Chapter 2

Valued abelian groups

2.1 Definition and basic properties

Let G be an abelian group and $\Gamma\infty$ as defined in section 1.1. A map

$$v : \; G \ni x \mapsto vx \in \Gamma\infty$$

from G onto $\Gamma\infty$ is called a **valuation of** G if for all $x, y \in G$,

(V 0) $\quad vx = \infty \Longleftrightarrow x = 0\,,$

(VT) $\quad v(x-y) \geq \min\{vx, vy\}\,.$

Axiom (VT) can be viewed as the **ultrametric triangle inequality** for valued abelian groups. If we set $x = 0$ in (VT) and then replace y once by x and once by $-x$, we obtain $v(-x) \geq vx \geq v(-x)$, whence we obtain the "symmetry":

(VS) $\quad v(-x) = vx$

for all $x \in G$. Using this and axiom (VT), the following axiom scheme can be deduced by induction:

(VZ) $\quad v(nx) \geq vx \qquad\qquad (0 \neq n \in \mathbb{Z})\,.$

We will frequently omit the brackets and write "vnx". We will always omit brackets if the range of the symbol "v" is sufficiently clear.

If v is a valuation on the abelian group G, then we will write (G, v) and call it a **valued abelian group**, or shorter a **valued group**. Also, we will use $\mathbf{G}$ to denote a valued group, if the valuation was already specified before and there is no danger of confusion. We set $vG := \Gamma$ and call it the **value set of** G. Note that by our definition, vG does not contain the element ∞; by axiom (V 0), $vG = \{va \mid 0 \neq a \in G\}$. Following our above definition, we write $vG\infty$ for the ordered set $\{va \mid a \in G\} = vG \cup \{\infty\}$.

For torsion elements, we can improve (VZ) as follows:

Lemma 2.1 *Let (G, v) be a valued abelian group with a torsion element a. Let $m \in \mathbb{N}$ be such that $ma = 0$ and assume $n \in \mathbb{N}$ to be prime to m. Then $vna = va$.*

Proof: Since n is prime to m, we can find integers r, s such that $1 = rn + sm$. It follows that $va = v(rn + sm)a = v(rna + sma) \geq vrna \geq vna \geq va$. So equality must hold everywhere, and we obtain that $vna = va$. $\square$

Two valuations $v : G \to \Gamma\infty$ and $v' : G \to \Gamma'\infty$ (resp. ultrametrics $u : X \times X \to \Gamma\infty$ and $u' : X \times X \to \Gamma'\infty$) are called **equivalent** if there is an order isomorphism $\rho : \Gamma \to \Gamma'$ such that $\rho \circ v = v'$ (resp. $\rho \circ u = u'$). See our remark in section 0.1. We will say that v' is **coarser than** v or a **coarsening of** v if $v'a > v'b$ implies that $va > vb$, for all $a, b \in G$. In this case, v is said to be **finer than** v'.

Lemma 2.2 *Let G be an abelian group. If $v : G \to \Gamma\infty$ is a valuation on G, then $u(x, y) := v(x - y)$ is an ultrametric on G, satisfying $u(x + z, y + z) = u(x, y)$ for all $x, y, z \in G$. Conversely, if $u : G \times G \to \Gamma\infty$ is an ultrametric on G satisfying*

(UMG) $\quad u(x - y, 0) = u(x, y)$

for all $x, y \in G$, then $vx := u(x, 0)$ is a valuation on G.

Proof: Let v be a valuation on G and set $u(x, y) := v(x - y)$. Then (UM 0) follows immediately from (V 0), and (UMS) follows from (VS). Further, (VT) and (VS) give

$$v(x - y) = v((x - z) - (y - z)) \geq \min\{v(x - z), v(y - z)\} = \min\{v(x - z), v(z - y)\}\ ,$$

which translates to (UMT). Finally, $u(x + z, y + z) = u(x, y)$ is a trivial consequence of $u(x, y) = v(x - y)$.

For the converse, let u be an ultrametric on G satisfying $u(x - y, 0) = u(x, y)$, and set $vx := u(x, 0)$. Then (V 0) follows immediately from (UM 0). Further, $v(x - y) = u(x - y, 0) = u(x, y) \geq \min\{u(x, 0), u(0, y)\} = \min\{u(x, 0), u(y, 0)\} = \min\{vx, vy\}$, which is (VT). $\square$

One may call the axiom (UMG) the **compatibility of the ultrametric with the group structure**. As a consequence of this lemma and Lemma 1.5, where we set $z = 0$, we obtain:

(V≠) $\quad vx \neq vy \Longrightarrow v(x - y) = \min\{vx, vy\}\ ,$

(V=) $\quad v(x - y) > \min\{vx, vy\} \Longrightarrow vx = vy\ .$

From (V≠), replacing y by $-y$ and using (VS), we deduce by induction:

(VM) $\quad v(\sum_{1\leq i\leq n} x_i) = \min_{1\leq i\leq n} vx_i$ if all nonzero x_i have distinct values.

Axiom (UMG) can be equivalently interpreted as to say that addition by an arbitrary element of G is a automorphism of the ultrametric space (G, u); that is, it preserves the ultrametric: $u(x, y) = u(x + z, y + z)$.

Furthermore, (UMG) implies homogeneity (UMH). Indeed, since addition by an arbitrary element of G preserves the ultrametric, it shifts maximal equilateral polygons onto maximal equilateral polygons of the same distance. If $\alpha \in vG$ and x_1 and x_2 are members of two maximal equilateral polygons which reduce to different α-polygons, then addition of $x_2 - x_1$ shifts the first onto the second polygon, showing that the respective α-polygons are of equal cardinality. Consequently, in view of the above lemma, *every valuation of an abelian group has the homogeneity property* (UMH).

A subset $\mathcal{S}$ of a valued group $\mathbf{G} = (G, v)$ will be called **v-convex** if it satisfies

(vCONV) $\quad x, y \in \mathcal{S} \wedge v(x-y) \leq v(x-z) \implies z \in \mathcal{S}$.

Viewing $\mathbf{G}$ as an ultrametric space as we have done above, we see that all balls in $\mathbf{G}$ are v-convex. A subset $\mathcal{S}$ of $\mathbf{G}$ is v-convex if and only if for all $a, b \in \mathcal{S}$, the ball $B_{v(a-b)}(a)$ is contained in $\mathcal{S}$. Given a subgroup H of G, the reader may show that H is a **v-convex subgroup of G** if H is the preimage $v^{-1}(S)$ of a final segment S of $vG\infty$. Let γ be an initial segment of $vG\infty$ (recall that we identify an element γ of vG with the initial segment having γ as its maximal element, cf. section 1.1). We will consider

$$\mathcal{O}_{\mathbf{G}}^{\gamma} := \{a \in G \mid va \geq \gamma\} \text{ and } \mathcal{M}_{\mathbf{G}}^{\gamma} := \{a \in G \mid va > \gamma\} . \tag{2.1}$$

which are v-convex subgroups of G, except for $\mathcal{M}_{\mathbf{G}}^{\infty} = \emptyset$. (The definition works equally well for any subset of vG in the place of γ.) If γ does not represent an element of vG, then it contains no maximal value, implying that every element of $\mathcal{O}_{\mathbf{G}}^{\gamma}$ has value $> \gamma$ and consequently, $\mathcal{O}_{\mathbf{G}}^{\gamma} = \mathcal{M}_{\mathbf{G}}^{\gamma}$. If γ is an element of $vG\infty$, then we can take γ' to be the initial segment $\{\alpha \in vG \mid \alpha < \gamma\}$ and obtain $\mathcal{O}_{\mathbf{G}}^{\gamma} = \mathcal{M}_{\mathbf{G}}^{\gamma'}$. More generally, this works for every v-convex subgroup (H, v): if we take $\gamma' = vG \setminus vH$, then we get $H = \mathcal{M}_{\mathbf{G}}^{\gamma'}$. We have thus shown that all v-convex subgroups can be represented in the form $\mathcal{M}_{\mathbf{G}}^{\gamma}$. Nevertheless, also the notation $\mathcal{O}_{\mathbf{G}}^{\gamma}$ will be useful. If $a \in G$, then we set $\mathcal{O}_{\mathbf{G}}(a) := \mathcal{O}_{\mathbf{G}}^{va}$ and call it a **principal subgroup**. Similarly, we set $\mathcal{M}_{\mathbf{G}}(a) := \mathcal{M}_{\mathbf{G}}^{va}$. In particular, $\mathcal{O}_{\mathbf{G}}(0) = \{0\}$ and $\mathcal{M}_{\mathbf{G}}(0) = \emptyset$.

Lemma 2.3 *The set v-Conv$(\mathbf{G})$ of all v-convex subgroups of $\mathbf{G}$ is ordered by inclusion. The map $\gamma \mapsto \mathcal{M}_{\mathbf{G}}^{\gamma}$ is an order reversing bijection between the ordered set of initial segments of vG and v-Conv$(\mathbf{G})$, with inverse $H \mapsto vG \setminus vH$. In particular, it induces an order reversing bijection between initial segments of the form $\{\alpha \in vG \mid \alpha < \gamma\}$ with $\gamma \in vG$ and the principal subgroups of $\mathbf{G}$.*

The proof is straightforward. Just recall that for initial segments γ_1, γ_2 of T, $\gamma_1 < \gamma_2$ implies that γ_2 contains an element α such that $\alpha > \gamma_1$ and thus, $\mathcal{M}_{\mathbf{G}}^{\gamma_1}$ contains all elements of value α while $\mathcal{M}_{\mathbf{G}}^{\gamma_2}$ does not. This shows the injectivity of the map.

The balls with center 0 in $\mathbf{G}$ appear as v-convex subgroups: For $\gamma \in vG$, $B_{\gamma}(0) = \mathcal{O}_{\mathbf{G}}^{\gamma}$ and $B^{\gamma}(0) = \mathcal{M}_{\mathbf{G}}^{\gamma}$. By virtue of addition, we actually know all balls: For every $a \in G$,

$$B_{\gamma}(a) = a + \mathcal{O}_{\mathbf{G}}^{\gamma} \text{ and } B^{\gamma}(a) = a + \mathcal{M}_{\mathbf{G}}^{\gamma}$$

(and $B_{\infty}(a) = a + \mathcal{O}_{\mathbf{G}}^{\infty} = \{a\}$), which are cosets modulo the respective v-convex subgroups. Indeed, $b \in B_{\gamma}(a) \Leftrightarrow u(b, a) = u(a, b) \geq \gamma \Leftrightarrow b - a \in \mathcal{O}_{\mathbf{G}}^{\gamma} \Leftrightarrow b \in a + \mathcal{O}_{\mathbf{G}}^{\gamma}$, and similarly for the balls $B^{\gamma}(a)$. Further, the equivalence relation $\sim_{\gamma}$ now has an easy interpretation: $a \sim_{\gamma} b \Leftrightarrow a - b \in \mathcal{M}_{\mathbf{G}}^{\gamma}$.

Lemma 2.4 *Let γ be an initial segment of vG. The group $G/\sim_{\gamma} = G/\mathcal{M}_{\mathbf{G}}^{\gamma}$ carries a valuation $v/\gamma : G/\sim_{\gamma} \to \gamma$ which for all $a \in G$ satisfies*

$$va \in \gamma \implies v/\gamma\,(a/\sim_{\gamma}) = va . \tag{2.2}$$

If $u(a, b) = v(a - b)$ is the ultrametric associated with v, then $v/\gamma\, c = u/\sim_{\gamma}(c, 0)$ is the valuation associated with the ultrametric $u/\sim_{\gamma}$.

In particular, the value set of $(\mathcal{O}_{\mathbf{G}}^{\gamma}/\mathcal{M}_{\mathbf{G}}^{\gamma}, v/\gamma)$ consists just of the one element γ if $\gamma \in vG$; the group is trivial if $\gamma \notin vG$.

Proof: Let v/γ on $G/\sim_\gamma$ be given as the valuation associated with $u/\sim_\gamma$. We know that u satisfies axiom (UMG). Now $u/\sim_\gamma(a/\sim_\gamma - b/\sim_\gamma, 0) = u/\sim_\gamma((a-b)/\sim_\gamma, 0) = u(a-b,0) = u(a,b) = u/\sim_\gamma(a/\sim_\gamma, b/\sim_\gamma)$, showing that also $u/\sim_\gamma$ satisfies axiom (UMG). Hence by Lemma 2.2, v/γ is a valuation. Assertion (2.2) follows directly from our construction.

Now consider the subgroup $\mathcal{O}_{\mathbf{G}}^\gamma$ of G. Every element of it has value $\geq \gamma$. If γ does not represent a value in vG, then it contains no maximal value, implying that every element of $\mathcal{O}_{\mathbf{G}}^\gamma$ has value $> \gamma$; in this case, $\mathcal{O}_{\mathbf{G}}^\gamma = \mathcal{M}_{\mathbf{G}}^\gamma$ and the factor group is trivial. If on the other hand, $\gamma \in vG$, then the elements in $\mathcal{O}_{\mathbf{G}}^\gamma$ which are not equivalent to zero modulo $\sim_\gamma$ are precisely those of value γ. By the first part of the lemma, this yields that γ is the only element in $v/\gamma\,(\mathcal{O}_{\mathbf{G}}^\gamma/\mathcal{M}_{\mathbf{G}}^\gamma)$. □

2.2 The skeleton

Let us have a look at the polygons in $\mathbf{G}$. By virtue of addition, we can shift them around without loosing the property of being equilateral (since $v((a+c)-(b+c)) = v(a-b)$). In particular, if $\alpha \in vG$ and P_α is an equilateral polygon of distance α, then for every $a \in P_\alpha$, the set $P_\alpha - a$ is an equilateral polygon of distance α whose nonzero elements all have value α. Moreover, $P_\alpha - a$ is maximal if and only if P_α is. That is, every maximal equilateral polygon of distance α can be shifted onto some maximal equilateral polygon P_α^0 of distance α which is entirely contained in $\mathcal{O}_{\mathbf{G}}^\alpha$. Its reduction $\overline{P_\alpha^0}$ modulo $\sim_\alpha$ (that is, modulo $\mathcal{M}_{\mathbf{G}}^\alpha$) is a maximal equilateral polygon in $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$. But the last lemma shows that the value set of this group contains but one element, which yields that every maximal equilateral polygon in the group is equal to the group itself. It follows that P_α^0 is a set of representatives for the factor group $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$, and that any maximal equilateral polygon of distance α containing x is of the form $x+\mathcal{R}$ where $\mathcal{R}$ is some set of representatives. We see that in our search for "horizontal" invariants, we can now replace the α-polygons by the factor group $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$. This leads to the following definitions.

Let $\alpha \in vG$. The α**–component of** (G,v) is the factor group

$$\mathrm{C}^\alpha\mathbf{G} := \mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha\ ,$$

which again is a non-trivial abelian group. Further, we define $\mathrm{C}^\infty\mathbf{G} := \{0\}$. For $a \in G$, the va-component $\mathrm{C}_{\mathbf{G}}(a) := \mathrm{C}^{va}\mathbf{G}$ will be called the **component of** a. The natural epimorphism $\mathcal{O}_{\mathbf{G}}^\alpha \to \mathrm{C}^\alpha\mathbf{G}$ will be denoted by $\mathrm{co}_{\,\alpha}$ and called the α**-coefficient map of G**. If γ is an initial segment of vG not representing an element of vG, then $\mathcal{O}_{\mathbf{G}}^\gamma/\mathcal{M}_{\mathbf{G}}^\gamma$ is the trivial group and $\mathrm{co}_{\,\gamma}$ the trivial epimorphism. A direct consequence of our definition is:

$$\text{if}\ \ va = vb = \alpha < \infty\,,\ \text{then}\quad \mathrm{co}_{\,\alpha}a = \mathrm{co}_{\,\alpha}b \iff v(b-a) > \alpha\ . \tag{2.3}$$

For simplicity, we set $\mathrm{co}\,a := \mathrm{co}_{\,va}a$ and call co the **coefficient map of G** $= (G,v)$. It satisfies

(CO<) $\qquad vx < vy \implies \mathrm{co}\,(x+y) = \mathrm{co}\,x\,,$

(CO+) $\qquad vx = vy = v(x+y) \implies \mathrm{co}\,(x+y) = \mathrm{co}\,x + \mathrm{co}\,y\,.$

The pair $(va, \mathrm{co}\, a)$ will be called the **bone of** a. In view of (2.3) and (V=), we have the following criterion for the equality of bones:

$$b \neq 0 \wedge (va, \mathrm{co}\, a) = (vb, \mathrm{co}\, b) \iff v(b-a) > vb \,. \tag{2.4}$$

The collection

$$\mathrm{sk}\, \mathbf{G} := \bigcup_{\alpha \in vG\infty} \{\alpha\} \times \mathrm{C}^{\alpha}\mathbf{G}$$

of all bones of $\mathbf{G}$ is called the **skeleton of G** and is also denoted by $\mathrm{sk}\,(G, v)$. We use $\mathrm{sk}\, G$ if there is no danger of confusion. In general, $\bigcup_{\gamma\in\Gamma}\{\gamma\} \times C_\gamma$ will be called a **skeleton** if Γ is an ordered set and all C_γ are abelian groups. We write shortly $\mathrm{sk}_{\,\gamma\in\Gamma} B_\gamma$ for such a skeleton. In the literature, the reader will also find notations like B_γ for the components and $[\Gamma\,;\, B_\gamma\,,\, \gamma \in \Gamma]$ for the skeleton. We will not use "B_γ" in order not to get confused with balls.

The sum of two bones (γ, ζ) and (γ', ζ') is defined whenever $\gamma' = \gamma$ and is then equal to $(\gamma, \zeta + \zeta')$, according to (CO+). We will frequently identify $\{\gamma\} \times \mathrm{C}^{\gamma}\mathbf{G}$ with the group $\mathrm{C}^{\gamma}\mathbf{G}$ and call it the **γ-component of the skeleton**. Note that the ∞–component is the trivial group, it consists of the single element $(\infty, 0)$.

Let $\mathcal{S} = \mathrm{sk}_{\,\gamma\in\Gamma} C_\gamma$ and $\mathcal{S}' = \mathrm{sk}_{\,\gamma\in\Gamma'} B'_\gamma$ be two skeletons. If $\Gamma \subset \Gamma'$ and for every $\gamma \in \Gamma$, the γ–component C_γ is a subgroup of the γ–component B'_γ, then we say that $\mathcal{S}$ is a **subskeleton of** $\mathcal{S}'$ and write $\mathcal{S} \subset \mathcal{S}'$. If $\gamma \in \Gamma' \setminus \Gamma$ then we define the γ-component of $\mathcal{S}$ to be the trivial group 0. In this way, we obtain that C_γ is a subgroup of B'_γ for every $\gamma \in \Gamma'$. An **isomorphism of** $\mathcal{S}$ onto $\mathcal{S}'$ is a system $\sigma = (\sigma_\Gamma, \{\sigma_\gamma \mid \gamma \in \Gamma\})$ such that $\sigma_\Gamma : \Gamma \to \Gamma'$ is an isomorphism of ordered sets and $\sigma_\gamma : C_\gamma \to B'_{\sigma_\Gamma \gamma}$ is an isomorphism of abelian groups for every $\gamma \in \Gamma$. An **embedding of** $\mathcal{S}$ in $\mathcal{S}'$ is an isomorphism of $\mathcal{S}$ onto a subskeleton of $\mathcal{S}'$.

At this point, we should discuss the possible definitions for what we will call an **embedding** resp. **isomorphism of valued groups**. The meaning of "**preserve the valuation**" actually depends on the language that we use for valued groups (cf. 20.15). Let us use a binary predicate which expresses the relation $vx < vy$. Then $\iota : (G, v) \to (G', v')$ is an embedding of valued abelian groups if it is an embedding of G in G' and satisfies

$$vx < vy \iff v'\iota x < v'\iota y \,.$$

The reader may show that such an embedding induces an embedding $\sigma : \mathrm{sk}\, \mathbf{G} \to \mathrm{sk}\, \mathbf{G}'$ satisfying

$$(v\iota a, \mathrm{co}\, \iota a) = \sigma(va, \mathrm{co}\, a) \quad \text{for all } a \in G \,.$$

If this holds, then we say that ι **preserves** σ (or **induces** σ). If ι is an isomorphism, then so is σ. An embedding (resp. isomorphism) of valued abelian groups is at the same time an embedding (resp. isomorphism) of the underlying ultrametric spaces.

On the other hand, we could take v as a map. In this case, we would work with embeddings of structures of the form (G, vG, v) or, if we add the coefficient map, of the form $(G, \mathrm{sk}\, G, (v, \mathrm{co}\,))$. But our above consideration shows that this is only a formal difference; in all cases, we obtain the same class of group embeddings.

The following lemma determines the skeleton of a v-convex subgroup and the corresponding valued factor group:

Lemma 2.5 *Let* $\mathbf{H}$ *be a* v*-convex subgroup of* $\mathbf{G}$. *Then* $\mathrm{sk}\,\mathbf{H} = \mathrm{sk}_{\alpha\in vH\infty}\mathrm{C}^\alpha\mathbf{G}$. *Up to isomorphism, the skeleton of the factor group* G/H *endowed with the valuation induced by* v*, is the subskeleton* $\mathrm{sk}_{\alpha\in(vG\setminus vH)\infty}\mathrm{C}^\alpha\mathbf{G}$ *of* $\mathrm{sk}\,\mathbf{G}$.

Proof: Let us first consider $\mathbf{H}$. For every $\alpha\in vH$, we have $\mathcal{O}^\alpha_{\mathbf{G}} = \mathcal{O}^\alpha_{\mathbf{H}}$ and $\mathcal{M}^\alpha_{\mathbf{G}} = \mathcal{M}^\alpha_{\mathbf{H}}$ and consequently, $\mathrm{C}^\alpha\mathbf{G} = \mathrm{C}^\alpha\mathbf{H}$. This proves our first assertion.

Now we compute the skeleton of G/H. By Lemma 2.3, we can write $H = \mathcal{M}^\gamma_{\mathbf{G}}$ for some initial segment γ of vG. Then the valuation induced by v on G/H is v/γ. We know already from Lemma 2.4 that $v/\gamma\,(G/H) = \gamma = vG\setminus vH$. To determine the components of $(G/H, v/\gamma)$, let $\alpha\in vG\setminus vH = \gamma$. Then $\mathcal{M}^\gamma_{\mathbf{G}}\subset\mathcal{M}^\alpha_{\mathbf{G}}\subset\mathcal{O}^\alpha_{\mathbf{G}}$. Since the α-component of $(G/H, v/\gamma)$ is just $(\mathcal{O}^\alpha_{\mathbf{G}}/\mathcal{M}^\gamma_{\mathbf{G}})/\,(\mathcal{M}^\alpha_{\mathbf{G}}/\mathcal{M}^\gamma_{\mathbf{G}})$, it is thus equal to $\mathrm{C}^\alpha\mathbf{G} = \mathcal{O}^\alpha_{\mathbf{G}}/\mathcal{M}^\alpha_{\mathbf{G}}$ by the isomorphism theorem. □

The valued factor group $(G/H, v/\gamma)$ may also be denoted by $\mathbf{G}/\mathbf{H}$. Let H, H' be subgroups of (the additively written abelian group) G. Then H' is called a **group complement** for H in G if G is the direct sum of H and H', or equivalently, if $G = H + H'$ and $H\cap H' = \{0\}$. The reader may show the following: *If the* v*-convex subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *admits a group complement* H' *in* G*, then with the restriction of the valuation from* G *to* H'*, the latter is isomorphic to* $\mathbf{G}/\mathbf{H}$. Given two valued groups (H,v) and (H',v'), we let $(H',v')\amalg(H,v)$ be the direct sum of H' and H with the uniquely determined valuation w having value set $v'H' + vH$ and satisfying $wa' = v'a' < va = wa$ for all $a'\in H'$ and $a\in H$. Then (H,v) is a v-convex subgroup of $(H',v')\amalg(H,v)$. The α-component of $(H',v')\amalg(H,v)$ is that of (H',v') if $\alpha\in v'H'$, and that of (H,v) if $\alpha\in vH$. We leave it to the reader to show: *If the* v*-convex subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *admits a group complement* H' *in* G*, then with the restriction of the valuation* v *from* G *to* H'*, we have* $\mathbf{G}\cong(H',v)\amalg(H,v)$.

Let $\mathbf{H}\subset\mathbf{G}$ be an extension of valued groups. Then $vH\subset vG$ and for every $\alpha\in vH$, we have $\mathcal{O}^\alpha_{\mathbf{H}}\subset\mathcal{O}^\alpha_{\mathbf{G}}$ and $\mathcal{M}^\alpha_{\mathbf{G}}\cap H = \mathcal{M}^\alpha_{\mathbf{H}}$, showing that we have a canonical embedding

$$\mathrm{C}^\alpha\mathbf{H}\longrightarrow\mathrm{C}^\alpha\mathbf{G}\,,\qquad a+\mathcal{M}^\alpha_{\mathbf{H}}\mapsto a+\mathcal{M}^\alpha_{\mathbf{G}}\tag{2.5}$$

of groups. We have proved that $\mathrm{sk}\,\mathbf{H}\subset\mathrm{sk}\,\mathbf{G}$. Observe that because of $vH\subset vG$ we can form $\mathcal{O}^\alpha_{\mathbf{H}}$ and $\mathcal{M}^\alpha_{\mathbf{H}}$ also for all $\alpha\in vG$, and if $\alpha\in vG\setminus vH$ then $\mathcal{O}^\alpha_{\mathbf{H}} = \mathcal{M}^\alpha_{\mathbf{H}}$, which gives that $\mathrm{C}^\alpha\mathbf{H} = 0$, in accordance with our convention just introduced for abstract skeletons.

Given an arbitrary extension $\mathbf{H}\subset\mathbf{G}$ of valued groups, the question arises whether we can give an estimate for $(G:H)$ in terms of the indices of the corresponding components.

Lemma 2.6 *Let* $\mathbf{H}\subset\mathbf{G}$ *be an extension of valued groups. Reading the indices as finite or infinite cardinals,*

$$(G:H)\,\geq\,\sum_{\alpha\in vG}(\mathrm{C}^\alpha\mathbf{G}:\mathrm{C}^\alpha\mathbf{H})\,.$$

Proof: For every $\alpha\in vG$, take representatives for the different cosets of $\mathrm{C}^\alpha\mathbf{G}$ modulo $\mathrm{C}^\alpha\mathbf{H}$. We have to show that the representatives so obtained all belong to different cosets of G modulo H. It suffices to show the following: if $a,b\in G$ satisfy $\mathrm{co}\,a\notin\mathrm{C}^{va}\mathbf{H}$ and $\mathrm{co}\,b\notin\mathrm{C}^{vb}\mathbf{H}$ and $va = vb\Rightarrow\mathrm{co}_{\,va}a-\mathrm{co}_{\,va}b\notin\mathrm{C}^{va}\mathbf{H}$, then $a-b\notin H$. But this is almost trivial: if $va\neq vb$, say $va<vb$, then $(v(a-b),\mathrm{co}\,(a-b)) = (va,\mathrm{co}\,a)$ and $a-b\notin H$ because of $\mathrm{co}\,a\notin\mathrm{C}^{va}\mathbf{H}$; if $va = vb$, then $a-b\notin H$ because of $\mathrm{co}\,(a-b) = \mathrm{co}_{\,va}a-\mathrm{co}_{\,va}b\notin\mathrm{C}^{va}\mathbf{H}$. □

An extension $H \subset G$ of abelian groups will be called **algebraic** if G/H is a torsion group. The extension is called **finite** if the index $(G : H)$ is finite. A finite extension is algebraic. An extension is algebraic if and only if it is the union of finite extensions. By the foregoing lemma, all extensions $\mathrm{C}^\alpha\mathbf{H} \subset \mathrm{C}^\alpha\mathbf{G}$ are finite if $(G : H)$ is finite. If G is the union of finite extensions of H then for every $\alpha \in vG$, the component $\mathrm{C}^\alpha\mathbf{G}$ is the union of finite extensions of $\mathrm{C}^\alpha\mathbf{H}$. We have thus proved:

Corollary 2.7 *If* $\mathbf{H} \subset \mathbf{G}$ *is an algebraic extension of valued groups, then* $\mathrm{C}^\alpha\mathbf{H} \subset \mathrm{C}^\alpha\mathbf{G}$ *is an algebraic extension of abelian groups, for every* $\alpha \in vG$.

Exercise 2.1 *Discuss the possible definitions for "homomorphism of valued groups".*

2.3 Immediate extensions and maximal groups

The extension $\mathbf{H} \subset \mathbf{G}$ is called **immediate** if for every $\alpha \in vG$, the canonical embedding (2.5) is onto, that is, if the induced embedding $\mathrm{sk}\,\mathbf{H} \subset \mathrm{sk}\,\mathbf{G}$ is actually an isomorphism. Loosely speaking, "immediate" means that the skeletons are equal. At this point, we have to distinguish between "isomorphic" and "equal"; non-immediate extensions may still have isomorphic skeletons, the isomorphism *not* being the induced embedding (see Exercise 2.9). If $\mathbf{H} \subset \mathbf{G}$ is an immediate extension, then for every $\alpha \in vG$, the component $\mathrm{C}^\alpha\mathbf{H} = \mathrm{C}^\alpha\mathbf{G}$, which says that $\alpha \in vH$. Hence, $vH = vG$ for every immediate extension $\mathbf{H} \subset \mathbf{G}$. As a direct consequence of our definition, we obtain:

Lemma 2.8 *Let* $\mathbf{G} \subset \mathbf{G}_1 \subset \mathbf{H}$ *be valued groups. Then* $\mathbf{G} \subset \mathbf{H}$ *is immediate if and only if* $\mathbf{G} \subset \mathbf{G}_1$ *and* $\mathbf{G}_1 \subset \mathbf{H}$ *are.*

Observe that this result is stronger than the corresponding result for ultrametric spaces. (It is actually axiom (UMG) that gives this stronger result). Since the above definition of immediate extensions is quite different from that of immediate extensions of ultrametric spaces, we will now show the relation between them. Let $b \in G$. Assume that there is some $a \in H$ such that $v(b-a) > vb$. Then by (2.4), the bone of b is equal to the bone of a and thus lies in $\mathrm{sk}\,\mathbf{H}$. Conversely, if the latter holds, then there is some $a \in H$ such that a and b have equal bones. Again by (2.4), this yields that $v(b-a) > vb$. We have shown:

$$\forall b \in G \setminus \{0\} : \quad (vb, \mathrm{co}\, b) \in \mathrm{sk}\,\mathbf{H} \iff \exists a \in H : \ v(b-a) > vb \,. \tag{2.6}$$

This gives the following important characterization of immediate extensions:

Lemma 2.9 *The extension* $\mathbf{H} \subset \mathbf{G}$ *is immediate if and only if for every* $b \in G \setminus \{0\}$ *there is some* $a \in H$ *such that* $v(b-a) > vb$.

A valued group is called **maximal** or **maximally valued** if it does not admit a proper immediate extension. It is called a **spherically complete group** if its underlying ultrametric space is spherically complete.

Corollary 2.10 *The extension* $\mathbf{H} \subset \mathbf{G}$ *is immediate if and only if the underlying extension of ultrametric spaces is immediate. Consequently, a spherically complete valued group is maximal.*

Proof: Assume $y \in G$ and $x' \in H$ and set $b := y - x'$. If $a \in H$ is chosen such that $v(b-a) > vb$, then for $x := x' + a$ we have that $v(y-x) = v(b-a) > vb = v(y-x')$. For the converse, given $b \in G$, then set $y := b$ and $x' = 0$. If $x \in H$ is chosen such that $v(y-x) > v(y-x')$, then for $a := x$ we have that $v(b-a) = v(y-x) > v(y-x') = vb$.

From Lemma 1.19 it follows that a spherically complete ultrametric space does not admit proper immediate extensions. We conclude that a spherically complete valued group does not admit proper immediate valued group extensions and is thus a maximal group.

□

By an **approximation type over G** we mean an approximation type over the underlying ultrametric space. We know from section 2.1 that in the underlying ultrametric space of a valued group, the balls are cosets with respect to suitable v-convex subgroups. We find that for an approximation type $\mathbf{A}$ over $\mathbf{G}$, every nonempty $\mathbf{A}_\alpha$ can just be presented as the coset $\mathbf{A}_\alpha = c_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha$ for an arbitrary $c_\alpha \in \mathbf{A}_\alpha$, and every nonempty $\mathbf{A}_\alpha^\circ$ can be presented as the coset $\mathbf{A}_\alpha^\circ = c_\alpha^\circ + \mathcal{M}_{\mathbf{G}}^\alpha$ for an arbitrary $c_\alpha^\circ \in \mathbf{A}_\alpha^\circ$.

The following lemma is just the transposition of (ATVI) and (ATRI) to the case where $\mathbf{A} = \mathrm{appr}(x,G)$ is an approximation type over the valued group $\mathbf{G}$.

Lemma 2.11 *Let* $\mathbf{A} = \mathrm{appr}(x,G)$. $\mathbf{A}$ *is value-immediate if and only if*

$$\forall c \in G:\ v(x-c) \in vG\infty\ .$$

$\mathbf{A}$ *is residue-immediate if and only if*

$$\forall c \in G: \quad v(x-c) \in vG\ \Rightarrow\ \exists c' \in G:\ v(x-c') > v(x-c)\ .$$

This in turn is equivalent to:

$$\forall c \in G: \quad v(x-c) = \alpha \in vG\ \Rightarrow\ \mathrm{co}\,(x-c) \in C_\alpha G\ .$$

Consequently, $\mathbf{A}$ *is immediate if and only if*

$$\forall c \in G: \quad \exists c' \in G:\ v(x-c') > v(x-c)\ .$$

This lemma together with Lemma 2.9 shows that $\mathbf{H} \subset \mathbf{G}$ is immediate if and only if every element in G has an immediate approximation type over H. But this is already a consequence of Lemma 1.36, which is the corresponding assertion for ultrametric spaces.

As we are dealing with the ultrametric space underlying a valued group (G,v), the question arises whether its completion also carries the structure of a valued group. To show this, we will employ the following simple observation: if $B_\alpha(x_\alpha)$ and $B_\alpha(x'_\alpha)$ are balls in a valued group, then the same holds for the sum $B_\alpha(x_\alpha) + B_\alpha(x'_\alpha)$ since it is equal to $x_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha + x'_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha = x_\alpha + x'_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha = B_\alpha(x_\alpha + x'_\alpha)$. The same holds for balls of the form B_α°. Given two completion types $\mathbf{A}$ and $\mathbf{A}'$, we define the sum $\mathbf{A} + \mathbf{A}'$ by setting $(\mathbf{A} + \mathbf{A}')_\alpha := \mathbf{A}_\alpha + \mathbf{A}'_\alpha$ and $(\mathbf{A} + \mathbf{A}')_\alpha^\circ := \mathbf{A}_\alpha^\circ + \mathbf{A}'^\circ_\alpha$ (if B is some ball, then $B + \emptyset = \emptyset$). Now $\mathbf{A} + \mathbf{A}'$ is again an approximation type. Since $\mathbf{A}$ and $\mathbf{A}'$ are completion types, all values in vG appear as radii of balls in $\mathbf{A} + \mathbf{A}'$, which proves that it is also a completion type. We leave it to the reader to verify that G^c with this addition is an abelian group extension of G.

Now recall how we extend the ultrametric to the completion. We set $u(\mathbf{A},\mathbf{A}') = u(x_\alpha, x'_\alpha)$ if $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$ and $x_\alpha \in \mathbf{A}_\alpha$, $x'_\alpha \in \mathbf{A}'_\alpha$ (cf. page 26). But $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$ is equivalent to $B_\alpha(x_\alpha - x'_\alpha) \neq B_\alpha(0)$, that is, $(\mathbf{A}-\mathbf{A}')_\alpha \neq \mathrm{at}\,(0,X)_\alpha$. Moreover, $u(x_\alpha, x'_\alpha) = u(x_\alpha - x'_\alpha, 0)$ since the ultrametric is induced by the group valuation v. This shows that the extension of u to G^c again satisfies (UMG), proving that $vx := u(x,0)$ is a group valuation on G^c which extends the valuation of G. The valued group (G^c, v) is called the **completion of** (G,v) and is also denoted by $(G,v)^c$. Furthermore, (G,v) is dense in $(G,v)^c$ in the sense of ultrametric spaces, which is equivalent to $\forall y \in G^c\, \forall \beta \in vG^c\, \exists x \in G:\ v(x-y) \geq \beta$. In particular, by the previous corollary, (G^c, v) is an immediate extension of (G,v).

Maximal immediate extensions of a valued group are maximal. Let us show their existence. The idea is to apply Zorn's Lemma. Indeed, if we have an increasing chain of valued abelian groups which are immediate extensions of $\mathbf{G}$, with index set an ordinal, then the union is again a valued group which is an immediate extension of $\mathbf{G}$ (this holds since valued abelian groups can be axiomatized by universal axioms). To obtain that *every* increasing chain of immediate extensions of $\mathbf{G}$ admits a union (which is a set and not a proper class), we just need to know that the cardinality of the extension groups is bounded from above. As a corollary to Lemma **??**, we get such a bound depending on $|vG|$ and the size of the polygons in $\mathbf{G}$. But we have obtained by our above arguments that for every $\alpha \in vG$, all α-polygons of $\mathbf{G}$ are of equal size, namely $|\mathrm{C}^\alpha\mathbf{G}|$. This shows:

Lemma 2.12 *Let κ be a cardinal which is an upper bound for the cardinality of all components of the valued group $\mathbf{G}$. Then*

$$|G| \leq \kappa^{|vG|}\ .$$

Since immediate extensions do not enlarge the value set or the components, this lemma yields:

Corollary 2.13 *For every valued group there exists a maximal immediate extension.*

Given an ordered index set Γ and for every $\gamma \in \Gamma$ an arbitrary abelian group C_γ, we can form the **Hahn sum** $\coprod_{\gamma\in\Gamma} C_\gamma$. As an abelian group, this is the direct sum of the groups C_γ, represented as the set of all tuples $(\zeta_\gamma)_{\gamma\in\Gamma}$ with only finitely many of the $\zeta_\gamma \in C_\gamma$ nonzero. For a given $0 \neq c = (\zeta_\gamma)_{\gamma\in\Gamma}$, let γ_0 be the smallest index such that $\zeta_{\gamma_0} \neq 0$; we let $vc = \gamma_0$. This defines the valuation on $\coprod_{\gamma\in\Gamma} C_\gamma$.

In a similar way, we define a group called the **Hahn product**, denoted by $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. Consider the product $\prod_{\gamma\in\Gamma} C_\gamma$ and an element $c = (\zeta_\gamma)_{\gamma\in\Gamma}$ of this group. Then the **support of** c is the set of all $\gamma \in \Gamma$ for which $\zeta_\gamma \neq 0$. As a set, the Hahn product is the subset of $\prod_{\gamma\in\Gamma} C_\gamma$ containing all elements whose support is a well-ordered subset of Γ. In particular, the support of every element c in the Hahn product has a minimal element γ_0, which again enables us to define $vc = \gamma_0$. Finally, the Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two wellordered sets is again wellordered.

Let us take $\mathbf{G}$ to be the Hahn sum or the Hahn product as defined above. In both cases, for arbitrary $\gamma \in \Gamma$, the group $\mathcal{O}_{\mathbf{G}}^\gamma$ consists of all elements whose minimal element in the support is γ, whereas $\mathcal{M}_{\mathbf{G}}^\gamma$ contains all elements whose support "starts later". At γ, the tuples assume values in C_γ. This shows that $\mathcal{O}_{\mathbf{G}}^\gamma/\mathcal{M}_{\mathbf{G}}^\gamma$ is isomorphic to C_γ. Consequently, the skeleton of the Hahn sum $\coprod_{\gamma\in\Gamma} C_\gamma$ and of the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$ (and of every group in between) is just $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$. Consequently, they are also called the **Hahn sum** (resp. **Hahn product**) **over the skeleton** $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$. Note that we have just constructed an example of an immediate extension. (It may happen that Hahn sum and Hahn product coincide, e.g. if Γ is finite, but in general, the extension is non-trivial.)

Lemma 2.14 *Every Hahn product is spherically complete (as an ultrametric space). Consequently, it is a maximal group.*

Proof: Let $\mathbf{B}$ be a nest of balls in the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. We have to show that its intersection is nonempty. Since $\mathbf{B}$ is ordered by inclusion, we can choose a coinitial decreasing sequence $(B_\nu)_{\nu<\lambda}$ of balls $B_\nu = B_{\alpha_\nu}(a_\nu) \in \mathbf{B}$. Then $(\alpha_\nu)_{\nu<\lambda}$ will be a cofinal increasing sequence in $\Lambda(\mathbf{B})$. Let us write $a_\nu = (\zeta_{\nu,\gamma})_{\gamma\in\Gamma}$ with $\zeta_{\nu,\gamma} \in C_\gamma$.

Suppose that $\nu < \mu < \lambda$. Since $B_\mu \subset B_\nu$, we have $v(a_\nu - a_\mu) \geq \alpha_\nu$. That is, $\zeta_{\nu,\gamma} = \zeta_{\mu,\gamma}$ for all $\gamma < \alpha_\nu$. So we can define an element $a = (a_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} C_\gamma$ by setting $a_\gamma = \zeta_{\nu,\gamma}$ if there exists $\nu < \lambda$ with $\gamma < \alpha_\nu$, and setting $a_\gamma = 0$ otherwise. The latter guarantees that every subset of the support of a which is (strictly) bounded from above by some α_ν, is already a subset of the support of $a_\nu \in \mathbf{H}_{\gamma\in\Gamma} C_\gamma$. This shows that the support of a is wellordered and $a \in \mathbf{H}_{\gamma\in\Gamma} C_\gamma$. By our construction, $v(a - a_\nu) \geq \alpha_\nu$ for every $\nu < \lambda$ and thus, $a \in B_\nu$. Consequently, a is an element of $\bigcap \mathbf{B} = \bigcap_\nu B_\nu$. $\Box$

This lemma shows that the Hahn product is a maximal immediate extension of every valued group which lies between the Hahn sum and the Hahn product over the same skeleton. With this pair of valued groups, we can give some examples for immediate approximation types and immediate extensions. In particular, we are now able to show that the maximal immediate extensions of valued groups are not necessarily unique up to isomorphism.

Example 2.15 Consider the Hahn sum $\coprod_{i\in\mathbb{N}} C_i$ and the Hahn product $\mathbf{H}_{i\in\mathbb{N}} C_i$ where all C_i are equal to $\mathbb{Z}$. We will write shorter: $\coprod_{\mathbb{N}} \mathbb{Z}$ and $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$. Since every subset of $\mathbb{N}$ is well-ordered, $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$ is the full product over the C_i, or in other words, it consists of all maps from $\mathbb{N}$ into $\mathbb{Z}$. Let us denote by b the element in $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$ which has a 1 at every entry. (Note: b is not an element of $\coprod_{\mathbb{N}} \mathbb{Z}$). Further, let b_i be the element of $\coprod_{\mathbb{N}} \mathbb{Z}$ which has a 1 at the first i entries and 0 everywhere else. We have $\coprod_{\mathbb{N}} \mathbb{Z} \subset \mathbf{H}_{\mathbb{N}} \mathbb{Z}$, so we can consider the approximation type of b over $\coprod_{\mathbb{N}} \mathbb{Z}$. Since $v(b - b_i) = i + 1$, we have that b_i is a center of every ball at $(b, \coprod_{\mathbb{N}} \mathbb{Z})_j$ for $j \leq i + 1$, but it is not contained in the balls at $(b, \coprod_{\mathbb{N}} \mathbb{Z})^\circ_{i+1}$ and at $(b, \coprod_{\mathbb{N}} \mathbb{Z})_{i+2}$. We find that dist $(b, \coprod_{\mathbb{N}} \mathbb{Z})$ is ∞; which is not assumed since $b \notin \coprod_{\mathbb{N}} \mathbb{Z}$. Further, for every element $c \in \coprod_{\mathbb{N}} \mathbb{Z}$ there is some j such that c is not contained in at $(b, \coprod_{\mathbb{N}} \mathbb{Z})_j$. We only have to choose j so big that the finitely many nonzero entries of c lie below $j - 1$, then it will follow that $v(b - c) \leq j - 1$. Hence by Lemma **??**, at $(b, \coprod_{\mathbb{N}} \mathbb{Z})$ is a non-trivial approximation type. Its distance being ∞, it is even a non-trivial completion type. In the same way, replacing b by any element of $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$, it is seen that $\coprod_{\mathbb{N}} \mathbb{Z}$ is dense in $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$. If we replace the index set $\mathbb{N}$ by the ordinal $\omega + 1$ and define b as above, then the approximation type of b will still be a non-trivial immediate approximation type, but its distance is not any longer ∞ (in fact, it is ω); so it is not a completion type.

All this remains true if we replace the components $\mathbb{Z}$ by, say, $\mathbb{Z}/2\mathbb{Z}$. The element b as defined above now satisfies $2b = 0$ like any other element in the group. But we can also construct elements which realize the same immediate approximation type as b but do not have the same torsion. Indeed, let us take index set $\omega + 1$; note that $\mathbf{H}_{\omega+1}\,\mathbb{Z}/2\mathbb{Z} \cong (\mathbf{H}_{\mathbb{N}}\,\mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/2\mathbb{Z}$. Let a be a generator of $\mathbb{Z}/4\mathbb{Z}$. We can embed $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$ by sending the 1 of $\mathbb{Z}/2\mathbb{Z}$ to $2a$. This induces an embedding of $(\mathbf{H}_{\mathbb{N}}\,\mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/2\mathbb{Z}$ over $\mathbf{H}_{\mathbb{N}}\,\mathbb{Z}/2\mathbb{Z}$ in $(\mathbf{H}_{\mathbb{N}}\,\mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/4\mathbb{Z}$. In the latter group, we define b' to be the element which has a (or $3a$) at the last entry and 1 everywhere else (so $b' = b - a'$ or $b' = b + a'$ for $a' = (0,\ldots,0,\ldots,a)$). Now $\mathrm{at}\,(b', \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z}) = \mathrm{at}\,(b, \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z})$; this can be verified by a direct computation, but it also follows from the fact that $\mathrm{at}\,(b, \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z})$ is immediate and $v(b - b') = \omega > \Lambda(b, \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z})$ (cf. Lemma 1.30). Since $2b' = 2a' \in \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z}$, the fact that $\mathrm{at}\,(b', \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z})$ is immediate implies that for $\mathbf{H} := \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z}$ and $\mathbf{G} := \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}b'$, the extension $\mathbf{H} \subset \mathbf{G}$ is immediate (cf. Exercise 2.3). In spite of this, $\mathbf{G}$ has an element of order 4 whereas $\mathbf{H}$ does not.

We know already that $\mathbf{H}_{\omega+1}\,\mathbb{Z}/2\mathbb{Z}$ is an immediate extension of $\mathbf{H} = \coprod_{\omega+1}\mathbb{Z}/2\mathbb{Z}$. By the last lemma, this group is maximal, and so it is a maximal immediate extension of $\mathbf{H}$. On the other hand, Corollary 2.13 asserts the existence of a maximal immediate extension $\mathbf{G}'$ of $\mathbf{G}$, which is consequently also a maximal immediate extension of $\mathbf{H}$. While $\mathbf{G}'$ contains an element of order 4, the group $\mathbf{H}_{\omega+1}\,\mathbb{Z}/2\mathbb{Z}$ does not, and it is even a vector space over the field $\mathbb{F}_2$ of two elements. So these two maximal immediate extensions of $\mathbf{H}$ are not isomorphic at all, even if one drops the condition that the isomorphism fixes the subgroup $\mathbf{H}$. In some sense, $\mathbf{H}_{\omega+1}\,\mathbb{Z}/2\mathbb{Z}$ is the "sound" maximal immediate extension of $\mathbf{H}$, when one considers the additional vector space structure of $\mathbf{H}$. $\diamondsuit$

If $\mathcal{S}$ is a v-convex subset of the valued group $\mathbf{G}$, then every ball in $\mathcal{S}$ is a ball in $\mathbf{G}$. This implies:

Lemma 2.16 *Every v-convex subset $\mathcal{S}$ of a spherically complete group $\mathbf{G}$ is spherically complete.*

For the conclusion of this section, we note the following lemma whose proof we leave to the reader:

Lemma 2.17 *a) Let $\mathbf{H}$ be a v-convex subgroup of the valued group $\mathbf{G}$. Then $\mathbf{G}$ is spherically complete if and only if $\mathbf{H}$ and $\mathbf{G}/\mathbf{H}$ are spherically complete.*
b) Let $\mathbf{H}$ and $\mathbf{H}'$ be two valued groups. Then $\mathbf{H}' \amalg \mathbf{H}$ is maximal if and only if $\mathbf{H}$ and $\mathbf{H}'$ are, and it is spherically complete if and only if $\mathbf{H}$ and $\mathbf{H}'$ are.

Exercise 2.2 *Under which condition is the completion of a Hahn sum equal to the Hahn product?*

Exercise 2.3 *Let $\mathbf{H} \subset \mathbf{G}$ be an extension of valued groups. Given $b \in G$ and $\mathrm{at}\,(b, \mathbf{H})$, compute $\mathrm{at}\,(y + a, \mathbf{H})$ for $a \in H$. Show that $\mathrm{at}\,(y + a, \mathbf{H})$ is immediate if and only if $\mathrm{at}\,(b, \mathbf{H})$ is immediate. Conclude that the extension $\mathbf{H} \subset \mathbf{G}$ is immediate if in a given set of representatives for the cosets of G modulo H every element has an immediate approximation type over $\mathbf{H}$.*

2.4 p-valuations of abelian groups

For the field theorist, examples of valued abelian groups are easily obtained by taking valued fields (see Chapter 4) and forgetting the multiplication. But this is not only a procedure to produce examples. It will be one aim of this book to show its various useful applications.

For the abelian group theorist, the first and main example is that of a p-valuation, where p is a prime. A map v from the abelian group G onto $\Gamma\infty$ is called a **p-valuation** if Γ is a set of ordinals and v satisfies (VT) together with

(pV) $$vx \neq \infty \Longrightarrow vpx > vx\,.$$

A **valuated group** is an abelian group together with a p-valuation for every prime p. The prototype of a p-valuation is the **p-height function**. By induction (possibly transfinite), we define the following **filtration of** G. Let $p^0G := G$ and $pG := \{pa \mid a \in G\}$; this is a subgroup of G. Having defined $p^\mu G$ for an ordinal μ, we set $p^{\mu+1}G := p(p^\mu G)$. If λ is a limit ordinal and we have defined $p^\mu G$ for all $\mu < \lambda$, we set $p^\lambda G := \bigcap_{\mu<\lambda} p^\mu G$. This definition establishes a descending chain of subgroups, which must eventually become constant (see Exercise 2.6). That is, there is some minimal ordinal ν such that $p(p^\nu G) = p^\nu G$. Hence, $H := p^\nu G$ is **p-divisible**, i.e., for every $a \in H$ there is some $b \in H$ such that $a = pb$. In fact, H is the maximal p-divisible subgroup of G. If G is a **p-group**, i.e., the order of every element is a power of p, then H is even **divisible**, i.e., for every $a \in H$ and every $n \in \mathbb{N}$, there is some $b \in H$ such that $a = nb$ (see Exercise 2.5). A p-group is called **reduced** if its maximal divisible subgroup is $\{0\}$. The ordinal ν is called the **p-height of** G.

The p-height function is introduced as follows. Given an element $a \in G \setminus p^\nu G$, there is a minimal ordinal $\alpha < \nu$ such that $a \in p^\alpha G \setminus p^{\alpha+1}G$; we set $h_{G,p}(a) = \alpha$ (we may say that a is divisible by p^α but not by $p^{\alpha+1}$; in the case of α being a natural number, this indeed holds in the usual sense). For $a \in p^\nu G$ we set $h_{G,p}(a) = \infty$. Then $h_{G,p} : G \to \nu\infty$ is onto (here, the ordinal ν is interpreted as the set of all ordinals $\mu < \nu$). We leave it to the reader to verify (VT) and (pV). Further, we see that (V 0) holds if and only if $p^\nu G = \{0\}$. In particular, we find that the height function in a reduced p-group G is a valuation of G.

Now p-valuations appear as the restrictions of height functions to subgroups; in that sense, they are a generalization of height functions. Indeed, if the height function is restricted to a subgroup, it may not coincide with the height function of that subgroup since the height of an element in a subgroup H of G may be lower than in G. An element which is divisible by p in G may not be so in H. We can only say that $h_{G,p}(a) \geq h_{H,p}(a)$ for all $a \in H$. The following properties of the height function may be lost in a subgroup:

(pH1) $$h_{G,p}(x) > h_{G,p}(y) \wedge x \neq 0 \Longrightarrow \exists z:\ h_{G,p}(y) \leq h_{G,p}(z) < h_{G,p}(x) \wedge pz = x\ ,$$

(pH2) $$h_{G,p}(x) = h_{G,p}(z) + 1 \Longrightarrow \exists z:\ h_{G,p}(y) = h_{G,p}(z) \wedge pz = x\ .$$

Nevertheless, the restriction of $h_{G,p}$ still satisfies (VT) and (pV) (since these axioms are universal). Conversely, it was shown that every p-valuation arises in this way: every abelian group H with a p-valuation v can be embedded in an abelian group G in a way that v coincides with the restriction of $h_{G,p}$ to H (cf. [RIC–WA]).

Remark 2.18 For the development of the theory of valuated groups see [RIC], [RIC–WA], [HU–WA]. A slightly different definition of a p-valuation was given in [HI–ME].

As we have seen, the p-height function $h_{G,p}$ is a valuation if and only if G is reduced. If G is not reduced, then it does not satisfy (V0). This is bad in view of the notion of valuation independence that will be developed in the next chapter (if (V0) does not hold, then we loose the fact that valuation independent elements are $\mathbb{Z}$-independent). So let us modify the definition of the height function in order to obtain a valuation v. Suppose that the value set of the p-height function of G is $\nu\infty$, that is, ν is the height of G. For $a \in G$, we set $va = h_{G,p}(a)$ if $h_{G,p}(a) < \infty$. If $a \neq 0$ and $h_{G,p}(a) = \infty$, then we set $va = \nu$. Finally, we let $v0 = \infty$. Then v is a valuation on G, and we will call it the **p-height valuation of G**. Its value set is still ν if G is reduced, and it is $\nu + 1$ if G is not reduced. In the latter case, v does not satisfy (pV) since then there are divisible elements $a \neq 0$ such that $pa \neq 0$. But v will satisfy the modified axiom

(pV′) $\qquad vx \neq \nu, \infty \implies vpx > vx\,.$

Since our valuation v on G is derived from the p-height function on G, it satisfies

(pH1′) $\qquad \infty > vx > vy \implies \exists z :\ vy \leq vz < vx \wedge pz = x\ ,$

(pH2′) $\qquad vx = vy + 1 \implies \exists z :\ vy = vz \wedge pz = x\ .$

Since descending sequences in ordinals are finite, axiom (pH1′) yields that for every element a of value $va < \nu$ there is some element b of value $vb = 0$ (this is, $h_{G,p}(b) = 0$) and some $n \in \mathbb{N}$ such that $a = p^n b$. Further, if vx is a limit ordinal and $vy < vx$, then there exists some y' such that $vy < vy' < vx$ and it follows from (pH1′) that there exists z such that $vy < vz < vx$ and $pz = x$.

There is a further p-valuation on a p-group. It is induced by the **order** $\mathrm{ord}(a)$ of the elements $a \in G$ in the following way. We set $va := -n$ if $\mathrm{ord}(a) = p^n$ for $a \neq 0$, and $v0 = \infty$. Then (V0) and (VT) hold. For convenience, we identify ∞ with the integer 0. Then the following stronger form of (pV) is true:

$$x \neq 0 \implies vpx = vx + 1\ .$$

In view of this, all components of (G, v) are $\mathbb{F}_p$-vector spaces. The value set of v is either $-\mathbb{N}$ or $\{m \in -\mathbb{N} \mid -n \leq m < 0\}$ for some natural number n. We will call v the **order valuation of** G.

In section 3.5, we will discuss applications of the p-height valuation and the order valuation of a p-group.

Exercise 2.4 *Prove (pH1) and (pH2). Give examples where these properties of the p-height function are not inherited by subgroups. For which p-groups can the conditions "$x \neq 0$" in (pH1) and "$\infty > vx$" in (pH1′) be omitted?*

Exercise 2.5 *Show that a p-group is q-divisible for every prime $q \neq p$ (hint: write $1 = rp^n + sq$ with $r, s \in \mathbb{Z}$). Conclude that a p-divisible p-group is divisible. For a p-valuation v, prove that $v(nx) = vx$ whenever $n \neq 0$ is not divisible by p (same hint).*

Exercise 2.6 *Show that the height of a p-group G is smaller than $|G|^+$.*

2.5 Ordered abelian groups and their natural valuations

For the ordered mathematician who is considering ordered abelian groups and ordered fields, the main example of a valuation is the "natural" one, which represents the archimedean classes.

An **ordered abelian group** is an abelian group $(G, +)$ equipped with an ordering $<$ which is compatible with the addition, that is, it satisfies

(OG) $\qquad x < y \implies x + z < y + z$

for all $x, y, z \in G$. We will write $(G, <)$ to indicate an ordered abelian group. From (OG) it follows that $a < b \Leftrightarrow 0 < b - a$. Hence, the ordering on G is already determined by the set $\mathbf{P} = \{a \in G \mid 0 \leq a\}$, which we call the **positive cone of** $(G, <)$. The positive cone satisfies

(PC+) $\qquad \mathbf{P} + \mathbf{P} \subset \mathbf{P}$

(PC∩) $\qquad \mathbf{P} \cap -\mathbf{P} = \{0\}$

(PC∪) $\qquad \mathbf{P} \cup -\mathbf{P} = G$.

Conversely, if a subset $\mathbf{P}$ of an abelian groups G satisfies these axioms, then it is called a **positive cone in** G and indeed, it is the positive cone of an ordering on G which is defined by $a \leq b \Leftrightarrow b - a \in \mathbf{P}$ (cf. Exercise 2.7).

We set $\mathrm{sign}(0) = 0$ and for $a \in G$, we set $\mathrm{sign}(a) = 1$ if $a > 0$, and $\mathrm{sign}(a) = -1$ if $a < 0$. If we use the sign with respect to the positive cone $\mathbf{P}$, then we will also write $\mathrm{sign}_{\mathbf{P}}$. Further, we set $|a| := \max\{a, -a\} = \mathrm{sign}(a) \cdot a$. Two elements $a, b \in G$ are called **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. The reader may show that this is indeed an equivalence relation. The ordered group $(G, <)$ is called **archimedean** if all nonzero elements are archimedean equivalent. Let va denote the equivalence class of a. The set of equivalence classes is ordered as follows: $va < vb$ if and only if $|a| > |b|$ and a and b are not archimedean equivalent; in this case, we will say that b is **archimedean smaller than** a. We write $\infty := v0$; this is the maximal element in the ordered set of equivalence classes. The map $a \mapsto va$ is a valuation on G, called the **natural valuation** (cf. Exercise 2.8). Here, the elements of the value set have a concrete interpretation as archimedean classes, but modulo order isomorphism, it looses this meaning. Since we are identifying equivalent valuations, it is better to define the natural valuation in the following way: "$va = vb$ if a and b are archimedean equivalent, and $va < vb$ if b is archimedean smaller than a". With this definition, it is immediately seen that the restriction of the natural valuation v to a subgroup H of G is (equivalent to) the natural valuation of $(H, <)$. (Although the equivalence class of $x \in H$ may be smaller in H than in G, there is an order isomorphism between $v_{|H}(H)$ and the value set of the natural valuation of (H, v).)

Alternatively, we define the natural valuation by the axiom

(NV) $\qquad vx \leq vy \iff \exists n \in \mathbb{N}:\ n|x| \geq |y|$.

This actually defines the valuation in terms of a relation rather than a map. (See the remarks in section 20.1.) We leave it to the reader to verify that a map $v : G \to vG\infty$ is the natural valuation if and only if it satisfies (NV); the only thing to prove is that the right side of (NV) expresses the fact that y is archimedean equivalent or archimedean smaller than x. Note that for every $a \in G$ and every $n \in \mathbb{Z} \setminus \{0\}$, the element $na \in G$ is archimedean equivalent to a and so, the natural valuation satisfies the axiom scheme

(NVZ) $\qquad v(nx) = vx \qquad\qquad (0 \neq n \in \mathbb{Z})$.

In particular, this shows that every ordered abelian group is torsion free. Further, it follows from (NV) that the natural valuation satisfies

$$\forall x, y :\ (vx < vy \Rightarrow |x| > |y|) \ \wedge\ (|x| > |y| \Rightarrow vx \leq vy)\,. \tag{2.7}$$

Observe that a valuation v of G is coarser than the natural valuation if and only if it satisfies the implication "$\Leftarrow$" of (NV). In this case, v is called **compatible with the ordering** or a **convex valuation of** $(G,<)$, and $<$ will be called a v**-compatible ordering**. Hence, the convex valuations of $(G,<)$ are precisely the coarsenings of its natural valuation. The reader may show that a valuation v of $(G,<)$ is convex if and only if for every $\gamma \in vG$, the subset $\{a \in G \mid a > 0 \wedge va = \gamma\}$ of the positive cone is convex.

In the following, let v be the natural valuation of the ordered group G. Then the order type of vG will be called the **principal rank of the ordered group** G. In particular, G is archimedean if and only if its principal rank is 1, that is, vG consists of just one element.

Let $(H,<)$ be a subgroup of $(G,<)$. It will be called a **convex subgroup** if it is convex as a subset of the ordered set G. We leave it to the reader to show that this is equivalent to: $\forall x \in H\, \forall y \in G:\ |x| \geq |y| \Rightarrow y \in H$. Here, we can replace "$|x| \geq |y|$" by "$\exists n \in \mathbb{N}:\ n|x| \geq |y|$". Indeed, since H is assumed to be a subgroup, $x \in H$ implies that $nx \in H$; moreover, $n|x| = |nx|$. By virtue of (NV), it now follows that the condition is equivalent to

$$\forall x \in H\, \forall y \in G:\ vx \leq vy \Rightarrow y \in H\ , \tag{2.8}$$

that is, (H,v) is a v-convex subgroup of (G,v). On the one hand, this shows that for every initial segment γ of vG, the subgroups $\mathcal{O}_{\mathbf{G}}^{\gamma}$ and $\mathcal{M}_{\mathbf{G}}^{\gamma}$ of G are convex subgroups. On the other hand, we deduce that every convex subgroup $(H,<)$ of $(G,<)$ is of the form $\mathcal{M}_{\mathbf{G}}^{\gamma}$ for some initial segment γ of vG. The reader may show that a valuation w of $(G,<)$ is convex if and only if for every $\gamma \in wG$, the subgroup $\mathcal{O}_{w}^{\gamma}$ is convex.

Let a be an element in the ordered abelian group $(G,<)$ with natural valuation v. Then $\mathcal{O}_{\mathbf{G}}(a)$ is called a **principal convex subgroup**. It is the set of all elements in $(G,<)$ which are archimedean smaller or equivalent to a. It is contained in every convex subgroup that contains a and thus, it is equal to the intersection of all convex subgroups containing a. Similarly, $\mathcal{M}_{\mathbf{G}}(a)$ is the set of all elements which are archimedean smaller than a, and it is equal to the union of all convex subgroups not containing a. The following lemma is a consequence of Lemma 2.3:

Lemma 2.19 *The set* $\mathrm{Conv}(G,<)$ *of proper convex subgroups of* $(G,<)$ *is ordered by inclusion. The map* $\gamma \mapsto \mathcal{M}_{\mathbf{G}}^{\gamma}$ *induces an order reversing bijection between the ordered set of nonempty initial segments of* vG *and* $\mathrm{Conv}(G,<)$. *Similarly, the map* $\alpha \mapsto \mathcal{O}_{\mathbf{G}}^{\alpha}$ *induces an order reversing bijection between* $vG\infty$ *and the set of principal convex subgroups of* $(G,<)$.

The order type of $\mathrm{Conv}(G,<)$ is called the **rank of** $(G,<)$ and denoted by $\mathrm{rk}\,(G,<)$. If $\mathrm{Conv}(G,<)$ is finite, then the rank is equal to the number of elements of $\mathrm{Conv}(G,<)$. The lemma shows that the principal rank is equal to the reversed order type of the nonzero principal convex subgroups of $(G,<)$. If this is finite, then vG is finite and every convex subgroup is already principal convex. Hence if rank or principal rank are finite, they are both equal to $|vG|$.

Let $(G,<)$ and $(G',<)$ be two ordered groups and $\rho : G \to G'$ a group homomorphism. Then we say that ρ **preserves** $\leq$ if $g \leq h$ implies $\rho g \leq \rho h$, for all $g,h \in G$. This holds if and only if ρ maps the positive cone of $(G,<)$ into the positive cone of $(G',<)$. Note that a group monomorphism preserving $\leq$ also preserves the order. If ρ preserves $\leq$ and g and h lie in the kernel of ρ and if $a \in G$ such that $g \leq a \leq h$, then $0 = \rho g \leq \rho a \leq \rho h = 0$, showing that a also lies in the kernel of ρ. This proves:

Lemma 2.20 *If $\rho : G \to G'$ is a group homomorphism of ordered groups $(G,<)$ and $(G',<)$ and if ρ preserves $\leq$, then its kernel* $\ker\rho$ *is a convex subgroup of $(G,<)$.*

Every convex subgroup is the kernel of a homomorphism which preserves $\leq$. Indeed, as an analogue to Lemma 2.4, we have

Lemma 2.21 *Let $(H,<)$ be a convex subgroup of $(G,<)$ and let $\mathbf{P}$ be the positive cone of $(G,<)$. Then $<$ induces an ordering on the factor group G/H whose positive cone is $\mathbf{P}/H$. The canonical epimorphism $G \to G/H$ preserves $\leq$, that is,*

(OFG) $$x \leq y \implies x+H \leq y+H\ .$$

Note that this ordering on G/H is in fact the one that you obtain when you read $x+H$, $y+H$ as subsets of the ordered set G and interpret $x+H < y+H$ in the usual way.

If $a,b \in G\setminus H$, then a and b are archimedean equivalent in G if and only if $a+H$ and $b+H$ are archimedean equivalent in G/H.

Proof: We have $\mathbf{P}/H+\mathbf{P}/H = (\mathbf{P}+\mathbf{P})/H \subset \mathbf{P}/H$ and $\mathbf{P}/H \cup -\mathbf{P}/H = (\mathbf{P}\cup -\mathbf{P})/H = G/H$. So it remains to prove property (PC$\cap$) for $\mathbf{P}/H$. Assume that $a+H$ is an element of $\mathbf{P}/H \cap -\mathbf{P}/H$, that is, there is $b \in -P$ such that $a+H = b+H$. But then $a-b \in H$, and since $a,-b \in P$, we have $0 \leq a \leq a-b$. By the convexity of H, we obtain $a \in H$ showing that $a+H$ is the zero in G/H, as required.

Further, we have $a \leq b \Rightarrow b-a \in \mathbf{P} \Rightarrow b-a+H \in \mathbf{P}/H \Rightarrow a+H \leq b+H$, which is (OFG). Since H is convex, so are the subsets $a+H$ and $b+H$ of G. If they are not equal, they are disjoint and by convexity, it follows that $a+H < b+H$ or $a+H > b+H$ as sets. But if $a+H < b+H$ holds, then by (OFG), we have $a+H < b+H$ as elements of $(G/H,<)$. Conversely, if $a+H < b+H$ does not hold for sets, then by what we have shown, we have $a+H = b+H$ or $a+H > b+H$ as sets and thus, $a+H = b+H$ or $a+H > b+H$ as elements of $(G/H,<)$. This shows that both interpretations of "$<$" coincide.

For our final assertion, note that a and b are archimedean equivalent in G if and only if $n|a|-|b| \in \mathbf{P}$ and $n|b|-|a| \in \mathbf{P}$. This in turn holds if and only if $n|a|-|b|+H \in \mathbf{P}/H$ and $n|b|-|a|+H \in \mathbf{P}/H$. Since $na+H = n(a+H)$ and $|a|+H = |a+H|$ (interpreting $a+H$ as an element of G/H, not as a set), the latter is equivalent to $n|a+H|-|b+H| \in \mathbf{P}/H$ and $n|b+H|-|a+H| \in \mathbf{P}/H$, which says that $a+H$ and $b+H$ are archimedean equivalent in G/H. □

The factor group G/H with its induced ordering will be denoted by $(G,<)/H$.

Corollary 2.22 *Let $(G,<)$ and $(G',<)$ be two ordered groups and $\rho : G \to G'$ a group homomorphism which preserves $\leq$. Then the embedding $G/\ker\rho \to G'$ induced by ρ is order preserving, that is, an embedding $(G,<)/\ker\rho \to (G',<)$.*

Proof: If η denotes the canonical epimorphism $G \to G/\ker\rho$, then the embedding $\overline{\rho} : G/\ker\rho \to G'$ may be written as $\rho\circ\eta^{-1}$. This maps the positive cone of $(G,<)/\ker\rho$ into that of $(G',<)$, hence $\overline{\rho}$ preserves $\leq$. Since $\overline{\rho}$ is injective, it thus preserves the order. □

We can determine the natural valuation of the factor group:

Corollary 2.23 *If we write $H = \mathcal{M}^{\gamma}_{\mathbf{G}}$ according to Lemma 2.3, then the natural valuation of $(G/H,<)$ is v/γ. If $a \in G \setminus H$, then the component of a in G is isomorphic to that of $a+H$ in G/H, as ordered abelian groups.*

Proof: The fact that v/γ is the natural valuation of $(G/H,<)$ follows from the last assertion of Lemma 2.21 and formula (2.2) of Lemma 2.4. The isomorphism between the components follows from Lemma 2.5. We only have to show that it preserves the ordering. Let $\mathbf{P}_\alpha$ be the positive cone of $\mathcal{O}^{\alpha}_{\mathbf{G}}$. Applying Lemma 2.21 several times, we find that $(\mathbf{P}_\alpha/\mathcal{M}^{\gamma}_{\mathbf{G}})/(\mathcal{M}^{\alpha}_{\mathbf{G}}/\mathcal{M}^{\gamma}_{\mathbf{G}})$ is the positive cone of $(\mathcal{O}^{\alpha}_{\mathbf{G}}/\mathcal{M}^{\gamma}_{\mathbf{G}})/(\mathcal{M}^{\alpha}_{\mathbf{G}}/\mathcal{M}^{\gamma}_{\mathbf{G}})$, and that $\mathbf{P}_\alpha/\mathcal{M}^{\alpha}_{\mathbf{G}}$ is the positive cone of $\mathcal{O}^{\alpha}_{\mathbf{G}}/\mathcal{M}^{\alpha}_{\mathbf{G}}$. Both sets are sent onto each other by the isomorphism theorem, showing that the isomorphism preserves the ordering. $\square$

Let $\alpha \in vG$. Then all elements in $\mathcal{O}^{\alpha}_{\mathbf{G}} \setminus \mathcal{M}^{\alpha}_{\mathbf{G}}$ have value α and are thus archimedean equivalent. Since they are precisely those elements which are not reduced to 0 modulo $\mathcal{M}^{\alpha}_{\mathbf{G}}$, we can deduce from the foregoing lemma:

Corollary 2.24 *Every component of an ordered abelian group carries an induced ordering, and this ordering is archimedean.*

This fact gives rise to the following definition. The skeleton $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$ is called an **ordered skeleton** if every component C_γ is an ordered abelian group. An isomorphism (resp. embedding) $\{\sigma_\Gamma, \sigma_\gamma \mid \gamma \in \Gamma\}$ of skeletons is called **isomorphism** (resp. **embedding**) **of ordered skeletons** if σ_γ preserves the ordering of C_γ for every $\gamma \in \Gamma$. The **natural skeleton of an ordered abelian group** $(G,<)$ is the skeleton of G with respect to the natural valuation where the components $\mathrm{C}^\alpha\mathbf{G}$ carry the induced ordering, and these components are called the **archimedean components of** $(G,<)$. So the natural skeleton is an ordered skeleton, and if we say that natural skeletons are isomorphic we will mean that they are isomorphic as ordered skeletons. Every isomorphism of ordered groups induces an isomorphism of their natural skeletons.

As a consequence of the definition of the induced ordering, we obtain that the ordering of a group G is uniquely determined by its ordered skeleton:

Corollary 2.25 *Let (G,v) be a valued abelian group whose skeleton $\mathrm{sk}_{\alpha\in vG}\mathrm{C}^\alpha\mathbf{G}$ is an ordered skeleton with archimedean groups $\mathrm{C}^\alpha\mathbf{G}$. Then there is a unique ordering on G whose natural valuation is v and which satisfies, for all $a \in G$,*

$$a > 0 \iff \mathrm{co}\, a > 0\,.$$

In particular, there is a unique extension of this ordering to every immediate extension of (G,v).

Corollary 2.26 *Let $(G,<)$ and $(G',<)$ be ordered abelian groups and let v be their natural valuation. Assume σ is an isomorphism of their natural skeletons and ι is an isomorphism of the valued groups $\mathbf{G}$ and $\mathbf{G}'$ which preserves σ. Then ι also preserves the ordering.*

Proof: Since ι preserves σ, which is an isomorphism of ordered skeletons, we have $(v\iota a, \mathrm{co}\,\iota a) = (\sigma_{vG} va, \sigma_{va}\mathrm{co}\, a)$. Now we compute: $a > 0 \Leftrightarrow \mathrm{co}\, a > 0 \Leftrightarrow \sigma_{va}\mathrm{co}\, a > 0 \Leftrightarrow \mathrm{co}\,\iota a > 0 \Leftrightarrow \iota a > 0$. $\square$

Remark 2.27 The value set of the natural valuation reflects the archimedean incomparability of elements, while the components reflect the ordering between archimedean equivalent elements. Thinking of the "size" of elements (for example "nonstandard big", "standard", "infinitesimally small") we may thus say that the value set encodes the "vertical structure" of the ordered group, while the components encode its "horizontal structure".

Remark 2.28 The author has learned the above systematic treatment of ordered abelian groups with their natural valuation from S. Kuhlmann (cf. [KUS1], [KUS2]). In particular, her notion of an "ordered skeleton" and her characterization of order preserving isomorphisms (Corollary 2.26) turned out to be very useful.

The theory of ordered groups with their natural valuations and ordered skeletons was extensively exploited by S. Kuhlmann in the investigation of the structure of non-archimedean ordered exponential fields (cf. [KUS1], [KUS2], [KU–KUS1], [KU–KUS2], [KU–KUS–SH1]). Some of the results will be presented in section 10.1 below.

Given two ordered abelian groups $\mathbf{G}_1$ and $\mathbf{G}_2$, the **lexicographic product** of $\mathbf{G}_1$ and $\mathbf{G}_2$, denoted by $\mathbf{G}_1 \amalg \mathbf{G}_2$, is the direct sum $G_1 \oplus G_2$, endowed with the lexicographic product of their orderings. We leave it to the reader to show that this ordering is compatible with the addition of the product group. Then $\mathbf{G}_2$ is a convex subgroup of $\mathbf{G}_1 \amalg \mathbf{G}_2$, and $\mathbf{G}_1$ is isomorphic to the ordered group $(\mathbf{G}_1 \amalg \mathbf{G}_2)/\mathbf{G}_2$. For the groups endowed with their natural valuations, $\mathbf{G}_1 \amalg \mathbf{G}_2$ coincides with the definition that we have already given for valued groups. Hence, the value set of $\mathbf{G}_1 \amalg \mathbf{G}_2$ is the sum $vG_1 + vG_2$ of ordered sets, and the α-component of $\mathbf{G}_1 \amalg \mathbf{G}_2$ is that of $\mathbf{G}_1$ if $\alpha \in vG_1$ and that of $\mathbf{G}_2$ if $\alpha \in vG_2$. (We may call $\mathrm{sk}\,\mathbf{G}_1 \amalg \mathbf{G}_2$ the **sum** of $\mathrm{sk}\,\mathbf{G}_1$ and $\mathrm{sk}\,\mathbf{G}_2$.) The reader may prove the following: *If the convex subgroup $\mathbf{G}_2$ of an ordered group $\mathbf{G}$ admits a group complement G_1 in G (which is always the case if G is divisible), then with the induced ordering on G_1, we have $\mathbf{G} = \mathbf{G}_1 \amalg \mathbf{G}_2$.*

Example 2.29 The groups $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, endowed with the usual ordering are archimedean ordered groups. Every archimedean ordered group admits an order preserving embedding in $\mathbb{R}$ (cf. Lemma 3.55). The lexicographic products $\mathbb{Z} \amalg \mathbb{Q}$, $\mathbb{Q} \amalg \mathbb{Z}$ and $\mathbb{R} \amalg \mathbb{R}$ are ordered groups of rank 2. In the densely ordered group $\mathbb{Z} \amalg \mathbb{Q}$, $\mathbb{Q}$ is a proper convex subgroup. In the discretely ordered group $\mathbb{Q} \amalg \mathbb{Z}$, $\mathbb{Z}$ is a proper convex subgroup.

By induction, we form lexicographical products $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$. The two latter groups are divisible ordered abelian groups. Groups of the form $\mathbb{Z}^n$ are sometimes called **discrete groups** or **generalized discrete groups** (cf. [ZA–SA2]). $\diamondsuit$

More generally, we can form the **ordered Hahn sum**, also called **lexicographical sum**, again denoted by $\coprod_{\gamma\in\Gamma} C_\gamma$, if the components C_γ are arbitrary (not necessarily archimedean) ordered abelian groups. Similarly, we obtain the **ordered Hahn product**, also called **lexicographical product**, again denoted by $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. As abelian groups, these are the Hahn sum and the Hahn product as introduced in section 2.3. The ordering is then defined as follows. Given an element $c = (\zeta_\gamma)_{\gamma\in\Gamma}$, let γ_0 be the minimal element of its support. Then we let $c > 0$ if and only if $\zeta_{\gamma_0} > 0$ (which is the same as defining $\mathrm{sign}(c) := \mathrm{sign}(\zeta_{\gamma_0})$). If all C_γ are archimedean ordered, then the valuation v of the Hahn sum (resp. the Hahn product), as defined in section 2.3, is indeed the natural valuation of the ordered Hahn sum (resp. ordered Hahn product). But if there are non-archimedean ordered groups among the C_γ, then v will be a proper coarsening of the natural valuation. In this case, the natural skeleton will be obtained by replacing every non-archimedean ordered component C_γ in $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$ by the natural skeleton of C_γ (where we omit the ∞-component).

Every torsion free abelian group G admits a divisible extension group $\tilde{G}$ which has the universal property that it admits a unique embedding in every other divisible extension group. It is consequently unique up to isomorphism and is called the **divisible hull of** G. It is an algebraic extension of G. It can be represented as the tensor product $\mathbb{Q}\otimes_{\mathbb{Z}} G$. Like every torsion free divisible abelian group, it is a $\mathbb{Q}$-vector space, and its $\mathbb{Q}$-dimension is called the **rational rank of** G (which can be a finite or infinite cardinal), denoted by $\mathrm{rr}\, G$. It is equal to the cardinality of every maximal set of rationally independent elements in G. Recall that elements a_i, $i \in I$, in an abelian group G are called **rationally independent**, if for every choice of integers n_i, only finitely many of them nonzero, $\sum n_i a_i = 0$ implies that all n_i are zero. If H is a subgroup of G, then a_i, $i \in I$, are called **rationally independent over** H, if for every choice of integers n_i, only finitely many of them nonzero, and every $a \in H$, we have that $a + \sum n_i a_i = 0$ implies that a and all n_i are zero. We will see later (cf. Corollary 3.29) that the cardinality of the rank of an ordered abelian group can not exceed its rational rank.

Example 2.30 The ordered Hahn sum $\coprod_{\mathbb{N}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$ have rank and principal rank ω, which is the order type of $\mathbb{N}$; all proper convex subgroups are principal. Both ordered groups have no smallest nonzero convex subgroup. The rational rank of $\coprod_{\mathbb{N}} \mathbb{Q}$ is the countable cardinal $\aleph_0$, and the rational rank of $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$ is $2^{\aleph_0}$. If we replace $\mathbb{N}$ by any ordinal ν then it still follows that all proper convex subgroups are principal (and the group admits a smallest nonzero convex subgroup if and only if ν is a successor ordinal). In contrast to this, the ordered Hahn sum $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$ both contain precisely one proper non-principal convex subgroup, which is of the form $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$, resp. $\mathbf{H}_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$. Their principal rank is $\mathbb{Z}+\mathbb{Z}$, while their rank is $\mathbb{Z}+1+\mathbb{Z}$. In the ordered Hahn sum $\coprod_{\mathbb{R}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{R}} \mathbb{Q}$, for Λ a cofinal segment of $\mathbb{R}$ the convex subgroups of the form $\coprod_{\Lambda} \mathbb{Q}$ resp. $\mathbf{H}_{\Lambda} \mathbb{Q}$ are non-principal if and only if Λ has no smallest element.

There are natural order preserving isomorphisms $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q} \cong \coprod_{\mathbb{Z}} \mathbb{Q} \amalg \coprod_{\mathbb{Z}} \mathbb{Q}$ and $\coprod_{\mathbb{Z}}(\mathbb{Q} \amalg \mathbb{Q}) \cong \coprod_{\mathbb{Z}} \mathbb{Q}$. In a natural way, $\coprod_{\mathbb{Z}} \mathbb{Q}$ is an ordered subgroup of the ordered groups $\coprod_{\mathbb{Z}} \mathbb{R}$, $\coprod_{\mathbb{Q}} \mathbb{Q}$, $\coprod_{\mathbb{Q}} \mathbb{R}$ and $\coprod_{\mathbb{R}} \mathbb{R}$. The same holds for the respective Hahn products. The reader should determine in which sense these isomorphisms and embeddings also preserve the natural valuation. Observe that on the valued Hahn sum $\coprod_{\mathbb{Z}}(\mathbb{Q} \amalg \mathbb{Q})$ whose components are all equal to the group $\mathbb{Q} \amalg \mathbb{Q}$ of rank 2, the valuation is a proper coarsening of the natural valuation. $\diamondsuit$

At present, we are interested in the extension of the ordering to the divisible hull.

Lemma 2.31 *Every ordered group $(G,<)$ admits a unique extension of the ordering to its divisible hull $\tilde{G}$. Under this ordering, every element of $\tilde{G}$ is archimedean equivalent to some element of G and thus, $vG = v\tilde{G}$ for the natural valuation v. Moreover, the α-component of $\tilde{G}$ is precisely the divisible hull of the α-component of G, for every $\alpha \in vG$.*

Proof: Let $\mathbf{P}$ be the positive cone of $(G,<)$. We leave it to the reader to show that $\overline{\mathbf{P}} := \{a \in \tilde{G} \mid \exists n \in \mathbb{N} : \ na \in \mathbf{P}\}$ is a positive cone in $\tilde{G}$ and that in view of (PC+), it is the only one containing $\mathbf{P}$. Given $a \in \tilde{G}$, there is an integer $n \neq 0$ such that $na \in G$, and a and na are archimedean equivalent; in particular, $va = v(na)$.

Now consider an element in the α-component of $\tilde{G}$; we can represent it as $\mathrm{co}_{\alpha}a$ with suitable $a \in \tilde{G}$. Let $n \neq 0$ such that $na \in G$; then we have $n\mathrm{co}_{\alpha}a = \mathrm{co}_{\alpha}na \in \mathrm{C}^{\alpha}\mathbf{G}$ showing that $\mathrm{co}_{\alpha}a$ lies in the divisible hull of $\mathrm{C}^{\alpha}\mathbf{G}$. Conversely, since $\tilde{G}$ is divisible, the same holds for the subgroup $\mathcal{M}^{\alpha}_{\mathbf{G}}$ and its factor group $\mathrm{C}^{\alpha}\mathbf{G}$. □

Using that a and na are archimedean equivalent for $n \neq 0$, we can also deduce that every convex subgroup H of G is **pure in** G; that is, for every $a \in G$, $na \in H$ implies that $a \in H$. The converse is certainly not true: $\mathbb{Q}$ is pure but not convex in $\mathbb{R}$. For every subgroup H of G, $\{a \in G \mid \exists n \neq 0 : \ na \in H\}$ is a pure subgroup of G, called the **relative divisible closure of** H **in** G. Since it is a subgroup of the divisible hull of H, it also has the same natural value set as H, and its α-component is contained in the divisible hull of the α-component of H. However, it may be smaller than the relative divisible closure of the α-component of H in the α-component of G:

Example 2.32 Let r be a real number $\notin \mathbb{Q}$. Consider $H = \mathbb{Z} \amalg \mathbb{Z}$ and $G = H + \mathbb{Z}(\frac{1}{2}, r) \subset \mathbb{R} \amalg \mathbb{R}$. Then for $\alpha = v(1,0)$, we have that $\mathrm{C}^{\alpha}H = \mathbb{Z}$ is not pure in $\mathrm{C}^{\alpha}G = \frac{1}{2}\mathbb{Z}$, although H is pure in G. ♢

The relative divisible closure is an algebraic extension. By the previous lemma, every algebraic extension of an ordered abelian group admits a unique extension of the ordering.

While $\mathbb{Z}$ is an ordered subgroup of every non-trivial ordered abelian group, it may not be a convex subgroup, as already the examples $\mathbb{Q}$ and $\mathbb{R}$ show. In fact, $\mathbb{Z}$ is a convex subgroup of the ordered group $\mathbf{G}$ if and only if $\mathbf{G}$ admits a **smallest positive element** (here, we mean "positive" in the sense "> 0"), and this is the case if and only if $\mathbf{G}$ is discretely ordered. Indeed, if $g \in G$ is the smallest positive element, then $\mathbb{Z}g$ (the cyclic group generated by g) is a convex subgroup of $\mathbf{G}$. If moreover, the factor group $G/\mathbb{Z}g$ is divisible, then $\mathbf{G}$ is called a **$\mathbb{Z}$-group**. In some sense that will be made precise by model theory in section 20.11, the $\mathbb{Z}$-groups are the groups which are similar to $\mathbb{Z}$ (cf. Theorem 20.69). The prototype of a $\mathbb{Z}$-group of rank 2 is $\mathbb{Q} \amalg \mathbb{Z}$. It is the smallest $\mathbb{Z}$-group of rank 2 since its rational rank is 2. The largest $\mathbb{Z}$-group of rank 2 is $\mathbb{R} \amalg \mathbb{Z}$. But note: it is not the case that the convex subgroup $\mathbb{Z}$ is a direct summand in every $\mathbb{Z}$-group. The reason for this is the fact that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ is non-trivial and that every extension of $\mathbb{Z}$ by $\mathbb{Q}$ can be ordered such that it is a $\mathbb{Z}$-group; see [HI–MO], Theorem 3.1 and Proposition 3.2.

Let v be the natural valuation of the $\mathbb{Z}$-group $\mathbf{G}$. Then vG contains a largest element, namely the archimedean class vg of g. Further, co_{vg} is an isomorphism of $\mathbb{Z}g$ onto the vg-component $\mathrm{C}^{vg}\mathbf{G}$ (as ordered groups).

The relative divisible closure of an ordered abelian group in a divisible ordered abelian group is again a divisible ordered abelian group. $\mathbb{Z}$-groups have a similar property:

Lemma 2.33 *Let* $\mathbf{G}$ *be a* $\mathbb{Z}$*-group with least positive element* g*. Then for every* $n \in \mathbb{N}$ *and* $a \in G$ *there is some* $m \in \mathbb{N}$ *such that* $a + mg$ *is divisible by* n *in* G*. In particular, if* $\mathbf{H}$ *is a subgroup of* $\mathbf{G}$ *containing* g*, then the relative divisible closure of* $\mathbf{H}$ *in* $\mathbf{G}$ *is again a* $\mathbb{Z}$*-group. On the other hand, if* $\mathbf{H} \subset \mathbf{G}$ *is an extension of* $\mathbb{Z}$*-groups, both having the same least positive element, then* G/H *is torsion free.*

Proof: Let $n \in \mathbb{N}$ and $a \in G$. Since $G/\mathbb{Z}g$ is divisible, there is some $b \in G$ such that $nb + \mathbb{Z}g = a + \mathbb{Z}g$. Hence, $nb = a + mg \in G$ for some $m \in \mathbb{Z}$, showing that $a + mg$ is divisible by n in G. Writing $m = m' + kn$ with $m', k \in \mathbb{Z}$ and $0 \leq m' < n$, we find that also $a + m'g = a + mg - kng$ is divisible by n. Thus, we can replace m by m' to obtain that $0 \leq m < n$.

Denote by $\mathbf{H}'$ the relative divisible closure of $\mathbf{H}$ in $\mathbf{G}$. Suppose that in addition to the previous assumption, we have that $a \in H$. Since $g \in H$ and hence $a + mg \in H$, we obtain $b \in H'$. This shows that $a + \mathbb{Z}g$ is divisible by n in $H'/\mathbb{Z}g$. We have thus proved that $H'/\mathbb{Z}g$ is divisible.

Now assume that $\mathbf{H} \subset \mathbf{G}$ is an extension of $\mathbb{Z}$-groups, both having the same least positive element g. Let $a \in G$ and $n \in \mathbb{N}$ such that $na \in H$. Choose $m \in \mathbb{N}$ such that $0 \leq m < n$ and $na + mg$ is divisible by n in H. Write $na + mg = nb$ with $b \in H$. Then $0 \leq b - a = \frac{m}{n} g < g$ shows that $a = b \in H$. We have thus proved that G/H is torsion free. $\square$

When we consider $\wp$-adically closed fields, we will need:

Lemma 2.34 *Let $(H, <)$ be an ordered abelian group with least positive element g. Then there is an algebraic extension $(G, <)$ of $(H, <)$ such that $(G, <)$ is a $\mathbb{Z}$-group with least positive element g.*

Proof: Let $(G, <)$ be a subgroup of the divisible hull of $(H, <)$ containing $(H, <)$ and maximal with the property that g is the least positive element in $(G, <)$. (Such maximal subgroups exist by Zorn's Lemma). Suppose that $G/\mathbb{Z}\alpha$ is not divisible, and choose $a \in G$ and a prime p such that $a + \mathbb{Z}g$ is not divisible by p. Take $b \in \tilde{H} = \tilde{G}$ such that $pb = a$. Then $(G + \mathbb{Z}b : G) = p = ((G + \mathbb{Z}b)/\mathbb{Z}g : G/\mathbb{Z}g)$. This yields that $\mathbb{Z}g$ is still a convex subgroup of $G + \mathbb{Z}b$, contrary to the maximality of G. This proves that $G/\mathbb{Z}\alpha$ is divisible, i.e., $(G, <)$ is a $\mathbb{Z}$-group. $\square$

Ordered abelian groups play an important role as value groups of valued fields. The relation between a valued field and its value group is a "one level higher analogue" to the relation between a valued group and its value set. Consequently, initial segments of ordered abelian groups will also play a role. For later use, let us state some auxiliary results.

Lemma 2.35 *Let Λ be an initial segment of the ordered abelian group Γ.*
a) For every $\Lambda' \subset \Gamma$, $\Lambda + \Lambda' = \{\alpha + \beta \mid \alpha \in \Lambda\,,\ \beta \in \Lambda'\}$ is again an initial segment of Γ.
b) For every natural number $n > 0$, the set $n \cdot \Lambda = \{n \cdot \alpha \mid \alpha \in \Lambda\}$ is cofinal in the n–fold sum $\Lambda + \Lambda + \ldots + \Lambda$ (which is an initial segment by part a)).
c) If Δ is a subgroup of Γ, then $\Lambda \cap \Delta$ is an initial segment of Δ.

Proof: a): Let $\alpha \in \Lambda$, $\beta \in \Lambda'$ and $\gamma \in \Gamma$ such that $\gamma < \alpha + \beta$. Then $\gamma - \beta < \alpha$, and since Λ is an initial segment of Γ, we have $\gamma - \beta \in \Lambda$. Hence, $\gamma = \gamma - \beta + \beta \in \Lambda + \Lambda'$.
b): Given $\alpha_i \in \Lambda$, $1 \leq i \leq n$, let $\alpha = \max_i \alpha_i$. Then $\alpha_1 + \ldots + \alpha_n \leq n\alpha$. Since the latter is again an element of the n–fold sum of Λ's, this shows our assertion.
c): An easy exercise. $\square$

Exercise 2.7 *Let $\mathbf{P}$ be a positive cone of G. Show that the relation defined by $x \leq y \Leftrightarrow y - x \in \mathbf{P}$ is an ordering on G.*

Exercise 2.8 *Let v denote the natural valuation that we have defined above. We wish to show that v is indeed a valuation.*

a) Show that no nonzero element is archimedean equivalent to 0 and deduce that v satisfies (V0).

b) Prove that $|a+b| \leq |a| + |b|$. Use this and (NV) to show that v satisfies (VT).

c) Deduce the following properties of the natural valuation:

(NV1) $\forall x, y \in G: \ v(x-y) > vx \Rightarrow \mathrm{sign}(x) = \mathrm{sign}(y)\,,$

(NV2) $\mathrm{sign}(\sum_{1\leq i\leq n} x_i) = \mathrm{sign}(x_m)$ *if* $vx_m < vx_i$ *for all* $i \neq m\,,$

(NV3) $x_m < x'_m \Rightarrow \sum_{1\leq i\leq n} x_i < \sum_{1\leq i\leq n} x'_i$ *if* $vx_m < vx_i$ *and* $vx'_m < vx'_i$ *for all* $i \neq m\,.$

Exercise 2.9 *Show that $\mathbb{Z} \amalg \mathbb{Z}$ does not admit proper immediate extensions. On the other hand, prove that every finite extension is again isomorphic to $\mathbb{Z} \amalg \mathbb{Z}$ and thus, their skeletons are isomorphic. Do the same with $\mathbb{Q} \amalg \mathbb{Z}$ and other examples of ordered groups of different ranks.*

Exercise 2.10 *Show that the lexicographic product of two Hahn sums (resp. Hahn products) is a Hahn sum (resp. a Hahn product).*

2.6 Immediate maps on valued abelian groups

Observe that in a valued abelian group, any ball around 0 is a subgroup. Since balls are unions of closed balls, this has only to be proved for closed balls. Note that

$$B_\alpha(0) \ = \ \{z \in G \mid u(0,z) \geq \alpha\} \ = \ \{z \in G \mid vz \geq \alpha\}$$

since $u(0,z) = v(0-z) = v(-z) = vz$. Take $a, b \in B_\alpha(0)$. Then $va \geq \alpha$ and $vb \geq \alpha$, whence $v(a-b) \geq \alpha$ by (V2), that is, $a - b \in B_\alpha(0)$. This proves that every $B_\alpha(0)$ and every other ball B containing 0 is a subgroup of G. Let us note that since every ball B containing 0 is a union of closed balls $B_\alpha(0)$, it follows that

$$y \in B \text{ and } vz \geq vy \ \Rightarrow \ z \in B\,.$$

Every ball $\tilde{B}$ in (G,v) can be written in the form $b + B$ where $b \in \tilde{B}$ and $B = \{a - b \mid a \in \tilde{B}\}$ is a ball around 0. Hence the balls in (G,v) are precisely the cosets with respect to the subgroups that are balls.

2.6.1 Immediate homomorphisms

In this section we will give a handy criterion for group homomorphisms to be immediate. Throughout, let (G,v) and (G',v') be valued abelian groups.

Proposition 2.36 *Let $f: G \rightarrow G'$ be a map such that $f0 = 0$. If f is immediate, then for every $a' \in G' \setminus \{0\}$ there is some $a \in G$ such that*

(IH1) $v'(a' - fa) > v'a'$,

(IH2) *for all $b \in G$, $va \leq vb$ implies $v'fa \leq v'fb$.*

The converse is true if f is a group homomorphism.

Proof: Suppose first that f is immediate, and take any $a' \in G'$, $a' \neq 0$. Set $z' := a'$ and $y := 0$. Take $z \in G$ such that conditions (AT1) and (AT2) hold, and set $a := z$. Then $v'(a' - fa) = u'(z', fz) > u'(z', fy) = v'(a' - f0) = v'a'$. Hence, (IH1) holds. Also, we obtain from the ultrametric triangle law that $v'a' = v'fa$. Further, condition (AT2) shows that

$$\begin{aligned} f(\{b \mid vb \geq va\}) &= f(B(0,a)) \;=\; f(B(y,z)) \\ &\subseteq B(fy, z') \;=\; B(0,a') \;=\; \{b' \mid v'b' \geq v'a' = v'fa\}. \end{aligned}$$

That is, $va \leq vb \Rightarrow v'fa \leq v'fb$, i.e., (IH2) holds.

For the converse, take any $y \in G$ and $z' \in G' \setminus \{fy\}$. Set $a' := z' - fy \neq 0$. Choose $a \in G$ such that conditions (IH1) and (IH2) hold, and set $z := y + a$. Then $u'(z', fz) = v'(z' - fz) = v'(z' - fy - fa) = v'(a' - fa) > v'a' = v'(z' - fy) = u'(z', fy)$. So (AT1) holds. Also, we obtain from the ultrametric triangle law that $v'fa = v'(z' - fy)$. To show that (AT2) holds, take any $x \in B(y,z)$. Then $v(x - y) \geq v(z - y) = va$. Hence by (IH2), $v'(fx - fy) = v'f(x - y) \geq v'fa = v'(z' - fy)$, so $fx \in B(fy, z')$. □

By Theorem 1.26, we obtain:

Theorem 2.37 *Let $f : G \to G'$ a group homomorphism which satisfies (IH1) and (IH2). Assume further that (G, v) is spherically complete. Then f is surjective and (G', v') is spherically complete.*

Lemma 2.38 *Let $f, \tilde{f} : G \to G'$ be group homomorphisms. Suppose that f is immediate and for all $a \in G$,*

$$v'(\tilde{f}a - fa) \;>\; v'fa \quad \text{or} \quad \tilde{f}a = fa = 0\,. \tag{2.9}$$

Then also $\tilde{f}$ is immediate.

Proof: If f satisfies (IH1) of Proposition 2.36, then

$$v'(a' - \tilde{f}a) \;\geq\; \min\{v'(a' - fa), v'(\tilde{f}a - fa)\} \;>\; v'\tilde{f}a = v'a'\,,$$

showing that also $\tilde{f}$ satisfies (IH1). Since (2.9) implies that $v'\tilde{f}a = v'fa$, $\tilde{f}$ will satisfy (IH2) whenever f does. Hence by Proposition 2.36, $\tilde{f}$ is immediate whenever f is. □

For an arbitrary map $f : G \to G'$ we will say that $a \in G$ is **f-regular** if it is non-zero and satisfies condition (IH2). We will denote the set of all f-regular elements by $\mathrm{Reg}\,(f)$. Then the following holds:

Proposition 2.39 *If $f : G \to G'$ is an immediate group homomorphism, then*

$$va \;\mapsto\; v'fa$$

for $a \in \mathrm{Reg}\,(f)$ induces a well defined and $\leq$-preserving map from $\{va \mid a \in \mathrm{Reg}\,(f)\}$ onto $v'G'$.

Proof: If $a, b \in \mathrm{Reg}\,(f)$ such that $va = vb$, then by (IH2), $v'fa \leq v'fb$ and $v'fa \geq v'fb$, whence $v'fa = v'fb$. This shows that the map is well defined. Again because of (IH2), it preserves $\leq$. Now take any $a' \in v'G'$, $a' \neq 0$. Then by (IH1), there is $a \in G$ such that $v'(a' - fa) > v'a'$, whence $v'a' = v'fa$ by the ultrametric triangle law. This proves that the map is onto. □

2.6.2 Basic criteria

Even if the map f that we consider on a valued abelian group is not a homomorphism, the presence of addition helps us to give handy and natural criteria for the map to be immediate. We just have to work a little harder. In this section, we present basic criteria that will cover all our applications in the non-additive case.

Proposition 2.40 *Take valued abelian groups (G,v) and (G',v'), an element $b\in G$, a ball B around 0 in G, a ball B' around 0 in G', and a map $f: b+B\rightarrow fb+B'$. Assume that $\phi: B\rightarrow B'$ is a map such that for all $a'\in B'\setminus\{0\}$ there is $a\in \mathrm{Reg}\,(\phi)$ with the following properties:*

$$v'(a'-\phi a)\;>v'a'\;=\;v'\phi a\;, \tag{2.10}$$

and

$$\left\{\begin{array}{l} v'(fy-fz\;-\;\phi(y-z))\;>\;v'\phi a\\ \text{for all } y,z\in b+B \text{ such that } v(y-z)\geq va\,. \end{array}\right. \tag{2.11}$$

Then f is immediate.

If $\phi 0=0$ then (2.11) needs to be checked only for $y\neq z$.

Proof: Take $z'\in fb+B'$ and $y\in b+B$ such that $z'\neq fy$. Applying our assumption to $a':=fy-z'$ we find that there is some $a\in \mathrm{Reg}\,(\phi)$ such that by (2.10),

$$v'(fy-z'\;-\;\phi a)\;>\;v'(fy-z')\;=\;v'\phi a\;, \tag{2.12}$$

and such that (2.11) holds. Set $z:=y-a\in y-B=y+B=b+B$. Then $y-z=a$ and hence by (2.11) and (2.12),

$$v'(fy-fz-\phi(y-z))\;>\;v'\phi a\;=\;v'(fy-z')\;.$$

Consequently,

$$\begin{array}{rcl} v'(z'-fz) & \geq & \min\{v'(z'-fy\;+\;\phi a)\,,\,v'(fy-fz-\phi a)\}\\ & = & \min\{v'(fy-z'\;-\;\phi a)\,,\,v'(fy-fz-\phi(y-z))\}\\ & > & v'(fy-z')\;=\;v'(z'-fy)\;. \end{array}$$

Hence (AT1) holds. Now take $x\in B(y,z)\subseteq b+B$, i.e., $v(y-x)\geq v(y-z)=va$. Then $v'\phi(y-x)\geq v'\phi a$ because $a\in \mathrm{Reg}\,(\phi)$, and $v'(fy-fx\;-\;\phi(y-x))>v'\phi a$ by (2.11). Therefore,

$$\begin{array}{rcl} v'(fy-fx) & \geq & \max\{v'(fy-fx\;-\;\phi(y-x))\,,\,v'\phi(y-x)\}\\ & \geq & v'\phi a\;=\;v'(fy-z')\,, \end{array}$$

whence $fx\in B(fy,z')$. Hence (AT2) holds.

Assume that $\phi 0=0$. Observe that $\phi a\neq 0$ since $a'\neq 0$ and $v'a'=v'\phi a$. Hence if $y=z$ then $v'(fy-fz\;-\;\phi(y-z))=v'0=\infty>v'\phi a$, which shows that (2.11) need only be checked for $y\neq z$. □

Note that by the ultrametric triangle law, the equality in (2.10) is a consequence of the inequality. Further, observe that this proposition proves the direction "⇐" of Proposition 2.36: if we take $B=G$, $B'=G'$ and $\phi=f$, then (IH1) implies (2.10) and (IH2) implies that $a \in \mathrm{Reg}\,(\phi)$, while (2.11) is trivially satisfied. Hence if for every $a' \in G' \setminus \{0\}$ there is $a \in G$ such that (IH1) and (IH2) hold, then the above proposition shows that f is immediate.

The following is a special case of the above criterion, with nicer properties.

Proposition 2.41 *Take valued abelian groups (G, v) and (G', v'), an element $b \in G$, a ball B in G around 0, a ball B' in G' around 0, and a map $f : b + B \to G'$. Assume that*

(PC1) *$\phi : B \to B'$ is immediate,*

(PC2) *for all $y, z \in b + B$, $fy - fz = \phi(y - z) = 0$ or*

$$v'(fy - fz - \phi(y - z)) \;>\; v'(fy - fz) \;=\; v'\phi(y - z)\,.$$

Then $f(b + B) \subseteq fb + B'$, and $f :\; b + B \;\to\; fb + B'$ is immediate.

If in addition ϕ is injective, then so is f, and if ϕ is an embedding of ultrametric spaces with value map φ, then so is f.

Proof: Taking $y = z$, we obtain from (PC2) that $\phi(0) = 0$. So we can apply Proposition 2.36 to find that ϕ satisfies (IH1) and (IH2). Therefore, for $a' \in B' \setminus \{0\}$ we can choose $a \in \mathrm{Reg}\,(\phi) \setminus \{0\}$ such that $v'(a' - \phi a) > v'a'$.

Take $y, z \in b + B$ such that $v(y - z) \geq va$. By the regularity of a, $v'\phi(y - z) \geq v'\phi a$. Hence by (PC2), $v'(fy - fz - \phi(y - z)) > v'\phi(y - z) \geq v'\phi a$. Now it follows from Proposition 2.40 that f is immediate. If in addition, ϕ is injective, it follows from (PC2) that also f is injective. If ϕ is an embedding of ultrametric spaces with value map φ, then $v'\phi(y - z) = \varphi v(y - z)$ shows that also f is an embedding with value map φ. □

If the map ϕ satisfies the conditions (PC1) and (PC2) of the foregoing proposition, it will be called a **pseudo-companion of f on $b + B$**.

We will later need the following fact:

Lemma 2.42 *Let the situation be as in Proposition 2.41 and let $\phi, \tilde{\phi} : B \to B'$ be group homomorphisms. Suppose that $v'(\tilde{\phi}a - \phi a) \;>\; v'\phi a$ or $\tilde{\phi}a = \phi a = 0$ for all $a \in G$. If ϕ is a pseudo companion for f on $b + B$, then so is $\tilde{\phi}$.*

Proof: Assume that ϕ is a pseudo-companion of f on $b + B$. Then by Proposition 2.38, also $\tilde{\phi}$ is immediate. Now take $y, z \in b + B$. If $\phi(y - z) = 0$ then by assumption, $\tilde{\phi}(y - z) = 0$. Otherwise, $v'(fy - fz - \tilde{\phi}(y - z)) \geq \min\{v'(fy - fz - \phi(y - z)), v'(\phi(y - z) - \tilde{\phi}(y - z))\} > v'\phi(y - z) = v'(fy - fz)$. This shows that also $\tilde{\phi}$ is a pseudo-companion of f on $b + B$. □

2.7 Sums of spherically complete valued abelian groups

Let $(\mathcal{A}, v)$ be a valued abelian group and $A_1, \ldots, A_n$ be subgroups of $\mathcal{A}$. The restrictions of v to every A_i will again be denoted by v. We call the sum $A_1 + \ldots + A_n \subseteq \mathcal{A}$ **pseudo-direct** if for every $a' \in A_1 + \ldots + A_n$, $a' \neq 0$, there are $a_i \in A_i$ such that

$$v \sum_{i=1}^{n} a_i \;=\; \min_{1 \leq i \leq n} va_i \quad \text{and} \quad v\left(a' - \sum_{i=1}^{n} a_i\right) \;>\; va'\,. \tag{2.13}$$

Proposition 2.43 *The sum $A_1 + \ldots + A_n \subseteq \mathcal{A}$ is pseudo-direct if and only if the group homomorphism $f : A_1 \times \ldots \times A_n \to A_1 + \ldots + A_n$ defined by $f(a_1, \ldots, a_n) := a_1 + \ldots + a_n$ is immediate.*

Proof: $\Rightarrow$: Assume that the sum $A_1 + \ldots + A_n$ is pseudo-direct. Take any $a' \in \sum_i A_i$ and choose $a_i \in A_i$ such that (2.13) holds. Then $a := (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ satisfies (IH1). If $b = (b_1, \ldots, b_n) \in A_1 \times \ldots \times A_n$ such that $vb \geq va$, then

$$vfb \;=\; v\sum_i b_i \;\geq\; \min_i vb_i \;=\; vb \;\geq\; va \;=\; \min_i va_i \;=\; v\sum_i a_i \;=\; vfa\;.$$

This shows that a also satisfies (IH2).

$\Leftarrow$: Assume that f is immediate. Take any $a' \in \sum_i A_i$, $a' \neq 0$. Choose $a := (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ such that (IH1) and (IH2) hold. Then $v\,(a' - \sum_i a_i) = v(a' - fa) > va'$. Now choose some j such that $va_j = \min_i va_i$. Then set $b_j = a_j \in A_j$ and $b_i = 0 \in A_i$ for $i \neq j$. For $b = (b_1, \ldots, b_n)$, we thus have that $va = \min_i va_i = va_j = vb_j = \min_i vb_i = vb$. Hence by (IH2), $v\sum_i a_i = vfa \leq vfb = vb_j = \min_i va_i$. We have proved that the elements a_i satisfy (2.13). □

If the groups (A_i, v) are spherically complete, then by Proposition 1.11, the same is true for their direct product $A := A_1 \times \ldots \times A_n$, endowed with the minimum valuation as defined in (**??**). Hence, the foregoing proposition, Theorem 1.26 and Corollary **??** show:

Theorem 2.44 *Assume that the subgroups (A_i, v) of $(\mathcal{A}, v)$, $1 \leq i \leq n$, are spherically complete. If the sum $A_1 + \ldots + A_n$ is pseudo-direct, then it is also spherically complete. and has the optimal approximation property.*

Recall that when the sum $A_1 + \ldots + A_n$ is spherically complete, it follows that it is spherically closed and has the optimal approximation property in $(\mathcal{A}, v)$.

Open Problem 2.1 Is the sum of spherically complete subgroups always spherically complete? Under which conditions does a sum of subgroups with the optimal approximation property again have the optimal approximation property?