Chapter 3

Valued modules and vector spaces

3.1 Basic definitions

Let R be a ring, not necessarily commutative. **By an R-module, we will always mean a left R-module**. A **(general) valued R-module** (M, v) is a valued group (M, v) which also carries an R-module structure (with $(M, +)$ as its additive group). Every $r \in R$ can be viewed as an endomorphism of the group $(M, +)$; however, it may not respect the valuation or the coefficient map. On purpose, our definition does not say anything about the compatibility between valuation and module structure. So it enables us to view a valued abelian group (G, v) as a valued module over the endomorphism ring of G, and it is general enough to permit a study of the different possible compatibility properties. In almost all cases that have been studied in the literature, valued modules satisfy the following compatibility axiom

(VM ≥) $\quad \forall a \in M\, \forall r \in R: \ vra \geq va\,.$

A valued module satisfying this axiom will be called **component-compatibly valued**. Every valued abelian group is a component-compatibly valued $\mathbb{Z}$-module. Indeed, as every abelian group is a $\mathbb{Z}$-module, every valued abelian group is a valued $\mathbb{Z}$-module. Indeed, for $R = \mathbb{Z}$, property (VM ≥) is the same as the law (VZ), which we have deduced from the ultrametric triangle law. As we will see later, this does not mean that (VM ≥) remains true for larger rings.

The following lemma shows the reason for the notion "component-compatibly valued"; its proof is straightforward:

Lemma 3.1 *Let $\mathbf{M} = (M, v)$ be a component-compatibly valued R-module. Then for every initial segment γ of vM,*

$$R\mathcal{O}^{\gamma}_{\mathbf{M}} \subset \mathcal{O}^{\gamma}_{\mathbf{M}} \ \ \textit{and} \ \ R\mathcal{M}^{\gamma}_{\mathbf{M}} \subset \mathcal{M}^{\gamma}_{\mathbf{M}}\,.$$

Hence, $\mathcal{O}^{\gamma}_{\mathbf{M}}$, $\mathcal{M}^{\gamma}_{\mathbf{M}}$ and $\mathrm{C}^{\gamma}\mathbf{M} = \mathcal{O}^{\gamma}_{\mathbf{M}}/\mathcal{M}^{\gamma}_{\mathbf{M}}$ are R-modules, and co_{γ} is R-linear, i.e.

$$\forall r \in R\, \forall a \in M: \ r\,\mathrm{co}_{\gamma}a = \mathrm{co}_{\gamma}\,ra\,.$$

In particular, for every $a \in M$ the annihilator of a is contained in the annihilator of $\mathrm{co}\,a$.

If we are given a skeleton $\mathrm{sk}_{\gamma\in\Gamma}C_\gamma$ where all C_γ are R-modules, then in a natural way, Hahn sum and Hahn product over this skeleton are also component-compatibly valued R-modules.

The above lemma has shown that every component of a component-compatibly valued R-module $\mathbf{M}$ is an R-module, which in particular means that every $r \in R$ induces an endomorphism of every component. Analogously, we will say that $\mathbf{M}$ is **value-compatible** if every $r \in R$ induces an $\leq$-preserving map of vM into itself, given by $va \mapsto vra$ for $0 \neq a \in M$. Using the terminology introduced in Section **??**, $\mathbf{M}$ is value-compatible if every $r \in R$ induces a value-compatible map on $\mathbf{M}$. In a value-compatibly valued module, the value of vra only depends on va and not on $\mathrm{co}\, a$. Observe that for every invertible $r \in R$, the induced map $va \mapsto vra$ will then be an order preserving bijection. An important example for such valued modules is given by extensions $(L, v) \supset (K, v)$ of valued fields (see Part II). Here, (L, v) may be considered as a value-compatibly valued K-vector space. It is then of interest to study the valued K-subspaces of (L, v).

To classify a larger but still well-behaved class of valued modules, we invoke the skeleton. In short, we want to talk about those valued modules in which the scalar multiplication induces maps on the skeleton (or at least on almost all elements of the skeleton). Given a valued R-module $\mathbf{M}$ and any map f of M into itself, a bone $(\alpha, \zeta) \in \mathrm{sk}\,\mathbf{M}$ will be called an **exceptional bone for** f if there exist two elements a, b of M both having (α, ζ) as their bone, such that $f(a)$ and $f(b)$ have distinct bones. Hence f induces a map on all bones that are not exceptional for f. If $(va, \mathrm{co}\, a)$ is an exceptional bone for f, then a is called an **exceptional element for** f, and va is called an **exceptional value for** f. Further, $\mathrm{Exb}_f\mathbf{M}$ will denote the set of all exceptional bones for f, $\mathrm{Exe}_f\mathbf{M}$ will denote the set of all exceptional elements for f, and $\mathrm{Exv}_f\mathbf{M}$ will denote the set of all exceptional values for f. Every element of the ring R can be identified with the endomorphism that it induces on M. This leads to the following definitions. A bone will be called **exceptional bone of M** if it is exceptional for some $r \in R$. The set of all exceptional bones of $\mathbf{M}$ will be denoted by $\mathrm{Exb}\mathbf{M}$. Further, an element of $\mathbf{M}$ is called **exceptional element of M** if it is exceptional for some $r \in R$, and the set of all exceptional elements of $\mathbf{M}$ is denoted by $\mathrm{Exe}\mathbf{M}$. Similarly, we define the **exceptional values of M** and the set $\mathrm{Exv}\mathbf{M}$.

Example 3.2 The exceptional elements in the valued $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}\amalg\mathbb{Z}/3\mathbb{Z}$ are precisely $(1, 0)$, $(1, 1)$ and $(1, 2)$. All have the same bone, but we have $2(1, 0) = (0, 0)$, whereas $2(1, 1) \neq (0, 0)$ and $2(1, 2) \neq (0, 0)$. On the other hand, the trivially valued $\mathbb{Z}$-module $\mathbb{Z}/3\mathbb{Z}$ has no exceptional elements and correspondingly, the elements $(0, 0)$, $(0, 1)$ and $(0, 2)$ of $\mathbb{Z}/2\mathbb{Z} \amalg \mathbb{Z}/3\mathbb{Z}$ are not exceptional. If Γ is any ordered set and p a prime, then $\coprod_\Gamma \mathbb{Z}/p\mathbb{Z}$ is a valued $\mathbb{Z}$-module and a valued $\mathbb{Z}/p\mathbb{Z}$-vector space without exceptional elements. $\diamondsuit$

Bones, elements or values which are not exceptional will be called **ordinary**. A valued module without exceptional elements will be called an **ordinary valued module**. For example, an ordered abelian group equipped with its natural valuation is an ordinary valued $\mathbb{Z}$-module. Also, the valued field extensions mentioned above are ordinary valued vector spaces. The following is an important characterization of ordinary valued modules:

Lemma 3.3 *Let* $\mathbf{M}$ *be a valued* R*-module. Then* $a \in M$ *is ordinary for* $r \in R$ *if and only if*

$$va < va' \implies vra < vra' \vee ra = ra' = 0 \tag{3.1}$$

for every $a' \in M$. Hence, $\mathbf{M}$ *is ordinary if and only if (3.1) holds for all $a, a' \in M$ and every $r \in R$. Furthermore,* $\mathrm{Exe}_r\mathbf{M}$ *is the set of all $a \in M$ for which (3.1) does not hold for all $a' \in M$.*

Proof: Suppose that $a \in M$ is ordinary for $r \in R$ and let $a' \in M$ such that $ra' \neq 0$ and $va < va'$. Then by (2.4), a and $b := a' + a$ have the same bone. By assumption, it follows that ra and rb have the same bone. Since $ra' \neq 0$, this implies that $ra \neq 0$ and again by (2.4), $vra < v(ra - rb) = v(-ra') = vra'$.

For the converse, let $a \neq 0$ and b have the same bone. Then by (2.4), $va < va'$ for $a' := a - b$. If this implies that $vra < vra'$, then $vra < v(ra - rb)$, and (2.4) shows that ra and rb have the same bone. □

A map f on the valued R-module $\mathbf{M}$ will be called a **finitely exceptional map** if there are only finitely many exceptional bones for f on $\mathbf{M}$. So $\mathbf{M}$ will be called a **finitely exceptional valued module** if every $r \in R$ has only finitely many exceptional bones on $\mathbf{M}$. In the same spirit, $\mathbf{M}$ will be called an **almost value-compatible valued module** if every $r \in R$ is an almost value-compatible map on M (the latter notion has been introduced in Section **??**). At present, these definitions may seem a bit weird. But for the study of spherical completeness and spherical continuity, the finitely exceptional and the almost value-compatible modules provide an appropriate generalization of the notions of ordinary and value-compatible modules. Moreover, from the model theoretical point of view, finitely many “exceptions” are not too bad since they can be pinned down by axioms. We will meet our most important example of finitely exceptional almost value-compatible modules in Section 12.4 when we study valued fields of characteristic $p > 0$ as valued modules over rings of additive polynomials. For the time being, it may suffice to present an infinite but finitely exceptional $\mathbb{Z}$-module by just replacing “$\mathbb{Z}/3\mathbb{Z}$” by “$\mathbb{Z}$” in our last example.

Every element $r \in R$ operates on the set $\mathrm{sk}\,\mathbf{M} \setminus \mathrm{Exb}_r\mathbf{M}$ by $(va, \mathrm{co}\, a) \mapsto (vra, \mathrm{co}\, ra)$. Recall that two bones (α_1, ζ_1) and (α_2, ζ_2) can be added if and only if they lie in the same component (i.e., $\alpha_1 = \alpha_2$). So we will call a finite formal sum $\sum_i r_i(\alpha_i, \zeta_i)$ with $r_i \in R$ an **admissible linear combination** if for every i, the bone (α_i, ζ_i) is not exceptional for r_i and if all bones $r_i(\alpha_i, \zeta_i)$ lie in the same component. If $\mathcal{B} \subset \mathbf{M}$, then $\underline{\mathcal{B}} := \{(vb, \mathrm{co}\, b) \mid b \in \mathcal{B}\}$ will be the set of the bones of all elements of $\mathcal{B}$. Sets of bones will be denoted by $\underline{\mathcal{B}}$ since it is always possible to find $\mathcal{B} \subset \mathbf{M}$ such that $\underline{\mathcal{B}}$ is precisely the set of bones of all elements in $\mathcal{B}$.

A valued group or module is called **spherically complete** resp. **complete** if it has these properties as an ultrametric space.

Remark 3.4 A special type of valued vector spaces is studied in non-archimedean analysis. Here, the value set is a subset of the reals, the field K carries a valuation whose value group is a subgroup of the reals, and on the valued K-vector space (V, v), the scalar multiplication acts as follows: $vrx = vr + vx$ for all $r \in K$ and all $x \in V$. (The scalar multiplication behaves in a similar way for the valued vector spaces that we will derive from valued field extensions.) For details on these **real-valued vector spaces**, see [GRU1], [MON], [VDP] and their references. In particular, A. W. Ingleton [ING] shows that real-valued vector spaces satisfy the Hahn-Banach Theorem if and only if they are spherically complete.

An **embedding** (resp. **isomorphism**) **of valued modules** is an embedding (resp. isomorphism) of valued abelian groups which also preserves the module structure. An **extension of valued modules** is an extension of valued groups which at the same time is an extension of modules. In this case, we will write $\mathbf{M} \subset \mathbf{N}$ and call $\mathbf{M}$ a **valued submodule of N**. We will say that $\mathbf{M} \subset \mathbf{N}$ is a **finite extension** (or that $\mathbf{N}$ is **finitely generated over M**) if N/M is finitely generated as an R-module. An extension of valued modules is called **immediate** if it is immediate as an extension of valued groups. Thus, the same cardinality bound as for valued groups also holds for valued modules, showing that

Lemma 3.5 *Every valued module admits some maximal immediate extension.*

Such maximal immediate extension are **maximal modules** in the sense that they do not admit proper immediate module extensions. But note that for $R \neq \mathbb{Z}$ it is not a priori clear whether every maximal R-module is also a maximal group. In the next section, we will show that Hahn products are always maximal.

Exercise 3.1 *Let Γ be an ordered set, n_γ a natural number for every $\gamma \in \Gamma$ and $C_\gamma = \mathbb{Z}/n_\gamma\mathbb{Z}$. Give conditions on the numbers n_γ and the set Γ which yield that $\coprod_{\gamma\in\Gamma} C_\gamma$ is an ordinary $\mathbb{Z}$-module. More generally, give conditions on the components and the value set of a valued R-module which yield that it is ordinary. (Hint: consider annihilators.)*

3.2 Spherically closed submodules

We will say that $\mathbf{M}$ is an **spherically closed submodule of N** or just **spherically closed in N** if as an ultrametric space, it is spherically closed in $\mathbf{N}$. This is equivalent to the condition that for every $b \in N \setminus M$, the subset $v(b+M) = \{v(b+c) \mid c \in M\} = \{v(b-c) \mid c \in M\}$ of vN admits a maximum. If vb is this maximum, then b is called a **proper representative for the coset** $b+M$. By equivalence (2.6) on page 39 we see that b is a proper representative for the coset $b+M$ if and only if $(vb, \mathrm{co}\, b) \notin \mathrm{sk}\,\mathbf{M}$.

Note that $\mathbf{M}$ is a spherically closed submodule of $\mathbf{N}$ if and only if for every $b \in N$, $\mathbf{M}$ is a spherically closed submodule of $\mathbf{M} + Rb$. If $\mathbf{M}$ is a spherically closed submodule of $\mathbf{N}$, then it is also a spherically closed submodule of every $\mathbf{N}'$ such that $\mathbf{M} \subset \mathbf{N}' \subset \mathbf{N}$. From Lemma 1.17 we know that both properties "spherically closed in" and "immediate extension" are transitive. From Lemma 1.19 and Corollary 2.10, we obtain:

Theorem 3.6 *A spherically complete valued module is spherically closed in every valued module extension, and in particular is maximal.*

Together with Lemma 2.14, this gives:

Theorem 3.7 *Every Hahn product is maximal.*

Remark 3.8 It is an interesting question to ask whether the converse of Theorem 3.6 holds: is every maximal module spherically complete? Or at least: does every valued module admit some immediate extension which is a spherically complete module? For certain classes of valued modules, this is known to be true. If a valued abelian group is embedded between the Hahn sum and the Hahn product over its skeleton, then the Hahn product is an immediate spherically complete extension. We will show that component-compatibly valued vector spaces can be embedded in such a way (this is the Hahn Embedding Theorem for valued vector spaces, cf. Theorem 3.51). Moreover, by a simple model theoretic construction, it can be shown that every maximal ordinary valued vector space is spherically complete.

According to I. Fleischer [FL1], every maximal component-compatibly valued R-module is spherically complete if R is a principal ideal ring. (We will prove this result in Section 3.8 below.) In the same article, an example is given showing that for valued modules, the maximal immediate extensions are not necessarily unique if torsion interferes. It is essentially the example that we have given in 2.15. The groups that we have constructed there are valued $\mathbb{Z}$-modules. The group H of that example has two nonisomorphic maximal immediate extensions, one of which is even a valued $\mathbb{F}_2$-vectorspace like **H** while the other is not.

For valued modules with value preserving scalar multiplication (cf. section 3.4), N. Sankaran and R. Venkataraman [SAN–VE] have shown that there is always a maximal immediate extension which is spherically complete. For component-compatibly valued modules in general, they are still able to show the existence of what they call an **almost spherically complete** immediate extension.

For valued fields, the above question is answered in the affirmative by Ostrowski's and Kaplansky's theory of immediate extensions of valued fields; see section **??**.

The quotient by a spherically closed submodule can be endowed with a canonical valuation, which we will call the **quotient valuation**:

Lemma 3.9 *Suppose that* $\mathbf{M}$ *is a spherically closed submodule of* $\mathbf{N}$*, and set*

$$w(b+M) := \max_{c\in M} v(b-c) \quad \textit{for every } b\in N\,.$$

Then w is well-defined, and it is a valuation of N/M. Further, $w(N/M)\subset vN$, and $w(b+M)=vb$ holds if and only if b is a proper representative. If v is component-compatible, then so is w.

Proof: Since $\max_{c\in M} v(b-c)$ does not depend on the representative b of the coset $b+M$, it follows that w is well-defined. We have $w(0+M) = \max_{c\in M} v(0-c) = \infty$, but if $b\notin M$, then $0\notin b+M$ and $w(b+M) = \max_{c\in M} v(b-c) \neq \infty$; the maximum exists by our assumption that $\mathbf{M}$ be spherically closed in $\mathbf{N}$. By definition, b is a proper representative if and only if this maximum is equal to vb. To prove that $w(b+b'+M)\geq \min\{w(b+M), w(b'+M)\}$, we can assume without loss of generality that b, b' are proper representatives. Then

$$\begin{aligned} w(b+b'+M) &= \max_{c\in M} v(b+b'-c) \;\geq\; v(b+b') \\ &\geq \min\{vb, vb'\} \;=\; \min\{w(b+M), w(b'+M)\}\,. \end{aligned}$$

(Note that in general, $b+b'$ is not a proper representative of $b+b'+M = (b+M)+(b'+M)$.) If we assume in addition that v is component-compatible, then we obtain for every $r\in R$:

$$w(rb+M)\geq vrb\geq vb = w(b+M)\,,$$

which shows that w is component-compatible. □

Lemma 3.10 *Let* $\mathbf{M}\subset\mathbf{N}'\subset\mathbf{N}$ *be valued modules with* $\mathbf{M}$ *spherically closed in* $\mathbf{N}$.
a) $\mathbf{N}'$ *is spherically closed in* $\mathbf{N}$ *if and only if* $(N'/M, w)$ *is spherically closed in* $(N/M, w)$.
b) $\mathbf{N}'\subset\mathbf{N}$ *is immediate if and only if* $(N'/M, w)\subset(N/M, w)$ *is.*

Proof:
a): $\Rightarrow$: Suppose that $\mathbf{N}'$ is spherically closed in $\mathbf{N}$. Then for every $b\in N$ there is $b'\in N'$ such that $v(b-b') = \max_{c'\in N'} v(b-c')$. Hence, $w(b-b'+M) = \max_{c\in M} v(b-b'-c) = v(b-b') = \max_{c'\in N'}\max_{c\in M} v(b-c'-c) = \max_{c'\in N'} w(b-c'+M)$, showing that $(N'/M, w)$ is spherically closed in $(N/M, w)$.

$\Leftarrow$: Suppose that $(N'/M, w)$ is spherically closed in $(N/M, w)$. Then for every every $b \in N$ there is $b' \in N'$ (which we can choose to be a proper representative of $b' + M$) such that $v(b - b') = w(b - b' + M) = \max_{c' \in N'} w(b - c' + M) = \max_{c' \in N'} \max_{c \in M} v(b - c' - c) = \max_{c' \in N'} v(b - c')$, showing that $\mathbf{N}'$ is spherically closed in $\mathbf{N}$.

b): $\Rightarrow$: Suppose that $\mathbf{N}' \subset \mathbf{N}$ is immediate. We show that for every $b \in N$ there exists some $b' \in N'$ such that $w(b - b' + M) > w(b + M)$. We assume that b is a proper representative of $b + M$. By hypothesis, there exists some $b' \in N'$ such that $v(b - b') > vb$. This implies that $w(b - b' + M) = \max_{c \in M} v(b - b' - c) \geq v(b - b') > vb = w(b + M)$.
$\Leftarrow$: Suppose that $(N'/M, w) \subset (N/M, w)$ is immediate, and let $b \in N$. There exists some $c' \in N'$ such that $w(b - c' + M) > w(b + M)$, i.e. $\max_{c \in M} v(b - c' - c) > \max_{c \in M} v(b - c) \geq vb$. Hence there exists $c \in M$ such that $v(b - c' - c) > vb$, so $b' = c' + c \in N'$ is an element which satisfies $v(b - b') > vb$. $\square$

Exercise 3.2 *Assume that* $\mathbf{M}$ *is spherically closed in* $\mathbf{N}$.
a) Show that the converse of the last assertion of Lemma 3.9 does not hold in general; that is, w *may be component-compatible while* v *is not.*
b) Assume that for every $r \in R$, *if* b *is a proper representative of* $b + M$ *then* rb *is a proper representative of* $rb + M$. *Use Lemma 3.3 to show that* $(N/M, w)$ *is ordinary if* $\mathbf{N}$ *is ordinary. Does this also hold without our additional assumption? Show that the assumption holds if* R *is a (skew) field.*

3.3 Valuation independence

In this section, we will consider the following situation:

R	a ring
$\mathbf{N} = (N, v)$	a valued R-module
$\mathbf{M}$	an R-submodule of $\mathbf{N}$, equipped with the restriction of v (which we will again denote by v).

If $\mathcal{B}$ is a subset of N, then we let $M + R\mathcal{B}$ denote the submodule of N and $\mathbf{M} + R\mathcal{B}$ the valued submodule of $\mathbf{N}$ generated over M by the elements of $\mathcal{B}$. Similarly, if $\underline{\mathcal{B}}$ is a subset of $\operatorname{sk}\mathbf{N}$, then $\operatorname{sk}\mathbf{M} + R\underline{\mathcal{B}}$ denotes the subskeleton of $\operatorname{sk}\mathbf{N}$ generated over $\operatorname{sk}\mathbf{M}$ by the elements of $\underline{\mathcal{B}}$; that is, it is the set of all admissible linear combinations of bones in $\operatorname{sk}\mathbf{M} \cup \underline{\mathcal{B}}$. If $\mathbf{N}$ is component-compatibly valued, then for every $\alpha \in vN$, the α-component of $\operatorname{sk}\mathbf{M} + R\underline{\mathcal{B}}$ is just the R-module generated over the α-component of $\operatorname{sk}\mathbf{M}$ by $\{\zeta \mid (\alpha, \zeta) \in \underline{\mathcal{B}}\}$.

A subset $\mathcal{B} \subset N$ is called **R-independent over $\mathbf{M}$** if $0 \notin \mathcal{B}$ and for every choice of finitely many $r_i \in R$, $b_i \in \mathcal{B}$, where we let i range over some finite index I, and for every $a \in M$,

$$\sum_{i \in I} r_i b_i + a = 0 \implies \forall i \in I : \; r_i b_i = 0 \;\text{ and }\; a = 0 \,.$$

Note: whenever we write "$\sum r_i b_i$ with $b_i \in \mathcal{B}$", then we implicitly mean that all b_i are distinct.

We will say that $\mathcal{B} \subset N$ is **R-valuation independent over $\mathbf{M}$** if $0 \notin \mathcal{B}$ and for every choice of finitely many $r_i \in R$, $b_i \in \mathcal{B}$, $i \in I$, and every $a \in M$,

$$v\left(\sum_{i \in I} r_i b_i + a\right) = \min_{i \in I}\{v r_i b_i, va\} \,.$$

Further, $\mathcal{B} \subset N$ is called **R-valuation independent** if it is R-valuation independent over the zero module. Recall that we have already defined the notion of valuation independent subsets of a valued abelian group. Using this notion, our above definition reads as follows: $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$ if and only if $0 \notin \mathcal{B}$ and for every n and every choice of $b_1, \ldots, b_n \in \mathcal{B}$, the submodules $Rb_1, \ldots, Rb_n, M$ are valuation independent in $\mathbf{N}$. The reader may prove the following characterization:

Lemma 3.11 *A subset $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if every finite subset of $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$. A finite subset $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if the ultrametric space $(M + R\mathcal{B}, v)$ is the product of the ultrametric spaces (M, v) and (Rb, v), $b \in \mathcal{B}$.*

If $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$, then it is R-independent over $\mathbf{M}$ since otherwise, there would exist finitely many $r_i \in R$, $b_i \in \mathcal{B}$, $i \in I$, and $a \in M$ such that some $r_i b_i \neq 0$ and

$$v\left(\sum_{i\in I} r_i b_i + a\right) = \infty \neq \min_{i\in I}\{vr_i b_i, va\} .$$

This motivates the next definition. If a set $\mathcal{B}$ of generators of the extension $\mathbf{M} \subset \mathbf{N}$ is R-valuation independent over $\mathbf{M}$, then it will be called an **R-valuation basis of $\mathbf{N}$ over $\mathbf{M}$** or an **R-valuation basis of $\mathbf{M} \subset \mathbf{N}$**. Further, $\mathbf{M} \subset \mathbf{N}$ will be called a **defectless extension of valued modules** if every finite subextension admits an R-valuation basis over $\mathbf{M}$. Note that this does not imply that $\mathbf{N}$ itself admits an R-valuation basis over $\mathbf{M}$. For instance, the extension $0 \amalg \mathbb{Z} \subset \mathbb{Q} \amalg \mathbb{Z}$ is a defectless extension of valued $\mathbb{Z}$-modules, but $\mathbb{Q} \amalg \mathbb{Z}$ does not even admit a $\mathbb{Z}$-basis over $0 \amalg \mathbb{Z}$. Later, we will meet examples of another type (cf. Examples 3.62). Note that it is not really necessary to exclude 0 from R-independent and R-valuation independent sets. We have done that in view of the above definition of an R-valuation basis and our later definition of the valuation basis of an extension of valued fields, since a "basis" is usually not supposed to contain 0. However, with the above definition, it may still contain elements $b \in N$ such that $Rb = \{0\}$, but this is only possible if R does not contain 1.

Lemma 3.12 *The set $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if for every finite sum $b = \sum_{i\in I} r_i b_i + a$ of elements $b_i \in \mathcal{B}$ with $r_i \in R$ and $a \in M$,*

$$\min_{i\in I} vr_i b_i \;=\; \max_{c\in M} v(b-c) . \tag{3.2}$$

If this equation holds, then $v(b-a) = \max_{c\in M} v(b-c)$.

Proof: Suppose first that $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$. Then for every $c \in M$, $v(b-a) = \min_{i\in I} vr_i b_i \geq \min_{i\in I}\{vr_i b_i, v(a-c)\} = v(b-c)$, which yields equation (3.2). For the converse, suppose that (3.2) holds for all b. Let $\gamma = \min_{i\in I} vr_i b_i$. If $v(a-c) < \gamma$, then

$$v\left(\sum_{i\in I} r_i b_i + a - c\right) = v(a-c) = \min_{i\in I}\{vr_i b_i, v(a-c)\} . \tag{3.3}$$

Now let $v(a-c) \geq \gamma$. Then by (3.2), $v(b-c) \leq \min_{i\in I} vr_i b_i = \min_{i\in I}\{vr_i b_i, v(a-c)\} \leq v(b-c)$, hence equality holds everywhere.

Now suppose that (3.2) holds for b. Then $v(b-a) \geq \min_{i\in I} vr_ib_i = \max_{c\in M} v(b-c) \geq v(b-a)$, hence equality holds everywhere. This completes our proof. $\square$

Corollary 3.13 *If* $\mathbf{N}$ *admits an* R*-valuation basis* $\mathcal{B}$ *over* $\mathbf{M}$*, then* $\mathbf{M}$ *is spherically closed in* $\mathbf{N}$*. The same holds if* $\mathbf{M} \subset \mathbf{N}$ *is a defectless extension.*

Proof: First, let $\mathcal{B}$ be an R-valuation basis of $\mathbf{N}$ over $\mathbf{M}$. If $b \in N \setminus M$ then write $b = \sum_{i\in I} r_ib_i + a$ with elements $b_i \in \mathcal{B}$, $r_i \in R$, $a \in M$. Then by the previous lemma, $b-a$ is a proper representative for $b+M$.

Now assume that $\mathbf{M} \subset \mathbf{N}$ is a defectless extension. Then for every $b \in N$, the module $\mathbf{M} + Rb$ admits an R-valuation basis over $\mathbf{M}$. By what we have shown, it follows that $\mathbf{M}$ is spherically closed in $\mathbf{M} + Rb$. Hence, $\mathbf{M}$ is spherically closed in $\mathbf{N}$. $\square$

Lemma 3.14 *Assume that* $\mathbf{M}$ *is spherically closed in* $\mathbf{N}$*, and let* $\mathcal{B} \subset N$*. Then* $\mathcal{B}$ *is* R*-valuation-independent with respect to* v *over* $\mathbf{M}$ *if and only if* $\{b+M\,;\, b\in\mathcal{B}\}$ *is* R*-valuation-independent with respect to* w *and for all* $r \in R$ *and* $b \in \mathcal{B}$*, the elements* rb *are proper representatives of* $rb + M$*. In this case, every finite sum* $\sum_{i\in I} r_ib_i$ *with* $b_i \in \mathcal{B}$ *is a proper representative of* $\sum_{i\in I} r_i(b_i+M)$*.*

Proof: If b is R-valuation-independent with respect to v over $\mathbf{M}$, then it is a proper representative of $b+M$ by virtue of Lemma 3.12. Since

$$\max_{c\in M} v\left(\sum_{i\in I} r_ib_i + a - c\right) = w\left(\sum_{i\in I} r_ib_i + M\right) = w\left(\sum_{i\in I} r_i(b_i+M)\right)$$

and

$$\min_{i\in I} vr_ib_i = \min_{i\in I} w(r_ib_i + M)$$

for the proper representatives r_ib_i, our assertion follows from Lemma 3.12. $\square$

From this lemma and the foregoing corollary, we obtain:

Corollary 3.15 *If* $\mathbf{N}$ *admits an* R*-valuation basis over* $\mathbf{M}$*, then* $\mathbf{M}$ *is spherically closed in* $\mathbf{N}$ *and* $(N/M, w)$ *admits an* R*-valuation basis. If* $\mathbf{M} \subset \mathbf{N}$ *is defectless, then* $\mathbf{M}$ *is spherically closed in* $\mathbf{N}$ *and* $(N/M, w)$ *is a defectless extension of* 0*.*

For ordinary valued vector spaces, also the converses of these assertions hold; cf. Corollary 3.23 below.

The proof of the following lemma is left to the reader.

Lemma 3.16 *Let* $\mathcal{B}$*,* $\mathcal{B}'$ *be subsets of* N*. Then* $\mathcal{B} \cup \mathcal{B}'$ *is* R*-valuation independent over* $\mathbf{M}$ *if and only if* $\mathcal{B}$ *is* R*-valuation independent over* $\mathbf{M}$ *and* $\mathcal{B}'$ *is* R*-valuation independent over* $\mathbf{M} + R\mathcal{B}$*. Further, the union of an ascending chain of* R*-valuation independent sets over* $\mathbf{M}$ *is again* R*-valuation independent over* $\mathbf{M}$*.*

A subset $\underline{\mathcal{B}} \subset \operatorname{sk}\mathbf{N}$ will be called R**-independent over** $\operatorname{sk}\mathbf{M}$ if for every admissible R-linear combination $\sum_i r_i(\alpha_i, \zeta_i)$ of bones in $\underline{\mathcal{B}} \cup \operatorname{sk}\mathbf{M}$,

$$\sum_i r_i(\alpha_i, \zeta_i) = (\alpha, 0) \implies \forall i: \; r_i(\alpha_i, \zeta_i) = (\infty, 0) \; .$$

Lemma 3.17 *Let $\mathcal{B} \subset N \setminus \mathrm{Exe}\mathbf{N}$. Then $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$ if and only if $\underline{\mathcal{B}}$ is R-independent over $\operatorname{sk}\mathbf{M}$.*

Proof: $\Rightarrow$: Suppose there are $b_1, \ldots, b_n \in \mathcal{B}$ and $r_1, \ldots, r_n \in R$, not all equal to zero, and $a \in M$ such that $\sum r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$ is an admissible linear combination which equates to zero. Then $vr_1b_1 = \ldots = vr_nb_n = va$ and $\sum r_i \operatorname{co}_{va} b_i + \operatorname{co}_{va} a = 0$. We obtain

$$v(r_1b_1 + \ldots + r_nb_n + a) > va = \min_{i \in I}\{vr_ib_i, va\} \; ,$$

which shows that $\mathcal{B}$ is not R-valuation independent over $\mathbf{M}$.
$\Leftarrow$: Let $\sum_{i \in I} r_ib_i$ be a finite sum of elements $b_i \in \mathcal{B}$ with $r_i \in R$ and let $a \in M$. Let $\gamma = \min_{i \in I} vr_ib_i$. If $va < \gamma$, then (3.3) holds. Assume now that $va \geq \gamma$. Let $J \subset I$ contain precisely the indices for which $r_ib_i \neq 0$ and $vr_ib_i = \gamma$. By assumption, the bones of b_j, $j \in J$, are R-independent over $\operatorname{sk}\mathbf{M}$. Since all b_i are ordinary, this implies

$$\operatorname{co}_\gamma \left(\sum_{i \in J} r_ib_i + a \right) = \sum_{i \in J} \operatorname{co}_\gamma r_ib_i + \operatorname{co}_\gamma a \neq 0 \; ,$$

whence

$$v \left(\sum_{i \in I} r_ib_i + a \right) = \gamma = \min_{i \in I} vb_i = \min_{i \in I}\{vb_i, va\} \; .$$

□

Corollary 3.18 *Let $\mathcal{B} \subset N \setminus \mathrm{Exe}\mathbf{N}$ be R-valuation independent over $\mathbf{M}$. Then $\underline{\mathcal{B}}$ is an R-independent set of generators of $\operatorname{sk}(\mathbf{M} + R\mathcal{B})$ over $\operatorname{sk}\mathbf{M}$.*

Proof: In view of the last lemma, we only have to show that $\mathcal{B}$ generates $\operatorname{sk}(\mathbf{M} + R\mathcal{B})$ over $\operatorname{sk}\mathbf{M}$. For $b \in M + R\mathcal{B}$, let us write

$$b = \sum_{i=1}^{n} r_ib_i + a \quad \text{with} \quad r_i \in R, \; b_i \in \mathcal{B}, \; a \in M \; . \tag{3.4}$$

We assume (after a suitable renumeration) that precisely $r_1b_1, \ldots, r_mb_m, a$ are the summands of least value γ in (3.4). If va is smaller than the value of the other summands, then omit the elements r_ib_i and their bones in the following. On the other hand, if va is greater than the value of some summand r_ib_i, then omit a and its bone. Since all b_i are ordinary, the linear combination $\sum_{i=1}^m r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$ is admissible. In view of our assumption and the foregoing lemma, $\sum_{i=1}^m \operatorname{co}_\gamma r_ib_i + \operatorname{co}_\gamma a \neq 0$. Hence the bone of b is equal to $\sum_{i=1}^m r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$. □

For the conclusion of this section, we will consider ordinary valued K-vector spaces, where K is a field or a skew field. We need the following characterization of ordinary valued vector spaces:

Lemma 3.19 *Let (V, v) be a valued K-vector space. Then (V, v) is ordinary if and only if*

$$va < va' \iff vra < vra' \tag{3.5}$$

holds for all $a, a' \in V$ and every $r \in K \setminus \{0\}$.

This is a direct consequence of Lemma 3.3 since every $r \in K \setminus \{0\}$ is invertible. For the same reason, for an extension $(W, v) \subset (V, v)$ of ordinary valued K-vector spaces every bone in $\operatorname{sk}(V, v) \setminus \operatorname{sk}(W, v)$ is already K-independent over $\operatorname{sk}(W, v)$. This shows:

Lemma 3.20 *Let $(W, v) \subset (V, v)$ be an extension of ordinary valued K-vector spaces and $b \in V$. If $(vb, \operatorname{co} b) \notin \operatorname{sk}(W, v)$, that is, if $vb = \max\{v(b - c) \mid c \in W\}$, then $\{b\}$ is a valuation basis of $(W + Kb, v)$ over W. Hence if (W, v) is spherically closed in (V, v), then every K-subspace (V', v) of (V, v) of dimension 1 over W admits a valuation basis over W.*

In view of Lemma 3.17 and Corollary 3.18 we deduce that if $\mathcal{B} \subset V$ is a maximal subset which is K-valuation independent over W then $\operatorname{sk}(W + K\mathcal{B}) = \operatorname{sk} W + K\underline{\mathcal{B}} = \operatorname{sk} V$. The existence of such maximal $\mathcal{B}$ is shown by Zorn's Lemma together with Lemma 3.16. We have proved:

Lemma 3.21 *For every extension $(W, v) \subset (V, v)$ of ordinary valued K-vector spaces there exists a subextension $(W, v) \subset (V', v)$ such that (V', v) admits a K-valuation basis over W and $(V', v) \subset (V, v)$ is immediate. Every subset of V which is maximal K-valuation independent over W, generates a subextension V' with these properties.*

For ordinary valued vector spaces, we can improve our results on quotients:

Lemma 3.22 *Let $(W, v) \subset (V, v)$ be ordinary valued K-vector spaces. If $b \in V$ and $0 \neq r \in K$, then b is a proper representative of $b + W$ if and only if rb is a proper representative of $rb + W$.*

Proof: From Lemma 3.19, we obtain that $vb = \max_{c \in W} v(b - c)$ if and only if $vrb = \max_{c \in W} v(rb - rc)$. Since $rW = W$, we have $\max_{c \in W} v(rb - rc) = \max_{c \in W} v(rb - c)$, which proves our assertion. □

From this lemma together with Lemma 3.14, we deduce:

Corollary 3.23 *Assume $(W, v) \subset (V, v)$ to be ordinary valued K-vector spaces with (W, v) spherically closed in (V, v). Let $\mathcal{B} \subset V$. If $\{b + W\,;\, b \in \mathcal{B}\}$ is K-valuation-independent in $(V/W, w)$ and every $b \in \mathcal{B}$ is a proper representative of $b + W$, then $\mathcal{B}$ is K-valuation independent over W. Consequently, if $(V/W, w)$ admits an K-valuation basis, then (V, v) admits a K-valuation basis over W, and if $(V/W, w)$ is a defectless extension of 0, then $(W, v) \subset (V, v)$ is defectless.*

See Section 3.7 below for further details on defectless extensions of ordinary valued vector spaces.

3.4 Value preserving scalar multiplication

A valued R-module $\mathbf{M}$ will be said to have **value preserving scalar multiplication** if it satisfies

(VPSM) $\quad \forall x \in M\, \forall r \in R \setminus \{0\}:\ vrx = vx\,.$

In particular, every valued R-module with value preserving scalar multiplication is component-compatibly valued. Conversely, if a component-compatibly module is a vector space, then it has value preserving scalar multiplication. Indeed, for every nonzero invertible $r \in R$, we must have $vx = vr^{-1}rx \geq vrx \geq vx$, hence equality. We can generalizes this observation as follows. Let $\mathbf{M}$ be a component-compatibly valued R-module, $a \in M$ and $0 \neq r \in R$. Then $r\operatorname{co} a = 0$ if and only if $vra > va$. This proves:

Lemma 3.24 *A valued R-module has value preserving scalar multiplication if and only if it is component-compatible and every component is a torsion free R-module. In particular, a valued abelian group has value preserving scalar multiplication if and only if every component is torsion free.*

Examples 3.25 Every ordered group with its natural valuation is a valued $\mathbb{Z}$-module with value preserving scalar multiplication since it satisfies (NVZ). If in addition, this group is divisible, then it is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication.

The **socle of a p-group** is the set of all elements of order p. It is consequently an $\mathbb{F}_p$-vector space. Restricting the height function to the socle, we obtain a p-valuation on this $\mathbb{F}_p$-vector space with value preserving scalar multiplication.

The set $K^\times/K^{\times 2}$ of square classes of a field K is an $\mathbb{F}_2$-vector space. If the field has a valuation, then under certain conditions, it induces a valuation on $K^\times/K^{\times 2}$ $\diamondsuit$

As a direct consequence of Lemma 3.3, we get:

Lemma 3.26 *Every valued module with value preserving scalar multiplication is ordinary.*

In the case of value preserving scalar multiplication, a linear combination of bones is admissible if and only if all bones show the same value in their first component. Then we have $\sum_i r_i(\alpha,\zeta_i) = (\alpha, \sum_i r_i\zeta_i)$. Consequently, the condition that a set $\underline{\mathcal{B}}$ of bones be R-independent over $\operatorname{sk}\mathbf{M}$ is equivalent to the condition that for all $\alpha \in vN$, the set $\underline{\mathcal{B}}_\alpha := \{\zeta \mid (\alpha,\zeta) \in \underline{\mathcal{B}}\}$ is R-independent over $\mathrm{C}^\alpha\mathbf{M}$. Hence, the following is a corollary to Lemma 3.17:

Corollary 3.27 *Suppose that $\mathbf{N}$ has value preserving scalar multiplication. Then $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if for every $b_1,\ldots,b_n \in \mathcal{B}$ of equal value γ, the elements $\operatorname{co}_\gamma b_1,\ldots,\operatorname{co}_\gamma b_n \in \mathrm{C}^\gamma\mathbf{N}$ are R-independent over $\mathrm{C}^\gamma\mathbf{M}$.*

Observe that in view of Lemma 3.24 which tells us that the components are torsion free R-modules, R-independence is equivalent to R-linear independence: The elements $\zeta_1,\ldots,\zeta_n$ are said to be **R-linear independent** if $\sum r_i\zeta_i = 0$ implies $\forall i:\ r_i = 0$.

The above corollary yields that $\mathcal{B} \subset N$ is maximal with the property of being R-valuation independent over $\mathbf{M}$ if and only if for every $\alpha \in vN$, $\underline{\mathcal{B}}_\alpha$ is maximal with the property of being R-independent over $\mathrm{C}^\alpha\mathbf{M}$. In the case of vector spaces, where R is a field (or a skew field), a maximal R-independent set is just a basis. So we obtain the following corollary to Lemma 3.17:

Corollary 3.28 *Let K be a field or a skew field and $\mathbf{W} \subset \mathbf{V}$ an extension of valued K-vector spaces with value preserving scalar multiplication.*

a) A subset $\mathcal{B} \subset V$ is maximal with the property of being K-valuation independent over W if and only if for every $\alpha \in vV$, the set $\underline{\mathcal{B}}_\alpha = \{\mathrm{co}_\alpha\, b \mid b \in \mathcal{B} \text{ and } vb = \alpha\}$ forms a basis of $\mathrm{C}^\alpha\mathbf{V}$ over $\mathrm{C}^\alpha\mathbf{W}$.

b) Let $V' \subset V$ be as in Lemma 3.21. Reading the dimensions as (finite or infinite) cardinals,

$$\dim_K V/W \geq \dim_K V'/W = \sum_{\alpha\in vV} \dim_K \mathrm{C}^\alpha\mathbf{V}\,/\mathrm{C}^\alpha\mathbf{W} \geq |vV \setminus vW|\,. \tag{3.6}$$

The last inequality of (3.6) holds since for every $\alpha \in vV \setminus vW$, we have $\mathrm{C}^\alpha\mathbf{W} = \{0\}$ whereas $\mathrm{C}^\alpha\mathbf{V}$ contains at least one copy of K, showing that $\dim_K \mathrm{C}^\alpha\mathbf{V}\,/\mathrm{C}^\alpha\mathbf{W} \geq 1$.

We apply this corollary to the divisible hulls of ordered abelian groups.

Corollary 3.29 *Let $(H,<) \subset (G,<)$ be an extension of ordered abelian groups and let v be the natural valuation of G. Then*

$$\mathrm{rr}\, G/H \geq \sum_{\alpha\in vG} \mathrm{rr}\ (\mathrm{C}^\alpha\mathbf{G}\,/\mathrm{C}^\alpha\mathbf{H}) \geq |vG \setminus vH|\,. \tag{3.7}$$

In particular,

$$\mathrm{rr}\, G \geq \sum_{\alpha\in vG} \mathrm{rr}\, \mathrm{C}^\alpha\mathbf{G} \geq |vG| \tag{3.8}$$

and if G has finite rank, then $\mathrm{rr}\, G \geq \mathrm{rk}\, G$.

Proof: The reader may verify that for every extension $H \subset G$ of abelian groups, $\mathbb{Q}\otimes_{\mathbb{Z}} G/H \cong (\mathbb{Q}\otimes_{\mathbb{Z}} G)/(\mathbb{Q}\otimes_{\mathbb{Z}} H)$, whence $\dim_{\mathbb{Q}} \mathbb{Q}\otimes_{\mathbb{Z}} G/H = \dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}} G)/(\mathbb{Q}\otimes_{\mathbb{Z}} H)$. Applying this to the given extension as well as to the extensions of components, inequality (3.7) follows from part b) of the previous corollary.

Inequalities (3.8) follow from (3.7) with $H = \{0\}$. For groups of finite rank, the rank is equal to the principal rank $|vG|$, so the inequality $\mathrm{rr}\, G \geq \mathrm{rk}\, G$ follows from (3.8). $\square$

It can be shown that also for infinite rank, the cardinality of the rank is equal to the cardinality of the principal rank. So replacing the rank by its cardinality, the last inequality remains true for arbitrary ordered groups.

Using the upper bound for $vV \setminus vW$ given in part b) of Corollary 3.28, we can show the following important theorem. It was exploited by R. Brown [BRW1] to classify countably-dimensional valued vector spaces with value preserving scalar multiplication by their skeletons (see Theorem 3.46 below).

Theorem 3.30 *Every countably-dimensional valued vector space with value preserving scalar multiplication admits a valuation basis.*

Proof: Let (V,v) be such a valued K-vector space. Then it is the union of finite-dimensional valued vector spaces (V_i,v), $i \in \mathbb{N}$, such that $V_1 = 0$ and $\dim_K V_{i+1}/V_i = 1$ for every i. By part b) of Corollary 3.28, every value set vV_i is finite and thus, every (V_i,v) is spherically complete. Hence by Lemma 3.6, every (V_i,v) is maximal and consequently, the extension $(V_i, v) \subset (V_{i+1}, v)$ is not immediate. In view of Lemma 3.21 and $\dim_K V_{i+1}/V_i = 1$, it follows that every (V_{i+1}, v) admits a K-valuation basis $\{b_i\}$ over V_i. Now it follows from Lemma 3.16 that $\{b_i \mid i \in \mathbb{N}\}$ is a K-valuation basis of (V, v) over $V_1 = 0$. $\square$

This theorem was generalized by S. Kuhlmann [KUS1] in the following way: *If $(W, v) \subset (V, v)$ is a countably generated extension of valued K-vector spaces with value preserving scalar multiplication and if (W, v) admits a K-valuation basis, then (V, v) also admits a K-valuation basis.* Note that this does not mean that (V, v) would admit a K-valuation basis over W: this is not true in general. However, it is true if (W, v) is spherically closed in (V, v):

Corollary 3.31 *Let $(W, v) \subset (V, v)$ be valued vector spaces with value preserving scalar multiplication. If (W, v) is spherically closed in (V, v) and V/W is of countable dimension, then (V, v) admits a valuation basis over W.*

Proof: We endow V/W with the quotient valuation w. By Theorem 3.30, $(V/W, w)$ admits a valuation basis. From Lemma 3.26 we know that (V, v) is ordinary valued. Now our assertion follows from Corollary 3.23. $\square$

From this corollary together with Corollary 3.13, we obtain the following result that was proved by S. Kuhlmann in [KUS1]:

Theorem 3.32 *Let $(W, v) \subset (V, v)$ be valued vector spaces with value preserving scalar multiplication. Then (W, v) is spherically closed in (V, v) if and only if the extension $(W, v) \subset (V, v)$ is defectless.*

Note that Theorem 3.30 does not remain true if the assumption of value preserving scalar multiplication is omitted. The simplest counterexample is that of a proper immediate extension $(K, v) \subset (L, v)$ of valued fields (see below): if we choose $a \in L \setminus K$ then the 2-dimensional valued K-vector space $(K + Ka, v)$ does not admit a K-valuation basis.

We should mention that in a category of valued vector spaces with value preserving scalar multiplication with suitable morphisms, those admitting a valuation basis are precisely the **free** ones. See L. Fuchs [FU2], p. 29.

If all R-modules C_γ are torsion free, then the Hahn sum $\coprod_{\gamma\in\Gamma} C_\gamma$ and the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$ are valued R-modules with value preserving scalar multiplication. If R is a field, then the Hahn sum admits a natural R-valuation basis. Indeed, for every γ we take an R-basis $\underline{\mathcal{B}}_\gamma$ of C_γ and for every element ζ of this basis, we take e_ζ to be the tuple which has ζ at the γ-th entry and zeros everywhere else. Then the elements $\{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\gamma \wedge \gamma \in \Gamma\}$ generate $\coprod_{\gamma\in\Gamma} C_\gamma$. On the other hand, they are R-valuation independent since their bones are R-independent. This shows that $\{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\gamma \wedge \gamma \in \Gamma\}$ is an R-valuation basis of $\coprod_{\gamma\in\Gamma} C_\gamma$.

Remark 3.33 For a more detailed investigation of valued vector spaces with value preserving scalar multiplication, see [BRW1], [FU2], [FU3], [GRA1], [HI1], [HI2], [HI–WH], [KUS1] and their references. Much of the material on valued vector spaces as exposed in this chapter, is taken from S. Kuhlmann's thesis [KUS1].

Exercise 3.3 *Let $(W, v) \subset (V', v) \subset (V, v)$ be valued vector spaces with value preserving scalar multiplication. Suppose that (W, v) is spherically closed in (V, v) and that V'/W is of countable dimension. Prove that (V', v) is spherically closed in (V, v).*

Exercise 3.4 *Prove that every finitely generated valued R-module $\mathbf{M}$ with value preserving scalar multiplication has a finite value set. More precisely, assume that M is generated by n elements, and show that $|vM| \leq n$. Assume in addition that all components are free R-modules, and show that $\mathbf{M}$ admits a valuation basis with at most n elements.*

3.5 p-bases and straight bases of p-groups

If we consider an abelian group with a p-valuation as a valued $\mathbb{Z}$-module, then we do not have value preserving scalar multiplication. Nevertheless, there is an analogue of Brown's Theorem (3.30), which we shall discuss now. See Section 2.4 for the definition of the p-height function, the p-height valuation and the order valuation.

Throughout this section, let G be an abelian p-group. We will consider G with its p-height valuation v. If ν is the p-height of G, i.e. $\nu\infty$ is the set of heights in G, then $vG = \nu$ or $vG = \nu + 1$ depending on whether G is reduced or not. If $va < \nu$, then $vpa > va$; that is, $p \operatorname{co}_{va} a = 0$. This shows that for every $\mu < \nu$, the μ-component $\mathrm{C}_\mu(G, v)$ is an $\mathbb{F}_p$-vector space. This is not true for $\mathrm{C}_\nu(G, v)$ if it is non-trivial.

In the following, we will prove some facts for arbitrary valuations v on the p-group G. For this, we need some preparation.

Lemma 3.34 *Let (G, v) be an arbitrary valued group, $a \in G$ and p a prime.*

a) If $\mathrm{C}_{va}(G, v)$ is an $\mathbb{F}_p$-vector space, then $vpa > va$ and $vna = va$ for every integer n which is prime to p.

b) If G is a p-group, then $vna = va$ for every integer n which is prime to p.

Proof: a): Let $\mathrm{C}_{va}(G, v)$ be an $\mathbb{F}_p$-vector space. Then $p \operatorname{co}_{va} a = 0$ and thus $vpa > va$. If n prime to p, then there exist integers r, s such that $1 = rn + sp$. By (VZ) and what we have already shown, $vspa \geq vpa > va$. Hence $va = v(rn + sp)a = v(rna + spa) = vrna$. Again by (VZ), $vrna \geq vna \geq va$. Altogether, we obtain that $va = vna = nrna$.

b): Since in a p-group every element has order a power of p, the assertion follows from Lemma 2.1. $\square$

Note that the assertion of part b) in particular holds for the p-height valuation v, even if $\mathrm{C}_\nu(G, v)$ is non-trivial; that is, G is not reduced.

Lemma 3.35 *Let G be an abelian p-group and v an arbitrary valuation on G. Let $\mathcal{B} \subset G \setminus \{0\}$ and assume that for every $\mu \in \{vb \mid b \in \mathcal{B}\}$, the component $\mathrm{C}_\mu(G, v)$ is an $\mathbb{F}_p$-vector space. Then the following conditions are equivalent:*

1) For every $\mathbb{Z}$-linear combination $\sum_{i=1}^m n_i b_i$ with $b_i \in \mathcal{B}$ and n_i integers prime to p,

$$v\left(\sum_{i=1}^m n_i b_i\right) = \min vn_i b_i \ .$$

(Since n_i is prime to p, we could write "vb_i" in the place of "$vn_i b_i$".)

2) Condition 1) holds whenever $0 < n_i < p$.

3) For every $\mu \in \{vb \mid b \in \mathcal{B}\}$, *the coefficients* $\mathrm{co}_{\mu} b$, $b \in \mathcal{B}$ *of value* $vb = \mu$, *are* $\mathbb{F}_p$*-independent (we could say: the bones of the elements* $b \in \mathcal{B}$ *are* $\mathbb{F}_p$*-independent).*

Recall that whenever we write "$\sum n_i b_i$ with $b_i \in \mathcal{B}$", then we implicitly mean that all b_i are distinct (unless stated otherwise).

Proof: 1)⇒2) is trivial. We prove 2)⇒3). Take a linear combination $\sum \overline{n}_i \mathrm{co}_{\mu} b_i$, where b_i are elements in $\mathcal{B}$ of value μ. We choose natural numbers $0 < n_i < p$ whose residues modulo p are $\overline{n}_i$. Condition 2) tells us that $v(\sum n_i b_i) = \mu$. Hence, $\sum \overline{n}_i \mathrm{co}_{\mu} b_i = \mathrm{co}_{\mu} \sum n_i b_i \neq 0$.

Now we prove 3)⇒1). Take a linear combination $\sum n_i b_i$, where b_i are elements of $\mathcal{B}$, and n_i are integers prime to p. Let $\mu = \min_i v n_i b_i = \min_i v b_i$. We have $\mathrm{co}_{\mu} \sum n_i b_i = \sum \overline{n}_i \mathrm{co}_{\mu} b_i$, where $\overline{n}_i$ denotes the residue of n_i modulo p. Since n_i is assumed to be prime to p, we have $\overline{n}_i \neq 0$ for all i. By our choice of μ, there is at least one i such that $\mathrm{co}_{\mu} b_i \neq 0$. By condition 3), $\sum \overline{n}_i \mathrm{co}_{\mu} b_i \neq 0$, showing that $v \sum n_i b_i = \mu = \min_i v n_i b_i$. □

A subset $\mathcal{B} \subset G$ will be called p'**-valuation independent in** (G, v) (read: "p-prime valuation independent") if it satisfies condition 1) of this lemma. This looks like valuation independence, but it is restricted to scalar multiplication with scalars which are prime to p (that is, preserve the value). Further, $\mathcal{B} \subset G \setminus \{0\}$ will be called a v-p**-basis of** G if

(vpB1) the elements of $\mathcal{B}$ are p'-valuation independent in (G, v),

(vpB2) $\mathcal{B}$ generates G,

(vpB3) $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$.

A subset $\mathcal{B} \subset G \setminus \{0\}$ of a p-group G is called a p**-basis of** G if it is a v-p-basis of G with respect to the p-height valuation v of G. A p-group admitting a p-basis is called **simply presented**.

We have used condition 1) of the above lemma for the definition of p'-valuation independence and thus also in the definition of a p-basis. In the literature, condition 2) in the place of condition 1) is commonly used for the definition of a p-basis. Nevertheless, this is equivalent to our definition. To show this, we establish the equivalence of conditions 1) and 2) for every subset $\mathcal{B}$ of a p-group, provided that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$. Even without this provision, the equivalence holds for all reduced p-groups by means of the foregoing lemma since for them every component is an $\mathbb{F}_p$-vector space. For the general case, we now give a proof that does not use Lemma 3.35.

Lemma 3.36 *Let* G *be a* p*-group and* $\mathcal{B} \subset G \setminus \{0\}$ *such that* $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$. *Then every finite sum* $a = \sum_{i=1}^{m} n_i b_i$, *with* b_i *(in this case not necessarily distinct) elements from* $\mathcal{B}$ *and* n_i *arbitrary integers, can be written as a (possibly trivial) sum* $a = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ *with* $0 < \tilde{n}_i < p$ *and* $\tilde{b}_i$ *distinct elements from the finite set*

$$\{p^j b_i \mid 1 \leq i \leq m \text{ and } j \geq 0 \text{ with } p^j b_i \neq 0\} \subset \mathcal{B}\,.$$

If all b_i *are distinct and all* n_i *are prime to* p, *then the latter sum is non-trivial (i.e.,* $\tilde{m} \geq 1$*); in particular, every element of maximal order among the* b_i *will appear as some* $\tilde{b}_i$.

As a consequence, conditions 1) and 2) of Lemma 3.35 are equivalent for every valued p*-group* (G, v) *and* $\mathcal{B} \subset G \setminus \{0\}$, *provided that* $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$.

Proof: We shall prove the first assertion by induction on the maximal order of the elements b_i appearing in the sum. Beforehand, if the elements b_i are not all distinct, we rewrite the sum so that all appearing b_i are distinct. Further, we can assume without loss of generality that all n_i are positive. Indeed, if $n_i < 0$, then we choose ν_i such that $p^{\nu_i} b_i = 0$ and add a multiple of p^{ν_i} to n_i in order to obtain a positive coefficient (which is still prime to p if n_i was). Now we use the p-adic expansion of every n_i: We represent n_i as a finite sum $\sum n_{i,j} p^j$ in which $0 \leq n_{i,j} < p$ for $j \geq 0$. Then $a = \sum_{i,j} n_{i,j} p^j b_i$. If all b_i have order p, then this sum is equal to $\sum_{i,0} n_{i,0} b_i$. If all b_i were distinct from the start and if all n_i were prime to p, then all $n_{i,0}$ are nonzero. This proves our assertion in the case where all b_i have order p.

Let us assume that we have proved our assertion for all sums in which the order of all elements b_i is smaller than p^k. Consider a sum $a = \sum_{i=1}^m n_i b_i$ where the elements $b_1, \ldots, b_{m'}$ have order p^k and the elements $b_{m'+1}, \ldots, b_m$ have smaller order, with $m' \leq m$. Proceeding as before, we find that

$$a = \sum_{i,j} n_{i,j} p^j b_i = \sum_{i=1}^{m'} n_{i,0} b_i + \sum_{i>m' \vee j>0} n_{i,j} p^j b_i \,.$$

Since $\mathrm{ord}(pb) < \mathrm{ord}(b)$ for every element b in a p-group, all $p^j b_i$ appearing in the latter sum have order smaller than p^k. So we can apply the induction hypothesis to find that the latter sum can be written as a sum $\sum_{i=m'+1}^{m''} \tilde{n}_i \tilde{b}_i$ with $0 < \tilde{n}_i < p$ and $\tilde{b}_i$ distinct elements from $\{p^j b_i \mid 1 \leq i \leq m \text{ and } j \geq 0 \text{ with } p^j b_i \neq 0 \text{ and } i \leq m' \Rightarrow j > 0\} \subset \mathcal{B}$. This latter set contains only elements of order smaller than p^k, so it does not contain $b_1, \ldots, b_{m'}$. Putting $\tilde{b}_i := b_i$ and $\tilde{n}_i := n_{i,0}$ for $1 \leq i \leq m'$, then omitting $\tilde{n}_i \tilde{b}_i$ if $\tilde{n}_i = 0$ and renumbering suitably, we arrive at a sum $a = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ which is of the required form. Again, if all b_i were distinct from the start and if all n_i were prime to p, then all $n_{i,0}$ are nonzero. This completes the proof of our first assertion.

Now we turn to the equivalence of conditions 1) and 2) of Lemma 3.35. 1)$\Rightarrow$2) is trivial. For the proof of 2)$\Rightarrow$1), take a finite sum $\sum_{i=1}^m n_i b_i$, where $b_i \in \mathcal{B}$ and n_i are integers prime to p. Let $\mu = \min_i v n_i b_i = \min_i v b_i$. If we are able to prove that $v \sum_{vb_i=\mu} n_i b_i = \mu$, then also $v \sum n_i b_i = \mu = \min_i v n_i b_i$. So we can assume from the start that $\mu = v b_i$ for all i. We write $\sum_{i=1}^m n_i b_i = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ according to our first assertion. Then all $\tilde{b}_i$ have a value $\geq \mu$ (this follows from (VZ) since every $\tilde{b}_i$ is a multiple of some b_j). On the other hand, at least one b_i (of maximal order) appears among the $\tilde{b}_i$, so condition 1) yields that $v \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i = \mu$, as required. □

We will also need the following corollary:

Corollary 3.37 *Assume that $\mathcal{B}$ is a v-p-basis of the valued p-group G. Suppose that*

$$v(nb - p \sum_{i=1}^m n_i b_i) > vnb$$

with b, b_i (not necessarily distinct) elements in $\mathcal{B}$, n_i arbitrary integers and n prime to p. Then there is some $j \geq 1$ and some i such that $b = p^j b_i$.

Proof: According to the foregoing lemma, we write $p \sum_{i=1}^m n_i b_i = \sum_{i=1}^m n_i p b_i = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ with $0 < \tilde{n}_i < p$ and $\tilde{b}_i$ distinct elements from the set $\{p^j p b_i \mid 1 \leq i \leq m \text{ and } j \geq$

0 with $p^j pb_i \neq 0\} \subset \mathcal{B}$. But $b \in \mathcal{B}$ and $v(nb - \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i) > vnb$ then show that some $\tilde{b}_i$ must be equal to b, by property (vpB1) of the v-p-basis $\mathcal{B}$. On the other hand, $\tilde{b}_i = p^j b_i$ with $j \geq 1$. □

In the common definitions of p-bases, a stronger version of condition (vpB2) is used. Namely, it is required that every element $a \in G$ *has a unique representation* $a = \sum n_i b_i$ with $b_i \in \mathcal{B}$ and $0 < n_i < p$. However, in the presence of (vpB1) and (vpB3), this condition is equivalent to condition (vpB2). Indeed, (vpB2) and (vpB3) imply the existence of such a representation, as follows from the first assertion of the foregoing lemma. The uniqueness of this representation follows from condition (vpB1): Assume that $a = \sum n_i b_i = \sum m_i b_i$ with $0 \leq n_i < p$ and $0 \leq m_i < p$ (we admit zero coefficients in order to let the sums run over the same set of elements b_i in $\mathcal{B}$). Then $0 = \sum (n_i - m_i) b_i$ with $n_i - m_i$ either zero or prime to p. If $n_i \neq m_i$ would hold for some i, then by p'-valuation independence, the value of this sum would be equal to the value of some b_i and thus $< \infty$, a contradiction. This proves the uniqueness of the representation.

For later use, we give a procedure for extending the p-basis of a subgroup of a p-group:

Lemma 3.38 *Let (G, v) be a valued p-group and H a subgroup of G with v-p-basis $\mathcal{B}_H$. If $a \in G$ is a proper representative of $a + H$ such that $pa \in \mathcal{B}_H \cup \{0\}$, then $\mathcal{B}_H \cup \{a\}$ is a v-p-basis of $H + \mathbb{Z}a$.*

Proof: Since $\mathcal{B}_H$ generates H, we see that $\mathcal{B}_H \cup \{a\}$ generates $H + \mathbb{Z}a$. Since $p\mathcal{B}_H \subset \mathcal{B}_H \cup \{0\}$ and $pa \in \mathcal{B}_H \cup \{0\}$, it follows that $p(\mathcal{B}_H \cup \{a\}) \subset \mathcal{B}_H \cup \{0\}$. The p'-valuation independence is seen as follows. We only have to show that va is the value of every linear combination $na + \sum n_i b_i$ with $b_i \in \mathcal{B}_H$ of value $vb_i = va$ and n, n_i prime to p. To see this, we multiply by an integer n' which satisfies $n'n \equiv 1$ modulo the order p^ν of a. Then $n'na = a$, and we obtain that $va \leq v(na + \sum n_i b_i) \leq v(a + \sum n' n_i b_i) \leq va$, where the latter inequality holds since a was assumed to be a proper representative. This proves that the value of the sum is indeed equal to va. □

The elements of a p-basis $\mathcal{B}$ will in general not be $\mathbb{Z}$-valuation independent unless all of them have order p, since if $b \in \mathcal{B}$ with $pb \neq 0$, then the linear combination $1 \cdot (pb) + (-p) \cdot b$ is zero. But throwing away the multiples, one can select subsets of $\mathcal{B}$ which are almost $\mathbb{Z}$-valuation independent: A subset $\mathcal{B} \subset G$ will be called **weakly $\mathbb{Z}$-valuation independent** if every $\mathbb{Z}$-independent subset of $\mathcal{B}$ is $\mathbb{Z}$-valuation independent. If in addition $\mathcal{B}$ generates G, then it will be called a **weak $\mathbb{Z}$-valuation basis of** G.

Lemma 3.39 *If $\mathcal{B}$ is a p-basis of the p-group G, then the subset $\mathcal{B}_0 \subset \mathcal{B}$ of elements of p-height 0 is weakly $\mathbb{Z}$-valuation independent and generates all elements in $\mathcal{B}$ of p-height $< \infty$. In particular, if G is reduced, then a weak $\mathbb{Z}$-valuation basis can be selected from every p-basis. Conversely, if $\mathcal{B}'$ is a weak $\mathbb{Z}$-valuation basis of the reduced p-group G, then $\mathcal{B} = \bigcup_{i=0}^{\infty} p^i \mathcal{B}'$ is a p-basis of G.*

Proof: Let $\mathcal{B}$ be a p-basis of the p-group G. Let $\mathcal{B}_0 \subset \mathcal{B}$ be the set of all elements of p-height 0 in $\mathcal{B}$. Since $p^j \mathcal{B}_0 \subset \mathcal{B}$ for all $j \in \mathbb{N}$, every $\mathbb{Z}$-linear combination of elements in $\mathcal{B}_0$ can be written as a $\mathbb{Z}$-linear combination of elements in $\mathcal{B}$ with coefficients prime to p. Since $p^j b = p^j b'$ is possible for $b, b' \in \mathcal{B}_0$ even if $b \neq b'$, we have to admit that an element of $\mathcal{B}$ appears more than once in the latter linear combination. If no element of $\mathcal{B}$ appears more than once, then the value of this linear combination is equal to the minimal value of its summands, because $\mathcal{B}$ is a p-basis. But if some element of $\mathcal{B}$ appears at least twice, then this means that the elements of $\mathcal{B}_0$ appearing in our original $\mathbb{Z}$-linear combination are not $\mathbb{Z}$-independent. This proves that the elements of $\mathcal{B}_0$ are weakly $\mathbb{Z}$-valuation independent. Further, $\bigcup_{i=0}^{\infty} p^i\mathcal{B}_0$ contains all elements in $\mathcal{B}$ of p-height $< \infty$. Indeed, let $b \in \mathcal{B}$ be of p-height $< \infty$, and assume that our assertion is already proved for all elements of $\mathcal{B}$ of value $< vb$. By (pH1′) there is some a such that $pa = b$ and $va < vb$. We write $a = \sum_{i=1}^{m} n_i b_i$ with $b_i \in \mathcal{B}$ and arbitrary integers n_i. Consequently, $b = p\sum_{i=1}^{m} n_i b_i$. According to Corollary 3.37, $b = p^j b_i$ for some i and $j \geq 1$. Since b is of p-height $< \infty$, the latter implies $vb_i < vb$. By assumption, $b_i \in p^k\mathcal{B}_0$ for some integer $k \geq 0$. Consequently, $b \in p^{k+j}\mathcal{B}_0$.

The proof of the remaining assertions is left to the reader. □

This procedure of selecting a weak valuation basis from a p-basis is made possible by the finite descent from an element of arbitrary height $< \infty$ to an element of height 0. The same principle can be found in very different settings; see e.g. the selection of a $K[\varphi]$-basis from a Frobenius-closed K-basis in section 12.3.

If a p-group G admits a $\mathbb{Z}$-valuation basis $\mathcal{B}'$ (with respect to its p-height valuation), then the elements of $\mathcal{B}'$ are $\mathbb{Z}$-independent and thus, G is the direct sum $\bigoplus_{b\in\mathcal{B}'} \mathbb{Z}b$. Such a group is reduced, and all elements have finite p-height. Consequently, a p-group with elements of infinite height can not admit a $\mathbb{Z}$-valuation basis.

The following theorem is the promised analogue of Brown's Theorem. Its first assertion is a result of R. Hunter and E. Walker ([HU–WA], Corollary 4.11).

Theorem 3.40 *If G is a countable p-group, then it admits a p-basis. If in addition, G is reduced and v is the p-height valuation, then (G, v) admits a weak $\mathbb{Z}$-valuation basis. If all elements of the reduced p-group G have height $< \omega$, then (G, v) admits a $\mathbb{Z}$-valuation basis.*

Proof: Let v be the p-height valuation of G. Let a_j , $j \in \mathbb{N}$, be an enumeration of the elements of G. It may be chosen in such a way that for every $j \in \mathbb{N}$, the element pa_j is contained in the group generated by $a_1, \ldots, a_{j-1}$. We set $G_1 := \{0\}$ and $\mathcal{B}_1 = \emptyset$. Suppose that we have already constructed a finite subgroup G_j of G containing $a_1, \ldots, a_{j-1}$ and admitting a v-p-basis $\mathcal{B}_j$ with the following property:

$$\forall b \in \mathcal{B}_j :\ vb = \mu + 1 < \nu \ \Rightarrow\ \exists b' \in \mathcal{B}_j :\ pb' = b \wedge vb' = \mu\ , \tag{3.9}$$

where ν is the p-height of G. We show how to construct a finite subgroup G_{j+1} of G containing G_j and a_j and admitting a v-p-basis $\mathcal{B}_{j+1}$ containing $\mathcal{B}_j$ and having again property (3.9).

Since G_j is finite, its value set vG_j is also finite. Consequently, G_j is spherically closed in (G, v). Hence, there is a proper representative a'_j for $a_j + G_j$. Since also $pa'_j \in G_j$ and a_j and a'_j generate the same abelian group extension of G_j , we can assume from the start that a_j is itself a proper representative. If $pa_j \in \mathcal{B}_j$, then $\mathcal{B}''_j := \mathcal{B}_j \cup \{a_j\}$ is a p-basis of $G''_j := G'_j + \mathbb{Z}a_j$, by virtue of Lemma 3.38.

If $pa_j \notin \mathcal{B}_j$, then we carry through the following construction. Let $pa_j = \sum n_i b_i$ with $b_i \in \mathcal{B}_j$ and n_i prime to p. We distinguish two cases.

Case 1: $va_j < \nu$. Then $vb_i \geq vpa_j > va_j$ for all i. By property (3.9), for all i with $vb_i < \nu$ a successor ordinal, there are elements $b'_i \in \mathcal{B}_j$ such that $pb'_i = b_i$ and $vb'_i \geq va_j$. For b_i with vb_i a limit ordinal, (pH1′) allows us to choose an element b'_i of value $vb'_i \geq va_j$ such that $pb'_i = b_i$. If $\mathcal{B}_j$ does not contain such an element, then we just choose it in G. Since vb_i is a limit ordinal and vG_j is finite, there are values in vG which lie properly between vb_i and all values of vG_j that are smaller than vb_i. By virtue of (pH1′), we can choose b'_i with such a value. Then $vb'_i \notin vG_j$ and $vb'_i > va_j$. From the former property, it follows that $\mathcal{B}_j \cup \{b'_i\}$ is a v-p-basis of $G'_j = G_j + \mathbb{Z}b'_i$.

Finally, for b_i with $vb_i = \nu$, we choose b'_i such that $pb'_i = b_i$ and $vb'_i = vb_i = \nu$ (there is such an element in G since $p^\nu G$ is divisible). If possible, we choose $b'_i \in \mathcal{B}_j$. If $\mathcal{B}_j$ does not contain such an element, then we just choose it in G, and we have to show that $\mathcal{B}_j \cup \{b'_i\}$ is a v-p-basis of $G'_j = G_j + \mathbb{Z}b'_i$. If this were not the case, then there would exist a $\mathbb{Z}$-linear combination $nb'_i + \sum_k n''_k b''_k$ with $b''_k \in \mathcal{B}_j$ of value ν and n''_k, n prime to p, such that $v(nb'_i + \sum_k n''_k b''_k > vnb'_i = \nu$. But then, the value of $nb_i + p\sum_k n''_k b''_k = p(nb'_i + \sum_k n''_k b''_k)$ is bigger than $\nu = vnb_i$. Now Corollary 3.37 shows that $b_i = p^\ell b''_k$ for some k and $\ell \geq 1$, hence $b_i = pp^{\ell-1}b''_k$ with $p^{\ell-1}b''_k \in \mathcal{B}_j$ of value ν, which contradicts our assumption on b_i.

Since $vb'_i > va_j$ in all cases where we have to adjoin b'_i to $\mathcal{B}_j$, we find that a_j will still be a proper representative of $a_j + G'_j$. If necessary, we repeat this procedure for other elements b_i. After a finite number of repetitions, we arrive at a group which we will again call G'_j, with a v-p-basis $\mathcal{B}'_j$ such that for every i there is $b'_i \in \mathcal{B}'_j$ satisfying $vb'_i \geq va_j$ and $pb'_i = b_i$. Moreover, a_j will still be a proper representative of $a_j + G_j$. We set $\tilde{a}_j = a_j - \sum n_i b'_i$. Then $v\tilde{a}_j \geq va_j$, which yields that $v\tilde{a}_j = va_j$ and that $\tilde{a}_j$ is still a proper representative of $a_j + G_j$. Since $p\tilde{a}_j = 0$ by construction, it now follows from Lemma 3.38 that $\mathcal{B}''_j := \mathcal{B}'_j \cup \{\tilde{a}_j\}$ is a v-p-basis of $G''_j := G'_j + \mathbb{Z}\tilde{a}_j$.

Case 2: $va_j = \nu$. As in case 1, we choose b'_i such that $pb'_i = b_i$ and $vb'_i = vb_i = \nu$. Again, $\mathcal{B}_j \cup \{b'_i\}$ will be a v-p-basis of $G'_j = G_j + \mathbb{Z}b'_i$, but we can not guarantee that a_j is still a proper representative of $a_j + G'_j$. After a finite number of repetitions, we arrive at a group which we will again call G'_j, with a p-basis $\mathcal{B}'_j$ such that for every i there is $b'_i \in \mathcal{B}'_j$ satisfying $vb'_i = va_j = \nu$ and $pb'_i = b_i$. We set $\tilde{a}_j = a_j - \sum n_i b'_i$. Then $v\tilde{a}_j \geq va_j$. If "=" holds, then we proceed as in case 1. If ">" holds, then that means that $a_j = \sum n_i b'_i$. In this case, we just set $\mathcal{B}''_j := \mathcal{B}'_j$ and $G''_j := G'_j$.

Now we have to enlarge $\mathcal{B}''_j$ and G''_j to obtain a v-p-basis that has property (3.9). Assume that $b \in \mathcal{B}''_j$ has value $\mu + 1 < \nu$ and that there is no $b' \in \mathcal{B}''_j$ of value μ such that $pb' = b$. (Note that by our construction of $\mathcal{B}''_j$, there are only finitely many such b.) We pick an element $b' \in G$ with the above properties. If there is no element in $\mathcal{B}''_j$ of value μ, then we can just adjoin b' to $\mathcal{B}''_j$ to obtain a p-basis of the enlarged group $G''_j + \mathbb{Z}b'$. Now assume that there are elements of value μ in $\mathcal{B}''_j$. Suppose that $\mathcal{B}''_j \cup \{b'\}$ is not p'-valuation independent. Then there is a linear combination $nb' + \sum n_i b_i$ with elements $b_i \in \mathcal{B}''_j$ of value μ and n, n_i prime to p, whose value is $> \mu$. But then, the value of $nb + p\sum n_i b_i = p(nb' + \sum n_i b_i)$ is $> \mu + 1 = vb$ (using the assumption that $\mu + 1 = vb < \nu$). Hence by virtue of Corollary 3.37, $b = p^j b_i$ for some i and $j \geq 1$. Then $j = 1$ since otherwise, $vb = vp^j b_i \geq \mu + 2 > vb$, a contradiction. But $b = pb_i$ contradicts our assumption on b. This shows that $\mathcal{B}''_j \cup \{b'\}$ is p'-valuation independent. So again, we can adjoin b' to $\mathcal{B}''_j$. The descent from a successor ordinal to the next lower limit ordinal (or 0) is finite. Hence, after a finite repetition of this procedure we will arrive at a finite subgroup G_{j+1} of G with a v-p-basis $\mathcal{B}_{j+1}$ that has property (3.9). This establishes the induction step. The union $\mathcal{B} := \bigcup_{j \in \mathbb{N}} \mathcal{B}_j$ is the desired

p-basis of G.

The second assertion of our theorem follows from what we just proved, together with the previous lemma. The third assertion is seen as follows. If G is reduced and only ordinals $< \omega$ appear as values of elements of G, then for every b being put into the p-basis by the above procedure, we only need to add an element b' once so that $pb' = b$, namely, the one satisfying $vb = vb' + 1$. That is, in the constructed p-basis $\mathcal{B}$ there will be no two elements $b_1 \neq b_2$ with $pb_1 = pb_2$. This implies that the elements of $\mathcal{B}_0 = \{b \in \mathcal{B} \mid vb = 0\}$ are $\mathbb{Z}$-independent, and thus are $\mathbb{Z}$-valuation independent. If G is reduced, hence generated by $\mathcal{B}_0$, this implies that $\mathcal{B}_0$ is a $\mathbb{Z}$-valuation basis of (G, v). □

From now on, let v be the order valuation of the abelian p-group G. With this valuation, all components of (G, v) are $\mathbb{F}_p$-vector spaces. A p'-valuation independent set $\mathcal{B}$ of generators of (G, v) is called a **straight basis of** G (we do not require that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$). For details on straight bases, see K. Benabdallah and K. Honda [BE–HO]. Here, we prove:

Theorem 3.41 *Every p-group G has a straight basis.*

Proof: For every $m \in vG$, we choose elements $b_{m,i} \in G$, $i \in I_m$, such that $vb_{m,i} = m$ and the elements $\mathrm{co}_m b_{m,i}$, $i \in I_m$, form an $\mathbb{F}_p$-basis of $\mathrm{C}_m(G, v)$. Then by Lemma 3.35, the elements $b_{m,i}$ are p'-valuation independent. Let $a \in G$ and $m = va$. Then there are finitely many elements $b_{m,i}$ and integers $0 < n_{m,i} < p$ such that $a - \sum n_{m,i} b_{m,i}$ has value $\geq m + 1$. By a finite repetition of this procedure, we find finitely many elements $b_{j,i}$ and integers $0 < n_{j,i} < p$, $m \leq j < 0$, such that $a = \sum_{j,i} n_{j,i} b_{j,i}$. Hence, $\mathcal{B} = \{b_{m,i} \mid m \in vG \wedge i \in I_m\}$ generates G. By what we have shown before, $\mathcal{B}$ is thus a straight basis of G. □

Finally, let us say some words about linear independence of the elements of p-bases and straight bases. Given elements $b_1, \ldots, b_m \in G$, we will call them **p'-independent** if no non-trivial $\mathbb{Z}$-linear combination $\sum_{i=1}^{m} n_i b_i$, with n_i integers prime to p, equates to zero. p'-valuation independence implies p'-independence. Thus, the elements of p-bases and straight bases are p'-independent. But contrary to the case of a p-basis, for fixed $m \in vG$, the elements of value m of a straight basis $\mathcal{B}$ are even $\mathbb{Z}$-independent. This is seen as follows. Let $\sum_{i=1}^{k} n_i b_i$ be any $\mathbb{Z}$-linear combination with $b_i \in \mathcal{B}$ of value m and $n_i b_i \neq 0$. Let p^n be the highest power of p which divides all n_i, and write $n_i = n_i' p^n$. Note that $m + n = vp^n b_i \leq vn_i b_i < \infty$. Without loss of generality, we may assume that there is j with $1 < j \leq k$ such that $n_1', \ldots, n_j'$ are prime to p and $n_{j+1}', \ldots, n_m'$ are divisible by p. Now $v \sum_{i=1}^{j} n_i' b_i = m$ and $v \sum_{i=j+1}^{k} n_i' b_i \geq m + 1$, which shows that $v \sum_{i=1}^{k} n_i' b_i = m$. Consequently, $v \sum_{i=1}^{k} n_i b_i = m + n < \infty$, which proves that $\sum_{i=1}^{k} n_i b_i \neq 0$.

Exercise 3.5 *Show that p'-independence of the elements of $\mathcal{B}$ does not imply the uniqueness of a representation $a = \sum n_i b_i$ with $b_i \in \mathcal{B}$ and n_i prime to p.*

Exercise 3.6 *Let G be a divisible p-group. Construct a p-basis $\mathcal{B}$ of G (hint: start from an $\mathbb{F}_p$-basis of the socle of G). Show that in the case of divisible p-groups, condition 1) for a p-basis is equivalent to 1′) the elements of $\mathcal{B}$ are p'-independent.*

3.6 Embedding lemmas

Valuation independent sets can serve to obtain isomorphisms between valued modules, as the following lemma shows.

Lemma 3.42 *Let* $\mathbf{N}$ *and* $\mathbf{N}'$ *be valued* R*-modules containing* $\mathbf{M}$ *as a common valued submodule. Let* $\mathcal{B} \subset N \setminus \mathrm{Exe}\mathbf{N}$ *and* $\mathcal{B}' \subset N' \setminus \mathrm{Exe}\mathbf{N}$ *be* R*-valuation independent over* $\mathbf{M}$*. Suppose that there exists an* R*-linear embedding*

$$\sigma :\ \mathrm{sk}\,\mathbf{N} \longrightarrow \mathrm{sk}\,\mathbf{N}'$$

over $\mathrm{sk}\,\mathbf{M}$ *and a bijection*

$$\psi :\ \mathcal{B} \longrightarrow \mathcal{B}'$$

preserving σ*, i.e.*

$$\forall b \in \mathcal{B} : (v\psi b, \mathrm{co}\,\psi b) = \sigma(vb, \mathrm{co}\,b)\ .$$

Then ψ *extends* R*-linearly to an isomorphism*

$$\iota :\ M + R\mathcal{B} \longrightarrow M + R\mathcal{B}'$$

of valued R*-modules which also preserves* σ*.*

If in addition, $\mathbf{N}$ *and* $\mathbf{N}'$ *are ordered and* v *is their natural valuation, and if* σ *is an embedding of ordered skeletons (i.e. it preserves the ordering on the components), then* ι *also preserves the ordering.*

Proof: Since $\mathcal{B}$ and $\mathcal{B}'$ are R-independent over M, ψ extends R-linearly to an isomorphism ι between $M + R\mathcal{B}$ and $M + R\mathcal{B}'$. We have to show that it is an isomorphism of valued R-modules which preserves σ. Given $b \in M + R\mathcal{B}$, let us write b in the form (3.4). We assume (after a suitable renumeration) that precisely $r_1b_1, \ldots, r_mb_m, a$ are the summands of least value γ in (3.4). If va is smaller than the value of the other summands, then omit the elements r_ib_i in the following; if on the other hand, va is greater than the value of some summand r_ib_i, then omit a everywhere. By our hypothesis on ψ it follows that precisely $r_1\psi b_1, \ldots, r_m\psi b_m, a$ are the summands of least value in $\iota b = \sum r_i\psi b_i + a$. Since $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$, we have $(vb, \mathrm{co}\,b) = r_1(vb_1, \mathrm{co}\,b_1) + \ldots + r_m(vb_m, \mathrm{co}\,b_m) + (va, \mathrm{co}\,b)$. Since also $\mathcal{B}'$ is R-valuation independent over $\mathbf{M}$,

$$\begin{aligned}
(v\iota b, \mathrm{co}\,\iota b) &= r_1(v\psi b_1, \mathrm{co}\,\psi b_1) + \ldots + r_m(v\psi b_m, \mathrm{co}\,\psi b_m) + (va, \mathrm{co}\,b)\\
&= r_1\sigma(vb_1, \mathrm{co}\,b_1) + \ldots + r_m\sigma(vb_m, \mathrm{co}\,b_m) + (va, \mathrm{co}\,b)\\
&= \sigma(r_1(vb_1, \mathrm{co}\,b_1) + \ldots + r_m(vb_m, \mathrm{co}\,b_m) + (va, \mathrm{co}\,b)) = \sigma(vb, \mathrm{co}\,b)\ .
\end{aligned}$$

This shows that ι preserves σ, which in turn yields that ι is an isomorphism of valued R-modules.

The last assertion follows directly from Corollary 2.26. □

Let us apply this lemma to ordered abelian groups. As we have mentioned already, with their natural valuation they are ordinary valued $\mathbb{Z}$-modules.

Corollary 3.43 *Let $(G,<)$ and $(G',<)$ be ordered abelian groups, equipped with their natural valuation v and containing $(H,<)$ as a common ordered subgroup. Let $\mathcal{B} \subset G$ and $\mathcal{B}' \subset G'$ be $\mathbb{Z}$-valuation independent over $\mathbf{H}$. Suppose that there exists a $\mathbb{Z}$-linear embedding $\sigma:\ \operatorname{sk}\mathbf{G} \to \operatorname{sk}\mathbf{G}'$ of ordered skeletons over $\operatorname{sk}\mathbf{H}$ and let ψ be as in the lemma. Then ψ extends linearly to an isomorphism $\iota:\ H+\mathbb{Z}\mathcal{B} \to H+\mathbb{Z}\mathcal{B}'$ of ordered abelian groups which also preserves σ.*

By Lemma 2.31, there is a unique extension of the orders from $M+\mathbb{Z}\mathcal{B}$ to $M+\mathbb{Q}\mathcal{B}$ and from $M+\mathbb{Z}\mathcal{B}'$ to $M+\mathbb{Q}\mathcal{B}'$. Consequently, ι also extends to an isomorphism $\iota:\ M+\mathbb{Q}\mathcal{B} \to M+\mathbb{Q}\mathcal{B}'$ of ordered abelian groups, and if they are contained in $\mathbf{N}$ resp. $\mathbf{N}'$ then ι preserves σ on their skeletons.

The following is an interesting special case of the foregoing corollary; for an application, see [KU4].

Corollary 3.44 *Let $(G,<)$, $(G',<)$, $(H,<)$ and v be as in the foregoing corollary. Suppose that b_i and b'_i, $i\in I$, are elements of G and G' respectively, such that*

1) all vb_i, $i\in I$, are distinct and not contained in vH,

2) $\operatorname{sign}(b_i)=\operatorname{sign}(b'_i)$ for all $i\in I$, and the assignment $vb_i\mapsto vb'_i$ establishes an order isomorphism

$$\rho:\ vH\cup\{vb_i\mid i\in I\}\to vH\cup\{vb'_i\mid i\in I\}\ .$$

Then the assignment $b_i\mapsto b'_i$ defines order and valuation preserving isomorphisms

$$\begin{array}{ccc} H+\sum_{i\in I}\mathbb{Z}b_i & \longrightarrow & H+\sum_{i\in I}\mathbb{Z}b'_i \\ H+\sum_{i\in I}\mathbb{Q}b_i & \longrightarrow & H+\sum_{i\in I}\mathbb{Q}b'_i \end{array}$$

both preserving ρ. (Here, the groups are endowed with the restrictions of the orders of the divisible hulls of G and G' which are uniquely determined by those of G and G'.) Note that $vH\cup\{vb_i\mid i\in I\}$ is the value set of both groups on the left hand side, and $vH\cup\{vb'_i\mid i\in I\}$ is the value set of both groups on the right hand side.

Alternatively, assume 1) and the following condition:

2') the assignment $b_i\mapsto b'_i$ establishes an isomorphism

$$\iota:\ H\cup\{b_i\mid i\in I\}\to H\cup\{b'_i\mid i\in I\}$$

as ordered sets.

Then ι extends linearly to order and valuation preserving isomorphisms of the above groups.

The equivalence of the conditions 2) and 2') is seen as follows. Let be given an $i\in I$. Condition 1) says that $vb_i\notin vH\cup\{vb_j\mid i\neq j\in I\}$. Hence, (2.7) gives $|b_i|>|a|\Leftrightarrow vb_i<va$ for all $a\in H\cup\{b_j\mid i\neq j\in I\}$. Note that by virtue of $H\cap\{b_i\mid i\in I\}=\emptyset$, the order on $H\cup\{b_i\mid i\in I\}$ is already determined by the cuts induced in H by the elements b_i and by the order on the set $\{b_i\mid i\in I\}$. Further, observe that every element of G of the same sign and value as b_i will induce the same cut as b_i in H. Cuts induced in H by elements of value not in vH, have special properties; such cuts are studied in the preliminaries of [KU4] under the name ***v*-cuts**.

Another corollary to Lemma 3.42 is the following characterization of those valued vector spaces which are isomorphic to the Hahn sum over their skeleton:

Corollary 3.45 *Let (V,v) be a valued K-vector space with value preserving scalar multiplication which admits a K-valuation basis. Then there is an isomorphism from (V,v) onto the valued Hahn sum over its skeleton. This isomorphism can be chosen so that it preserves the coefficient map.*

If in addition, V is an ordered K-vector space and v is its natural valuation, then the isomorphism can be chosen as to preserve the ordering.

Conversely, every valued Hahn sum whose components are K-vector spaces is a valued K-vector space with value preserving scalar multiplication which admits a K-valuation basis.

Proof: Assume that (V,v) admits a K-valuation basis $\mathcal{B}$. Then by Corollary 3.28 where we take $W = 0$, the set $\underline{\mathcal{B}}_\alpha = \{\mathrm{co}_\alpha b \mid b \in \mathcal{B} \text{ and } vb = \alpha\}$ forms a basis of $\mathrm{C}^\alpha\mathbf{V}$ for every $\alpha \in vV$. Using these bases of the components, we form a K-valuation basis $\mathcal{B}' = \{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\alpha\mathbf{V} \wedge \alpha \in vV\}$ of the Hahn sum $\coprod_{\alpha\in vV} \mathrm{C}^\alpha\mathbf{V}$. Now our first and second assertion follow from Lemma 3.42, where we take σ to be the identity.

The converse was already proved in the last section. □

According to Theorem 3.30, every countably-dimensional valued vector space with value preserving scalar multiplication admits a valuation basis. Hence, the foregoing corollary yields the following theorem of R. Brown [BRW1]:

Theorem 3.46 *Two countably-dimensional valued vector spaces with value preserving scalar multiplication are isomorphic if and only if their skeletons are isomorphic. Similarly, two countably-dimensional ordered vector spaces are isomorphic if and only if their natural skeletons are isomorphic.*

The following example is meant to prevent a common error:

Example 3.47 Given an element a in a valued module, let us call the representation $a = a_1 + \ldots + a_n$ a **convex decomposition of a of length n** if all values $va_1, \ldots, va_n$ are distinct. It is *not* true that every element in every Hahn sum has a convex decomposition of maximal length. As soon as the value set admits an infinite ascending chain β_i, $i \in \mathbb{N}$ of values starting from $\beta_1 = va$, we can carry out the following construction. We choose elements b_i such that $vb_i = \beta_i$ and $b_1 = a$. To obtain a decomposition of a of length n for arbitrary $n \in \mathbb{N}$, we just have to set $a_i = b_i - b_{i+1}$ for $i < n$ and $a_n = b_n$. The sum of the a_i's is a, and their values $va_i = vb_i$ are all distinct.

For instance, this construction works for every nonzero element in the Hahn sum $\coprod_{\mathbb{N}} \mathbb{Q}$, which is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication. ♢

Remark 3.48 This example shows a theorem of Banaschewski ([BAN], Satz, p. 435) to be false. This theorem characterizes the Hahn sums to be those valued vector spaces with value preserving scalar multiplication in which every element admits a convex decomposition of maximal length. (Note that Banaschewski uses the names “schwache Hahnsche Summe” for the Hahn sum and “Hahnsche Summe” for the Hahn product.) The gap in Banaschewski's proof is that he assumes as obvious that a tuple of the Hahn product with only one nonzero entry admits only convex decompositions of length 1. (His verification reads “wie man sofort sieht”. We have carefully avoided using corresponding english phrases in this book, in order not to mark our errors.)

Most remarkably, Fleischer [FL2] cites the wrong theorem of Banaschewski in order to disprove a theorem of Hill and Mott ([HI–MO], Theorem 5.1). This theorem states that a countable ordered abelian group G whose components are all isomorphic to $\mathbb{Z}$, can be embedded in the ordered Hahn sum $\coprod_{vG} \mathbb{Z}$

(without claiming that the resulting extension is immediate). Fleischer gives an interesting example which, because of Banaschewski's error, does not show what he wants but rather lends credibility to the theorem of Hill and Mott. Consider the Hahn product $\mathbf{H}_{\mathbb{N}}\mathbb{Z}$ and its subgroup of all cofinitely constant tuples. The latter is generated over the Hahn sum $\coprod_{\mathbb{N}}\mathbb{Z}$ by just one element: the tuple $(1,1,1,\ldots)$ which has a 1 at every entry. According to Banaschewski's theorem, this group can not be isomorphic to the Hahn sum over its skeleton, and hence cannot admit a $\mathbb{Z}$-valuation basis. But it does: the elements $(0,\ldots,0,1,1,1,\ldots)$ which are constantly 1 from the i-th entry on and 0 before, for $i\in\mathbb{N}$. (I am endebted to S. Kuhlmann for bringing my attention to the cited articles, for detecting the mentioned errors and for providing the valuation basis.)

Let us consider once more the assertion of Lemma 3.21. It says that under the assumptions of that lemma, an extension $(W,v)\subset(V,v)$ can be broken up into one which is generated by a system of valuation independent elements and one which is immediate. This principle will play an important role throughout this book. Having treated extensions of the first type in the last lemma and corollary, the question arises what we can say about immediate extensions. By the characterization of immediate extensions of ultrametric spaces given in Lemma 1.36 we know that the approximation type of every element in such an extension is immediate. On the other hand, we have Lemma **??** at hand. If $\mathbf{M}$ is a valued module, $\mathbf{N}$ an immediate and $\mathbf{N}'$ an arbitrary extension of $\mathbf{M}$, and if $b\in N$ and $b'\in N'$ have the same approximation type over $\mathbf{M}$, then this lemma tells us that $v(b-a)=v(b'-a)$ for all $a\in M$. The question is, whether also $v(rb-a)=v(rb'-a)$ for all $a\in M$ and all $r\in R$, because then $b\mapsto b'$ will define an isomorphism of the valued modules $\mathbf{M}+Rb$ and $\mathbf{M}+Rb'$ over $\mathbf{M}$. If both $\mathbf{N}$ and $\mathbf{N}'$ are valued vector spaces with value preserving scalar multiplication, then we are done: if $r=0$, there is nothing to show, and if $r\neq 0$, then $v(rb-a)=v(b-r^{-1}a)=v(b'-r^{-1}a)=v(rb'-a)$. But we want to prove a little bit more.

Lemma 3.49 *Let $\mathbf{N}$, $\mathbf{N}'$ be torsion free ordinary valued R-modules, containing a common valued submodule $\mathbf{M}$ such that $\mathbf{M}\subset\mathbf{N}$ is an immediate extension. Assume that for every $r\in R$, the R-module $r\mathbf{M}$ is spherically closed in $\mathbf{M}$ (note that this condition is void if R is a field). Given elements $b\in N\setminus M$ and $b'\in N'$ such that $\operatorname{at}(b,M)=\operatorname{at}(b',M)$, then the assignment $b\mapsto b'$ induces an embedding of $(M+Rb,v)$ in $\mathbf{N}'$ over $\mathbf{M}$.*

If in addition, $\mathbf{N}$ and $\mathbf{N}'$ are ordered and v is their natural valuation, then the embedding also preserves the ordering.

Proof: From Lemma **??** we know that $v(b-a)=v(b'-a)$ for all $a\in M$. We show that $v(rb-a)=v(rb'-a)$ for all $r\in R\setminus\{0\}$ and $a\in M$. Since $r\mathbf{M}$ is assumed to be spherically closed in $\mathbf{M}$, there is some $c\in M$ such that $v(a-rc)=\max\{v(a-rd)\mid d\in M\}$. Since $\mathbf{M}\subset\mathbf{N}$ is an immediate extension, Lemma 2.9 shows that there is some $d\in M$ such that $v(b-d)>v(b-c)$ (note that $b-c\neq 0$ since $b\notin M$). By assumption, the R-module N is torsion free and $r\neq 0$, so we have $r(b-c)\neq 0$. Hence by Lemma 3.3, $v(b-d)>v(b-c)$ yields that $v(rb-rd)>v(rb-rc)$. Now we compute: $v(rb-rd)>v(rb-rc)\geq\min\{v(rb-a),v(a-rc)\}\geq\min\{v(rb-a),v(a-rd)\}$, showing that $v(rb-a)=v(a-rd)$ by virtue of (V=). Since we know already that $v(b-c)=v(b'-c)$ and $v(b-d)=v(b'-d)$, we can replace b by b' everywhere in the above computation. This gives $v(rb-a)=v(a-rd)=v(rb'-a)$, as desired. Furthermore, we observe that $v(rb-a)=v(rb'-a)\leq v(rb-rc)<\infty$ for all $r\in R\setminus\{0\}$ and $a\in M$. This proves $rb\notin M$ and $rb'\notin M$. Thus, $b\mapsto b'$ induces an embedding of $M+Rb$ in N' over M, and our computation of the values shows that this embedding preserves the valuation.

Since $\mathbf{M} \subset \mathbf{M}+Rb$ is immediate as a subextension of the immediate extension $\mathbf{M} \subset \mathbf{N}$, the same is true for the extension $\mathbf{M} \subset \mathbf{M}+Rb'$. Hence trivially, the embedding preserves the identity on their skeletons, which are equal to the skeleton of $\mathbf{M}$. If $\mathbf{N}$ and $\mathbf{N}'$ are ordered and v is their natural valuation, then by Corollary 2.26 the embedding will also preserve the ordering. □

The significance of this lemma is that it deduces information about the valued modules $M+Rb$ and $M+Rb'$ from information that is encoded in the ultrametric spaces $M\cup\{b\}$ and $M\cup\{b'\}$. This principle of reduction to simpler structures is the basic idea in the application of approximation types and will be met several times in different forms in this book (cf. e.g. the Ax–Kochen–Ershov Principle introduced in Section 21.2 and Lemma 21.8).

Let us consider for a moment the assumptions of the lemma. If R is a (skew) field, then for all $r \in R \setminus \{0\}$ we have $r\mathbf{M} = \mathbf{M}$, which is spherically closed in $\mathbf{M}$. Having in mind the ordered abelian groups, let us also consider (left) valued R-modules $\mathbf{M}$ with value preserving scalar multiplication, and assume in addition that the ring R admits a field K of (left) fractions (cf. [COHN2], chapter 0, section 0.8). Then we can consider the K-module $K \otimes_R M$. Because of value preserving scalar multiplication, M is R-torsion free and thus a submodule of $K \otimes_R M$ (cf. [COHN2], chapter 0, Proposition 9.1). There is a unique extension of the valuation from M to $K \otimes_R M$ such that the scalar multiplication remains value preserving. Indeed, for every element $b \in K \otimes_R M$ there is some $r \in R$ such that $rb \in M$, and we have $vb = vrb$. Equipped with this extension of the valuation, the K-module $K \otimes_R M$ will be denoted by $K \otimes_R \mathbf{M}$. (Without the assumption of value preserving scalar multiplication, existence and uniqueness of such an extension are not a priori guaranteed.) Now if $\mathbf{M}$ is spherically closed in $K\otimes_R\mathbf{M}$, then for every $r \in R\setminus\{0\}$ and every $a \in M$ there is some $c \in M$ such that $v(r^{-1}a-c) = \max\{v(r^{-1}a-d) \mid d \in M\}$. Since the scalar multiplication is value preserving, we find $v(a-rc) = \max\{v(a-rd) \mid d \in M\}$. Hence: *if* $\mathbf{M}$ *is spherically closed in* $K \otimes_R \mathbf{M}$*, then* $r\mathbf{M}$ *is spherically closed in* $\mathbf{M}$ *for every* $r \in R \setminus \{0\}$. With the assumptions modified correspondingly, the lemma can now be applied to ordered abelian groups, with v the natural valuation, $R = \mathbb{Z}$ and $K = \mathbb{Q}$. In this case, $K \otimes_R \mathbf{M}$ is just the divisible hull of M, equipped with the natural valuation with respect to the unique extension of the ordering (cf. Lemma 2.31). For divisible ordered abelian groups, the spherical closedness condition is trivially satisfied. Another important class of ordered abelian groups satisfying this condition is the class of $\mathbb{Z}$-groups:

Lemma 3.50 *Every* $\mathbb{Z}$*-group, equipped with its natural valuation, is spherically closed in its divisible hull.*

Proof: Let $(G,<)$ be a $\mathbb{Z}$-group with smallest positive element g. Let $a \in G$ and $n \in \mathbb{N}$; we wish to show that $\{v(\frac{a}{n}-d) \mid d \in G\}$ admits a maximum. According to Lemma 2.33 we write $a = nb - mg$ with $b \in G$ and $m \in \mathbb{N}$. If n divides m, then $\frac{a}{n} \in G$ and the assertion is trivial. If n does not divide m, then $\frac{a}{n} = b - \frac{m}{n}g \notin G$. In this case, $\infty \notin \{v(\frac{a}{n}-d) \mid d \in G\}$, and hence, the maximum is $v\frac{m}{n}g = vg = v(\frac{a}{n} - b)$ (which is the maximum of vG). □

Now we are able to prove the Hahn Embedding Theorem for valued and ordered vector spaces. In particular, it shows that every maximal component-compatibly valued vector space is isomorphic to the Hahn product over its skeleton, and thus, it is spherically complete. (Recall that every component-compatibly valued vector space has value preserving scalar multiplication).

Theorem 3.51 *Let K be a field or a skew field and (V,v) an ordinary valued K-vector space with value preserving scalar multiplication. Then there is an isomorphism ι of the valued Hahn sum $\coprod_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$ onto a valued subspace (V',v) of (V,v) which also preserves the coefficient map. On the other hand, ι^{-1} can be extended to an embedding of (V,v) in the valued Hahn product $\mathbf{H}_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$ which also preserves the coefficient map. This embedding is onto if and only if (V,v) is maximal. Hence, for component-compatibly valued vector spaces the properties "maximal", "spherically complete" and "being (isomorphic to) a Hahn product" coincide.*

If in addition, V is an ordered K-vector space and v is its natural valuation, then the embeddings can be chosen as to preserve the ordering.

Proof: By Lemma 3.21 where we take $W = 0$, we obtain the existence of some valued subspace (V',v) of (V,v) which admits a K-valuation basis $\mathcal{B}$ and such that $(V',v) \subset (V,v)$ is an immediate extension. Hence, (V',v) has the same skeleton as (V,v). By Corollary 3.45, (V',v) is isomorphic to the Hahn sum over this skeleton, that is, to $\coprod_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$. The isomorphism can be chosen as to preserve the coefficient map and, in the ordered case, to preserve the ordering.

We identify the Hahn sum with its isomorphic image (V',v). We wish to show that (V,v) is embeddable in $\mathbf{H}_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$ over V'. Consider the set of all such embeddings of subspaces of (V,v). A union over an ascending chain of such embeddings is again an embedding of a subspace of (V,v). By Zorn's Lemma, there is a maximal subspace (V'',v) which is embeddable in this way. We have to show $V'' = V$. Otherwise, there is some $b \in V \setminus V''$. Let us identify (V'',v) with its image in $\mathbf{H}_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$. Since the extension $(V'',v) \subset (V,v)$ is immediate, $\mathrm{at}\,(b,V'')$ is immediate by Lemma 1.36. Since $\mathbf{H}_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$ is spherically complete (Lemma 2.14), Lemma 1.40 shows that $\mathrm{at}\,(b,V'')$ is realized by some element b' in the Hahn product, i.e., $\mathrm{at}\,(b,V'') = \mathrm{at}\,(b',V'')$. Now Lemma 3.49 shows that also $(V''+Kb,v)$ can be embedded, contrary to our maximality assumption. This shows that (V,v) can be embedded over V' in the Hahn product. Since $(V',v) \subset (V,v)$ is immediate, for every $b \in V$ there is some $a \in V'$ such that $v(b-a) > vb$. If b' is the image of b under the embedding, then also $v(b'-a) > vb = vb'$. It follows that $\mathrm{co}_{\,vb} b = \mathrm{co}_{\,vb} a = \mathrm{co}_{\,vb'} a = \mathrm{co}_{\,vb'} b'$, showing that the embedding also preserves the coefficient map. In the ordered case, all embeddings are taken to be order preserving, and we use the corresponding additional assertion of Lemma 3.49.

The Hahn product $\mathbf{H}_{\alpha\in vV} \mathrm{C}^\alpha \mathbf{V}$ is an immediate extension of the embedded image of (V,v). If (V,v) is maximal, then this extension must be trivial. Conversely, if the embedding is an isomorphism onto the Hahn product, then in view of Theorem 3.7 and Lemma 2.14, (V,v) must be maximal and spherically complete. □

As a corollary, we obtain the following counterpart to Brown's Theorem (Theorem 3.46):

Theorem 3.52 *Two maximal valued vector spaces with value preserving scalar multiplication are isomorphic if and only if their skeletons are isomorphic. Similarly, two maximal ordered vector spaces are isomorphic if and only if their natural skeletons are isomorphic.*

Since a divisible ordered abelian group equipped with its natural valuation is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication, we obtain:

Corollary 3.53 *Every divisible ordered abelian group is embeddable between the ordered Hahn sum and the ordered Hahn product over its natural skeleton. The embedding is an* *isomorphism onto the Hahn product if and only if the group is maximal with respect to its natural valuation. Hence, for divisible ordered abelian groups the properties "maximal", "spherically complete" and "being (isomorphic to) an ordered Hahn product" coincide.*

Given an arbitrary ordered abelian group $(G,<)$, the ordering can be extended to the divisible hull of $\tilde{G}$, according to Lemma 2.31. From the preceding Corollary, we thus obtain

Corollary 3.54 *Every ordered abelian group is embeddable in the ordered Hahn product over the natural skeleton of its divisible hull.*

From Lemma 2.31 we also know the natural skeleton of $\tilde{G}$. In fact, we have $v\tilde{G} = vG$ and for every $\alpha \in vG$, the α-component of $\tilde{G}$ is the divisible hull of the α-component of G. So the corollary shows that $(G,<)$ is embeddable in the ordered Hahn product $\coprod_{\alpha\in vG} \mathrm{C}^\alpha\tilde{\mathbf{G}}$. If we wish to replace these components by a uniform one, we can employ the following theorem which is due to Hölder [HÖL]:

Theorem 3.55 *Every archimedean ordered abelian group can be embedded in the ordered additive group of the reals.*

For the proof and additional remarks, see [PC1], Chapter I, §3. Cf. also Exercise 3.7. This lemma shows that every component $\mathrm{C}^\alpha\tilde{\mathbf{G}}$ can be viewed as an ordered subgroup of $\mathbb{R}$. Thus, the ordered Hahn product $\coprod_{\alpha\in vG} \mathrm{C}^\alpha\tilde{\mathbf{G}}$ is an ordered subgroup of the ordered Hahn product $\coprod_{\alpha\in vG} \mathbb{R}$. (At this point, we should note that all embeddings of ordered abelian groups that have ocurred so far can be chosen as to preserve also the natural valuation.) We have now deduced the original Hahn Embedding Theorem:

Corollary 3.56 *Every ordered abelian group $(G,<)$ with natural valuation v is embeddable in the ordered Hahn product $\mathbf{H}_{\alpha\in vG}\,\mathbb{R}$.*

Remark 3.57 Hahn proved this theorem in 1907, cf. [HAHN]. For a brief history of generalizations and different proofs and for a proof following Banaschewski's approach, see Prieß-Crampe [PC1]. Corollary 3.53 is due to Banaschewski [BAN]. Our own valuation theoretical approach follows the spririt of Gravett's papers [GRA1], [GRA2].

It is not true that an arbitrary ordered abelian group is embeddable in the ordered Hahn product over its own natural skeleton. Apparently, the first counterexample was given by B. Gordon, as cited by A. H. Clifford in [CLI]. Contrary to the statements in P. Ribenboim's papers [RIB4] and [RIB6], it is not even true if the group is regular in the sense of these papers. Counterexamples were given by P. Hill and J. L. Mott in [HI–MO]. For instance, there are $\mathbb{Z}$-groups of rank 2 and rational rank 2 with archimedean components $\mathbb{Z}$ and $\mathbb{Q}$ such that the convex subgroup $\mathbb{Z}$ is not a direct summand (cf. Theorem 3.1 and Proposition 3.2 of [HI–MO]). It should be noted that if the group is represented as a subgroup of $\mathbb{Q}\times\mathbb{Q}$, the projections on the second component will not coincide with the archimedean component $\mathbb{Z}$. Otherwise, $\mathbb{Z}$ would be a direct summand, as was pointed out by G. Sabbagh. In this book, we use the word "component" in the sense of α-components and archimedean components, and these may differ significantly from the **components** which are understood as images under the coordinate projections of tuples.

For ordered abelian groups, the above lemmas are only of interest in the case where the natural valuation is non-trivial, i.e. where the ordering on the group is non-archimedean. The results are void for the archimedean case, which in particular is of interest for the embedding of components. So let us give an auxiliary embedding lemma for this case, without aiming at the best possible result.

Lemma 3.58 *Let $(G,<) \subset (H,<)$ and $(G,<) \subset (H',<)$ be two extensions of ordered abelian groups and $b \in H \setminus G$, $b' \in H' \setminus G$. Assume further that G is divisible. If b and b' induce the same cut in G, then the assignment $b \mapsto b'$ defines order preserving isomorphisms from $G + \mathbb{Z}b$ onto $G + \mathbb{Z}b'$ and from $G + \mathbb{Q}b$ onto $G + \mathbb{Q}b'$.*

Proof: Since $b, b' \notin G$, the assignment $b \mapsto b'$ defines isomorphisms from $G + \mathbb{Z}b$ onto $G + \mathbb{Z}b'$ and from $G + \mathbb{Q}b$ onto $G + \mathbb{Q}b'$. Let (Λ_1, Λ_2) be the cut induced by b and b' in G. Given $m, n \in \mathbb{N}$ and $a \in G$, then $\frac{m}{n}a \in G$ and $b > \frac{m}{n}a \Leftrightarrow -\frac{m}{n}a \in \Lambda_1 \Leftrightarrow b' > \frac{m}{n}a$. Hence, $\frac{n}{m}b > a \Leftrightarrow \frac{n}{m}b' > a$. This also yields that $-\frac{n}{m}b > a \Leftrightarrow -\frac{n}{m}b' > a$. We have thus shown that the isomorphisms are order preserving. □

Exercise 3.7 *a) Try to prove Lemma 3.58 without the hypothesis that G be divisible. What is the difficulty? Compare with our remark following Lemma 3.49. Prove Lemma 3.58 under the assumption that for every $n \in \mathbb{N}$, there is an element divisible by n between every two elements of G. Show that G has this property already if it is p-divisible for an arbitrary prime p.*

b) Prove Theorem 3.55, using Lemma 3.58, Lemma 2.31 and the fact that $\mathbb{R}$ is **Dedekind complete** *(that is, every cut in $\mathbb{R}$ is already realized in $\mathbb{R}$, provided that both sets in the cut are nonempty).*

3.7 Defectless extensions of valued vector spaces

We have defined defectless extensions of valued modules in section 3.3 to be those extensions for which every finite subextension admits a valuation basis. It is a direct consequence of this definition that every subextension of a defectless extension is again defectless. The most important questions arising in this connection are: Is an extension defectless if it admits a valuation basis? If an extension admits a valuation basis, does this also hold for every subextension? We will consider these questions in the case of ordinary valued vector spaces. In what follows, we let K be an arbitrary field (or a skew field).

Vector spaces have the following **basis exchange property**: If V is a K-vector space with basis $\mathcal{B}$ over W and if V' is a subspace of V with basis $\mathcal{B}'$ over W, then there is a subset $\mathcal{B}'' \subset \mathcal{B}$ which is a basis of V over V'. This implies that $\mathcal{B}' \cap \mathcal{B}'' = \emptyset$ and that $\mathcal{B}' \cup \mathcal{B}''$ is a basis of V over W. We will now show the same for valuation bases for the case where V' is finite-dimensional over W. Contrary to the cited situation, we now must also show the existence of the K-valuation basis $\mathcal{B}'$.

Let (V, v) be an ordinary valued K-vector space and (W, v) a valued subspace. Assume that $\mathcal{B}$ is a K-valuation basis of (V, v) over W. Let $b \in V \setminus W$. We write $b = a + \sum_i r_i b_i$ with $r_i \in K$, $b_i \in \mathcal{B}$ and $a \in W$. Set $b' := b - a$. Since the elements b_i are K-valuation independent over W, the same holds for $b' = \sum_i r_i b_i$. Hence, $\mathcal{B}' := \{b'\}$ is a K-valuation basis of $(W + Kb, v)$ over W. Let j be some index among the i for which the summand $r_j b_j$ is of minimal value. Set $\mathcal{B}'' := \mathcal{B} \setminus \{b_j\}$. We wish to show that $\mathcal{B}''$ is a K-valuation basis of (V, v) over $W + Kb$. This is seen as follows. Given a finite linear combination $a + rb' + \sum_{i \neq j} r_i' b_i$ with $r, r_i \in K$, $b_i \in \mathcal{B}''$ and $a \in W$, the only critical case appears if $vrb' = \min_{i \neq j}\{va, vr_i' b_i\}$. In this case, we have to verify that the value of the linear combination is equal to vrb'. By Lemma 3.3 we know that $rr_j b_j$ is a summand of minimal value in $rb' = \sum_i rr_i b_i$ since $r_j b_j$ was of minimal value in $b' = \sum_i r_i b_i$. Hence, $vrb' = vrr_j b_j$. Rewriting the given linear combination as $a + rr_j b_j + \sum_{i \neq j}(rr_i + r_i')b_i$, we now see that its value must be vrb' by the valuation independence of the b_i.

Repeating this argument and using Lemma 3.16, the following can be shown by induction on $\dim_K V'/W$:

Lemma 3.59 *Let (V,v) be an ordinary valued K-vector space admitting a K-valuation basis $\mathcal{B}$ over the subspace W. If $W \subset V'$ is a finite subextension of $W \subset V$ (i.e., $\dim_K V'/W < \infty$), then (V',v) admits a K-valuation basis $\mathcal{B}'$ over W and there is $\mathcal{B}'' \subset \mathcal{B}$ such that $\mathcal{B}''$ is a K-valuation basis of (V,v) over V'.*

The lemma shows that the first question stated at the beginning can be answered in the affirmative:

Corollary 3.60 *Let (V,v) be an ordinary valued K-vector space with subspace W. If (V,v) admits a K-valuation basis over W, then $(W,v) \subset (V,v)$ is a defectless extension of valued vector spaces.*

If in the situation of Lemma 3.59, $W \subset V'$ is a subextension of countably infinite dimension, then it admits a basis indexed by the natural numbers, and by the same induction procedure, it can be deduced that V' admits a K-valuation basis over W:

Corollary 3.61 *Let (V,v) be an ordinary valued K-vector space admitting a K-valuation basis $\mathcal{B}$ over the subspace W. Then every subspace (V',v) of (V,v) of countable dimension over W admits a K-valuation basis over W.*

On the other hand, if in the same situation, V' is not of finite dimension over W, then the other part of the assertion of Lemma 3.59 may not remain true:

Examples 3.62 Let K be an arbitrary field and consider the Hahn sum $V := \coprod_{\mathbb{N}} K$, which is an ordinary valued K-vector space with value preserving scalar multiplication. Let e_i be the tuple which has a 1 at the i-th entry and zeros everywhere else. Then e_i, $i \in \mathbb{N}$, is a K-valuation basis of (V,v) (over the zero subspace). Consider the countably dimensional subspace V' which is spanned by all elements $e_i - e_{i+1}$, $i \in \mathbb{N}$. These elements even form a K-valuation basis of V'. Then $V = V' \oplus Ke_1$. On the other hand, the skeleton of both V and V' is $\mathrm{sk}_{\mathbb{N}} K$ and thus, the extension $(V',v) \subset (V,v)$ is immediate. This shows that (V,v) does not admit a K-valuation basis over V'. Even more, V' is dense in (V,v). This follows from the fact that for every $i \in \mathbb{N}$, the element $e_1 - e_i$ lies in V' and we have $v(e_1 - (e_1 - e_i)) = ve_i = i$.

By a slight modification of this example, given by L. Fuchs in [FU3], one can show that Corollary 3.61 does not remain true if the subextension is not countably generated. Let us now take $V := \coprod_\lambda K$ where λ is a limit ordinal of cofinality type $> \omega$, and let us take V' to be the subspace generated by all $e_i - e_j$ for $i, j \in \lambda$. By way of contradiction, assume that V' admits a K-valuation basis of the form b_i, $i \in \lambda$ with $vb_i = i$. Consider the elements $e_0 - e_j$ and their unique representations as linear combinations $\sum r_i b_i$, $r_i \in K$. If $j \leq k$, then $v((e_0 - e_j) - (e_0 - e_k)) = v(e_k - e_j) = j$ showing that for all $i < j$, the i-th coefficient must be the same in the representations of both $e_0 - e_j$ and $e_0 - e_k$. We form an increasing sequence j_ν, $\nu \in \mathbb{N}$, of elements in λ as follows. Let $j_0 > 0$. Observe that for all $j > 0$ we have $v(e_0 - e_j) = 0$ and thus, b_0 must appear in the representation of $e_0 - e_j$. If $e_0 - e_{j_\nu} = \sum r_i b_i$, then $e_0 \neq \sum_{i<j_\nu} r_i b_i$ since $e_0 \notin V'$. Hence, we can choose $j_{\nu+1} > v(e_0 - \sum_{i<j_\nu} r_i b_i) =: k$. For $j \geq j_{\nu+1}$, we have $v(e_0 - (e_0 - e_j)) = ve_j = j > k$ showing that b_k must appear in the representation of $e_0 - e_j$. By construction, for every

$\nu \in \mathbb{N}$, the representation of $e_0 - e_{j_\nu}$ requires at least ν basis elements b_i of value $< j_\nu$, and these are also required in the representations of all $e_0 - e_j$ for $j \geq j_\nu$. Since the cofinality type of λ is not ω, there must be some $j \in \lambda$ which is larger than all j_ν. But then, the representation of $e_0 - e_j \in V'$ requires infinitely many basis elements b_i, a contradiction. We have proved that V' does not admit a K-valuation basis. Observe that as before, V' is dense in (V, v).

In this last example we have constructed a valued vector space not admitting a valuation basis. There is another method of constructing such vector spaces, using a cardinality argument. For example, consider $\coprod_{\mathbb{N}} \mathbb{Q}$ and $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$. The first is countable since it is a sum of countably many countable groups. Similarly, the skeleton is countable. The second $\mathbb{Q}$-vector space has cardinality $2^{\aleph_0}$ since this is the cardinality of all maps from $\mathbb{N}$ into $\mathbb{Q}$ (note that every subset of $\mathbb{N}$ is wellordered, so every map from $\mathbb{N}$ into $\mathbb{Q}$ is an element of the Hahn product). Since the Hahn product has the same countable skeleton as the Hahn sum, every $\mathbb{Q}$-valuation independent set contained in it must be countable; hence, it can not be a basis of the Hahn product. On the other hand, Theorem 3.30 shows that the Hahn product is a defectless extension of the trivial vector space 0. Even more, Theorem 3.30 in combination with Lemma 3.59 shows that the Hahn product is a defectless extension of all of its finite-dimensional subspaces. $\diamondsuit$

Our example has shown that there exist defectless extensions $(W, v) \subset (V, v)$ of ordinary valued K-vector spaces which admit a subextension $(W, v) \subset (V', v)$ such that $(V', v) \subset (V, v)$ is a proper immediate extension, and thus is not defectless. On the other hand, we can prove the following **transitivity of defectless extensions**:

Lemma 3.63 *Let $(W, v) \subset (V, v)$ be an extension of ordinary valued K-vector spaces and $(W, v) \subset (V', v)$ a subextension. If $(W, v) \subset (V', v)$ and $(V', v) \subset (V, v)$ are defectless extensions, then so is $(W, v) \subset (V, v)$.*

Proof: Let $(W, v) \subset (V_0, v)$ be a finite subextension of $(W, v) \subset (V, v)$. We have to show that it is defectless. Choose any K-basis $b_1, \ldots, b_n$ of V_0 over W and let V_1 be the K-subspace of V generated by $b_1, \ldots, b_n$ over V'. Since we have assumed $(V', v) \subset (V, v)$ to be a defectless extension, we know that the finite extension $(V', v) \subset (V_1, v)$ admits a K-valuation basis $\mathcal{B}_1$. To write down the elements $b_1, \ldots, b_n$ as K-linear combinations of the basis elements from $\mathcal{B}_1$ and elements from V', we only need n elements $b'_1, \ldots, b'_n$ from V'. Let V_2 be the K-subspace of V' generated by $b'_1, \ldots, b'_n$ over W. Since we have assumed $(W, v) \subset (V', v)$ to be a defectless extension, it follows that the finite extension $(W, v) \subset (V_2, v)$ admits a K-valuation basis $\mathcal{B}_2$. Since $\mathcal{B}_1$ is K-valuation independent over (V', v), it is also K-valuation independent over (V_2, v). Hence $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$ is K-valuation independent over (W, v). By construction, the K-subspace V_3 of V generated by $\mathcal{B}$ over W contains V_0. Since $\mathcal{B}$ is a K-valuation basis of the extension $(W, v) \subset (V_3, v)$, this extension is defectless by Corollary 3.60. Hence, its subextension $(W, v) \subset (V_0, v)$ is also defectless. $\square$

For the conclusion of this section, we wish to discuss the relation of "spherically closed" and "defectless" for ordinary valued vector spaces (which is not quite as spherically closed as for vector spaces with value preserving scalar multiplication).

Lemma 3.64 *Let $(W,v) \subset (V,v)$ be an extension of ordinary valued K-vector spaces. If every K-subspace (V',v) of (V,v) of arbitrary dimension over W is spherically closed in (V,v), then (V,v) admits a K-valuation basis over W.*

Proof: Take a well-ordered K-basis $\{b_\nu \mid \nu < \nu_0\}$ of V over W, where ν_0 is some ordinal. We construct a K-valuation basis of $(W,v) \subset (V,v)$ by induction on ν. Let V_μ be the K-vector space generated by $\{b_\nu \mid \nu \leq \mu\}$ over W. Assume that we have already constructed valuation bases $\mathcal{B}_\nu$ for all $\nu < \mu$ such that $\mathcal{B}_\nu \subset \mathcal{B}_{\nu'}$ if $\nu \leq \nu'$. If μ is a limit ordinal, then we set $\mathcal{B}_\mu := \bigcup_{\nu<\mu} \mathcal{B}_\nu$. This is a valuation basis over W since every finite set of elements in $\mathcal{B}_\mu$ is already included in some $\mathcal{B}_\nu$ with $\nu < \mu$. Now let $\mu = \mu' + 1$ be a successor ordinal. By hypothesis, $(V_{\mu'},v)$ is spherically closed in (V,v) and hence also in (V_μ,v). Since V_μ is of dimension 1 over $V_{\mu'}$, Lemma 3.20 shows that (V_μ,v) admits a K-valuation basis $\mathcal{B}'_\mu$ over $V_{\mu'}$. It follows that $\mathcal{B}_\mu := \mathcal{B}_{\mu'} \cup \mathcal{B}'_\mu$ is a K-valuation basis of (V_μ,v) over W, cf. Lemma 3.16. (For $\mu = 0$, just replace $(V_{\mu'},v)$ by (W,v) and $\mathcal{B}_{\mu'}$ by $\emptyset$ in this argument.) Now $\mathcal{B}_{\nu_0}$ is the desired K-valuation basis of $(W,v) \subset (V,v)$. $\square$

The following result was proved by F. Delon for the case of valued vector spaces which are induced by valued fields, cf. [DEL8], Théorème, equivalence of a) and b).

Corollary 3.65 *The extension $(W,v) \subset (V,v)$ of ordinary valued K-vector spaces is defectless if and only if every K-subspace (V',v) of (V,v) of finite dimension over W is spherically closed in (V,v).*

Proof: Let $(W,v) \subset (V,v)$ be defectless and (V',v) a K-subspace of (V,v) of finite dimension over W. We have to show that (V',v) is spherically closed in every K-subspace (V'',v) of (V,v) of dimension 1 over V'. But V'' is also of finite dimension over W, and thus, it admits a K-valuation basis over W by hypothesis. Hence, it admits a K-valuation basis over V' by virtue of Lemma 3.59. Corollary 3.13 now shows that (V',v) is spherically closed in (V'',v).

For the converse, assume that every K-subspace (V',v) of (V,v) of finite dimension over W is spherically closed in (V,v). Given a K-subspace (V'',v) of (V,v) of finite dimension over W, we have to show that it admits a K-valuation basis over W. By hypothesis, every K-subspace (V',v) of (V'',v) of finite dimension over W is spherically closed in (V,v) and thus also in (V'',v). Thus, the existence of the desired valuation basis follows from the foregoing lemma. $\square$

Exercise 3.8 *What happens to the above examples if $\mathbb{N}$ is replaced by $\omega + 1$ and λ is replaced by $\lambda + 1$?*

Exercise 3.9 *Let (V,v) be a valued K-vector space with value preserving scalar multiplication. Prove that every finite-dimensional subspace (W,v) is spherically closed in (V,v) (and consequently, $(W,v) \subset (V,v)$ is defectless). Show that this is not true for all ordinary valued vector spaces.*

3.8 Immediate extensions of valued modules

Let $\mathbf{M} = (M,v)$ be a component-compatibly valued left R-module. Assume $\mathbf{A}$ to be a non-trivial immediate approximation type over $\mathbf{M}$.

For $r \in R$ and $c \in M$, define

$$\Upsilon(\mathbf{A}, r, c) \;:\Leftrightarrow\; \forall \alpha \in \Lambda(\mathbf{A})\, \exists a \in \mathbf{A}_\alpha :\; v(c - ra) \geq \alpha$$

and

$$I_{\mathbf{A}} := \{r \in R \mid \Upsilon(\mathbf{A}, r, c) \text{ for some } c \in M\} .$$

Note that "$\exists a \in \mathbf{A}_\alpha :\; v(c - ra) \geq \alpha$" is equivalent to "$\forall a \in \mathbf{A}_\alpha :\; v(c - ra) \geq \alpha$". Indeed, if $a, a' \in \mathbf{A}_\alpha$, then $v(ra - ra') = vr(a - a') \geq v(a - a') \geq \alpha$ and if $v(c - ra) \geq \alpha$, then $v(c - ra') \geq \min\{v(c - ra), v(ra - ra')\} \geq \alpha$. Further, note that $I_{\mathbf{A}}$ always contains 0 since for $r = 0$, the condition is satisfied by $c = 0$.

In the case of value preserving scalar multiplication, the reader may show that the sets $r\mathbf{A}_\alpha$, $\alpha \in \Lambda(\mathbf{A})$, form a nest of balls in rM, and that its intersection is empty since $\mathbf{A}$ is assumed to be a non-trivial immediate approximation type. Then $I_{\mathbf{A}}$ is the collection of all $r \in R$ for which this nest of balls is realized in M. Hence, $r \notin I_{\mathbf{A}}$ if rM is spherically closed in M. Recall that by Lemma 1.19 this holds if and only if rM is spherically closed in M. Recall further that this is always the case if rM is spherically complete.

For arbitrary rings R, we have:

Lemma 3.66 *For every component-compatibly valued left R-module $\mathbf{M}$ and every non-trivial immediate approximation type $\mathbf{A}$ over $\mathbf{M}$, the set $I_{\mathbf{A}}$ is a proper left ideal of R.*

Proof: For every element $r \in I_{\mathbf{A}}$ we choose an element $c_r \in M$ such that $\Upsilon(\mathbf{A}, r, c_r)$. Let $r \in I_{\mathbf{A}}$ and $s \in R$. Then $v(sc_r - sry) \geq v(c_r - ry) \geq \alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, showing that $sr \in I_{\mathbf{A}}$. Now let $s \in I_{\mathbf{A}}$. Then $v(c_r + c_s - (r + s)y) \geq \min\{v(c_r - ry), v(c_s - sy)\} \geq \alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, showing that $r + s \in I_{\mathbf{A}}$. Finally, the intersection of all $\mathbf{A}_\alpha$ for $\alpha \in \Lambda(\mathbf{A})$ is empty by virtue of Lemma **??**, because $\mathbf{A}$ is a non-trivial immediate approximation type. This shows that there is no $c_1 \in M$ such that $\Upsilon(\mathbf{A}, 1, c_1)$ (recall that $\exists a \in \mathbf{A}_\alpha : v(c_1 - a) \geq \alpha$ implies that $c_1 \in \mathbf{A}_\alpha$). Consequently, $1 \notin I_{\mathbf{A}}$ and $I_{\mathbf{A}}$ is a proper left ideal of R. □

Lemma 3.67 *Assume that $I_{\mathbf{A}} = \{0\}$. Then there is a component-compatibly valued left R-module $(M \oplus Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M \oplus Rx, v)$ is immediate and x realizes $\mathbf{A}$ in $(M \oplus Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$, then there is an isomorphism $(M \oplus Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y. In particular, y is R-independent and torsion free over M.*

Proof: We let Rx be the free R-module on the generator x and $M \oplus Rx$ the direct sum of M with Rx. We have to extend the valuation v from M to $M \oplus Rx$. So let $c \in M$ and $r \in R$; we have to define $v(c - rx)$. By our assumption on $I_{\mathbf{A}}$, there is some $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c - ra) < \alpha$. Then we set $v(c - rx) = v(c - ra)$. This does not depend on the choice of α and a. Indeed, let $\alpha' \in \Lambda(\mathbf{A})$ and $a' \in \mathbf{A}_{\alpha'}$ such that $v(c - ra') < \alpha'$. Without loss of generality, we may assume that $\alpha \leq \alpha'$. Then $v(ra - ra') = vr(a - a') \geq v(a - a') \geq \alpha$, and $v(c - ra') = v(c - ra + ra - ra') = \min\{v(c - ra), v(ra - ra')\} = v(c - ra)$. In particular, this shows that $v(c - ra') < \alpha$ for all $a' \in \mathbf{A}_\alpha$.

Let us show that our definition yields a group valuation on $M \oplus Rx$. So let $c, c' \in M$ and $r, r' \in R$; we have to show that $v(c - rx - c' + r'x) \geq \min\{v(c - rx), v(c' - r'x)\}$. By what we have just shown, we may choose $\alpha \in \Lambda(\mathbf{A})$ large enough and $a \in \mathbf{A}_\alpha$ such that

$v(c-rx)=v(c-ra)$, $v(c'-r'x)=v(c-ra)$ and $v(c-c'-(r-r')x)=v(c-c'-(r-r')a)$. Then we have that $v(c-rx-c'+r'x)=v(c-c'-(r-r')a)\geq\min\{v(c-ra),v(c'-r'a)\}=\min\{v(c-rx),v(c'-r'x)\}$, as required. In the same way, it is shown that $vr'(c-rx)\geq v(c-rx)$, that is, the valuation on $M\oplus Rx$ is component-compatible.

We show that x realizes $\mathbf{A}$. Let $\alpha\in\Lambda(\mathbf{A})$ and $a\in\mathbf{A}_\alpha$. Then we choose $\alpha'>\alpha$ and $a'\in\mathbf{A}_{\alpha'}$ such that $v(a-x)=v(a-a')$. But $v(a-a')\geq\alpha$, hence $v(a-x)\geq\alpha$, as required. Now we can also show that the extension is immediate. Given $c-rx\in M\oplus Rx$, we choose $\alpha\in\Lambda(\mathbf{A})$ and $a\in\mathbf{A}_\alpha$ such that $v(c-rx)=v(c-ra)<\alpha$ and $v(a-x)\geq\alpha$. Then $v(c-rx-c+ra)=vr(a-x)\geq v(a-x)\geq\alpha>v(c-rx)$. This proves that for every $b\in M\oplus Rx$ there is some $b_0\in M$ such that $v(b-b_0)>vb$. By Lemma 2.9 it follows that the extension is immediate.

Now assume that y realizes $\mathbf{A}$. Given $c\in M$ and $r\in R$, we again choose $\alpha\in\Lambda(\mathbf{A})$ and $a\in\mathbf{A}_\alpha$ such that $v(c-ra)<\alpha$. On the other hand, $v(a-y)\geq\alpha$ by our assumption, which yields that $v(ra-ry)\geq\alpha$. Hence, we have $v(c-ry)=\min\{v(c-ra),v(ra-ry)\}=v(c-ra)=v(c-rx)$. That is, the map $c-rx\mapsto c-ry$ is value preserving. Consequently, $c-ry\neq 0$ if $c\neq 0$ or $r\neq 0$, showing that this map is an isomorphism of R-modules over M and that y is R-independent over M. □

Lemma 3.68 *Assume that $I_{\mathbf{A}}\neq\{0\}$ is principal. Let r_0 be a generator of $I_{\mathbf{A}}$ and assume that r_0 is not a zero divisor. Then there is a component-compatibly valued left R-module $(M+Rx,v)$ extending $\mathbf{M}$ such that $(M,v)\subset(M+Rx,v)$ is immediate, $r_0x\in M$ and x realizes $\mathbf{A}$ in $(M+Rx,v)$. Moreover, if $(M,v)\subset(M+Ry,v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$ and $r_0y=r_0x\in M$, then there is an isomorphism $(M+Rx,v)\cong(M+Ry,v)$ of valued R-modules over M which sends x to y.*

Proof: We choose $c_0\in M$ such that $\Upsilon(\mathbf{A},r_0,c_0)$. We let Rz be the free R-module on the generator z, and we define N to be the module $(M\oplus Rz)/R(c_0-r_0z)$. Observe that $r(c_0-r_0z)\in M$ will imply that $rr_0=0$. Since R is assumed to be a domain, this in turn will imply that $r=0$. That is, N contains M as a submodule. Choosing x to be the image of z under the canonical epimorphism, we find that x is a generator of N over M, and we can write $N=M+Rx$. By definition, $r_0x=c_0\in M$. We extend the valuation v from M to N as follows. Given the element $c-rx\in N$, we can assume that $r\notin I_{\mathbf{A}}$; otherwise, $c-rx\in M$ and there is nothing to define. But if $r\notin I_{\mathbf{A}}$ then we may define $v(c-rx)$ just as in the proof of the last lemma. The independence of the choice of α and a is shown as it was done there. To prove that $v(c-rx)$ is well-defined, it remains to show that our definition assigns the same value to $c-rx$ and $c'-r'x$ if $c-rz$ and $c'-r'z$ are equivalent modulo $R(c_0-r_0z)$ and if $r\neq 0$, $r'\neq 0$ and $r-r'\neq 0$. But the latter means that there is $r''\in R$ such that $r-r'=r''r_0$ and $c-c'=r''c_0$. We choose $\alpha\in\Lambda(\mathbf{A})$ and $a\in\mathbf{A}_\alpha$ such that $v(c-ra)<\alpha$, $v(c'-r'a)<\alpha$ and $v(c-c'-ra+r'a)=vr''(c_0-r_0a)\geq v(c_0-r_0a)\geq\alpha$. We find that in view of our definition, $v(c-rx)=v(c-ra)=\min\{v(c'-r'a),v(c-c'-ra+r'a)\}=v(c'-r'a)=v(c'-r'x)$, as required. Having proved this, the triangle inequality follows as in the proof of the last lemma, with one additional provision: We have to take the representation of the elements $c-rx$ and $c'-r'x$ in such a way that $r=0$ if $r\in I_{\mathbf{A}}$ and $r'=0$ if $r'\in I_{\mathbf{A}}$, and if $r-r'\in I_{\mathbf{A}}$, then we take $r'=r$. Again with the provision that $r=0$ if $r\in I_{\mathbf{A}}$, it is shown as in the proof of the last lemma that the extension is immediate, and the proof for the fact that x realizes $\mathbf{A}$ can be taken over literally.

To show that v is component-compatible on $M+Rx$, let $c\in M$ and $r,r'\in R$, $r\notin I_{\mathbf{A}}$. We choose $\alpha\in\Lambda(\mathbf{A})$ and $a\in\mathbf{A}_\alpha$ such that $v(c-ra)<\alpha$. Then $v(c-rx)=v(c-ra)\leq vr'(c-ra)=v(r'c-r'ra)$. If $r'r\notin I_{\mathbf{A}}$, then we may choose α as large as to guarantee that $v(r'c-r'ra)<\alpha$, and we obtain $v(r'c-r'ra)=v(r'c-r'rx)$. Now assume that $r'r\in I_{\mathbf{A}}$ and write $r'r=r''r_0$. Then $v(r''c_0-r'ra)=vr''(c_0-r_0a)\geq\alpha$, and in view of $v(c-ra)<\alpha$ we obtain that $v(c-rx)\leq\min\{v(r'c-r'ra),v(r''c_0-r'ra)\}=v(r'c-r''c_0)$. But since $r''c_0=r''r_0x=r'rx$, we obtain that $v(c-rx)\leq v(r'c-r''c_0)=vr'(c-rx)$. We have thus proved that v is component-compatible on $M+Rx$.

Now assume that y realizes $\mathbf{A}$. The map $c-rx\mapsto c-ry$ is well-defined by virtue of our assumption that $r_0y=c_0$. We leave it to the reader to show that it is value preserving. Consequently, $c-ry\neq 0$ if $c-rx\neq 0$, showing that this map is an isomorphism of R-modules over M. □

As a consequence of the last two lemmata, and in view of Theorem 3.6 and Lemma 1.38, we obtain the following result of I. Fleischer (cf. [FL1]):

Theorem 3.69 *Let R be a left principal ideal domain. Then a component-compatibly valued left R-module $\mathbf{M}$ is maximal if and only if every immediate approximation type over $\mathbf{M}$ is trivial, that is, if and only if $\mathbf{M}$ is spherically complete.*

As a consequence, every component-compatibly valued left R-module admits an immediate extension which is spherically complete, provided that R is a left principal ideal ring.

With $R=\mathbb{Z}$, we obtain:

Corollary 3.70 *A valued abelian group is maximal if and only if it is spherically complete.*

Remark 3.71 E. Schörner [SCHÖ] has obtained the assertion of this corollary, using an induction method to show that a valued abelian group admits an immediate extension if it admits a non-trivial immediate approximation type.

For the second half of this section, **let $\mathbf{M}=(M,v)$ be a finitely exceptional valued left R-module.** In fact, all that we are about to prove, will already work if for every $r\in R$, the set of exceptional values $\mathrm{Exv}_r\mathbf{M}$ is finite. This is always true in the case of finitely exceptional modules. By Lemma 3.3, we can write $\mathrm{Exv}_r\mathbf{M}=\{va\mid a\in M\wedge\exists a'\in M:\ (3.1)\text{ does not hold}\}$. In particular,

$$va<va'\wedge va\notin\mathrm{Exv}_r\mathbf{M}\implies vra<vra'\vee ra=ra'=0\,.$$

It turns out that in the case of finitely exceptional modules, it is convenient to work with pseudo Cauchy sequences, in view of the following fact. Let $\mathbf{S}:=(a_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in some valued module $\mathbf{M}$. If $r\in R$ is finitely exceptional on $\mathbf{M}$, then $\mathrm{Exv}_r\mathbf{M}$ is finite and hence, there is some $\nu_0<\lambda$ such that $\alpha_\nu:=v(a_\nu-a_{\nu+1})\notin\mathrm{Exv}_r\mathbf{M}$ for $\nu_0\leq\nu<\lambda$. From this, it follows that $v(ra_\nu-ra_{\nu+1})<v(ra_{\nu+1}-ra_\mu)$ or $ra_\nu=ra_{\nu+1}=ra_\mu$ whenever $\nu_0\leq\nu\leq\mu<\lambda$. Hence $(ra_\nu)_{\nu<\lambda}$ is also a pseudo Cauchy sequence, and we will denote it by $\mathbf{S}_r$. Let ν_0 be as in the definition of pseudo Cauchy sequences (Section 1.14).

Assume that $c\in M$ is not a limit of $\mathbf{S}$. Then in view of Lemma 1.48, there is some μ_0 such that $\nu_0\leq\mu_0<\lambda$ and $v(c-a_\nu)<v(a_\nu-a_{\nu+1})$ whenever $\mu_0<\nu<\lambda$. Suppose that $v(c-a_\nu)\notin\mathrm{Exv}_r\mathbf{M}$ for arbitrarily high $\nu\geq\nu_0$. From $v(c-a_\nu)<v(a_\nu-a_{\nu+1})$ with

such a $\nu > \mu_0$, we then find that $v(rc - ra_\nu) < v(ra_\nu - ra_{\nu+1})$ or $rc = ra_\nu = ra_{\nu+1}$. In the latter case, it follows that r is not injective on M and that $\mathbf{S}_r$ is eventually constant. Consequently, if $\mathbf{S}_r$ is not eventually constant, then rc will not be a limit of $\mathbf{S}_r$. Hence if c is not a limit of $\mathbf{S}$, but rc is a limit of $\mathbf{S}_r$ and this is not eventually constant, then there is some $\nu_1 \geq \nu_0$ such that $v(a_\nu - a_{\nu+1}) > v(c - a_\nu) = v(c - a_{\nu_1}) \in \mathrm{Exv}_r\mathbf{M}$ whenever $\nu_1 \leq \nu < \lambda$. Note that the same will then hold for every $c' \in M$ with $rc' = rc$. In particular, our consideration shows the following:
If r is ordinary and injective on $\mathbf{M}$ and $\mathbf{S}$ has no limit in M, then $\mathbf{S}_r$ has no limit in rM.

Now assume again that $\mathbf{A}$ is a non-trivial immediate approximation type over $\mathbf{M}$. We choose a pseudo Cauchy sequence $\mathbf{S} = \mathbf{S}_\mathbf{A}$ without a limit in $\mathbf{M}$, according to Lemma 1.52. For $r \in R$, we define

$$J_\mathbf{A} := \{r \in R \mid \mathbf{S}_r \text{ has a limit in } \mathbf{M}\} .$$

We leave it to the reader to show that $J_\mathbf{A}$ does not depend on the choice of the pseudo Cauchy sequence $\mathbf{S}$. Since $\mathbf{S}_0$ has 0 as a limit, $J_\mathbf{A}$ always contains 0. But $J_\mathbf{A}$ does not contain 1, because $\mathbf{S}$ has no limit in $\mathbf{M}$. Suppose that c is a limit of $\mathbf{S}_r$, and let $s \in R$. We wish to show that sc is a limit of $\mathbf{S}_{sr}$. By the definition of a limit, we have that $v(c - ra_\nu) < v(c - ra_\mu)$ for all large enough μ, ν such that $\nu < \mu < \lambda$. But for large enough ν we also have that $v(c - ra_\nu) \notin \mathrm{Exv}_s\mathbf{M}$. This yields that $v(sc - sra_\nu) < v(sc - sra_\mu)$ or $sc = ra_\nu = ra_\mu$ for all large enough μ, ν such that $\nu < \mu < \lambda$. In both cases, we obtain that sc is a limit of $\mathbf{S}_{sr}$. We have proved:

Lemma 3.72 *For every finitely exceptional valued left R-module $\mathbf{M}$ and every non-trivial approximation type $\mathbf{A}$ over $\mathbf{M}$, $J_\mathbf{A}$ is a proper subset of R containing 0 but not 1, and satisfying $RJ_\mathbf{A} \subset J_\mathbf{A}$. If $r \notin J_\mathbf{A}$ and the bone of $c - ra_\nu$ is ordinary in $\mathbf{M}$ for $s \in R$, then $sr \notin J_\mathbf{A}$ or sc is a limit of $\mathbf{S}_{sr}$ and $sc = sra_\nu$ for all large enough ν.*

Now we prove the following analogue of Lemma 3.67:

Lemma 3.73 *Assume that $J_\mathbf{A} = \{0\}$. Then there is a finitely exceptional valued left R-module $(M \oplus Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M \oplus Rx, v)$ is immediate and x realizes $\mathbf{A}$ in $(M \oplus Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$, then there is an isomorphism $(M \oplus Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y. In particular, y is R-independent and torsion free over M.*

Proof: We let Rx be the free R-module on the generator x and $M \oplus Rx$ the direct sum of M with Rx. We have to extend the valuation v from M to $M \oplus Rx$. So let $c \in M$ and $0 \neq r \in R$; we have to define $v(c - rx)$. By our assumption on $J_\mathbf{A}$, we know that c is not a limit of $\mathbf{S}_r$. So by Lemma 1.48, there exists some μ_0 such that $\nu_0 \leq \mu_0 < \lambda$ and $v(c - ra_\nu) = v(c - ra_{\mu_0})$ whenever $\mu_0 \leq \nu < \lambda$. We set $v(c - rx) = v(c - ra_{\mu_0})$.

Let us show that our definition gives a group valuation on $M \oplus Rx$. If $r \neq 0$, then $ra_\nu \neq c$ since $\mathbf{S}_r$ has no limit in $\mathbf{M}$. Hence $v(c - rx) = v(c - ra_\nu) \neq \infty$, showing that (V0) holds. Now let $c, c' \in M$ and $r, r' \in R$; we have to show that $v(c - rx - c' + r'x) \geq \min\{v(c - rx), v(c' - r'x)\}$. By what we have just shown, we may choose $\nu < \lambda$ large enough such that $v(c - rx) = v(c - ra_\nu)$, $v(c' - r'x) = v(c - ra_\nu)$ and $v(c + c' - (r + r')x) = v(c + c' - (r + r')a_\nu)$. Then we have $v(c - rx - c' + r'x) = v(c + c' - (r + r')a_\nu) \geq \min\{v(c - ra_\nu), v(c' - r'a_\nu)\} = \min\{v(c - rx), v(c' - r'x)\}$, as required.

We show that x is a limit of $\mathbf{S}$ and thus realizes $\mathbf{A}$. For $\nu < \mu < \lambda$, we have that $v(a_\nu - a_\mu) = v(a_\nu - a_{\nu+1})$. Hence our definition yields that $v(a_\nu - x) = v(a_\nu - a_{\nu+1})$, as required. Similarly, it is shown that rx is a limit of $\mathbf{S}_r$ for every $r \in R$. Now we can also prove that the extension is immediate. Given $c - rx \in M \oplus Rx$ with $r \neq 0$, we choose $\nu < \lambda$ such that $v(c-rx) = v(c-ra_\nu) < v(ra_\nu - ra_{\nu+1})$ and $v(ra_\nu - rx) = v(ra_\nu - ra_{\nu+1})$. Then $v(c - rx - c + ra_\nu) = v(ra_\nu - rx) = v(ra_\nu - ra_{\nu+1}) > v(c - rx)$. This proves that for every $b \in M \oplus Rx$ there is some $b_0 \in M$ such that $v(b - b_0) > vb$. By Lemma 2.9 it follows that the extension is immediate. Further, we note that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ for large enough ν. Trivially, this also holds if $r = 0$. Hence if in addition $s \in R$, then the bone of $sc - srx$ is equal to the bone of $sc - sra_\nu$, which in turn only depends on s and the bone of $c - ra_\nu \in M$, if the latter is ordinary. This shows that the bone of $c-rx$ is ordinary in $(M \oplus Rx, v)$ for s if the same holds for the bone of $c - ra_\nu$. We have thus proved that also $(M \oplus Rx, v)$ is finitely exceptional, with the same exceptional bones as $\mathbf{M}$.

Now assume that y also realizes $\mathbf{A}$. It follows that y is a limit of $\mathbf{S}$, like x. Given $c \in M$ and $r \in R$, we choose $\nu < \lambda$ such that $v(c - ra_\nu) < v(ra_\nu - ra_{\nu+1})$. Since ry is a limit of $\mathbf{S}_r$ (which is shown as for rx), we have that $v(ra_\nu - ry) = v(ra_\nu - ra_{\nu+1}) > v(c - ra_\nu)$. Hence, $v(c-ry) = \min\{v(c-ra_\nu), v(ra_\nu - ry)\} = v(c-ra_\nu) = v(c-rx)$. That is, the map $c - rx \mapsto c - ry$ is value preserving. Consequently, $c - ry \neq 0$ if $c \neq 0$ or $r \neq 0$, showing that this map is an isomorphism of R-modules over M and that y is R-independent and torsion free over M. $\square$

The following is the analogue of Lemma 3.68:

Lemma 3.74 *Assume that R is a left euclidean domain with respect to a degree function* deg*, and assume that $J_{\mathbf{A}} \neq \{0\}$. Let $r_0 \neq 0$ be an element of minimal degree* $\deg r_0$ *in $J_{\mathbf{A}}$. Then there is a finitely exceptional valued left R-module $(M + Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M + Rx, v)$ is immediate, $r_0 x \in M$ and x realizes $\mathbf{A}$ in $(M + Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$ and $r_0 y = r_0 x \in M$, then there is an isomorphism $(M + Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y.*

Proof: We choose $c_0 \in M$ to be a limit of $\mathbf{S}_{r_0}$. We let Rz be the free R-module on the generator z, and we define N to be the module $(M \oplus Rz)/R(c_0 - r_0 z)$. As in the proof of Lemma 3.68, N is shown to contain M as a submodule. Choosing x to be the image of z under the canonical epimorphism, we find that x is a generator of N over M, and we can write $N = M + Rx$. By construction, $r_0 x = c_0 \in M$. We extend the valuation v from M to N as follows. Take any element $c - rx \in N$. We can assume that $\deg r < \deg r_0$: Since R is a left euclidean domain by hypothesis, for every $r \in R \setminus \{0\}$ there is $r' \in R$ with $\deg r' < \deg r$ such that $r - r' \in Rr_0$. But then, $c - rz$ is equivalent modulo $R(c_0 - r_0 z)$ to $c' - r'z$ for some c'. So every element of N can be given in the form $c - rx$ with $\deg r < \deg r_0$. This yields that $r = 0$ or $r \notin J_{\mathbf{A}}$. In the first case, $c - rx = c \in M$ and there is nothing to define. In the second case, we can define $v(c - rx)$ just as in the proof of the last lemma. Again it follows that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ for large enough ν (which again proves that the extension is immediate). To prove that $v(c-rx)$ is well-defined, it remains to show that our definition assigns the same value to $c - rx$ and $c' - r'x$ if $c - rz$ and $c' - r'z$ are equivalent modulo $R(c_0 - r_0 z)$ and if $\deg r < \deg r_0$ and $\deg r' < \deg r_0$.

We shall show a little bit more. We just assume that $c - rz$ and $c' - r'z$ are equivalent modulo $R(c_0 - r_0 z)$. Then there is $r'' \in R$ such that $r - r' = r'' r_0$ and $c - c' = r'' c_0$. Hence,

$$c - ra_\nu - (c' - r'a_\nu) = c - c' - (r - r')a_\nu = r'' c_0 - r'' r_0 a_\nu .$$

Since c_0 is a limit of $\mathbf{S}_{r_0}$, we know that $r'' c_0$ is a limit of $\mathbf{S}_{r'' r_0}$. Hence by the definition of a limit, $v(r'' c_0 - r'' r_0 a_\nu)$ increases with ν, or $r'' c_0 = r'' r_0 a_\nu$ for all large enough ν. In the second case, $c - ra_\nu = c' - r'a_\nu$ for all large enough ν. Now assume the first case, and assume in addition that $v(c - ra_\nu)$ and $v(c' - r'a_\nu)$ are constant for all large enough ν. But this can only be the case if $v(r'' c_0 - r'' r_0 a_\nu) > v(c - ra_\nu)$ for large enough ν. Since this means that $v(c - ra_\nu - (c' - r'a_\nu)) > v(c - ra_\nu)$, it follows that the bones of $c - ra_\nu$ and $c' - r'a_\nu$ are equal for all large enough ν, as in the second case. Note that if $c' - r'a_\nu$ is constantly 0 for all large enough ν, then we must have the first case and also $c - ra_\nu$ is constantly 0 for all large enough ν.

Now if $r, r' \notin J_{\mathbf{A}}$, then c, c' are not limits of $\mathbf{S}_r$ resp. $\mathbf{S}_{r'}$. Hence by Lemma 1.48, $v(c - ra_\nu)$ and $v(c' - r'a_\nu)$ are constant for all large enough ν, and we obtain the equality of the bones of $c - ra_\nu$ and $c' - r'a_\nu$ are equal for large enough ν. If in addition $\deg r < \deg r_0$, then we know already that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ and thus also to the bone of $c' - r'a_\nu$ for large enough ν. If we assume further that $\deg r' < \deg r_0$, then we obtain that $v(c - rx) = v(c - ra_\nu) = v(c' - r'a_\nu) = v(c' - r'x)$ for large enough ν, which proves that $v(c - rx)$ is well-defined.

Having proved this, the triangle inequality follows as in the proof of the last lemma, with one additional provision: We have to take the representation of the elements $c - rx$ and $c' - r'x$ in such a way that $r = 0$ if $r \in Rr_0$ and $r' = 0$ if $r' \in Rr_0$; and if $r - r' \in Rr_0$, then we take $r' = r$. The proof for the fact that x realizes $\mathbf{A}$ can be taken over literally.

To prove that (N, v) is finitely exceptional, we need an additional argument. Assume that $c - rx \in N$ with $\deg r < \deg r_0$. Let $s \in R$ and assume that the bone of $c - ra_\nu$ is ordinary in $\mathbf{M}$ for s. Then by Lemma 3.72, $sr \notin J_{\mathbf{A}}$, or sc is a limit of $\mathbf{S}_{sr}$ and $sc = sra_\nu$ for all large enough ν. In the first case, by what we have shown above, the bone of $sc - srx$ is equal to the bone of $sc - sra_\nu$ for large enough ν. In the second case, let $\tilde{r} \in R$ and $\tilde{c} \in M$ be such that $\deg \tilde{r} < \deg r_0$ and that $sc - srz$ and $\tilde{c} - \tilde{r}z$ are equivalent modulo $R(c_0 - r_0 z)$. Since $\tilde{r} \notin J_{\mathbf{A}}$, we know that $v(\tilde{c} - \tilde{r}a_\nu)$ is constant for all large enough ν. In view of $sc - sra_\nu = 0$, we can infer from our above computation that $sc - srx = \tilde{c} - \tilde{r}a_\nu = c - ra_\nu = 0$ for all large enough ν. So also in this case, the bones of $sc - sra_\nu$ and $c - rx$ are equal. But the bone of $sc - sra_\nu$ only depends on s and the bone of $c - ra_\nu$ since this is assumed to be ordinary in $\mathbf{M}$ for s. Hence $(M \oplus Rx, v)$ is finitely exceptional, with the same exceptional bones as $\mathbf{M}$.

Now assume that y realizes $\mathbf{A}$. The map $c - rx \mapsto c - ry$ is well-defined by virtue of our assumption that $r_0 y = c_0$. We leave it to the reader to show that it is value preserving. Consequently, $c - ry \neq 0$ if $c - rx \neq 0$, showing that this map is an isomorphism of R-modules over M. □

As a consequence of Lemma 3.73 and Lemma 3.74, and in view of Theorem 3.6 and Lemma 1.38, we obtain:

Theorem 3.75 *Let R be a left euclidean domain. Then a finitely exceptional valued left R-module $\mathbf{M}$ is maximal if and only if every immediate approximation type over $\mathbf{M}$ is trivial; that is, if and only if $\mathbf{M}$ is spherically complete.*

As a consequence, every finitely exceptional valued left R-module admits an immediate extension which is spherically complete, provided that R is a left euclidean domain.

Let $M \subset N$ be an extension of left R-modules and $b \in R$. We will call b **R-algebraic over** M if there is some $r \in R \setminus \{0\}$ such that $rb \in M$, and we will call $M \subset N$ an **R-algebraic extension** if N is generated over M by a set of elements which are R-algebraic over M. If R is commutative, then this implies that every element of N is R-algebraic over M. If $R = \mathbb{Z}$, then our definition coincides with our notion of "algebraic extension" for abelian groups. We say that a valued R-module $\mathbf{M}$ is **algebraically maximal** if it does not admit a proper immediate R-algebraic extension. Now let $\mathbf{M}$ be a component-compatible resp. a finitely exceptional valued left R-module and $\mathbf{A}$ a non-trivial immediate approximation type over $\mathbf{M}$. Let us call $\mathbf{A}$ a **transcendental immediate approximation type** if $I_{\mathbf{A}} = \{0\}$ resp. $J_{\mathbf{A}} = \{0\}$; otherwise, we call it an **algebraic immediate approximation type**. Lemma 3.68 and Lemma 3.68 have shown that if $\mathbf{M}$ admits a non-trivial algebraic approximation type, then it admits an immediate R-algebraic extension. Let us show the converse. Let $(M, v) \subset (M + Rx, v)$ be immediate and $0 \neq r_0 \in R$ such $r_0 x = c_0 \in M$. Then by Lemma 1.36, $\mathbf{A} := \mathrm{at}\,(x, M)$ is immediate. On the other hand, $v(c_0 - r_0 x) = \infty \geq \alpha$ for all $\alpha \in \Lambda(x, M)$, showing that $r_0 \in I_{\mathbf{A}}$. We have proved:

Theorem 3.76
a) Let R be a left principal ideal domain. Then a component-compatibly valued left R-module $\mathbf{M}$ is algebraically maximal if and only if every algebraic immediate approximation type over $\mathbf{M}$ is trivial.
b) Let R be a left euclidean domain. Then a finitely exceptional valued left R-module $\mathbf{M}$ is algebraically maximal if and only if every algebraic immediate approximation type over $\mathbf{M}$ is trivial.

Example 3.77 Let G be a $\mathbb{Z}$-group and v its natural valuation. Then (G, v) is an ordinary valued $\mathbb{Z}$-module. If $0 \neq n \in \mathbb{Z}$, then nG is again a $\mathbb{Z}$-group. According to Lemma 3.50, nG is spherically closed in its divisible hull and thus also in G. In fact, this was verified directly in the argument preceding that lemma. Now it follows from the above theorem that (G, v) is algebraically maximal. $\diamondsuit$

If R is a field or a skew field, then $I_{\mathbf{A}} = \{0\}$ (resp. $J_{\mathbf{A}} = \{0\}$) for every non-trivial approximation type $\mathbf{A}$ over $\mathbf{M}$, because $I_{\mathbf{A}}$ is an ideal of R (resp. $1 \notin RJ_{\mathbf{A}}$). In particular, we obtain:

Corollary 3.78 *Every ordinary valued vector space (V, v) is algebraically maximal. Its maximal immediate extensions are spherically complete and unique up to valuation preserving isomorphism over V.*

Proof: Only the uniqueness remains to be proved. Let (V_1, v_1) and (V_2, v_2) be two ordinary valued K-vector spaces which are maximal immediate extensions of the ordinary valued K-vector space (V, v), and let (V', v) be a maximal valued subspace of (V_1, v_1) which admits a valuation preserving embedding in (V_2, v_2) over V. We identify (V', v) with its image in (V_2, v_2). Suppose that $V' \neq V_1$. Then we choose $x \in V_1 \setminus V'$. Since $(V', v) \subset (V_1, v_1)$ is immediate like $(V, v) \subset (V_1, v_1)$, we know by Lemma 1.36 that $\mathbf{A} := \mathrm{at}\,(x, V')$ is immediate. Since (V', v_1) is algebraically maximal, this approximation type is transcendental. Since (V_2, v_2) is maximal, it must contain an element y which realizes $\mathbf{A}$. Now Lemma 3.73 shows that there is a valuation preserving isomorphism of $(V' + Kx, v_1)$ and $(V' + Ky, v_2)$ over V'. This contradicts the maximality of V' and thus shows that $V' = V_1$. Let V_2' denote the isomorphic image of V_1 in V_2. Then (V_2', v_2) is maximal (cf. Exercise 3.10). On the other hand, the extension $(V_2', v_2) \subset (V_2, v_2)$ is immediate. This proves that $V_2' = V_2$ and that (V_1, v_1) and (V_2, v_2) admit a valuation preserving isomorphism over V. $\square$

Exercise 3.10 *Show that maximality (of valued abelian groups, valued modules etc.) is preserved under valuation preserving isomorphisms.*

Exercise 3.11 *Show that every component-compatibly (resp. finitely exceptional) valued module has an immediate extension which does not admit any transcendental immediate approximation type.*

Exercise 3.12 *Let $\mathbf{G}$ be a valued group such that only finitely many of its components are not divisible. Assume that $vna = va$ for all $n \in \mathbb{Z} \setminus \{0\}$. Prove that $\mathbf{G}$ is algebraically maximal. Try to do the same without the latter hypothesis.*