Recall that a valued field is called **inseparably defectless** if each of its purely inseparable extensions is defectless. Recall also that every purely inseparable algebraic extension admits a unique extension of the valuation, cf. Corollary 6.57. Every defectless field and in particular every trivially valued field is inseparably defectless. A valued field is called **inseparably maximal** if it does not admit proper immediate purely inseparable extensions. Note that a valued field can be inseparably maximal without being inseparably defectless. The field $(F, v)$ of Example 11.64 is of this kind.

Let us observe that for an inseparably defectless field $(K, v)$, every immediate extension is separable. Indeed, it follows from Lemma 6.8 that every immediate extension of $(K, v)$ is linearly disjoint from the defectless extension $(K^{1/p^\infty}|K, v)$. In the literature, one can find the expression **excellent** for those fields for which all immediate extensions are separable (cf. [DEL1], Définition 1.41). But there are also other properties of certain valuation rings for which this expression is used.

By definition, $(K, v)$ is an inseparably defectless field if and only if the extension $(K^{1/p^\infty}|K, v)$ is defectless. For this to hold, it is sufficient that the subextension $(K^{1/p}|K, v)$ is defectless:

**Lemma 14.1** *$(K, v)$ is an inseparably defectless field if and only if $(K^{1/p}|K, v)$ is a defectless extension, and this holds if and only if $(K|K^p, v)$ is a defectless extension.*

**Proof:** The first implication "$\Rightarrow$" follows from the fact that every subextension of a defectless extension is again defectless. Now assume that $(K^{1/p}|K, v)$ is a defectless extension. The Frobenius endomorphism sends the extension $(K^{1/p^2}|K^{1/p}, v)$ onto the extension $(K^{1/p}|K, v)$ and is valuation preserving. Consequently, also the former extension is defectless (cf. Lemma 6.27). By induction, we find that $(K^{1/p^m}|K^{1/p^{m-1}}, v)$ is defectless for every $m \geq 1$. Hence by the transitivity of defectless extensions (Lemma 6.5), also $(K^{1/p^m}|K, v)$ is defectless. Since every finite subextension of $K^{1/p^\infty}|K$ is already contained in $K^{1/p^m}$ for some $m$, it follows that $(K^{1/p^\infty}|K, v)$ is defectless.

The second equivalence is proved again by use of the Frobenius endomorphism. $\square$

We can now give a characterization of inseparably defectless fields in terms of extremality.

**Theorem 14.2** *A valued field $K$ of positive characteristic is inseparably defectless if and only if it is $K$-extremal with respect to every $p$-polynomial of the form*

$$b - \sum_{i=1}^{n} b_i X_i^p \tag{14.3}$$

*with $n \in \mathbb{N}$, $b, b_1, \ldots, b_n \in K$ such that $b_1, \ldots, b_n$ form a basis of a finite extension of $K^p$ (inside of $K$). If the value group $vK$ of $K$ is divisible or a $\mathbb{Z}$-group, then $K$ is inseparably defectless if and only if it is $\mathcal{O}$-extremal with respect to every $p$-polynomial (14.3) with $n \in \mathbb{N}$, $b, b_1, \ldots, b_n \in \mathcal{O}$ such that $b_1, \ldots, b_n$ form a valuation basis of a finite defectless extension of $K^p$ and $vb_1, \ldots, vb_n$ are smaller than every positive element of $vK$.*

**Proof:** The first assertion is an easy consequence of the last lemma. Given $L$ and $b$ as in that lemma, we take $b_1 \ldots, b_n$ to be a $K^p$-basis of $L$. Then $c \in L$ if and only if $c = \sum_{i=1}^n b_i c_i^p$ for some $c_1, \ldots, c_n \in K$. Hence, $v(b - L)$ has a maximum if and only if $(K, v)$ is $K$-extremal with respect to the polynomial (14.3).

To prove the second assertion of Theorem 14.2, we assume that $vK$ is divisible or a $\mathbb{Z}$-group. The same is then true for $vK^p$ and for $vL$ for every $L$ as in the previous lemma. Further, we note that in the previous lemma, we can restrict the scope to all $b \in \mathcal{O}$. As well, we can restrict the scope to all defectless extensions $(L|K^p, v)$. So we can choose $b_1, \ldots, b_n$ to be a valuation basis of $(L|K^p, v)$. If $vK^p$ is divisible, then $vL = vK^p$ and we can assume in addition that $vb_1 = \ldots = vb_n = 0$. If $vK^p$ is a $\mathbb{Z}$-group with least positive element $\alpha$, then we can assume in addition that for $1 \leq i \leq n$, $vb_i = \frac{\ell_i}{p^m}\alpha$ for some $\ell_i \in \{0, \ldots, p^m - 1\}$, with $m \geq 0$ fixed; so $0 \leq vb_i < \alpha$. Now it remains to show that $v(b - L)$ has a maximal element if and only if $(K, v)$ is $\mathcal{O}$-extremal with respect to the polynomial (14.3). We observe that $0 \leq v(b - 0) \in v(b - L)$. Take $c \in L$ such that $v(b - c) \geq 0$. We write $c = \sum_{i=1}^n b_i c_i^p$ with $c_1, \ldots, c_n \in K$. It follows that

$$0 \;\leq\; vc \;=\; v\sum_{i=1}^n b_i c_i^p \;=\; \min_i vb_i c_i^p\;.$$

Hence for $1 \leq i \leq n$, $vb_i + vc_i^p \geq 0$, and by our assumptions on the values $vb_i$, this implies that $vc_i^p \geq 0$ and hence $c_i \in \mathcal{O}$. This shows that the image of $\mathcal{O}^n$ under the polynomial (14.3) is a final segment of $v(b - L)$, hence one of the sets has a maximal element if and only if the other has. □

**Corollary 14.3** *Every extremal field with value group a divisible or a $\mathbb{Z}$-group is inseparably defectless.*

**Corollary 14.4** *The property "inseparably defectless" is elementary in the language of valued fields.*

**Proof:** The property can be axiomatized by an infinite scheme of axioms where $n$ runs through all powers $p^\nu$ of $p$. Each of the axioms quantifies over all $b \in K$ and all bases of finite extensions of $K^p$. The latter is done by quantifying over all choices of $a_1, \ldots, a_\nu \in K$ such that the elements $a_1^{e_1} \cdot \ldots \cdot a_\nu^{e_\nu}$, $0 \leq e_i < p$ are linearly independent over $K^p$ (which can be expressed by an elementary sentence). Also the additional conditions concerning the values of these elements and that they form a valuation basis are elementary in the language of valued fields. □

For a valued field of finite degree of inseparability, one knows several properties which are equivalent to "inseparably defectless". The following theorem is due to F. Delon [DEL1]:

**Theorem 14.5** *Let $K$ be a field of characteristic $p > 0$ and finite degree of inseparability $[K : K^p]$. Then for the valued field $(K, v)$, the property of being inseparably defectless is equivalent to each of the following properties:*

**a)** *$[K : K^p] = (vK : pvK)[\overline{K} : \overline{K}^p]$, i.e., $(K|K^p, v)$ is a defectless extension*

**b)** *$(K^{1/p}|K, v)$ is a defectless extension*

**c)** *every immediate extension of $(K, v)$ is separable*

**d)** *there is a separable maximal immediate extension of $(K, v)$.*

**Proof:** The equivalence of "$(K,v)$ inseparably defectless" with properties a) and b) follows readily from Lemma 14.1. We have already seen at the beginning of this section that for an inseparably defectless field, every immediate extension is separable. This proves that "$(K,v)$ inseparably defectless" implies property c). Since every valued field admits a maximal immediate extension by Theorem 8.22, it follows that property c) implies property d).

It now suffices to show that property d) implies property a). Let $(L,v)$ be a separable maximal immediate extension of $(K,v)$. According to Corollary 24.42, the separability implies that $[L:L^p]\geq[K:K^p]$. On the other hand, we have that $vL=vK$ and $Lv=Kv$. By our choice of $(L,v)$, it is a maximal field. Theorem **??** thus shows that $(L,v)$ is a defectless field. Hence, the extension $(L^{1/p}|L,v)$ is defectless, and by Lemma 14.1 we conclude that also $(L|L^p,v)$ is defectless. Since $(vL:pvL)=(vK:pvK)$ and $[Lv:Lv^p]=[Kv:Kv^p]$ are finite, it follows that $[L:L^p]$ is finite and equal to $(vL:pvL)[Lv:Lv^p]$. Consequently, using also the fundamental inequality for $(K|K^p,v)$, we obtain that

$$[L:L^p]=(vL:pvL)[Lv:Lv^p]=(vK:pvK)[Kv:Kv^p]\leq[K:K^p]\leq[L:L^p]\,.$$

Thus, equality holds everywhere, showing that a) holds. □

From the proof, we also obtain:

**Corollary 14.6** *A given maximal immediate extension of a valued field $(K,v)$ has the same degree of inseparability as $K$ if and only if $(K,v)$ is an inseparably defectless field.*

The very useful upward direction of the following lemma was also stated by F. Delon ([DEL1], Proposition 1.44):

**Lemma 14.7** *Let $(L|K,v)$ be a finite extension of valued fields. Then $(K,v)$ is an inseparably defectless field and of finite degree of inseparability if and only if $(L,v)$ is.*

**Proof:** By Lemma 24.32, the degree of inseparability of a field does not change under finite extensions. Assume that one and hence both fields have finite degree of inseparability. Since $[L:K]$ is finite, also $(vL:vK)$ and $[Lv:Kv]$ are finite, by virtue of Lemma 6.13. Hence, also the degree of inseparability of $Lv$ is equal to that of $Kv$. The same can be shown for ordered abelian groups: $(vL:pvL)=(vK:pvK)$ (the details are left to the reader). It follows that $[K:K^p]=(vK:pvK)[Kv:Kv^p]$ if and only if $[L:L^p]=(vL:pvL)[Lv:Lv^p]$, which by Theorem 14.5 means that $(K,v)$ is inseparably defectless if and only if $(L,v)$ is. □

In Lemma 15.36 in Section **??** we will generalize the upward direction to the case of arbitrary degree of inseparability.

If a $p$-basis of the field $K$ is at the same time a valuation basis of the extension $(K|K^p,v)$, then we will call it a **valuation $p$-basis**. Since by Lemma 6.17 every finite defectless extension admits a valuation basis and every valuation basis is a basis, we obtain from the foregoing theorem:

**Corollary 14.8** *A valued field of finite degree of inseparability is inseparably defectless if and only if it admits a valuation $p$-basis.*

The following very useful lemma was also proved by F. Delon ([DEL1], Proposition 1.44):

**Lemma 14.9** *For valued fields of finite degree of inseparability, the property of being inseparably defectless is inherited by every finite extension.*

**Proof:** By Lemma 24.32, the degree of inseparability of a field does not change under finite extensions. On the other hand, if $(L|K,v)$ is finite, then also $vL|vK$ and $\overline{L}|\overline{K}$ are finite by virtue of Lemma 6.13. Hence, also the degree of inseparability of $\overline{L}$ is equal to that of $\overline{K}$. The same can be shown for abelian groups: $(vL : pvL) = (vK : pvK)$ (the details are left to the reader). It follows that $[L : L^p] = [K : K^p] = (vK : pvK)[\overline{K} : \overline{K}^p] = (vL : pvL)[\overline{L} : \overline{L}^p]$, which by Theorem 14.5 means that $(L,v)$ is inseparably defectless. □

**Theorem 14.10** *Let $(K,v)$ be an inseparably defectless field. Assume that the valued field extension $(F|K,v)$ admits a (not necessarily finite) standard valuation transcendence basis $\mathcal{T}$ such that $F = K(\mathcal{T})$. Then also $(F,v)$ and $(F,v)^h$ are inseparably defectless fields.*

**Proof:** We prove our assertion for the case of $\mathcal{T}$ finite; the general case then follows by Lemma 11.96. In view of Theorem 11.12, it suffices to prove that $(F,v)$ is an inseparably defectless field.

We write $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\} \subset F$ such that the values $vx_i\,,\, i \in I$, are rationally independent over $vK$, and that the residues $\overline{y}_i\,,\, j \in J$, are algebraically independent over $\overline{K}$. Every finite purely inseparable extension $L$ of $K(\mathcal{T})$ is contained in an extension $E = K'(\mathcal{T}^{1/p^m}) = K'(t^{1/p^m} \mid t \in \mathcal{T})$ for a suitable $m \in \mathbb{N}$ and some finite purely inseparable extension $K'$ of $K$. Since $K'|K$ is algebraic, we know from Corollary 6.15 that $vK'/vK$ is a torsion group and $\overline{K'}|\overline{K}$ is algebraic. Consequently, the values $vx_i^{1/p^m} = \frac{vx_i}{p^m}\,,\, i \in I$, are still rationally independent over $vK'$, and the residues $\overline{y^{1/p^m}}_i = \overline{y}_i^{1/p^m}\,,\, j \in J$, are still algebraically independent over $\overline{K'}$. This proves that $\mathcal{T}^{1/p^m}$ is a standard valuation transcendence basis of $(E|K',v)$. Now Lemma 6.35 shows that

$$vE = vK' \oplus \mathbb{Z}vx_1^{1/p^m} \oplus \ldots \oplus \mathbb{Z}vx_r^{1/p^m} = vK' \oplus \mathbb{Z}\frac{vx_1}{p^m} \oplus \ldots \oplus \mathbb{Z}\frac{vx_r}{p^m}$$

and that

$$\overline{E} = \overline{K'}\left(\overline{y_1^{1/p^m}}, \ldots, \overline{y_s^{1/p^m}}\right) = \overline{K'}(\overline{y_1}^{1/p^m}, \ldots, \overline{y_s}^{1/p^m})\,,$$

whence

$$\begin{aligned}
[E : K(\mathcal{T})] &= [K'(\mathcal{T}^{1/p^m}) : K'(\mathcal{T})] \cdot [K'(\mathcal{T}) : K(\mathcal{T})] = p^{m(r+s)} \cdot [K' : K] \\
&= p^{mr} \cdot p^{ms} \cdot (vK' : vK) \cdot [\overline{K'} : \overline{K}] = p^{mr} \cdot (vK' : vK) \cdot p^{ms} \cdot [\overline{K'} : \overline{K}] \\
&= (vE : vK(\mathcal{T})) \cdot \left[\overline{E} : \overline{K(\mathcal{T})}\right]
\end{aligned}$$

since $(K'|K,v)$ is defectless by hypothesis. This equation shows that $(E|K(\mathcal{T}),v)$ and thus also its subextension $(L|K(\mathcal{T}),v)$ is defectless. □

Every valuation on the finite field $\mathbb{F}_p$ is trivial and thus defectless. Hence we can apply the foregoing theorem to $(\mathbb{F}_p(t),v_t)$ to obtain that $(\mathbb{F}_p(t),v_t)$ and $(\mathbb{F}_p(t),v_t)^h$ are inseparably defectless fields. Let us note:

**Corollary 14.11** *Let $(K,v)$ be one of the fields $(\mathbb{F}_p(t),v_t)$, $(\mathbb{F}_p(t),v_t)^h$ or $(\mathbb{F}_p((t)),v_t)$. Then $(K,v)$ is an inseparably defectless field with degree of inseparability $p$ and valuation $p$-basis $1,t,\ldots,t^{p-1}$. In particular,*

$$\mathbb{F}_p((t))^{1/p^\infty} = \mathbb{F}_p((t))(t^{1/p^i} \mid i\in\mathbb{N})\ .$$

**Proof:** It follows from the foregoing theorem that $(\mathbb{F}_p(t),v_t)$ and $(\mathbb{F}_p(t),v_t)^h$ are inseparably defectless fields. Corollary 11.28 tells us that $(\mathbb{F}_p((t)),v_t)$ is a defectless and hence also inseparably defectless field. From Corollary 14.6 we know that the degree of inseparabilitys of $\mathbb{F}_p(t)$ and $\mathbb{F}_p((t))$ are equal. Since $\mathbb{F}_p$ is perfect and $\mathbb{F}_p(t)$ is a function field in one variable over $\mathbb{F}_p$, its degree of inseparability is $p$. Since $\mathbb{F}_p(t)^h$ is a separable extension of $\mathbb{F}_p(t)$, Lemma 24.32 shows that its degree of inseparability is again $p$. Since the values $v1,vt,\ldots,(p-1)vt$ belong to distinct cosets modulo $vK^p=pvK$, it follows that $1,t,\ldots,t^{p-1}$ is a valuation $p$-basis for all three fields.

By definition, $K^{1/p^\infty}=\bigcup_{i\in\mathbb{N}} K^{1/p^i}$ for every field $K$ of characteristic $p$. Since the degree of inseparability of $\mathbb{F}_p((t))$ is $p$, we know that $\mathbb{F}_p((t))^{1/p}=\mathbb{F}_p((t))(t^{1/p})$ and by induction, $\mathbb{F}_p((t))^{1/p^i}=\mathbb{F}_p((t))(t^{1/p^i})$. It follows that $\mathbb{F}_p((t))^{1/p^\infty}=\bigcup_{i\in\mathbb{N}}\mathbb{F}_p((t))(t^{1/p^i})=\mathbb{F}_p((t))(t^{1/p^i}\mid i\in\mathbb{N})$. □

Now we see that the field $K$ of Example 11.47 is indeed equal to $\mathbb{F}_p((t))^{1/p^\infty}$. We conclude:

**Corollary 14.12** *For every element $a\in\widetilde{\mathbb{F}_p(t)}$ such that $a^p-a=1/t$, the Artin-Schreier extension of $\mathbb{F}_p((t))^{1/p^\infty}$ generated by $a$ is immediate with* $\mathrm{dist}\,(a,\mathbb{F}_p((t))^{1/p^\infty})=0$.

We leave it to the reader to prove the assertions of the last two corollaries for the case that $\mathbb{F}_p$ is replaced by any perfect field $k$ of characteristic $p$ (the valuation $v_t$ being trivial on $k$).

For the conclusion of this section, let us consider an example of an immediate Artin-Schreier extension which is *not* a defect extension.

**Example 14.13** We consider the Artin-Schreier polynomial $X^p-X-t$ over the valued field $(\mathbb{F}_p(t),v_t)$ and over its perfect hull. Extend $v_t$ to the algebraic closure of $\mathbb{F}_p(t)$. Let $a$ be a root of $X^p-X-t$. Then $a\notin\mathbb{F}_p(t)^{1/p^\infty}$. To show this, we observe that $\mathbb{F}_p(t)=\mathbb{F}_p(1/t)$. So we set $s=1/t$ and consider $\mathbb{F}_p(t)$ with the $s$-adic valuation. Then we can infer from the foregoing corollary that the root $a$ of the polynomial $X^p-X-1/s$ generates a proper immediate extension of $(\mathbb{F}_p((s))^{1/p^\infty},v_s)$. In particular, $a$ can not lie in $\mathbb{F}_p(s)^{1/p^\infty}=\mathbb{F}_p(t)^{1/p^\infty}$. So $X^p-X-t$ is irreducible over $\mathbb{F}_p(t)$ and $\mathbb{F}_p(t)^{1/p^\infty}$. So all roots of $X^p-X-t$ are conjugate to $a$ over these fields. But they are all of the form $a_j=a+j$ with $j=0,1,\ldots,p-1$. It is impossible that $v_ta<0$ since then $v_t(a^p-a-t)=v_ta^p<0$ in contradiction to $a^p-a-t=0$. Hence, $v_ta_j\geq 0$, and all elements of $\mathbb{F}_p$ appear as the residue of some root $a_j$. Without loss of generality, we can choose our enumeration of the roots such that $\overline{a}=0$. Let $\sigma_j$ denote the automorphism in $\mathrm{Gal}\,\mathbb{F}_p(t)$ which sends $a$ to $a_j$. Then if $j\neq\ell$, then $v_t\sigma_j(a-j)=v_t(\sigma_ja-j)=v_t(a+j-j)=v_ta>0=v_t(a+\ell-j)=v_t(\sigma_\ell a-j)=v_t\sigma_\ell(a-j)$. This proves that $v_t$ admits $p$ distinct extensions from $\mathbb{F}_p(t)$ to $\mathbb{F}_p(t,a)$ and from $\mathbb{F}_p(t)^{1/p^\infty}$ to $\mathbb{F}_p(t)^{1/p^\infty}(a)$. That is, $(\mathbb{F}_p(t,a),v_t)$ lies in the henselization of $(\mathbb{F}_p(t),v_t)$ in $(\widetilde{\mathbb{F}_p(t)},v_t)$ and in particular, $(\mathbb{F}_p(t,a)|\mathbb{F}_p(t),v_t)$ is an immediate extension. The same holds for the extension $(\mathbb{F}_p(t)^{1/p^\infty}(a)|\mathbb{F}_p(t)^{1/p^\infty},v_t)$.

We define $b_i := a + t + t^p + \ldots + t^{p^i}$. Since $v_t a > 0$, we obtain that also $v_t b_i > 0$ for every $i$. We compute

$$\begin{aligned} 0 &= a^p - a - t = (b_i - t - t^p - \ldots - t^{p^i})^p - b_i + t + t^p + \ldots + t^{p^i} - t \\ &= b_i^p - b_i - t^{p^i} . \end{aligned}$$

Since $v_t b_i > 0$, we have that $v_t b_i^p > v_t b_i$ and consequently, $v_t b_i = v_t t^{p^i} = p^i v_t t$. Since the values $p^i v_t t$ are cofinal in $\mathbb{Z} v_t t = v_t \mathbb{F}_p(t)$, we find that at $(a, \mathbb{F}_p(t))$ is a completion type. $\diamondsuit$

Trivially, every perfect field is inseparably defectless. Hence, our example shows that there are perfect and thus also inseparably defectless fields which are not henselian. See also Example 9.3 and Section **??** for a discussion of Artin-Schreier polynomials of the above type over henselian fields.

## 14.3 Classification of Artin-Schreier defect extensions

We will consider the following situation:

- $(L|K, v)$ an Artin-Schreier defect extension of valued fields of characteristic $p > 0$,
- $\vartheta \in L \setminus K$ an Artin-Schreier generator of $L|K$,
- $a = \wp(\vartheta) = \vartheta^p - \vartheta \in K$,
- $\delta = \mathrm{dist}\,(\vartheta, K)$.

Since $(L|K, v)$ is immediate and non-trivial, we know that $v(\vartheta - K) = \Lambda^{\mathrm{L}}(\vartheta, K)$ has no maximal element and that $\delta > v\vartheta$ (cf. Theorem **??**). An element $\vartheta' \in L$ is another Artin-Schreier generator of $L|K$ if and only if

$$\vartheta' = i\vartheta + c \quad \text{with} \quad c \in K \quad \text{and} \quad 1 \leq i \leq p - 1. \tag{14.4}$$

(cf. Lemma **??**). Consequently, using Lemma **??** we see that $\delta$ is an invariant of the extension $(L|K, v)$:

**Lemma 14.14** *The distance $\delta$ does not depend on the choice of the Artin-Schreier generator $\vartheta$.*

So we can call $\delta$ the **distance of the Artin-Schreier defect extension** $(L|K, v)$. From Corollary **??** we know that

$$\delta \leq 0^- .$$

We will now distinguish two types of Artin-Schreier defect extensions. We will call $(L|K, v)$ a **dependent Artin-Schreier defect extension** if there exists an immediate purely inseparable extension $K(\eta)|K$ of degree $p$ such that

$$\eta \sim_K \vartheta . \tag{14.5}$$

Otherwise, we will speak of an **independent Artin-Schreier defect extension**. For the definition and properties of the equivalence relation "$\sim_K$", see Section **??**. We will now show that independent Artin-Schreier defect extensions are characterized by idempotent distances $\delta$. See Lemma **??** for a bunch of different criteria which are all equivalent to "$\delta$ is idempotent".

**Proposition 14.15** *In the situation as described above, the Artin-Schreier defect extension $(L|K,v)$ is independent if and only if its distance $\delta$ is idempotent:*

$$\delta = p\delta\,.$$

**Proof:** Assume that $K(\eta)|K$ is purely inseparable of degree $p$, that is, $\eta^p \in K\setminus K^p$. By definition, (15.17) is equivalent to $v(\vartheta-\eta)>\delta$. Since $v(\vartheta^p-\eta^p)=v(\vartheta-\eta)^p=pv(\vartheta-\eta)$, this in turn is equivalent to

$$v(\vartheta^p-\eta^p)\;>\;p\delta\,.$$

Here, the left hand side is equal to $v(\vartheta+a-\eta^p)=v(\vartheta-(\eta^p-a))$ which is a value in $\Lambda^{\rm L}(\vartheta,K)$ and hence is $\leq\delta$. Consequently, if (15.17) holds with $K(\eta)|K$ a purely inseparable extension of degree $p$, then $p\delta<\delta$, that is, $\delta$ is not idempotent.

For the converse, assume that $\delta$ is not idempotent. Since $\delta\leq 0^-$, this implies that $p\delta<\delta$. Then there is $c\in K$ such that $p\delta<v(\vartheta-c)\leq\delta$. Choose $\eta\in\tilde{K}$ such that $\eta^p=a+c$. Then $v(\vartheta^p-\eta^p)=v(\vartheta+a-\eta^p)=v(\vartheta-c)>p\delta$. Hence, $v(\vartheta-\eta)>\delta$, and it follows that $\eta\sim_K\vartheta$. Consequently, $\eta\notin K$, and we obtain that $K(\eta)|K$ is a purely inseparable extension of degree $p$. Finally, we deduce from Lemma 19.26 that this extension is immediate. □

**Corollary 14.16** *If $K$ admits no proper immediate purely inseparable extension, then $K$ admits no dependent Artin-Schreier defect extension.* □

The converse of this corollary is not true: every separable-algebraically closed non-trivially valued field $K$ of characteristic $p>0$ which is not algebraically closed is a counterexample. Indeed, its value group is divisible and its residue field is algebraically closed (see, e.g., [Ku2], Lemma 2.16) and hence, the proper purely inseparable extension $\tilde{K}|K$ is immediate. But a closer look shows that the irreversibility comes only from immediate purely inseparable extensions which lie in the completion $K^c$ of $K$:

**Proposition 14.17** *Assume that $K$ admits an immediate purely inseparable extension $K(\eta)|K$ of degree $p$ such that $\eta\notin K^c$, and set*

$$\varepsilon\;:=\;\mathrm{dist}\,(\eta,K)\,.$$

*Then $K$ admits a dependent Artin-Schreier defect extension $K(\vartheta)|K$. More precisely, given any $b\in K^\times$, then*

$$(p-1)vb+v\eta\;>\;p\varepsilon \tag{14.6}$$

*if and only if there is an Artin-Schreier generator $\vartheta$ such that $\vartheta^p-\vartheta=(\eta/b)^p$ and*

$$\begin{aligned}\vartheta &\;\sim_K\; \frac{\eta}{b}\\ v\vartheta &= v\eta-vb\\ \mathrm{dist}\,(\vartheta,K) &= \mathrm{dist}\,(\eta,K)-vb\,.\end{aligned}$$

*All Artin-Schreier defect extensions obtained in this way are dependent.*

**Proof:** Let $\vartheta$ be a root of the polynomial

$$X^p - X - \left(\frac{\eta}{b}\right)^p \in K[X]. \tag{14.7}$$

Assume that (14.6) holds. Then we have

$$(p-1)vb + v\eta \;>\; p\varepsilon \;>\; pv\eta \tag{14.8}$$

where the last inequality holds since $\varepsilon > v\eta$ by Theorem **??**. This gives $vb > v\eta$, showing that

$$v\left(\frac{\eta}{b}\right)^p \;<\; 0\;.$$

Hence by Lemma **??**,

$$v\vartheta \;=\; v\,\frac{\eta}{b} \;=\; v\eta - vb\;. \tag{14.9}$$

Putting $Y = bX$ we find that $b\vartheta$ is a root of the polynomial

$$Y^p - b^{p-1}Y - \eta^p \in K[Y] \tag{14.10}$$

and thus satisfies

$$\eta^p + b^p\vartheta \;=\; \eta^p + b^{p-1}b\vartheta \;=\; (b\vartheta)^p\;.$$

Let $c$ be an arbitrary element of $K$. By (14.9), (14.6) and the definition of $\varepsilon$,

$$vb^p\vartheta \;=\; pvb + v\eta - vb \;=\; (p-1)vb + v\eta \geq p\varepsilon \;>\; pv(\eta - c) \;=\; v(\eta^p - c^p)$$

which yields, using the ultrametric triangle inequality,

$$\begin{aligned} v(\eta - c) \;&=\; \frac{1}{p}v(\eta^p - c^p) \;=\; \frac{1}{p}\min\{v(\eta^p - c^p)\,,\, vb^p\vartheta\} \\ &=\; \frac{1}{p}v(\eta^p + b^p\vartheta - c^p) \;=\; \frac{1}{p}v((b\vartheta)^p - c^p) \;=\; v(b\vartheta - c)\;. \end{aligned}$$

By Lemma **??** this implies that $b\vartheta \sim_K \eta$, which by Lemma **??** implies that

$$\vartheta \;\sim_K\; \frac{\eta}{b}\;.$$

From this, the assertion on the distance of $\vartheta$ follows by virtue of Lemma **??**, while the value $v\vartheta$ has already been determined in (14.9). By Lemma **??**, the extension of $v$ from $K$ to $K(\vartheta)$ is unique and $(K(\vartheta)|K,v)$ is an Artin-Schreier defect extension. By definition, it is dependent.

For the converse, assume that (14.6) does not hold, i.e., $(p-1)vb + v\eta \leq p\varepsilon$. If $v\left(\frac{\eta}{b}\right)^p > 0$, then by Lemma **??**, $v\vartheta = v\left(\frac{\eta}{b}\right)^p = pv\frac{\eta}{b} > v\frac{\eta}{b}$ and we cannot have $\vartheta \sim_K \frac{\eta}{b}$. If $v\left(\frac{\eta}{b}\right)^p \leq 0$, then again by Lemma **??**, (14.9) holds, and so we have

$$vb^p\vartheta \;=\; pvb + v\eta - vb \;=\; (p-1)vb + v\eta \;\leq\; p\varepsilon\;.$$

Therefore, and since $\Lambda^{\mathrm{L}}(\eta, K)$ has no last element, there is some $c \in K$ such that $vb^p\vartheta < pv(\eta - c) = v(\eta^p - c^p)$. But then, by the ultrametric triangle inequality,

$$v(\eta - c) \;>\; \frac{1}{p}vb^p\vartheta \;=\; \frac{1}{p}v(\eta^p + b^p\vartheta - c^p) \;=\; v(b\vartheta - c)\;,$$

which again shows that $\vartheta \sim_K \frac{\eta}{b}$ cannot be true. □

The following proposition shows an even stronger independence property than what is expressed in the definition:

**Proposition 14.18** *Let $(L|K,v)$ be an independent Artin-Schreier defect extension, and take any element $\zeta \in L \setminus K$. Then there exists no purely inseparable extension $K(\eta)|K$ such that $\zeta \sim_K \eta$. In particular, it follows that*

$$\mathrm{dist}\,(\zeta, K) = \mathrm{dist}\,(\zeta, K^{1/p^\infty})\,. \tag{14.11}$$

**Proof:** Since $\zeta \in L \setminus K$, $[K(\zeta) : K] = p = [K(\vartheta) : K]$ and therefore, there is a polynomial $f \in K[X]$ of degree smaller than $p$ such that $\vartheta = f(\zeta)$. Suppose that there exists a purely inseparable extension $K(\eta)|K$ such that $\zeta \sim_K \eta$. But then by Lemma **??**, $\vartheta = f(\zeta) \sim_K f(\eta)$. Since also $K(f(\eta))|K$ is a purely inseparable extension, this is impossible since $(L|K,v)$ is assumed to be independent.

Equation (15.21) is deduced as follows. If it does not hold, then $\mathrm{dist}\,(\zeta, K) < \mathrm{dist}\,(\zeta, K^{1/p^\infty})$ in view of $K \subset K^{1/p^\infty}$. But then by virtue of Lemma **??**, there would exist some $\eta \in K^{1/p^\infty}$ such that $\zeta \sim_K \eta$, which we have just shown not to be the case. □

## 14.4 Deformation of Artin-Schreier defect extensions

For the proof of Proposition 14.17, we have transformed an immediate purely inseparable extension into an immediate separable extension. This was done by changing the minimal polynomial $Y^p - \eta^p$ to the minimal polynomial (15.20) of $b\vartheta$ through addition of the summand $b^{p-1}Y$. The hypothesis on the value of $b$ just means that it is large enough to guarantee that $b\vartheta \sim_K \eta$. For this hypothesis, it is necessary that $\eta$ is not contained in the completion of $K$. On the other hand, an immediate purely inseparable extension with a generator $\eta$ in the completion of $K$ cannot be transformed into any immediate separable extension with a generator $\vartheta$ such that $\vartheta \sim_K \eta$. Indeed, if $\eta \in K^c$ and $\eta \sim_K \eta'$, then $v(\eta - \eta') > \widetilde{vK}$, that is, $\eta = \eta'$. Moreover, every henselian field $K$ is separable-algebraically closed in its completion (cf. [W], Theorem 32.19).

The general idea of the transformation of the minimal polynomial can be expressed as follows: if $y \notin K^c$ is a root of the polynomial $f \in K[X]$, then for a given polynomial $g \in K[X]$, a root $z$ of $g$ will satisfy $y \sim_K z$ as soon as the coefficients of the polynomial $f - g$ have large enough values. This follows in general from the principle of Continuity of Roots. But we wanted to give a self-contained proof for our special case, because it is particularly simple and explicit and leads to the following deformation theory.

For any fixed $a \in K$, we consider the following family of polynomials defined over $K$:

$$f_{a,b}(Y) \;:=\; Y^p - b^{p-1}Y - a\,, \qquad b \in K^\times\,. \tag{14.12}$$

This family can be viewed as a deformation of the polynomial $Y^p - a$, with this polynomial as its limit for $vb \to \infty$:

$$\begin{array}{rcl} Y^p - b^{p-1}Y - a & \longrightarrow & Y^p - a \\ vb & \longrightarrow & \infty\,. \end{array}$$

But it is not necessarily true that the ramification theoretical properties are preserved in the limit, as Example 14.30 in Section 14.9 will show.

Associated with this family through the transformation $Y = bX$ is the family

$$g_{a,b}(X) \;:=\; X^p - X - \frac{a}{b^p}\;, \qquad b \in K^\times\;, \tag{14.13}$$

where $\vartheta_{a,b}$ is a root of $g_{a,b}$ if and only if $b\vartheta_{a,b}$ is a root of $f_{a,b}$.

We summarize the properties of these families in the following theorem:

**Theorem 14.19** *a) If $pvb \geq va$, then the polynomial $g_{a,b}(X)$ induces a Artin-Schreier extension for which equality holds in the fundamental inequality (7.26); if $pvb > va$, then this extension lies in the henselization of $K$.*
*b) Suppose that the polynomial $Y^p - a$ induces an immediate extension which does not lie in the completion of $K$. Then for each $b \in K^\times$ of large enough value, the polynomial $g_{a,b}(X)$ induces a dependent Artin-Schreier defect extension; every root $b\vartheta_{a,b}$ of $f_{a,b}(X)$ will then satisfy*

$$b\vartheta_{a,b} \sim_K a^{1/p}\;.$$

*"Large enough value" means that*

$$(p-1)vb + \frac{va}{p} \;>\; p\,\mathrm{dist}\,(a^{1/p}, K)\;. \tag{14.14}$$

*If this condition is violated, then $b\vartheta_{a,b} \sim_K a^{1/p}$ does not hold.*
*c) Suppose that a root $\vartheta_{a,1}$ of the polynomial $f_{a,1}(X) = X^p - X - a$ satisfies*

$$v\vartheta_{a,1} \;>\; p\,\mathrm{dist}\,(\vartheta_{a,1}, K)\;. \tag{14.15}$$

*Then the polynomial $X^p - a$ induces an immediate extension which does not lie in the completion, and for every $b$ in the valuation ring $\mathcal{O}$ of $K$ and every root $\vartheta_{a,b}$ of $g_{a,b}$, $K(\vartheta_{a,b})|K$ is a dependent Artin-Schreier defect extension with $b\vartheta_{a,b} \sim_K a^{1/p}$. If condition (14.15) is violated, then $\vartheta_{a,1} \sim_K a^{1/p}$ does not hold.*

**Proof:** a): Both assertions follow from Lemma **??**.

b): All assertions follow from Proposition 14.17 where $\eta = a^{1/p}$.

c): Assume that condition (14.15) holds. Then it follows from the second part of the proof of Proposition 15.22, where we set $c = 0$ and $\vartheta = \vartheta_{a,1}$, that the polynomial $X^p - a$ induces an immediate extension which does not lie in the completion, and that $\vartheta_{a,1} \sim_K a^{1/p}$. The latter implies that $v\vartheta_{a,1} = va^{1/p} = \frac{va}{p}$ and that $\mathrm{dist}\,(\vartheta_{a,1}, K) = \mathrm{dist}\,(a^{1/p}, K)$; hence, it implies that (14.15) is equivalent to

$$\frac{va}{p} \;>\; p\,\mathrm{dist}\,(a^{1/p}, K)\;. \tag{14.16}$$

Consequently, (14.14) will hold for every $b \in \mathcal{O}$, so it follows from part b) that for every root $\vartheta_{a,b}$ of $g_{a,b}$, $K(\vartheta_{a,b})|K$ is a dependent Artin-Schreier defect extension with $b\vartheta_{a,b} \sim_K a^{1/p}$.

The last assertion of part c) is seen as follows. We have shown that if $\vartheta_{a,1} \sim_K a^{1/p}$ holds, then (14.15) and (14.16) are equivalent. But if (14.16) is violated, then by part b), $\vartheta_{a,1} \sim_K a^{1/p}$ cannot hold. □

Note that

$$p\,\mathrm{dist}\,(a^{1/p}, K) \;=\; \mathrm{dist}\,(a, K^p)\;. \tag{14.17}$$

A deformation which at first sight seems to be different from the above has been used by B. Teissier in [T]. Starting from the Artin-Schreier polynomial $X^p - X - a$, we set $X = aY$ and then divide the polynomial by $a^p$, which leads to the polynomial

$$Y^p - a^{1-p}Y - a^{1-p} \;=\; Y^p - a^{1-p}(1+Y)\;.$$

Hence, $\vartheta^p - \vartheta = a$ if and only if for $\tilde{\vartheta} = \vartheta/a$,

$$\tilde{\vartheta}^p - a^{1-p}(1+\tilde{\vartheta}) = 0\;. \qquad (14.18)$$

We assume that $va < 0$. Then $\vartheta^p - \vartheta = a$ implies that $va = pv\vartheta < v\vartheta$ and therefore,

$$v\tilde{\vartheta} \;=\; v\vartheta - va \;>\; 0\;.$$

That is, $1 + \tilde{\vartheta}$ is a 1-unit in $\mathcal{O}$. Reducing this 1-unit to 1 deforms equation (14.18) to

$$\overline{\tilde{\vartheta}}^p - a^{1-p} \;=\; 0\;,$$

viewed as an equation in an associated graded ring. In fact, we have reduced equation (14.18) modulo the $\mathcal{O}$-ideal

$$a^{1-p}\tilde{\vartheta}\mathcal{O} \;=\; a^{-p}\vartheta\mathcal{O} \;=\; \vartheta^{1-p^2}\mathcal{O}\;.$$

Analyzing the above transformation, one sees that its advantage is that it leeds to equations with integral coefficients. However, if we multiply the polynomial $Y^p - a^{1-p}$ by $a^p$ and then set $X = aY$, we obtain the polynomial $X^p - a$. So we have just replaced the polynomial $X^p - X - a$ by $X^p - a$. From Theorem 14.19 together with (14.17) we see that this procedure preserves the valuation theoretical behaviour of the associated roots if and only if

$$va \;>\; p\,\mathrm{dist}\,(a, K^p)\;.$$

## 14.5 Fields without dependent Artin-Schreier defect extensions

If $K$ admits any immediate purely inseparable extension that does not lie in the completion $K^c$ of $K$, then $K$ satisfies the hypothesis of Proposition 14.17. To show this, suppose that $\tilde{\eta} \in K^{1/p^\infty} \setminus K^c$ such that $K(\tilde{\eta})|K$ is an immediate extension. We may assume that $\tilde{\eta}^p \in K^c$ (otherwise, we replace $\tilde{\eta}$ by a suitable $p^\nu$-th power). Since $\tilde{\eta} \notin K^c$, we have that $\Lambda^{\mathrm{L}}(\tilde{\eta}, K)$ is bounded from above in $vK$ and $\Lambda^{\mathrm{L}}(\tilde{\eta}^p, K^p) \;=\; p\Lambda^{\mathrm{L}}(\tilde{\eta}, K)$ is bounded from above in $vK^p = pvK$. On the other hand, since $\tilde{\eta}^p \in K^c$, there is some $b \in K$ such that $v(\tilde{\eta}^p - b) > \Lambda^{\mathrm{L}}(\tilde{\eta}^p, K^p)$. We choose $\eta \in K^{1/p}$ such that $\eta^p = b$. Then $v(\tilde{\eta} - \eta) = \frac{1}{p}v(\tilde{\eta}^p - b) > \Lambda^{\mathrm{L}}(\tilde{\eta}, K)$, that is,

$$\eta \;\sim_K\; \tilde{\eta}\;.$$

By Lemma 19.26, this shows that $K(\eta)|K$ is an immediate extension; since $\Lambda^{\mathrm{L}}(\eta, K) = \Lambda^{\mathrm{L}}(\tilde{\eta}, K) \neq vK$, it is not contained in $K^c$. We may now apply Proposition 14.17 to obtain:

**Corollary 14.20** *Assume that $K$ does not admit any dependent Artin-Schreier defect extension. Then every immediate purely inseparable extension lies in the completion of $K$.*

□

**Lemma 14.21** *If $K$ is Artin-Schreier closed, then so is $K^c$. If $K$ admits no dependent (or no independent) Artin-Schreier defect extension, then the same holds for $K^c$.*

**Proof:** Assume that $K^c(\vartheta)|K$ is an Artin-Schreier extension generated by a root $\vartheta$ of the polynomial $X^p - X - a$ over $K^c$. Since $\vartheta \notin K^c$, we have that $\mathrm{dist}\,(\vartheta, K^c) < \infty$. Since $a \in K^c$, we may choose an element $\tilde{a} \in K$ such that $v(a-\tilde{a}) > \mathrm{dist}\,(\vartheta, K^c)$ with $v(a-\tilde{a}) \geq 0$. Let $\tilde{\vartheta}$ be a root of the polynomial $X^p - X - \tilde{a} \in K[X]$. By Lemma **??**, the root $\vartheta - \tilde{\vartheta}$ of the polynomial $X^p - X - (a-\tilde{a})$ has value $v(\vartheta - \tilde{\vartheta}) = v(a-\tilde{a}) > \mathrm{dist}\,(\vartheta, K^c) \geq \mathrm{dist}\,(\vartheta, K)$. Thus, $\mathrm{dist}\,(\tilde{\vartheta}, K) = \mathrm{dist}\,(\vartheta, K) \leq \mathrm{dist}\,(\vartheta, K^c) < \infty$, which shows that $K(\tilde{\vartheta})|K$ is non-trivial and hence an Artin-Schreier extension. This proves the first assertion of our lemma.

Now assume that $(K^c(\vartheta)|K, v)$ is an Artin-Schreier defect extension. By Corollary **??** we have that $\mathrm{dist}\,(\vartheta, K^c) \leq 0^-$. With $\tilde{\vartheta}$ as before, we obtain that $\mathrm{dist}\,(\tilde{\vartheta}, K) = \mathrm{dist}\,(\vartheta, K) \leq 0^-$. By Lemma **??**, this shows that also $(K(\tilde{\vartheta})|K, v)$ is an Artin-Schreier defect extension. The equality of the distances shows that $K^c(\vartheta)|K^c$ is independent if and only if $K(\tilde{\vartheta})|K$ is. □

An immediate consequence of this lemma and the preceding corollary is:

**Corollary 14.22** *If $K$ does not admit any dependent Artin-Schreier defect extension, then $K^c$ does not admit any proper immediate purely inseparable extension. In particular, this holds if $K$ is separable-algebraically maximal.* □

We can now give the

**Proof of Theorem ??:** Every Artin-Schreier closed non-trivially valued field $K$ of characteristic $p > 0$ has $p$-divisible value group and perfect residue field (cf. Corollary 2.17 of [Ku2]). Therefore, every purely inseparable extension of $K$ is immediate. Hence by the last corollary, the perfect hull of $K$ lies in the completion of $K$, i.e., $K$ lies dense in its perfect hull.

An alternative proof of this fact can be given in the following way. We represent the extension $K^{1/p^\infty}|K$ as an infinite tower of purely inseparable extensions $K_{\mu+1}|K_\mu$ ($\mu < \nu$ where $\nu$ is some ordinal). Then we only have to show that $(K_{\mu+1}, v)$ lies in $(K_\mu, v)^c$ for every $\mu < \nu$. In view of Proposition 14.17, it suffices to show that $K_\mu$ is Artin-Schreier closed. But this holds by Lemma **??**.

Since $K^c$ has the same value group and the same residue field as $K$, also every purely inseparable extension of $K^c$ is immediate. By the preceding corollary, this yields that $K^c$ must be perfect. □

## 14.6 Persistence results

Another property of independent Artin-Schreier defect extensions is their persistence in maximal immediate extensions, in the following sense:

**Lemma 14.23** *If $K$ admits an independent Artin-Schreier defect extension $(K(\vartheta)|K, v)$ with Artin-Schreier generator $\vartheta$ of distance $\delta = 0^-$, then every algebraically maximal immediate extension (and in particular, every maximal immediate extension) $M$ of $K$ contains also an independent Artin-Schreier defect extension of $K$ with an Artin-Schreier generator $\tilde{\vartheta}$ of distance $0^-$ such that $\tilde{\vartheta} \sim_K \vartheta$.*

**Proof:** If $\vartheta \in M$, there is nothing to show. Assume that $\vartheta \notin M$. Then $M(\vartheta)|M$ is also an Artin-Schreier extension with Artin-Schreier generator $\vartheta$. Since $M$ is algebraically maximal, Corollary **??** shows that there exists an element $u \in M$ satisfying

$$v(\vartheta - u) \;\geq\; \Lambda^{\mathrm{L}}(\vartheta, M)\;.$$

On the other hand, $K \subseteq M$ implies

$$\Lambda^{\mathrm{L}}(\vartheta, K) \subseteq \Lambda^{\mathrm{L}}(\vartheta, M)\;.$$

Since $vM = vK$, this shows that $v(\vartheta - u) \geq 0$. We put

$$a_u \;:=\; \wp(\vartheta - u) \;=\; \wp(\vartheta) - \wp(u) \in M$$

and note that $va_u \geq 0$. Since $M|K$ is immediate, there exists $b \in K$ such that

$$v(a_u - b) > v(a_u) \geq 0$$

and $vb = va_u \geq 0$. Consequently, the polynomial $X^p - X - (a_u - b) \in M[X]$ admits a root $\vartheta'$ in the henselian field $M$. But then,

$$\tilde{\vartheta} := \vartheta' + u \in M$$

is a root of the polynomial $X^p - X - (\wp(\vartheta) - b) \in K[X]$. We compute:

$$\wp(\vartheta - \tilde{\vartheta}) = \wp(\vartheta) - \wp(\vartheta' + u) = \wp(\vartheta) - (\wp(\vartheta) - b) = b\;.$$

This shows $v(\vartheta - \tilde{\vartheta}) \geq 0$, whence $\tilde{\vartheta} \sim_K \vartheta$. In particular, this shows that $\tilde{\vartheta} \notin K$ so that $K(\tilde{\vartheta})|K$ is non-trivial and hence an Artin-Schreier extension. By Lemma **??**, the extension of $v$ from $K$ to $K(\tilde{\vartheta})$ is unique and $K(\tilde{\vartheta})|K$ is an Artin-Schreier defect extension. Finally, $\tilde{\vartheta} \sim_K \vartheta$ implies that $\mathrm{dist}\,(\tilde{\vartheta}, K) = \mathrm{dist}\,(\vartheta, K) = 0^-$ (Lemma **??**) and therefore, $K(\tilde{\vartheta})|K$ is an independent Artin-Schreier defect extension. □

From this lemma, we deduce the following:

**Corollary 14.24** *If there exists a maximal immediate extension in which $K$ is separable-algebraically closed, then $K$ admits no independent Artin-Schreier defect extension of distance* $0^-$.

We will now consider independent Artin-Schreier defect extensions $(K(\vartheta)|K, v)$ with Artin-Schreier generator $\vartheta$ of distance $\delta < 0^-$. In this case, Lemma **??** shows that $\delta = H^-$ for some non-trivial convex subgroup $H$ of $\widetilde{vK}$. This means that $v(\vartheta - K) = \Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $(\widetilde{vK})^{<0} \setminus H$. We denote by $v_\delta$ the coarsening of $v$ on $\tilde{K}$ with respect to $H$. Then $v_\delta(\vartheta - K)$ is cofinal in $(\widetilde{vK})^{<0}/H = (\widetilde{v_\delta K})^{<0}$. Thus, $v_\delta(\vartheta - K)$ has no maximal element. Since the extension of $v$ from $K$ to $K(\vartheta)$ is unique, the same must hold for $v_\delta$; cf. the proof of Lemma **??**. Now Lemma 19.26 shows that also $(K(\vartheta)|K, v_\delta)$ is an immediate Artin-Schreier extension. As its distance is $0^-$, it is covered by the case treated in Lemma 15.30. From this, we obtain:

**Lemma 14.25** *Assume that for every coarsening $w$ of $v$ (including $v$ itself), there exists a maximal immediate extension $(M_w, w)$ of $(K, w)$ such that $K$ is separable-algebraically closed in $M_w$. Then $K$ admits no independent Artin-Schreier defect extensions.* □

The condition of Lemma 15.33 is preserved under finite defectless extensions:

**Lemma 14.26** *Assume that for every coarsening $w$ of $v$ (including $v$ itself), $K_0$ admits a maximal immediate extension $(N_w|K_0, w)$ such that $K_0$ is relatively algebraically closed (or separable-algebraically closed) in $N_w$. If the extension $(K|K_0, v)$ is finite and defectless, then for every coarsening $w$ of $v$ (including $v$ itself), $(M_w, w) = (N_w.K, w)$ is a maximal immediate extension of $(K, w)$ such that $K$ is relatively algebraically closed (or separable-algebraically closed, respectively) in $M_w$.*

**Proof:** Since $(K|K_0, v)$ is defectless by hypothesis, the same is true for the extension $(K|K_0, w)$ by Lemma **??**. We note that $(K_0, w)$ is henselian since it is assumed to be separable-algebraically closed in the henselian field $(N_w, w)$. So we may apply Lemma **??**: since $(N_w|K_0, w)$ is immediate and $(K|K_0, w)$ is defectless, $(N_w.K|K, w)$ is immediate and $N_w$ is linearly disjoint from $K$ over $K_0$. The latter shows that $K$ is relatively algebraically closed (or separable-algebraically closed, respectively) in $N_w.K$. On the other hand, $(M_w, w) = (N_w.K, w)$ is a maximal field, being a finite extension of a maximal field. □

**Proposition 14.27** *If $K_0$ is a separable-algebraically maximal field and $K|K_0$ is a finite defectless extension, then $K$ admits no independent Artin-Schreier defect extensions.*

**Proof:** Let $w$ be any coarsening of $v$. Since $(K_0, v)$ is separable-algebraically maximal, the same is true for $(K_0, w)$ since every finite separable immediate extension of $(K_0, w)$ would also be immediate for the finer valuation $v$. Now let $(N_w, w)$ be a maximal immediate extension of $(K_0, w)$. Since $(K_0, w)$ is separable-algebraically maximal, it is separable-algebraically closed in $N_w$. Hence, $K_0$ satisfies the condition of Lemma 15.35. So our proposition is a consequence of Lemma 15.35 together with Lemma 15.33. □

## 14.7 Finite extensions of inseparably defectless fields

For the generalization of Lemma 14.7 to the case of infinite degree of inseparability, we will need the following result:

**Lemma 14.28** *Let $K \subset K_1 \subset K_2$ be extensions of valued fields of characteristic $p > 0$ such that $K_1|K$ is finite and purely inseparable and $K_2|K_1$ is an independent Artin-Schreier defect extension. Then there exists an Artin-Schreier extension $L|K$ such that $K_2 = K_1.L$, and every such extension $L|K$ is an independent Artin-Schreier defect extension.*

**Proof:** Let $\tilde{\vartheta}$ be an Artin-Schreier generator of $K_2|K_1$ and choose $\nu \geq 1$ such that

$$K_1^{p^\nu} \subseteq K .$$

Then
$$\wp(\tilde{\vartheta}^{p^\nu}) = (\wp(\tilde{\vartheta}))^{p^\nu} \in K \;,$$
hence
$$K(\tilde{\vartheta}^{p^\nu})|K$$
is an Artin-Schreier extension: it is non-trivial since $K(\tilde{\vartheta})|K$ is not purely inseparable. Comparing degrees, we see that $K_2 = K_1(\tilde{\vartheta}^{p^\nu}) = K_1.K(\tilde{\vartheta}^{p^\nu})$.

Now let $L|K$ be any such Artin-Schreier extension. Let $\vartheta$ be an Artin-Schreier generator of $L|K$ and hence of $K_2|K_1$ too. Using $\vartheta^p = \vartheta + a$ with $a \in K$, we compute
$$\vartheta^{p^\nu} = \vartheta + a' \;\text{ where }\; a' = a + \ldots + a^{p^{\nu-1}} \in K \;. \tag{14.19}$$
Hence,
$$\mathrm{dist}\,(\vartheta^{p^\nu}, K_1) = \mathrm{dist}\,(\vartheta, K_1) \;.$$
Further,
$$\delta := \mathrm{dist}\,(\vartheta, K_1) = p^\nu\delta = \mathrm{dist}\,(\vartheta^{p^\nu}, K_1^{p^\nu})$$
since $\delta$ is idempotent by hypothesis;
$$\mathrm{dist}\,(\vartheta^{p^\nu}, K_1^{p^\nu}) \le \mathrm{dist}\,(\vartheta^{p^\nu}, K) \le \mathrm{dist}\,(\vartheta^{p^\nu}, K_1)$$
because $K_1^{p^\nu} \subseteq K \subset K_1$. Putting these three equations together, we find that equality holds everywhere. In particular,
$$\mathrm{dist}\,(\vartheta, K_1) = \mathrm{dist}\,(\vartheta^{p^\nu}, K) = \mathrm{dist}\,(\vartheta, K) \;,$$
where the second equality again holds because of (14.19). This shows that $\Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $\Lambda^{\mathrm{L}}(\vartheta, K_1)$. Since $K_1(\vartheta)|K_1$ is immediate, we know from Theorem **??** that $\Lambda^{\mathrm{L}}(\vartheta, K_1) = v(\vartheta - K_1)$ has no maximal element. Now we have that $\Lambda^{\mathrm{L}}(\vartheta, K) \subseteq v(\vartheta - K) \subseteq v(\vartheta - K_1)$ and that $\Lambda^{\mathrm{L}}(\vartheta, K)$ is cofinal in $v(\vartheta - K_1)$; this yields that $v(\vartheta - K)$ is cofinal in $v(\vartheta - K_1)$ and thus has no maximal element. Now Lemma 19.26 shows that $K(\vartheta)|K$ is immediate. Since $\mathrm{dist}\,(\vartheta, K) = \mathrm{dist}\,(\vartheta, K_1)$ is idempotent, $K(\vartheta)|K$ is independent. □

Here is the promised generalization of Lemma 14.7:

**Lemma 14.29** *Every finite extension of an inseparably defectless field of characteristic $p > 0$ is again an inseparably defectless field.*

**Proof:** From Corollary 11.9 it follows that every finite purely inseparable extension of an inseparably defectless field is again an inseparably defectless field. Thus it remains to show the lemma in the case of a finite separable extension $L$ of an inseparably defectless field $K$. We fix an extension of $v$ to $K^{\mathrm{sep}}$ and consider the ramification fields $K^r$ and $L^r$ of $K$ and $L$ with respect to that extension. By Proposition 13.4, we know that $K$ is inseparably defectless if and only if $K^r$ is inseparably defectless, and the same holds for $L$ and $L^r$. By Lemma **??**, we have $L^r = L.K^r$, and therefore $L^r|K^r$ is a finite separable extension. The same proposition shows that $K^{\mathrm{sep}}|K^r$ is a $p$-extension, so $L^r|K^r$ is a tower of Artin-Schreier extensions (cf. Lemma 7.17). Hence, replacing $K$ and $L$ by their ramification fields, we may assume from the start that they are henselian and that $L|K$ is a tower of Artin-Schreier extensions. Now it suffices to prove that $L$ is inseparably defectless under the additional assumption that $L|K$ itself is an Artin-Schreier extension since then, our assertion will follow by induction. Since $L^{1/p^\infty} = L.K^{1/p^\infty}$, it suffices to show for every finite purely inseparable extension $K_1|K$ (which itself is defectless by hypothesis), that $K_2 = K_1.L$ is a defectless extension of $L$. This follows immediately if $K_2|K_1$ and thus $K_2|K$ are defectless. Now assume that $K_2|K_1$ is immediate. Note that $K_1$ is an inseparably defectless field, being a finite purely inseparable extension of the inseparably defectless field $K$. In particular, this yields that $K_1$ admits no immediate purely inseparable extension and hence by virtue of Proposition 15.22, no dependent Artin-Schreier defect extension. The immediate Artin-Schreier extension $K_2|K_1$ is thus independent. An application of Lemma 15.29 now shows that $L|K$ is immediate. But then, it follows already from Corollary **??** that $K_2|L$ is defectless. Hence we have proved that $L$ is an inseparably defectless field. □

In both of the preceding lemmas, the finiteness conditions cannot be dropped, as Examples 14.30 and 14.33 in the next section will show.

## 14.8 A characterization of henselian defectless fields

We are now able to give the

**Proof of Theorem 15.38:**
Assume that the valued field $K$ of characteristic $p > 0$ is separable-algebraically maximal and inseparably defectless. We note that $K$ is henselian since it is separable-algebraically maximal. Let $(L|K, v)$ be a finite extension. We want to show that it is defectless. Since any subextension of a defectless extension is defectless too, we may assume w.l.o.g. that $L|K$ is normal. Hence there exists an intermediate field $K_1$ such that $L|K_1$ is separable and $K_1|K$ is purely inseparable. By hypothesis, we know that $K_1|K$ is defectless. It remains to prove that $L|K_1$ is defectless.

Using Lemma 7.17, choose a finite tame extension $N$ of $K_1$ such that $L.N|N$ is a tower of Artin-Schreier extensions. By Proposition 13.4, $L|K_1$ is defectless if and only if $L.N|N$ is defectless. Since $K_1|K$ is defectless and $N|K_1$ is tame and hence defectless, both extensions being finite, $N|K$ is finite and defectless. Using Lemma 15.36 we conclude that $N$ is inseparably defectless too and therefore does not admit immediate purely inseparable extensions. By Corollary 15.23, this shows that every immediate Artin-Schreier extension of the henselian field $N$ must be independent. Moreover, from Proposition 15.34 we infer that $N$ does not admit independent Artin-Schreier defect extensions. Consequently, given an Artin-Schreier extension $L'|N$ contained in $L.N|N$, this extension must be defectless. In view of Lemma 15.36 and Proposition 15.34, $L'$ will again be inseparably defectless and will not admit any independent Artin-Schreier defect extension. By induction, we conclude that all Artin-Schreier extensions in the tower $L.N|N$ are defectless, hence $L.N|N$ and thus $L|K_1$ and $L|K$ are defectless, as asserted.

Conversely, every defectless field is immediately seen to be separable-algebraically maximal and inseparably defectless. □

## 14.9 Examples

**Example 14.30 (for an independent Artin-Schreier defect extension with dis-**

**tance $0^-$):** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $K = k(t)^{1/p^\infty}$ the perfect hull of the rational function field $k(t)$. Further, let $v = v_t$ be the unique extension of the $t$-adic valuation from $k(t)$ to $K$; we write $vt = 1$. Note that $vK$ is $p$-divisible and $Kv = k$ is algebraically closed.

We consider the Artin Schreier extension $L_0 = k(t, \vartheta)$ of $k(t)$ generated by a root $\vartheta$ of the polynomial

$$X^p - X - \frac{1}{t}.$$

As $v\vartheta = -1/p \notin \mathbb{Z} = vk(t)$, we see that $[L_0 : k(t)] = p = (vL_0 : vk(t))$. Thus, the extension of $v$ from $k(t)$ to $L_0$ is unique. Further, the extension of $v$ from $L_0$ to its perfect hull is unique. But the latter is equal to $L_0.K$, so we find that the extension of $v$ from $K$ to $L := L_0.K$ is unique. On the other hand, the extension $L|K$ is immediate since $vK$ is $p$-divisible and $Kv = k$ is algebraically closed. Therefore, $L|K$ is an Artin-Schreier defect extension. Since $K$ is perfect, it is independent by definition.

For

$$a_n := \sum_{i=1}^{n} \frac{1}{t^{p^{-i}}}$$

we have

$$a_n^p - a_n = \frac{1}{t} - \frac{1}{t^{p^{-n}}},$$

whence

$$(\vartheta - a_n)^p - (\vartheta - a_n) = \vartheta^p - \vartheta - (a_n^p - a_n) = \frac{1}{t} - \left(\frac{1}{t} - \frac{1}{t^{p^{-n}}}\right) = \frac{1}{t^{p^{-n}}}.$$

By Lemma **??** this yields

$$v(\vartheta - a_n) = \frac{1}{p}\, v\, \frac{1}{t^{p^{-n}}} = -\frac{1}{p^{n+1}}.$$

Since this increases with $n$, we see that $(a_n)_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence with limit $\vartheta$. By Corollary **??**, $\mathrm{dist}\,(\vartheta, K) \leq 0^-$. On the other hand, the values $v(\vartheta - a_n)$ are cofinal in $\widetilde{vK}^{<0}$. Therefore,

$$\mathrm{dist}\,(\vartheta, K) = 0^-.$$

This example shows that the condition in Lemma 15.29 that $K_1|K$ be finite cannot be dropped. Indeed, it is known that $(k(t), v_t)$ is a defectless field (for instance, this is a consequence of the Generalized Stability Theorem, cf. [Ku4]). So it does not admit any Artin-Schreier defect extension. But the infinite extension $K$ of $k(t)$ admits an independent Artin-Schreier defect extension.

The example also shows that ramification theoretical properties of a polynomial are not necessarily preserved in the limit. As above, one shows that for every $n \in \mathbb{N}$, a root of the polynomial

$$X^p - X - \frac{1}{t^{np+1}}$$

generates a non-trivial immediate extension of $K$. The same is true for a root of the polynomial

$$Y^p - t^{n(p-1)}Y - \frac{1}{t}.$$

Under $n \to \infty$ (which implies $vt^{n(p-1)} \to \infty$), the limit of this polynomial is

$$Y^p - \frac{1}{t}\,.$$

But this polynomial does not induce a non-trivial extension of $K$ since $K$ is perfect. $\diamondsuit$

This example works even for non-algebraically closed fields $k$. In [Ku2] we presented it with $k = \mathbb{F}_p$. See also [Ku5].

**Example 14.31 (for an independent Artin-Schreier defect extension with distance smaller than $0^-$):** In the previous example, we may choose $k$ such that it admits a non-trivial valuation $\overline{v}$. Now we consider the valuation $v' := v \circ \overline{v}$ on $L$. As $(L|K, v)$ is immediate and $Lv = k = Kv$, it follows that also $(L|K, v')$ is immediate. The value group $\overline{v}k$ is canonically isomorphic to a non-trivial convex subgroup $H$ of $v'L$ (such that $v'L/H \cong vL$). If there would exist some $c \in K$ and an element $\beta \in H$ such that $v'(\vartheta - c) \geq \beta$, then $v(\vartheta - c) \geq 0$ which is impossible. On the other hand, the values $v'(\vartheta - a_n)$ are cofinal in $\{\alpha \in \widetilde{v'K} \mid \alpha < H\}$ since the values $v(\vartheta - a_n)$ are cofinal in $vK^{<0}$. This shows that the distance $\operatorname{dist}(\vartheta, K)$ with respect to $v'$ is the cut

$$H^- \;=\; (\{\alpha \in \widetilde{v'K} \mid \alpha < H\}\,,\,\{\alpha \in \widetilde{v'K} \mid \exists \beta \in H:\, \beta \leq \alpha\})$$

which is smaller than $0^-$ since $H$ is non-trivial. $\diamondsuit$

**Example 14.32 (for a dependent Artin-Schreier defect extension):** With $k(t)$ as before, we take $K_0$ to be the separable-algebraic closure of $k(t)$, with any extension $v_t$ of the $t$-adic valuation of $k(t)$. Being separable-algebraically closed, $K_0$ does in particular not admit any Artin-Schreier extension. But we can build a field admitting a dependent Artin-Schreier defect extension by taking $K = K_0(x)$ and endowing it with the (unique) extension $v$ of $v_t$ such that $vx > vK_0$. (This means that $K$ has the $x$-adic valuation $v_x$ with residue field $K_0$, and $v = v_x \circ v_t$ is the composition of $v_x$ with $v_t$.) We take any $\eta \in \tilde{K}_0 \setminus K_0$. Since $\eta$ lies in the completion of $(K_0, v)$ by Theorem **??**, we have $\Lambda^{\mathrm{L}}(\eta, K_0) = v_t K_0 = vK_0$. It follows that $\Lambda^{\mathrm{L}}(\eta, K)$ is the least initial segment of $vK$ containing $vK_0$. That is, the cut $\operatorname{dist}(\eta, K)$ is the cut $(vK_0)^+$ induced in $\widetilde{vK}$ by the upper edge of the convex subgroup $vK_0$ of $vK$. In particular, $\eta$ does not lie in the completion of $(K, v)$. Now Proposition 14.17 shows that $K$ admits a dependent Artin-Schreier defect extension. According to this proposition, it can for instance be generated by a root $\vartheta$ of the polynomial $X^p - X - (\eta/x)^p$, as $vx > \operatorname{dist}(\eta, K) = p \operatorname{dist}(\eta, K)$. Then $\operatorname{dist}(\vartheta, K) = \operatorname{dist}(\eta, K) - vx = (vK_0)^+ - vx = (-vx + vK_0)^+$ is the cut induced by the upper edge of the coset $-vx + vK_0$ in $\widetilde{vK}$. Note that in $vK$, which is the lexicographic product $\mathbb{Z}vx \times vK_0$, the cut $(-vx + vK_0)^+$ is equal to the cut $vK_0^-$ induced by the lower edge of the convex subgroup $vK_0$ of $vK$. Nevertheless, the cut $\operatorname{dist}(\vartheta, K)$ in $\widetilde{vK}$ is *not* equal to $H^-$ or $H^+$ for any convex subgroup $H$ of $vK$ or of $\widetilde{vK}$ (cf. Example **??** in Section **??**). $\diamondsuit$

Enlarging the rank of the valuation in order to obtain a dependent Artin-Schreier defect extension may appear to be a dirty trick. Therefore, we add a further example which shows that such extensions can also appear for valuations of rank one.

**Example 14.33 (for a dependent Artin-Schreier defect extension in rank 1):** With $(k(t),v)$ as before, we take $a_1$ to be a root of the Artin-Schreier polynomial $X^p - X - 1/t$. Then $va_1 = -1/p < 0$. By induction on $i$, we take $a_{i+1}$ to be a root of the Artin-Schreier polynomial $X^p - X + a_i$, for all $i \in \mathbb{N}$. Then $va_i = -1/p^i < 0$. Note that $t, a_1, \ldots, a_i \in k(a_{i+1})$ for every $i$, because $a_i = a_{i+1} - a_{i+1}^p$. We have $1/p \in vk(a_1) \setminus vk(t)$. Since $p \leq (vk(a_1) : vk(t)) \leq [k(a_1) : k(t)] \leq p$, equality holds everywhere and we find that $vk(a_1) = \frac{1}{p} vk(t)$. Repeating this argument by induction on $i > 1$, we obtain $1/p^i \in vk(a_i) \setminus vk(a_{i-1})$ and thus, $vk(a_i) = \frac{1}{p} vk(a_{i-1}) = \frac{1}{p^i} vk(t)$. Therefore, the value group of $K := k(a_i \mid i \in \mathbb{N})$ is the $p$-divisible hull $\frac{1}{p^\infty}\mathbb{Z}$ of $\mathbb{Z}$ (an ordered abelian group of rank 1).

Finally, we choose $\eta$ such that $\eta^p = 1/t$. Since $vK$ is $p$-divisible and $Kv = k$ is algebraically closed, the extension $K(\eta)|K$ with the unique extension of the valuation $v$ is immediate. We wish to determine $\mathrm{dist}\,(\eta, K)$. We set $c_i := a_1 + \ldots + a_{i-1} \in k(a_{i-1})$ for $i > 1$. Using that $a_1^p = \frac{1}{t} + a_1$ and $a_{i+1}^p = a_{i+1} - a_i$ for $i \in \mathbb{N}$, we compute:

$$\begin{aligned} 0 &= \eta^p - \frac{1}{t} = (\eta - c_i + a_1 + \ldots + a_{i-1})^p - \frac{1}{t} \\ &= (\eta - c_i)^p + a_1^p + \ldots + a_{i-1}^p - \frac{1}{t} = (\eta - c_i)^p + a_{i-1} \,. \end{aligned}$$

It follows that $v(\eta - c_i)^p = va_{i-1}$, that is, $v(\eta - c_i) = \frac{1}{p} va_{i-1} = va_i = -1/p^i$. Hence, $-1/p^i \in \Lambda^{\mathrm{L}}(\eta, K)$ for all $i$. Assume that there is some $c \in K$ such that $v(\eta - c) > -1/p^i$ for all $i$. Then $v(c - c_i) = \min\{v(\eta - c_i), v(\eta - c)\} = -1/p^i$ for all $i$. On the other hand, there is some $i$ such that $c \in k(a_{i-1})$ and thus, $c - c_i \in k(a_{i-1})$. But this contradicts the fact that $v(c - c_i) = -1/p^i \notin vk(a_{i-1})$. This proves that the values $-1/p^i$ are cofinal in $\Lambda^{\mathrm{L}}(\eta, K)$. Hence, $\Lambda^{\mathrm{L}}(\eta, K) = vK^{<0}$ and $\mathrm{dist}\,(\eta, K) = 0^-$.

Now Proposition 14.17 shows that $K$ admits a dependent Artin-Schreier defect extension. According to this proposition, it can for instance be generated by a root $\vartheta$ of the polynomial $X^p - X - (\eta/t)^p$, as $vt = 1 > \mathrm{dist}\,(\eta, K) = p\,\mathrm{dist}\,(\eta, K)$. Then $\mathrm{dist}\,(\vartheta, K) = \mathrm{dist}\,(\eta, K) - 1 = 0^- - 1 = (-1)^-$.

This example shows that the condition in Lemma 15.36 that the extension be finite cannot be dropped. Indeed, as we have noted in Example 14.30, $(k(t), v_t)$ is a defectless and hence inseparably defectless field. But the infinite extension $K$ of $k(t)$ is not an inseparably defectless field. ♢

**Example 14.34 (for a field having a dependent but no independent Artin-Schreier defect extension):** We do not know whether the field $K$ of the last example admits any independent Artin-Schreier defect extension; this an open problem. But in any case, we can construct from it a field which has a dependent but no independent Artin-Schreier defect extension. Indeed, by Zorn's Lemma there is an extension field of $K$ within its algebraic closure not admitting any independent Artin-Schreier defect extension; such an extension field can be found by a (possibly transfinitely) repeated extension by independent Artin-Schreier defect extensions. We choose such an extension field and call it $L$. Since it is a separable algebraic extension of $K$, the extension $L(\eta)|L$ is still non-trivial and purely inseparable, and by our hypothesis on the value group and residue field of $K$, it is also immediate.

We wish to show that $\mathrm{dist}\,(\eta, L) = \mathrm{dist}\,(\eta, K)$. Assume that this is not true. Then there is an element $\zeta \in L$ such that $v(\eta - \zeta) > \mathrm{dist}\,(\eta, K)$. We write $L = \bigcup_{\mu<\nu} K_\mu$ where

$\nu$ is some ordinal, $K_{\mu+1}|K_\mu$ is an independent Artin-Schreier defect extension whenever $0 \leq \mu < \nu$, and $K_\lambda = \bigcup_{\mu<\lambda} K_\mu$ for every limit ordinal $\lambda < \nu$. Let $\mu_0$ be the minimal ordinal for which $K_{\mu_0}$ contains such an element $\zeta$. Then $\mu_0$ must be a successor ordinal, and we have that $\text{dist}\,(\eta, K) = \text{dist}\,(\eta, K_{\mu_0-1})$. Hence, $v(\eta - \zeta) > \text{dist}\,(\eta, K_{\mu_0-1})$, that is, $\zeta \sim_{K_{\mu_0-1}} \eta$. But this is a contradiction since by construction, $K_{\mu_0}|K_{\mu_0-1}$ is an independent Artin-Schreier defect extension. This proves that

$$\text{dist}\,(\eta, L) \;=\; \text{dist}\,(\eta, K) \;=\; 0^- \;.$$

Now Corollary 15.27 shows that $L$ admits a dependent Artin-Schreier defect extension $L'|L$. On the other hand, by construction it does not admit any independent Artin-Schreier defect extension.

This example shows once more that Lemma 15.29 becomes false if the finiteness condition is dropped. To see this, note that $L'.L^{1/p^\infty}|L^{1/p^\infty}$ is still an Artin-Schreier defect extension, since $L'|L$ is linearly disjoint from $L^{1/p^\infty}|L$, $vL^{1/p^\infty}$ is $p$-divisible and $L^{1/p^\infty}v$ is algebraically closed, and the extension of $v$ from $L$ to $L'.L^{1/p^\infty}$ and thus also the extension of $v$ from $L^{1/p^\infty}$ to $L'.L^{1/p^\infty}$ is unique. On the other hand, $L^{1/p^\infty}$ admits no purely inseparable extensions at all, so by Corollary 15.23, such an Artin-Schreier defect extension can only be independent. We have thus shown that $L^{1/p^\infty}$ admits an independent Artin-Schreier defect extension whereas $L$ does not. In view of Lemma 15.29, this is only possible since $L^{1/p^\infty}|L$ is an infinite extension. In contrast to Example 14.30, here we have the case where the lower field is *not* defectless. $\diamondsuit$

**Example 14.35 (for a field which is not relatively algebraically closed in any maximal immediate extension, but has no independent Artin-Schreier defect extension):** If we replace $k(t)$ by its absolute ramification field $k(t)^r$ (with respect to an arbitray extension of $v$ to the separable-algebraic closure of $k(t)$), then the constructions of Example 14.33 and 14.34 can be taken over literally. Since $vk(t)^r$ is divisible by every prime different from $p$, the value groups of $K$, $L$ and $L'$ will then be divisible. Since their residue fields are algebraically closed and all fields are henselian, it follows that $K$, $L$ and $L'$ are equal to their ramification fields.

Observe that now $L'$ will be contained in every maximal immediate extension of $L$. This is true because $vL$ is divisible and $Lv$ is algebraically closed, which implies that every maximal immediate extension of $L$ is algebraically closed. We have thus shown that $L$ is not separable-algebraically closed in any of its maximal immediate extensions, whereas it doesn't admit independent Artin-Schreier defect extensions.

Since $L'|L$ is linearly disjoint from $L^c|L$, we may replace $L$ by its completion $L^c$. By Lemma 14.21, $L^c$ still cannot admit independent Artin-Schreier defect extensions. As the completion of a henselian field is again henselian (cf. [W], Theorem 32.19) and is an immediate extension, it follows that the completion of a field which is equal to its absolute ramification field has the same property. The same argument as before shows that again, $L'.L^c$ will be contained in every maximal immediate extension of $L^c$. Hence, $L^c$ is an example of a complete field, equal to its absolute ramification field, which is not relatively algebraically closed in any maximal immediate extension, but has no independent Artin-Schreier defect extension. $\diamondsuit$

## 14.10 Another characterization of defectless fields

**Theorem 14.36** *Let $(K,v)$ be a separably defectless field of characteristic $p>0$. If in addition $K^c|K$ is separable, then $(K,v)$ is a defectless field.*

**Proof:** Assume that $K^c|K$ is separable, but that $(K,v)$ is not a defectless field. We have to show that $(K,v)$ is not separably defectless. Let $(F|K,v)$ be a finite defect extension of minimal degree of inseparability. If this extension is separable, then we are done. Suppose it is not. We wish to deduce a contradiction by constructing a defect extension of smaller degree of inseparability. Let $E|K$ be the maximal separable subextension. By assumption, it is defectless, so the purely inseparable extension $(F|E,v)$ must be a defect extension. Using the arguments of the proof of Theorem 14.2 (with $K$ replaced by $E$), one shows that there exists a subextension $L|E$ of $F|E$ and an element $\eta\in L^{1/p}\setminus L$ such that the extension $(L(\eta)|L,v)$ is immediate.

Since a finite extension of a complete field is again complete and since $L^c$ must contain both $K^c$ and $L$, we find that $L^c=L.K^c$. Together with the fact that $K^c|K$ is separable, this yields that also $L^c|L$ is separable (see [L], Chapter X, \$6, Corollary 4). It follows that $\eta\notin L^c$. By an application of Proposition 14.17, we now obtain an immediate separable extension $(L(\vartheta)|L,v)$. Altogether, we have constructed a defect extension $(L(\vartheta)|K,v)$ which has smaller degree of inseparability than $(F|K,v)$. This is the desired contradiction. □

We use this theorem to show:

**Theorem 14.37** *Let $K$ be a henselian field of characteristic $p>0$. Then $K$ is a separably defectless field if and only if $K^c$ is a defectless field.*

**Proof:** Since $K$ is henselian, the same holds for $K^c$ (cf. [W], Theorem 32.19). By virtue of the preceding Theorem, $K^c$ is a defectless field if and only if it is a separably defectless field. Thus it suffices to prove that $K^c$ is a separably defectless field if and only if $K$ is.

Let $L|K$ be an arbitrary finite separable extension. The henselian field $K$ is separable-algebraically closed in $K^c$ (cf. [W], Theorem 32.19). Consequently, every finite separable extension of $K$ is linearly disjoint from $K^c$ over $K$, whence

$$[L.K^c:K^c]\;=\;[L:K]\,. \tag{14.20}$$

On the other hand, $L.K^c=L^c$ is the completion of $L$ and thus an immediate extension of $L$. Consequently,

$$\begin{aligned}(vL.K^c:vK^c)\cdot[L.K^cv:K^cv] &= (vL^c:vK^c)\cdot[L^cv:K^cv]\\ &= (vL:vK)\cdot[Lv:Kv]\,.\end{aligned} \tag{14.21}$$

Assume that $K^c$ is a separably defectless field. Then $L.K^c|K^c$ is defectless, i.e., $[L.K^c:K^c]=(vL.K^c:vK^c)\cdot[L.K^cv:K^cv]$. Hence, $[L:K]=(vL:vK)\cdot[Lv:Kv]$, showing that $L|K$ is defectless. Since $L|K$ was an arbitrary finite separable extension, we have shown that $K$ is a separably defectless field.

Now assume that $K^c$ is not a separably defectless field. Then there exists a finite Galois extension $L'|K^c$ with non-trivial defect. Take an irreducible polynomial $f=X^n+$

$c_{n-1}X^{n-1} + \ldots + c_0 \in K^c[X]$ of which $L'$ is the splitting field. For every $\alpha \in vK$ there are $d_{n-1}, \ldots, d_0 \in K$ such that $v(c_i - d_i) \geq \alpha$. If $\alpha$ is large enough, then by Theorem 32.20 of [W], the splitting fields of $f$ and $g = X^n + d_{n-1}X^{n-1} + \ldots + d_0$ over the henselian field $K^c$ are the same. Consequently, if $L$ denotes the splitting field of $g$ over $K$, then $L' = L.K^c = L^c$. We obtain

$$\begin{aligned} [L:K] &\geq [L.K^c : K^c] = [L' : K^c] \\ &> (vL' : vK^c)[L'v : K^c v] = (vL^c : vK^c)[L^c v : K^c v] \\ &= (vL : vK)[Lv : Kv] . \end{aligned}$$

That is, the separable extension $L|K$ is not defectless. Hence, $K$ is not a separably defectless field. □

## 14.11 Algebraically maximal fields

We will now give a characterization of algebraically maximal fields which has been presented by F. Delon [D1]. We need the following fact, which was proved by Yu. Ershov in [Er1] by a different method. Note that the proof in [D1] has gaps since it is not immediately clear that if $\sum_{i=1}^n \alpha_{i,\nu}$ is increasing with $\nu$, then there is an increasing cofinal subsequence of $(\alpha_{i,\nu})_\nu$ for some $i$. Ershov solves this problem by invoking Ramsey theory. We will avoid this by further analyzing the valuation theoretical situation.

**Lemma 14.38** *Let $(K, v)$ be any valued field with valuation ring $\mathcal{O}$, and $f \in K[X]$ a polynomial in one variable.*
*1) If $v\operatorname{im}_K(f)$ has no maximum, then there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $(K, v)$ without limit in $K$ but admitting a root of $f$ as a limit, and such that $(vf(c_\nu))_{\nu<\lambda}$ is a strictly increasing cofinal sequence in $v\operatorname{im}_K(f)$.*
*2) If $v\operatorname{im}_{\mathcal{O}}(f)$ has no maximum, then there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $\mathcal{O}$ without limit in $K$ but admitting a root of $f$ as a limit, and such that $(vf(c_\nu))_{\nu<\lambda}$ is a strictly increasing cofinal sequence in $v\operatorname{im}_{\mathcal{O}}(f)$.*

**Proof:** 1): We choose a sequence $(c_\nu)_{\nu<\lambda}$ of elements in $K$ such that the values $vf(c_\nu)$ are strictly increasing and cofinal in $v\operatorname{im}_K(f)$. We write $f(X) = \prod_{i=1}^n (X - a_i)$ with $a_1, \ldots, a_n \in \tilde{K}$ and choose some extension of $v$ to $\tilde{K}$.

We introduce a symbol $-\infty$ and define $-\infty < \alpha$ for all $\alpha \in v\tilde{K}$. Now we consider all balls $B^\circ_\alpha(a_i) = \{a \in \tilde{K} \mid v(a_i - a) > \alpha\}$ with center a root $a_i$ of $f$, $1 \leq i \leq n$, and radius $\alpha$ in the finite set $\mathcal{D} := \{v(a_i - a_j) \mid 1 \leq i < j \leq n\} \cup \{-\infty\}$; note that $B^\circ_{-\infty}(a_i) = \tilde{K}$. These are finitely many balls, with $\tilde{K}$ one of them, so there is at least one among them with $\alpha$ maximal in which there lies some cofinal subsequence of $(c_\nu)_{\nu<\lambda}$. After renaming our elements if necessary, we may assume that this ball is $B^\circ_\alpha(a_1)$, that the subsequence is again called $(c_\nu)_{\nu<\lambda}$, and that exactly $a_1, \ldots, a_m$ $(m \leq n)$ are the roots of $f$ which lie in $B^\circ_\alpha(a_1)$. Then for every $\nu < \lambda$ and $m < i \leq n$, we have that

$$v(c_\nu - a_i) = \min\{v(c_\nu - a_1), v(a_1 - a_i)\} = v(a_1 - a_i) .$$

On the other hand, by the maximality of $\alpha$ we have the following: if $\mathcal{D}$ contains elements $> \alpha$ (which is the case if $B^\circ_\alpha(a_1)$ contains at least two roots of $f$) and if $\beta$ is the least of these elements, then there is no cofinal subsequence of $(c_\nu)_{\nu<\lambda}$ which lies in any of the balls $B^\circ_\beta(a_i)$. This even remains true if we replace $B^\circ_\beta(a_i)$ by $B_\beta(a_i) = \{a \in \tilde{K} \mid v(a_i - a) \geq \beta\}$. Indeed, by our choice of $\beta$ we have for $1 \leq i \leq m$ that $B_\beta(a_i)$ contains $a_1, \ldots, a_m$ and thus, $c \in B_\beta(a_i)$ implies $v(c - a_j) \geq \beta$ for $1 \leq j \leq m$. If in addition $c$ does not lie in any $B^\circ_\beta(a_j)$, then $v(c - a_j) = \beta$ for $1 \leq j \leq m$. Hence if a cofinal subsequence of $(c_\nu)_{\nu<\lambda}$ would lie in $B_\beta(a_i)$, then the value

$$vf(c_\nu) \;=\; v\prod_{i=1}^{n}(c_\nu - a_i) \;=\; \sum_{i=1}^{n} v(c_\nu - a_i) \;=\; m\beta + \sum_{i=m+1}^{n} v(a_1 - a_i)$$

would be fixed for all $c_\nu$ in this subsequence, a contradiction.

After deleting elements from $(c_\nu)_{\nu<\lambda}$, we may thus assume that $v(c_\nu - a_i) < \beta \leq v(a_1 - a_i)$ for all $\nu$ and $1 \leq i \leq m$. It follows that $v(c_\nu - a_i) = \min\{v(c_\nu - a_1), v(a_1 - a_i)\} = v(c_\nu - a_1)$ for all $\nu$ and $1 \leq i \leq m$. Now we compute:

$$vf(c_\nu) \;=\; \sum_{i=1}^{n} v(c_\nu - a_i) \;=\; mv(c_\nu - a_1) + \sum_{i=m+1}^{n} v(a_1 - a_i)\;.$$

If $\mu < \nu < \lambda$, then $vf(c_\mu) < vf(c_\nu)$ and hence we must have $v(c_\mu - a_1) < v(c_\nu - a_1)$. This shows that $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence with limit $a_1$.

Any limit $a \in \tilde{K}$ of this sequence satisfies $v(a - a_1) > v(c_\nu - a_1)$ and hence also $v(a - a_i) \geq \min\{v(a - a_1), v(a_1 - a_i)\} > v(c_\nu - a_1)$ for $1 \leq i \leq m$ and all $\nu$. Thus,

$$vf(a) \;=\; \sum_{i=1}^{n} v(a - a_i) \;>\; mv(c_\nu - a_1) + \sum_{i=m+1}^{n} v(a_1 - a_i) \;=\; vf(c_\nu)\;.$$

for all $\nu$. This shows that $a$ cannot lie in $K$. Hence, $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence without limit in $K$, and by construction, it is of algebraic type.

2): We proceed as in 1), but choose the sequence $(c_\nu)_{\nu<\lambda}$ in $\mathcal{O}$ such that the values $vf(c_\nu)$ are strictly increasing and cofinal in $v\operatorname{im}_{\mathcal{O}}(f)$. We only have to note in addition that if $a \in K$ would be a limit of the sequence, than it would also lie in $\mathcal{O}$. □

**Corollary 14.39** *Assume that $(K, v)$ is not $K$-extremal with respect to the polynomial $f(X) \in K[X]$. Then for all $c \in K$ of large enough value, $(K, v)$ is not $\mathcal{O}$-extremal with respect to the polynomial $f(c^{-1}X)$. Hence, if $(K, v)$ is $\mathcal{O}$-extremal with respect to every polynomial in one variable, then $(K, v)$ is $K$-extremal with respect to every polynomial in one variable. The same holds for "separable polynomial" in the place of "polynomial".*

**Proof:** Take the pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ as in Lemma 14.38. For large enough $\nu_0 < \lambda$, the values of the $c_\nu$ with $\nu_0 < \nu < \lambda$ are constant, say, $\alpha$. For every $c$ of value $\geq -\alpha$, we have that $cc_\nu \in \mathcal{O}$ for $\nu_0 < \nu < \lambda$. Hence, $(K, v)$ is not $\mathcal{O}$-extremal with respect to the polynomial $f(c^{-1}X)$. □

The first part of the following result was proved by Yu. Ershov in [Er1]:

**Proposition 14.40** *A valued field is algebraically maximal if and only if it is henselian and $K$-extremal with respect to every polynomial in one variable. The same holds with "$\mathcal{O}$-extremal" in the place of "$K$-extremal".*

**Proof:** Suppose that $(K,v)$ is henselian, but not algebraically maximal. Then there is a proper immediate algebraic extension $L|K$. Take $a \in L \setminus K$. By Theorem 1 of [Ka], there is a pseudo Cauchy sequence in $K$ without limit in $K$, having $a$ as a limit. Let $f \in K[X]$ be the minimal polynomial of $a$ over $K$. Since $K$ is henselian, the extension of $v$ from $K$ to $K(a)$ is unique. Now it follows from Lemma **??** that $v\,\mathrm{im}_K(f)$ has no maximal element. That is, $K$ is not $K$-extremal with respect to $f$. Hence by Corollary 14.39, $K$ is also not $\mathcal{O}$-extremal with respect to every polynomial in one variable.

For the converse, suppose that there is a polynomial $f \in K[X]$ such that $v\,\mathrm{im}_K(f)$ or $v\,\mathrm{im}_{\mathcal{O}}(f)$ has no maximal element. Then by Lemma 14.38, $(K,v)$ admits a pseudo Cauchy sequence of algebraic type in $(K,v)$ without limit in $K$. Now Theorem 3 of [Ka] shows that there is a proper immediate algebraic extension of $(K,v)$, i.e., $(K,v)$ is not algebraically maximal. □

Theorem **??** and its "$K$-extremal" version follow from this proposition once we have proved the following proposition:

**Proposition 14.41** *If a valued field is $K$- or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable, then it is henselian.*

**Proof:** In view of Corollary 14.39 we only have to prove the assertion for "$K$-extremal". Suppose that the valued field $(K,v)$ with valuation ring $\mathcal{O}$ is not henselian. Then there is a polynomial $f \in \mathcal{O}[X]$ and an element $b \in \mathcal{O}$ such that $vf(b) > 2vf'(b)$, but $f$ has no root in $K$. We take $K_0$ to be a finitely generated subfield of $K$ containing $b$ and all coefficients of $f$, and $K_1$ to be the relative algebraic closure of $K_0$ in $K$. Then $f$ has no root in $K_1$, which shows that $K_1$ is not henselian. Since $K_1$ has finite transcendence degree over its prime field, it has finite rank, which means that $v|_{K_1}$ is a composition $v|_{K_1} = v_1 \circ \ldots \circ v_k$ of valuations $v_i$ with archimedean value groups. By a repeated application of Theorem 32.15 of [W], it follows that $(K_1,v_1)$ is not henselian or for some $i \leq k$ and $v^i := v_1 \circ \ldots \circ v_{i-1}$, $(K_1v^i, v_i)$ is not henselian. In the first case, there is a monic separable and irreducible polynomial $g \in K_1[X]$ with $v_1$-integral coefficients and a $v_1$-integral element $c \in K_1$ such that $v_1g(c) > 2v_1g'(c)$, but $g$ does not have a zero in $K_1$. It follows that $vg(c) > 2vg'(c)$.

In the second case, there is a monic separable and irreducible polynomial $\overline{g} \in K_1v^i[X]$ with $v_i$-integral coefficients and a $v_i$-integral element $\overline{c} \in K_1v^i$ such that $v_i\overline{g}(\overline{c}) > 2v_i\overline{g}'(\overline{c})$, but $\overline{g}$ does not have a zero in $K_1v^i$. We take some monic polynomial $g \in K_1[X]$ with $v^i$-integral coefficients such that its $v^i$-reduction is equal to $\overline{g}$. Also, we pick a $v^i$-integral element $c \in K_1$ whose $v^i$-reduction is $\overline{c}$. Then it follows that $v^{i+1}g(c) > 2v^{i+1}g'(c)$, whence $vg(c) > 2vg'(c)$.

It is well known that if $w$ is any valuation for which the polynomial $g$ has $w$-integral coefficients and $wg(c) > 2wg'(c)$ holds, then a repeated application of the Newton algorithm

$$c_{n+1} \;:=\; c_n - \frac{g(c_n)}{g'(c_n)}\;,$$

starting with $c_0 = c$, leads to a strictly increasing sequence of values $wg(c_n)$; this sequence is cofinal in the value group of $w$ in case this value group is archimedean. Hence in the first case, we obtain a sequence of elements $c_n \in K_1$ such that the sequence $v_1g(c_n)$ is cofinal in $v_1K_1$. This implies that if $d \in K$ is such that $vg(d)$ is the maximum of $v\,\mathrm{im}_K(g)$ and $H$ denotes the convex subgroup of $vK$ generated by $v_1K_1$, then $vg(d) > H$. Let $v_H$ be the coarsening of $v$ with respect to $H$. Then $v_H g(d) > 0$, i.e., $g(d)v_H = 0$. On the other hand, the reduction modulo $v_H$ induces an isomorphism on $K_1$, and since $g$ was chosen to be separable and irreducible, we thus have that $g'(d)v_H \neq 0$, i.e., $v_H g'(d) = 0$. But then by the Newton algorithm, if $g(d) \neq 0$, then there is some $d' \in K$ such that $v_H g(d') > v_H g(d)$ and hence, $vg(d') > vg(d)$. This contradiction shows that $g(d) = 0$. But this contradicts our choice of $g$. Hence, $v\,\mathrm{im}_K(g)$ does not have a maximum.

In the second case, the Newton algorithm provides elements $\bar{c}_n$ such that the sequence $v_i\bar{g}(\bar{c}_n)$ is cofinal in $v_i(K_1v^i)$. We choose $v^i$-integral elements $c_n \in K_1$ whose $v^i$-reductions are $\bar{c}_n$. Then it follows that the values $vg(c_n)$ are cofinal in a convex subgroup $H$ of $vK$ which is the convex hull of the convex subgroup of $vK_1$ which corresponds to the coarsening $v^i$ of $v|_{K_1}$. This implies that if $d \in K$ is such that $vg(d)$ is the maximum of $v\,\mathrm{im}_K(g)$, then $vg(d) > H$. Let $v_H$ be the coarsening of $v$ with respect to $H$. Then again, $v_H g(d) > 0$ and $g(d)v_H = 0$. On the other hand, the reduction of $g$ modulo $v_H$ is $\bar{g}$, so $0 = g(d)v_H = \bar{g}(dv_H)$. Since $\bar{g}$ was chosen to be separable and irreducible, we thus have that $g'(d)v_H = \bar{g}'(dv_H) \neq 0$, i.e., $v_H g'(d) = 0$. Arguing as in the first case, we show that $v\,\mathrm{im}_K(g)$ does not have a maximum. Hence we find that $K$ is not $K$-extremal with respect to every separable polynomial in one variable. □

The following are corollaries to Theorem **??**:

**Corollary 14.42** *The property "algebraically maximal" is elementary in the language of valued fields.*

**Corollary 14.43** *Every extremal field is algebraically maximal.*

We will now give the

**Proof of Theorem ??** and its "$K$-extremal" version:
In view of Theorem **??**, it suffices to prove that if $K$ is a henselian but not algebraically maximal field, then there is a $p$-polynomial $f$ in one variable with coefficients in $K$ with respect to which $K$ is not $K$-extremal. By Corollary 14.39, for suitable $c \in K$, $K$ is then also not $\mathcal{O}$-extremal with respect to the $p$-polynomial $f(c^{-1}X)$.

Take a proper immediate algebraic extension of $K$. Since $K$ is assumed henselian, it follows that this extension is purely wild and hence linearly disjoint over $K$ from the absolute ramification field $K^r$ of $K$. We may assume that this extension is minimal, that is, it does not admit any proper subextension. Then by Theorem 13 of [Ku3], it is generated by a root of a $p$-polynomial $f$. As in the first part of the proof of Proposition 14.40 it follows that $v\,\mathrm{im}_K(f)$ has no maximal element, that is, $K$ is not $K$-extremal with respect to the $p$-polynomial $f$. By Corollary 14.39, for suitable $c \in K$, $K$ is not $\mathcal{O}$-extremal with respect to the $p$-polynomial $f(c^{-1}X)$. □

## 14.12 Separable-algebraically maximal fields

The following is a further consequence of Proposition 14.17:

**Corollary 14.44** *Take a separable-algebraically maximal field $(K, v)$. Every immediate algebraic extension of $(K, v)$ is purely inseparable and lies in its completion. Every pseudo Cauchy sequence of algebraic type in $(K, v)$ without limit in $K$ has breadth $\{0\}$, and its unique limit in $\tilde{K}$ is purely inseparable over $K$.*

**Proof:** Every immediate algebraic extension of $K$ must be purely inseparable since otherwise, it would contain a proper immediate separable-algebraic subextension. Since $K$ in particular does not admit any dependent Artin-Schreier defect extensions, we thus obtain from Corollary 15.25 that every immediate algebraic extension of $K$ must lie in $K^c$.

Take a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $(K,v)$ without limit in $K$. By Theorem 3 of [Ka], this pseudo Cauchy sequence gives rise to a proper immediate algebraic extension of $K$, in which it has a limit. By what we have just shown, this extension is purely inseparable and lies in the completion of $K$. The latter shows that $(c_\nu)_{\nu<\lambda}$ has breadth $\{0\}$ and therefore has a unique limit in the algebraic closure of $K$. The former shows that this limit must be purely inseparable over $K$. □

The following result has been presented by F. Delon in [D1]:

**Corollary 14.45** *The completion of a separable-algebraically maximal field is algebraically maximal.*

**Proof:** Take any valued field $(K,v)$ and suppose that $K^c$ admits a proper immediate algebraic extension. Then by Corollary **??** there is a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $K^c$ without limit in $K^c$. This must have non-trivial breadth, that is, there is some $\gamma \in vK$ such that $v(c_{\nu+1}-c_\nu) < \gamma$ for all $\nu$ (because otherwise, Theorem 3 of [Ka] would render a proper immediate extension of $K^c$ within $K^c$, which is absurd). Since $c_\nu \in K^c$, there is $c_\nu^* \in K$ such that $v(c_\nu - c_\nu^*) \geq \gamma$ and hence $v(c_{\nu+1}^* - c_\nu^*) = v(c_{\nu+1} - c_\nu)$ for all $\nu$. It follows that $(c_\nu^*)_{\nu<\lambda}$ is a pseudo Cauchy sequence in $K$ without limit in $K$ and with the same non-trivial breadth as $(c_\nu)_{\nu<\lambda}$.

Let $f \in K^c[X]$ be a polynomial such that for some $\mu < \lambda$, the sequence $(vf(c_\nu))_{\mu<\nu<\lambda}$ is strictly increasing. Such a polynomial must exist since by assumption, $(c_\nu)_{\nu<\lambda}$ is of algebraic type. Since $(c_\nu)_{\nu<\lambda}$ has non-trivial breadth, it follows from Lemma 8 of [Ka] that the sequence $(vf(c_\nu))_{\mu<\nu<\lambda}$ is bounded from above in $vK$. Hence, we can choose a polynomial $f^* \in K[X]$ with coefficients so close to the corresponding coefficients of $f$ that $vf^*(c_\nu) = vf(c_\nu)$ whenever $\mu < \nu < \lambda$. This shows that also $(c_\nu^*)_{\nu<\lambda}$ is of algebraic type. Hence by the foregoing corollary, $K$ cannot be separable-algebraically maximal. □

Now we give the

**Proof of Theorem ??** and its "$K$-extremal" version:
Assume that $(K,v)$ is $K$-extremal or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. Then by Proposition 14.41, $K$ is henselian. Suppose that $(K,v)$ is not separable-algebraically maximal. Then there is a proper immediate separable-algebraic extension $L|K$. Take $a \in L \setminus K$, and let $f \in K[X]$ be the minimal polynomial of $a$ over $K$. By Theorem 1 of [Ka], there is a pseudo Cauchy sequence in $K$ without limit in $K$, having $a$ as a limit. Since $K$ is henselian, the extension of $v$ from $K$ to $K(a)$ is unique. Now it follows from Lemma **??** that $v\,\mathrm{im}_K(f)$ has no maximal element, that is, $K$ is not $K$-extremal with respect to $f$. By Corollary 14.39, it follows that $K$ is not $\mathcal{O}$-extremal with respect to the separable polynomial $f(c^{-1}X)$ for some $c \in K$. This contradicts our assumption that $K$ is $K$-extremal or $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. Hence, $K$ is separable-algebraically maximal.

For the converse, assume that $(K,v)$ is separable-algebraically maximal. Suppose that there is a separable polynomial $f \in K[X]$ such that $v\,\mathrm{im}_K(f)$ or $v\,\mathrm{im}_{\mathcal{O}}(f)$ has no maximal element. Then by Lemma 14.38, $(K,v)$ admits a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of algebraic type in $(K,v)$ without limit in $K$, but with a root $a \notin K$ of $f$ as a limit. By Corollary 14.44, $a$ is purely inseparable over $K$. But this contradicts the fact that $a$ is a root of a separable polynomial over $K$. Hence, $K$ is $K$-extremal and $\mathcal{O}$-extremal with respect to every separable polynomial in one variable. □

**Corollary 14.46** *The property "separable-algebraically maximal" is elementary in the language of valued fields.*

We turn to the

**Proof of Theorem ??:** The proof is the same as for Theorem **??**, except that the immediate algebraic extension of $K$ can be taken to be separable, and hence the $p$-polynomial $f$ is separable. □

Finally, we note that **Theorem ??** has also been proved, as our above proof of Theorems **??**, **??**, **??** and **??** have all dealt simultaneously with both $K$- and $\mathcal{O}$-extremality.

## 14.13 Henselian-by-finite and maximal-by-finite fields

We call a valued field $(K,v)$ **henselian-by-finite** if it is not henselian, but there is a finite extension $(L|K,v)$ such that $(L,v)$ is henselian. Since the henselization of $(K,v)$ is contained in $(L,v)$, it is consequently also a finite extension.

**Theorem 14.47** *Let $(K,v)$ be a henselian-by-finite field. Then $K$ is formally real, but $v$ is not compatible with the ordering of $K$. The henselization of $(K,v)$ is $K(\sqrt{-1})$, and it admits precisely two distinct extensions of $v$. With both of them, it is a defectless field. Further, $Kv_h^K$ is real closed and if $v_h^K < w \leq v$, then $Kw$ is algebraically closed. Moreover, $v_c^K \leq v_h^K$, and $v_c^K$ is the finest of all coarsenings $w$ of $v_h^K$ such that $Kw$ is real closed.*

**Proof:** Assume that $(K^h|K,v)$ is a finite non-trivial extension. Then $v/v_h^K$ is non-trivial and not henselian on $Kv_h^K$. Let $L$ be the normal hull of $K^h$ over $K$. Then also $L|K$ is finite. Since $v$ is henselian on $K^h$, it is henselian on $L$ and by **??**, $v/v_h^K$ is henselian on $K^hv_h^K$ and on $Lv_h^K$. By Lemma 9.49, there is an extension of $v/v_h^K$ from $Kv_h^K$ to $Lv_h^K$ which is independent from $v/v_h^K$ on $Lv_h^K$. Since it is conjugate to $v/v_h^K$ (cf. **??**) and $Lv_h^K|Kv_h^K$ is normal by **??**, this extension is also non-trivial and henselian on $Lv_h^K$. Now it follows from Theorem 9.44 that $Lv_h^K$ is separable-algebraically closed. Since $Lv_h^K|Kv_h^K$ is finite, Artin-Schreier Theory shows that $Kv_h^K$ is real closed and $Lv_h^K$ is algebraically closed. As $v/v_h^K$ is henselian on $K^hv_h^K$ but not on $Kv_h^K$, the extension $K^hv_h^K|Kv_h^K$ must be non-trivial. This proves that $K^hv_h^K$ is algebraically closed. Consequently, $v_c^K \leq v_h^K$.

We also conclude that $K^hv_h^K = Kv_h^K(\sqrt{-1}) = K(\sqrt{-1})v_h^K$. Since $v/v_h^K$ is henselian on $K^hv_h^K$ and $v_h^K$ is henselian already on $K$, we find that $v$ is henselian on $K(\sqrt{-1})$. So $K^h \subset K(\sqrt{-1})$, but as the latter is an extension of $K$ of degree 2 and $(K,v)$ is not henselian, we see that $K^h = K(\sqrt{-1})$. So there is one further extension $v'$ of $v$ from $K$ to $K^h$.

As an extension of the real closed field $Kv_h^K$, the field $K^hv_h^K$ has characteristic 0. Hence, $(K^h, v_h^K)$ is a defectless field. Since $K^hv_h^K$ is algebraically closed, also $(K^hv_h^K, v/v_h^K)$ is a defectless field. From **??** it follows that $(K^h, v)$ is a defectless field. The same is true for $(K^h, v')$ since $v'$ is conjugate to $v$ and $K(\sqrt{-1})|K$ is normal.

Assume that $v_h^K < w \leq v$. As $v/v_h^K$ has two independent extensions to $Lv_h^K = K^h v_h^K$, the extension of its non-trivial coarsening $w/v_h^K$ from $Kv_h^K$ to $K^h v_h^K$ can not be unique. By the fundamental inequality and since $[K^h v_h^K : Kv_h^K] = 2$, we find that $Kw = (Kv_h^K)(w/v_h^K) = (K^h v_h^K)(w/v_h^K) = K^h w$. The latter is algebraically closed since already $K^h v_h^K$ is (cf. **??**). This proves that $Kw$ is algebraically closed.

Since $Kv_h^K$ is formally real, also $K$ is formally real by **??**. Similarly, if $w \leq v_h^K$, then $Kv_h^K = (Kw)(v_h^K/w)$ shows that $Kw$ is formally real. If $v_c^K \leq w$, then by definition of the core valuation, $K^h w$ is separable-algebraically closed. Since $2 = [K^h : K] \geq [K^h w : Kw]$, it follows again by Artin-Schreier Theory that $Kw$ is real closed. Now assume that $w < v_c^K$. That means that $K^h w$ is not separable-algebraically closed. Since $[K^h w : Kw] \geq [K^h v_h^K : Kv_h^K] = 2$, it implies that $Kw$ is not real closed. □

**Corollary 14.48** *If $(L|K, v)$ is a finite extension such that $v$ is henselian on $L$ but not on $K$, then $v_c^K$ and $v_c^L$ are henselian on $K$.*

**Exercise 14.1** *Prove: If $K$ is a real closed field and $v$ is not compatible with the ordering on $K$, then the henselian part of $v$ is trivial. If $(K, v)$ is a henselian field and $(L|K, v)$ is a finite extension, such that $v_c^L < v_c^K$, then $K$ and $Kv$ are real closed, $v_c^K = v$ and $v_c^L$ is trivial.*

## 14.14 Maximal-by-finite and defectless-by-finite fields

For the decomposition theory of modules over valuation domains, the following notions are of importance. We will say that $(K, v)$ is **maximal-by-finite** if it is not maximal, but there is a finite extension $(L|K, v)$ such that $(L, v)$ is maximal. Further, $(K, v)$ is said to be **almost maximal** if the completion of $(K, v)$ is maximal. See [VA ] for the background and recent results. Since maximal fields are henselian, we can apply the foregoing theorem to the case where $(K, v)$ is not henselian. The following is proved in [VA ]:

**Theorem 14.49** *Suppose that $(K, v)$ is maximal-by-finite, but not henselian. Then $(K(\sqrt{-1}), v)$ is a maximal immediate extension of $(K, v)$. Further, $(K, v_h^K)$ is maximal. Consequently, $(K, v)$ is complete if $v_h^K$ is non-trivial. If $v_h^K$ is trivial, then $(K, v)$ is almost maximal.*

**Proof:** Let $(L|K, v)$ be a finite extension such that $(L, v)$ is maximal and $(K, v)$ is not henselian. In particular, $(L, v)$ is henselian and thus, $(K, v)$ is henselian-by-finite. By the foregoing theorem, $K(\sqrt{-1}), v)$ is the henselization of $(K, v)$ in $(L, v)$, and it is a defectless field. Hence, $(L|K(\sqrt{-1}), v)$ is vs-defectless and it follows from Corollary 6.10 that also $(K(\sqrt{-1}), v)$ is maximal. By Theorem **??**, it follows that also $(K(\sqrt{-1}), v_h^K)$ is maximal.

Again by the foregoing theorem, $Kv_h^K$ is real closed and thus of characteristic 0. Since $(K, v_h^K)$ is henselian, it follows that $(K(\sqrt{-1})|K, v_h^K)$ is vs-defectless. Hence again by Corollary 6.10, $(K, v_h^K)$ is maximal. In particular, $(K, v_h^K)$ is complete. If $v_h^K$ is non-trivial, then also $(K, v)$ is complete by virtue of Lemma **??**.

Now assume that $v_h^K$ is trivial. Then by Lemma 9.49, $v$ admits two independent extensions to $K^h$. From [BOU], Chapter VI, §8.2, Corollary 1 we infer that $(K, v)$ cannot be complete. Being maximal, $(K(\sqrt{-1}), v)$ is complete, and since $[K(\sqrt{-1}) : K] = 2$, it must be the completion of $(K, v)$, i.e., $(K, v)$ is almost complete. □

**Examples 14.50** Let us give examples of finite immediate extensions $(L|K,v)$, where $(L,v)$ is maximal and $(K,v)$ is not henselian. We take $K=\mathbb{R}$ and $L=\mathbb{C}$ and equip $\mathbb{C}$ with non-trivial maximal valuations. Their restrictions to $\mathbb{R}$ are not compatible with the order and thus not henselian (cf. Corollary 10.17). Consider the rational function field $\mathbb{Q}(x)$ with the following two valuations: let $v_1$ be the $x$-adic valuation, having residue field $\mathbb{Q}$, and $v_2$ the $p$-adic valuation induced by some embedding of $\mathbb{Q}(x)$ into $\mathbb{Q}_p$. Extend $v_1$ and $v_2$ to the algebraic closure of $\mathbb{Q}(x)$ and let $(K_1,v_1)$ and $(K_2,v_2)$ be maximal immediate extensions of the respective algebraically closed valued fields. By Corollary 6.45, they are algebraically closed too. We leave it to the reader to compute that their transcendence degree over $\mathbb{Q}$ is $2^{\aleph_0}$ in both cases (see also Lemma **??**). Hence as fields, they are isomorphic to $\mathbb{C}$. This isomorphism induces two maximal valuations on $\mathbb{C}$, one of them having residue characteristic 0, the other having mixed characteristic.

In these examples, the henselian parts of $v_1$ and $v_2$ are trivial because these are non-henselian valuations of rank 1. Examples with non-trivial henselian part can be obtained by constructing an arbitrary non-trivially valued maximal field $(K',v')$ with residue field $\mathbb{R}$. Then $(K'(\sqrt{-1})|K_1,v'\circ v_1)$ and $(K'(\sqrt{-1})|K_1,v'\circ v_1)$ are such examples with non-trivial henselian part $v'$. ♢

Now it remains to consider the case of a finite extension $(L|K,v)$ where $(L,v)$ is maximal and $(K,v)$ is henselian but not maximal. Then in view of Corollary 6.10, we find that $(L|K,v)$ cannot be defectless. Hence, $(K,v)$ is not a defectless field. On the other hand, $(L,v)$ is a defectless field, being maximal. So let us discuss the following more general situation (in which we may call $(K,v)$ **defectless-by-finite**):

$$\left.\begin{array}{rl}(L|K,v) & \text{a finite extension of henselian fields with}\\ (L,v) & \text{defectless and } (K,v) \text{ not defectless.}\end{array}\right\} \tag{14.22}$$

By Lemma 7.28 we have that $L^r=L.K^r$, hence $(L^r|K^r,v)$ is again finite. It follows from Lemma 13.4 that the absolute ramification field $(L,v)^r$ is defectless and that $(K,v)^r$ is not defectless. Thus, $(L^r|K^r,v)$ cannot be defectless. By Corollary 24.56, it is a tower of extensions of degree $p=\operatorname{charexp}\overline{K}$. Consequently, there exist intermediate fields $K^r\subset K'\subset L'\subset L^r$ such that $(L^r|L',v)$ is defectless and $(L',v)|(K',v)$ is immediate of degree $p$. This implies that $(L',v)$ is a defectless field and $(K',v)$ is not a defectless field. (If in addition $(L,v)$ is maximal, then so is $(L',v)$.) Hence, every extension (14.22) yields another such extension which in addition is immediate.

The next theorem shows that in (14.22), $(L,v)$ cannot be a tame field.

**Theorem 14.51** *If $(L|K,v)$ is a finite extension of henselian fields and $(L,v)$ is a tame field, then also $(K,v)$ is a tame field and $(L|K,v)$ is defectless.*

**Proof:** As shown above, $L^r|K^r$ is a finite extension. Since $(L,v)$ is assumed to be a tame field, we have that $L^r=\tilde{L}=\tilde{K}$. By Artin-Schreier Theory, either $\tilde{L}=K^r$ or $K^r$ is real closed. The latter is not possible: Since $v$ is henselian on $K^r$, it would have to be compatible with the ordering, and in particular, of residue characteristic 0. But then, $(K^r,v)$ would be a tame field and thus algebraically closed. We conclude that $K^r$ is algebraically closed, showing that $(K,v)$ is a tame field. By definition, it follows that the finite extension $(L|K,v)$ is defectless. □

If $(K,v)$ is a henselian perfect field of finite characteristic or a henselian Kaplansky field, then the same is true for every algebraic extension $(L,v)$. Hence if $(L,v)$ is algebraically maximal or even defectless, then it follows from Corollary 13.38 resp. Lemma 13.53 that $(L,v)$ is a tame field. So the above theorem yields the following

**Corollary 14.52** *Let $(K,v)$ be a henselian perfect field of finite characteristic or a henselian Kaplansky field. If there exists a finite extension $(L|K,v)$ such that $(L,v)$ is algebraically maximal, then $(K,v)$ is a tame field.*

Note that if $L|K$ is a finite extension, then $L$ is perfect if and only if $K$ is. If in addition $\mathrm{char}\overline{L} > 0$, then $(L,v)$ is a Kaplansky field if and only if $(K,v)$ is; this follows from our definition of Kaplansky fields. Indeed, $vL|vK$ being a finite extension, $vL$ is $p$-divisible if and only if $vK$ is. Further, $\overline{L}|\overline{K}$ being a finite extension, $\overline{L}$ is perfect if and only if $\overline{K}$ is. For condition (KAP3), one uses the fact that the $p$-Sylow subgroups of the absolute Galois group of the field $\overline{K}$ of characteristic $p$ are $p$-free (cf. [SER], page II–5, cor. 1). Hence, they are either trivial or infinite. Consequently, $\overline{L}$ admits a finite extension of degree divisible by $p = \mathrm{char}\,\overline{L}$ if and only if $\overline{K}$ does.

**Corollary 14.53** *In the situation of (14.22), $(K,v)$ as well as $(L,v)$ cannot be a tame field, a Kaplansky field or a perfect field of positive characteristic.*

Having treated the case of perfect fields of positive characteristic, we turn to fields with non-trivial degree of inseparability $[K:K^p]$.

**Example 14.54** There is a discretely valued field $(K,v)$ of characteristic $p>0$ admitting a finite immediate extension $(L,v)$ which is complete and hence maximally valued. To construct $(K,v)$, we take a field $k$ of characteristic $p>0$ and infinite degree of inseparability $[k:k^p]$, e.g. $k=\mathbb{F}_p(t_i|i\in\mathbb{N})$ where the $t_i$ are algebraically independent elements over $\mathbb{F}_p$. Taking $t$ to be another transcendental element over $k$ we consider the power series fields $k((t))$ and $k^p((t)) = k^p((t^p))(t) = k((t))^p(t)$. Since $[k:k^p]$ is not finite, we have that $k((t))|k^p((t)).k$ is a non-trivial immediate purely inseparable algebraic extension. In fact, a power series in $k((t))$ is an element of $k^p((t)).k$ if and only if its coefficients generate a finite extension of $k^p$. Since $k^p((t)).k$ contains $k((t))^p$, this extension is generated by a set $X=\{x_i \mid i\in I\}\subset k((t))$ such that $x_i^p\in k^p((t)).k$ for every $i\in I$. Assuming this set to be minimal, or in other words, the $x_i$ to be $p$-independent over $k^p((t)).k$, we pick some element $x\in X$ and put $K := k^p((t)).k(X\setminus\{x\})$. Then $k((t))|K$ is a purely inseparable extension of degree $p$. Moreover, it is an immediate extension; in fact, $k((t))$ is the completion of $K$. ♢

Let us note that one can derive two other examples where $L|K$ is separable of degree $p$, one where $K,L$ are fields of characteristic $p$ and one where $K,L$ are characteristic 0 with residue characteristic $p$. To do so, choose $(L|K,w)$ to be any extension of degree $p$ of defectless fields whose corresponding residue field extension is just the extension of the above example; here, $L|K$ may always be chosen to be separable. If $v$ is the composition of $w$ with the valuation on the residue fields given by the above example, we get that $(L,v)$ is defectless and $(L|K,v)$ is an immediate extension of degree $p$. However, it is not possible to get $(L,v)$ to be complete while $(K,v)$ is not, and moreover, the constructed valuations are of rank at least 2. This leads to the following

**Open Problem 14.1** Does there exist an extension (14.22) with a rank 1 discrete valuation $v$ of mixed characteristic?

In the above example, both $K$ and $L$ have infinite degree of inseparability. This is indeed necessary, in view of Corollary **??**, which tells us that if $(L|K, v)$ is a finite extension of valued fields of finite degree of inseparability, then $(L, v)$ is inseparably defectless if and only if $(K, v)$ is. Hence if $L|K$ is purely inseparable in (14.22), then $(K, v)$ is not inseparably defectless and thus, neither $K$ nor $L$ have finite degree of inseparability.

**Open Problem 14.2** Does there exist a (separable) extension (14.22) where $(K, v)$ is inseparably defectless and of positive characteristic? Can we obtain in addition that $(L, v)$ is maximal?

**Exercise 14.2** *Prove: If there exists an extension (14.22) with* $(K, v)$ *inseparably defectless of positive characteristic, then there also exists an extension (14.22) which is an independent immediate Artin-Schreier extension.*