Chapter 6

Valued field extensions

6.1 Immediate extensions and density

An extension $(L|K,v)$ is called **immediate** if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are both onto. Instead of the latter, we will also say "if (K,v) and (L,v) have the same value group and the same residue field" or just "in $vL = vK$ and $Lv = Kv$" (recall that we are identifying equivalent valuations and places, so we may view vK as a subgroup of vL and Kv as a subfield of Lv). But the reader should note that this is less precise and can be misunderstood. For instance, if $vK \cong \mathbb{Z}$ and $L|K$ is finite, then still, $vK \cong \mathbb{Z}$ even if the embedding of vK in vL is not onto.

Lemma 6.1 *The following are equivalent:*

1) the extension $(L|K,v)$ is immediate,

2) for every $a \in L^\times$ there is $c \in K$ such that $v(a-c) > va$,

3) the underlying extension of valued abelian groups is immediate,

4) the underlying extension of ultrametric spaces is immediate.

Proof: Suppose that $(L|K,v)$ is immediate, and let $0 \neq a \in L$. Then $va \in vL = vK$ and thus, there is some $b \in K$ such that $va = vb$. Hence, $v\frac{a}{b} = 0$. Then $\frac{a}{b}v \in Lv = Kv$ and thus, there is some $d \in K$ such that $\frac{a}{b}v = dv$. That is, $v\left(\frac{a}{b} - d\right) > 0$, which yields that $v(a-bd) > vb = va$. Hence $c = bd$ is the element that we have looked for. We have proved that 1) implies 2).

Now we show that 2) implies 1). Take $\alpha \in vL$ and $a \in L$ such that $va = \alpha$. If there is $c \in K$ such that $v(a-c) > va$, then $\alpha = va = vc \in vK$. Now let $\zeta \in \overline{L}$ and $a \in L$ such that $av = \zeta$. If there is $c \in K$ such that $v(a-c) > va = 0$, then $\zeta = av = cv \in Kv$.

For the equivalence of 2) and 3), see Lemma 2.9. For the equivalence of 3) and 4), see Corollary 2.10. □

Some classes of valued fields are closed under immediate extensions. Let us give two important examples.

Lemma 6.2 *Every immediate extension of a finitely ramified field is again a finitely ramified field. Every immediate extension of a formally $\wp$-adic (resp. formally p-adic) field is again a formally $\wp$-adic (resp. formally p-adic) field.*

Proof: If (K, v) is a finitely ramified field and $(L|K, v)$ is an immediate extension, then because of $vL = vK$, the prime element of (K, v) will also be a prime element of (L, v), showing that also (L, v) is a finitely ramified field. If in addition, $\overline{K}$ is finite, then so is $\overline{L}$, and (L, v) is a formally $\wp$-adic field (resp. a formally p-adic field if $\overline{L} = \overline{K} = \mathbb{F}_p$. □

Recall that (K, v) is said to be **dense** in (L, v) if for every $a \in L$ and all $\beta \in vL$ there is some $c \in K$ such that $v(a - c) > \beta$. It follows directly from the definition that $(L|K, v)$ is immediate if (K, v) is dense in (L, v). The converse is not true, cf. Example 11.59. We leave the proof of the following lemma as an exercise to the reader.

Lemma 6.3 *The following are equivalent:*

1) the valued field (K, v) is dense in (L, v),

2) the underlying valued additive group of (K, v) is dense in the underlying valued additive group of (L, v).

3) the underlying ultrametric space of (K, v) is dense in the underlying ultrametric space of (L, v).

The properties discussed above are transitive:

Lemma 6.4 *Take an extension $(L|K, v)$ of valued fields and is an arbitrary subextension $E|K$ of $L|K$.*

a) The extension $(L|K, v)$ is immediate if and only if the extensions $(L|E, v)$ and $(E|K, v)$ are immediate.

b) The valued field (K, v) is dense in (L, v) if and only if (E, v) is dense in (L, v) and (K, v) is dense in (E, v).

Proof: a): This is trivial since $vK \subset vE \subset vL$ and $Kv \subset Ev \subset Lv$ for every extension $(L|K, v)$.

b): The implication "⇒" is trivial. Suppose that (K, v) is dense in (E, v) and (E, v) is dense in (L, v). Let $a \in L$ and $\beta \in vL$. Choose $c' \in E$ such that $v(a - c') > \beta$. Without loss of generality, we may assume that $\beta \geq va$. Then it follows that $va = vc' \in vE$. So we can choose $c \in K$ such that $v(c' - c) > \beta$. Then $v(a - c) \geq \min\{v(a - c'), v(c' - c)\} > \beta$. We have proved that (K, v) is dense in (L, v). □

6.2 Vector space defectless extensions

After having treated immediate extensions in the last section, we shall now consider the contrary case: one might call it the "anti-immediate case", where the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are the maximal possible. But we have to make precise what we mean by that. To this end, we use our notion of a defectless extension of valued vector spaces.

As usual in field theory, we may view L as a K-vector space. Even more, we may view (L, v) as a valued K-vector space. If v is nontrivial on K, then the scalar multiplication is not value preserving and also not component-compatible. On the other hand, (L, v) is an ordinary valued K-vector space: if a and b are elements of L having equal bones and

if $r \in K$, then $v(a-b) > va$ and thus, $v(ra-rb) = v(a-b)+vr > va+vr = vra$, showing that ra and rb have equal bones. But this is not the only basic property of (L,v) as a valued K-vector space. We also see that every 1-dimensional subvector space of (L,v) is isomorphic to (K,v) as a valued K-vector space. Indeed, it is of the form (Ka,v) for some $a \in L$, and the map $\iota : c \mapsto ca$ is an isomorphism of the K-vector space K onto the K-vector space Ka, satisfying $vb < vc \Leftrightarrow v\iota b < v\iota c$ for every $b,c \in K$. This gives rise to the following definition: A valued K-vector space (V,v) will be called a **(K,v)-vector space** if every 1-dimensional subvector space of (V,v) is isomorphic to (K,v) as a valued K-vector space. We have seen:
If $(L|K,v)$ is an extension of valued fields, then (L,v) is a (K,v)-vector space.

An extension $(L|K,v)$ of valued fields will be called **vs-defectless** if $(K,v) \subset (L,v)$ is a vs-defectless extension of valued K-vector spaces. Alternatively, the name **separated pair** was introduced by W. Baur in [BAUR1], [BAUR2] and used by F. Delon in [DEL8].

In view of the fact that valued fields are ordinary valued vector spaces, the following **transitivity of vs-defectless extensions** is a direct consequence of Lemma 3.63.

Lemma 6.5 *Let $(L|K,v)$ be an extension of valued fields and $E|K$ an arbitrary subextension of $L|K$. If $(L|K,v)$ is a vs-defectless extension, then so is $(E|K,v)$. Conversely, if $(L|E,v)$ and $(E|K,v)$ are vs-defectless extensions, then so is $(L|K,v)$.*

But note that if $(L|K,v)$ is a vs-defectless extension and $E|K$ is not finite, then it will in general not follow that $(L|E,v)$ is a vs-defectless extension. The examples of Section 11.5 will show that there are fields (K,v) admitting algebraic extensions (L,v) such that $(\tilde{K}|K,v)$ is a vs-defectless extension (for every extension of the valuation), while $(\tilde{K}|L,v)$ is not a vs-defectless extension.

Algebraic vs-defectless extensions may be characterized as follows:

Lemma 6.6 *An algebraic extension $(L|K,v)$ of valued fields is vs-defectless if and only if every finite subextension $(E|K,v)$ of $(L|K,v)$ is vs-defectless.*

Proof: The implication "⇒" is just a special case of Lemma 6.5, and the converse holds since every finite K-subvector space of an algebraic extension $L|K$ already lies in a finite subextension $E|K$. □

At the beginning of this section, we have mentioned the expression "anti-immediate". The vs-defectless extensions are indeed anti-immediate in the following sense:

Lemma 6.7 *Let $(L|K,v)$ be a vs-defectless extension and $b \in L \setminus K$. Then the set $v(b-K) = \{v(b-c) \mid c \in K\}$ has a maximum. Hence, in every coset $b-K$ there is an element a for which the condition in part 2) of Lemma 6.1 fails.*

Proof: It follows from the definition of vs-defectless extensions that the K-subvector space $(K \oplus Kb, v)$ of (L,v) admits a K-valuation basis. Corollary 3.13 shows that the set $v(b-K)$ has a maximum $v(b-c_0)$. For the element $b-c_0 \in b-K$ the condition in part 2) of Lemma 6.1 fails; otherwise, we would have that $v(b-c_0) < v((b-c_0)-c = v(b-(c_0+c)) \in v(a-K)$, a contradiction. □

The vs-defectless extensions are anti-immediate also in the sense that they are linearly disjoint from all immediate extensions. Recall that if $b_1,\ldots,b_n$ are K-valuation independent elements in a valued field extension of (K, v), then they are also K-linearly independent (cf. page 65).

Lemma 6.8 *Let $(\Omega|K, v)$ be an arbitrary valued field extension with $(L|K, v)$ a vs-defectless and $(F|K, v)$ an immediate subextension. Then $L|K$ is linearly disjoint from $F|K$, the extension $(L.F|L, v)$ is immediate and the valuation on $L.F$ is uniquely determined by the valuations on L and F. If $b_1, \ldots, b_n \in L$ are K-valuation independent, then they are also F-valuation independent.*

Proof: We have to show that for every choice of finitely many K-linearly independent elements $x_1, \ldots, x_n \in L$, these elements are also F-linearly independent. Since $(L|K, v)$ is assumed to be vs-defectless, the K-vector space generated by $x_1, \ldots, x_n$ admits a valuation basis. This must consist of n elements, say $b_1, \ldots, b_n \in L$. Since they generate the same vector space as the x_i also over F, it now suffices to show that $b_1, \ldots, b_n$ are F-linearly independent. This in turn will follow if we can show that the elements $b_1, \ldots, b_n$ are F-valuation independent. So let $a_1, \ldots, a_n \in F$ such that $a_i \neq 0$ for at least one i. Since $(F|K, v)$ is assumed to be immediate, we can choose $c_1, \ldots, c_n \in K$ such that $c_i = 0$ if $a_i = 0$, and $v(a_i - c_i) > va_i$ otherwise. Then we have that $va_i = vc_i$ and

$$v\left(\sum_{i=1}^{n} c_i b_i\right) = \min vc_i b_i < \min v(a_i - c_i)b_i$$

and therefore,

$$v\left(\sum_{i=1}^{n} a_i b_i\right) = v\left(\sum_{i=1}^{n} c_i b_i + \sum_{i=1}^{n}(a_i - c_i)b_i\right) = v\left(\sum_{i=1}^{n} c_i b_i\right) = \min vc_i b_i = \min va_i b_i\ .$$

This proves that the elements $b_1, \ldots, b_n$ are F-valuation independent. □

Corollary 6.9 *If $(L|K, v)$ is a vs-defectless and $(F|K, v)$ is an immediate extension, then the compositum of (L, v) and (F, v) is unique up to isomorphism of valued fields over K.*

Corollary 6.10 *Let $(L|K, v)$ be a vs-defectless extension. If (L, v) is a maximal field, then so is (K, v).*

Proof: If $(F|K, v)$ were a non-trivial immediate extension, then by Lemma 6.8, every compositum $(L.F, v)$ were a nontrivial immediate extension of (L, v). There is no such extension since (L, v) is maximal. Hence there is no nontrivial immediate extension of (K, v), showing that also (K, v) is maximal. □

In the following, let us use on (K, v) the coefficient map as defined on page **??**, with factors $\zeta_{\alpha,\beta}$ as given in (**??**). Then the scalar multiplication acts on the bones as follows. Given a bone (α, c), write it as $(\alpha, c) = (va, \mathrm{co}\, a)$ for some $a \in L$. Then $r \cdot (\alpha, c) = (vra, \mathrm{co}\, ra) = (vr + va\, , \overline{rat^{-vra}}) = (vr + va\, , \mathrm{co}\, r \cdot \mathrm{co}\, a \cdot \zeta_{vr,va})$. So we have:

$$r \cdot (\alpha, c) = (vr + \alpha\, , \mathrm{co}\, r \cdot c \cdot \zeta_{vr,\alpha})\ . \tag{6.1}$$

Note that the factor $\zeta_{vr,\alpha}$ only depends on α and the value of r. If the coefficient map is multiplicative, then this factor is always equal to 1.

Given bones $(\alpha_1,\zeta_1),\ldots,(\alpha_n,\zeta_n)$, we may deduce from (6.1) the following facts. If the values $\alpha_1,\ldots,\alpha_n$ belong to distinct cosets modulo vK, then the bones are K-independent. On the other hand, if all α_i are equal and the elements ζ_i are $\overline{K}$-independent, then again, the bones are K-independent. More precisely:

Lemma 6.11 *Let $(L|K,v)$ be an extension of valued fields and (α_i,ζ_{ij}), $i\in I$, $j\in J_i$, bones in the skeleton of (L,v). If the values α_i, $i\in I$, belong to distinct cosets modulo vK and for every i, the elements ζ_{ij}, $j\in J_i$, are $\overline{K}$-linearly independent, then these bones are K-independent.*

Proof: Suppose that $\sum_{i,j} r_{ij}(\alpha_i,\zeta_{ij})$, $r_{ij}\in K$, is an admissible finite linear combination, that is, all values $vr_{ij}+\alpha_i$ with $r_{ij}\neq 0$ are equal. Since all α_i are generating distinct cosets modulo vK, this yields that there is some i_0 such that for all $i\neq i_0$, we must have $r_{ij}=0$. Further, all $r_{i_0j}\neq 0$ will have the same value, which we will denote by β. Let $\alpha:=\alpha_{i_0}$. Then our linear combination is equal to the bone $(\beta+\alpha\,,\,\zeta_{\beta,\alpha}\cdot\sum_j \mathrm{co}\, r_{i_0j}\cdot\zeta_{i_0j})$. Since all $\mathrm{co}\, r_{i_0j}\in\overline{K}$ and since the elements ζ_{i_0j}, $j\in J_{i_0}$, are $\overline{K}$-independent by assumption, the sum $\sum_j \mathrm{co}\, r_{i_0j}\cdot\zeta_{i_0j}$ is not equal to 0. This proves that the bones (α_i,ζ_{ij}) are K-independent. □

6.3 Valuation independence

For extensions of valued modules we have developed a notion of valuation independence of elements. We will now transfer this notion to extensions of valued fields. In what follows, let $(L|K,v)$ be an arbitrary extension of valued fields.

Lemma 6.12 *Let $(L|K,v)$ be an extension of valued fields and $\{z_i,u_j\mid i\in I\,,\,j\in J\}\subset L$ such that the values vz_i, $i\in I$, belong to distinct cosets modulo vK and that u_j, $j\in J$, are elements of $\mathcal{O}_L^\times$ whose residues $\overline{u_j}$ are $\overline{K}$-independent. Then the elements*

$$z_iu_j \quad i\in I\,,\ j\in J$$

are K-valuation independent.

Proof: Let $r_{ij}\in K$. Consider the summands of least value α in the (nontrivial) sum

$$\sum_{i,j} r_{ij}z_iu_j\ .$$

We have to show that α is also the value of the whole sum. So let i_0,j_0 be such that $r_{i_0j_0}z_{i_0}u_{j_0}$ is a summand of least value α. Then $\alpha\in vz_{i_0}+vK$ and by hypothesis on the elements z_i, we find $\alpha\notin vz_i+vK$ for $i\neq i_0$. Hence, all summands $r_{ij}z_iu_j$ of least value must satisfy $i=i_0$. So if we are able to show that the value of the sum $\sum_j r_{i_0j}z_{i_0}u_j$ is α, then we are done since the remaining summands are all of value $>\alpha$. After a division by the element $r_{i_0j_0}z_{i_0}$ (which is of value α), it remains to show that the sum $a:=\sum_j r_{i_0j_0}^{-1}r_{i_0j}u_j$ is of value 0. By construction, all $s_j:=r_{i_0j_0}^{-1}r_{i_0j}$ are of value ≥ 0, and $s_{j_0}=1$. Hence,

$\overline{a} = \sum_j \overline{s_j}\,\overline{u_j}$ is a nontrivial linear combination with $\overline{s_j} \in \overline{K}$, and by our assumption on the elements u_j, it must be nonzero. That is, $va = 0$ as required. □

For the convenience of the reader, we have given a direct proof for this lemma. But it also follows from Lemma 6.11. Indeed, from Lemma 3.17 we know that elements of L are K-valuation independent if and only if their bones are K-independent. Further, note that if $a_i \in L$, $i \in I$, are elements of equal value α whose coefficients are K-independent, then for arbitrary $a \in L$, the elements aa_i, $i \in I$, have coefficients $\zeta_{va,\alpha} \cdot \mathrm{co}\, a \cdot \mathrm{co}\, a_i$ which are still K-independent. In view of these facts, the foregoing lemma implies:

Consider an arbitrary extension $(L|K, v)$ of valued fields. We choose elements $z_i \in L$ such that vz_i, $i \in I$, is a system of representatives for the cosets of vL modulo vK. Hence, $|I| = |vL/vK|$ (which may be a finite or infinite cardinal). Further, we may choose elements $u_j \in L$ such that $\overline{u_j}$, $j \in J$, is a $\overline{K}$-basis of $\overline{L}$. Hence, $|J| = \dim_{\overline{K}} \overline{L}$ (which again may be a finite or infinite cardinal). Then by the last corollary, the elements $z_i u_j$, $i \in I$, $j \in J$, are K-valuation independent. Hence, $[L : K]$ is an upper bound for the number $|I| \cdot |J|$ of these elements $z_i u_j$. This proves:

Lemma 6.13 *If $(L|K, v)$ is a finite extension of valued fields, then $\overline{L}|\overline{K}$ is finite and the quotient vL/vK is finite and hence a torsion group. Moreover,*

$$[L : K] \geq (vL : vK) \cdot [\overline{L} : \overline{K}] \,. \tag{6.2}$$

The (finite or infinite) cardinal $e(L|K, v) := (vL : vK)$ is called the **ramification index of** $(L|K, v)$, and $f(L|K, v) := [\overline{L} : \overline{K}]$ is called the **inertia degree of** $(L|K, v)$. Writing shortly $n = [L : K]$, $e = e(L|K, v)$ and $f = f(L|K, v)$, equation (6.2), called the **fundamental inequality**, can be expressed in the following form which is easy to remember:

$$n \geq e \cdot f \,.$$

Note that the extension $(L|K, v)$ is immediate if and only if $e = 1$ and $f = 1$.

As a first application of the fundamental inequality, let us give two important examples of classes of valued fields which are closed under finite extensions. We know already from Lemma 6.2 that the same classes are closed under immediate extensions.

Lemma 6.14 *Every finite extension of a finitely ramified field is again a finitely ramified field. Every finite extension of a formally $\wp$-adic field is again a formally $\wp$-adic field.*

Proof: Take a finitely ramified field (K, v) and a finite extension $(L|K, v)$. Then by Lemma 6.13, $(vL : vK)$ is finite, and since vK has a least positive element, the same holds for vL, showing that also (L, v) is a finitely ramified field. Again by Lemma 6.13, $[Lv : Kv]$ is finite. Hence, if in addition $\overline{K}$ is finite, then so is $\overline{L}$, and (L, v) is a formally $\wp$-adic field. □

Now let $(L|K, v)$ be an algebraic extension, not necessarily finite. For every value $\alpha \in vL$ and every residue $c \in \overline{L}$ we may pick $a, b \in L$ such that $va = \alpha$ and $\overline{b} = c$. Since $L|K$ is algebraic, $K(a, b)|K$ is a finite extension. Consequently, $(vK + \mathbb{Z}\alpha)/vK$ is a torsion group and $\overline{K}(c)|\overline{K}$ is finite, by virtue of the Lemma 6.13. This shows:

Corollary 6.15 *If $(L|K,v)$ is an algebraic extension of valued fields, then also $\overline{L}|\overline{K}$ and $vL|vK$ are algebraic extensions.*

Let us observe that ramification index and inertia degree are multiplicative, since the same holds for the degree of group and field extensions.

Lemma 6.16 *Let $(L|K,v)$ be a valued field extension and $(E|K,v)$ a subextension. Then*

$$e(L|K,v) = e(L|E,v)\cdot e(E|K,v) \quad \text{and} \quad f(L|K,v) = f(L|E,v)\cdot f(E|K,v)\;.$$

If $b_i \in L$ are K-valuation independent elements, then we will also say that they are **(linearly) valuation independent over** (K,v). This is not meant in the sense that they are K-valuation independent over the subvector space K. This would for instance not be true for the elements that we have constructed preceding to Lemma 6.13. Indeed, one of the values represents the coset vK, and some linear combination of the residues is an element of $\overline{K}$. In this case, however, we only have to omit one suitably chosen element b_i to obtain a set of elements which are K-valuation independent over the subvector space K. Vice versa, adding the element 1 to a set of the latter type, we obtain again a set of K-valuation independent elements. So there is no real danger of confusion in our abuse of notation. In the same spirit, a K-valuation basis of (L,v) (over the zero vector space) will also be called a **valuation basis of** (L,v) **over** (K,v) (or **of** $(L|K,v)$). Hence if the valued K-vector space (L,v) admits a K-valuation basis over (K,v), then the valued field extension $(L|K,v)$ admits a valuation basis.

Observe that in our construction preceding to Lemma 6.13, we may take the representative of the zero coset vK just to be 0 and the element of value 0 just to be 1. Similarly, we may choose a $\overline{K}$-basis of $\overline{L}$ which contains 1, and the element having residue 1 may also taken to be 1. Then, 1 will be an element of our valuation independent set of elements z_iu_j. A valuation independent set of this form will be called a **standard valuation independent set of** (L,v) **over** (K,v). If it is finite, we will assume in addition that it is of the form $\{z_iu_j, \mid 1\leq i\leq e(L|K,v)\,,\, 1\leq j\leq f(L|K,v)\}$, numbered such that $z_1 = 1$ and $u_1 = 1$. If a standard valuation independent set is a basis of $L|K$, then it will be called a **standard valuation basis of** (L,v) **over** (K,v). By the definition of a standard valuation independent set $\mathcal{B}$ of (L,v) over (K,v), the K-subvector space (V,v) generated by $\mathcal{B}$ has the same skeleton as the valued K-vector space (L,v). That is, the extension $(V,v)\subset(L,v)$ is immediate. We have $V = L$ if and only if $\mathcal{B}$ is a standard valuation basis of (L,v) over (K,v). If the latter holds, then the set $\mathcal{B}\setminus\{1\}$ is a K-valuation basis of the vector space (L,v) over the subspace (K,v), and Corollary 3.60 shows that $(K,v)\subset(L,v)$ is thus a vs-defectless extension of valued K-vector spaces, that is, every finitely generated subextension admits itself a K-valuation basis over the subspace (K,v).

Finite vs-defectless extensions are characterized as follows:

Lemma 6.17 *Let the extension $(L|K,v)$ be finite. Then the following conditions are equivalent:*

1) $(L|K,v)$ is a vs-defectless extension,
2) $(L|K,v)$ admits a standard valuation basis,
3) $(L|K,v)$ admits a valuation basis,
4) $[L:K] = (vL:vK)\cdot[\overline{L}:\overline{K}]$.

Proof: Let again V denote the vector space generated by a standard valuation independent set of (L,v) over (K,v). We have seen above that the equality $V=L$ implies condition 2), and that condition 2) implies condition 1).

Now assume that the extension $L|K$ is finite. Then V is a K-vector space of dimension $e(L|K,v)\cdot f(L|K,v)$. Hence $V=L$ if and only if $[L:K]=e(L|K,v)\cdot f(L|K,v)$, which is condition 4). If $V\neq L$, then $(V,v)\subset(L,v)$ is a proper immediate extension of valued K-vector spaces, and Lemma 3.59 shows that the valued K-vector space (L,v) can not admit a K-valuation basis over (K,v). This shows that condition 1) implies $V=L$. We have thus proved that conditions 1), 2) and 4) are equivalent.

Since 2)⇒3) is trivial, it now suffices to prove 3)⇒1). But a valuation basis of $(L|K,v)$ is a K-valuation basis of the valued K-vector space (L,v) over the zero vector space. Now Lemma 3.59 shows that (L,v) also admits a K-valuation basis over (K,v). By definition, this is condition 1). □

Let us remark that every K-valuation independent set in a valued field extension $(L|K,v)$ can be transformed by multiplication with elements from K into a valuation independent set where every two elements have already equal value if their values belong to the same coset modulo vK.

We will show later that a vs-defectless algebraic extension $(L|K,v)$ admits only one extension of v from K to L (cf. Lemma 11.15). In the case of an extension $(L|K,v)$ which admits a standard valuation basis, we can show the following result here:

Lemma 6.18 *Let $(L|K,v)$ be an algebraic extension which admits a standard valuation basis $\{z_i,u_j\mid i\in I\,,\,j\in J\}$ (where the values $\alpha_i=vz_i\,,\,i\in I$, form a set of representatives for the cosets of vL modulo vK, and the residues $\zeta_j=\overline{u}_j\,,\,j\in J$, form a basis of $\overline{L}$ over $\overline{K}$). Then the valuation v on L is uniquely determined by its restriction to K and the data*

$$vz_i=\alpha_i\ (i\in I)\quad\text{and}\quad u_jv=\zeta_j\ (j\in J)\,. \tag{6.3}$$

Proof: Every element of L is a sum $\sum_{i,j}c_{ij}z_iu_j$. If a valuation w satisfies $w=v$ on K and (6.3), then the value of this sum is equal to that of the summands of minimal value. But the value of a summand $c_{ij}z_iu_j$ is uniquely determined by vc_{ij} and vz_i. □

So far, we have slipped through without a general characterization of valuation independence. But we will need it later, so we have to append it here.

Lemma 6.19 *Let $(L|K,v)$ be a valued field extension and $\mathcal{B}\subset L$. Then $\mathcal{B}$ is valuation independent over (K,v) if and only if the following holds: For every choice of distinct elements $b_1,\dots,b_n\in\mathcal{B}$ such that the values $vb_1,\dots,vb_n$ belong to the same coset modulo vK, there exist elements $c_2,\dots,c_n\in K$ such that the elements $1,c_2b_2b_1^{-1}\dots,c_nb_nb_1^{-1}$ are of value 0 and their residues are $\overline{K}$-linearly independent.*

Proof: "⇒": Let $b_1,\dots,b_n\in\mathcal{B}$ such that the values $vb_1,\dots,vb_n$ belong to the same coset modulo vK. Then there exist $c_2\dots,c_n\in K$ such that the elements $b_1,c_2b_2\dots,c_nb_n$ are all of the same value, namely vb_1. Since $b_1,\dots,b_n$ are assumed to be valuation independent over (K,v), we have that for all elements $c'_1,\dots,c'_n\in\mathcal{O}_\mathbf{K}$, not all of them in $\mathcal{M}_\mathbf{K}$, the value of $c'_1b_1+c'_2c_2b_2+\ldots+c'_nc_nb_n$ is equal to vb_1, i.e., the value of $c'_1+c'_2c_2b_2b_1^{-1}+\ldots+c'_nc_nb_nb_1^{-1}$ is 0. This shows that for every choice of elements $\zeta_i\in\overline{K}$, not all of them zero, $\zeta_1 + \zeta_2\,\overline{c_2 b_2 b_1^{-1}} + \ldots + \zeta_n\,\overline{c_n b_n b_1^{-1}} \neq 0$. We have proved that the residues of $1, c_2 b_2 b_1^{-1} \ldots, c_n b_n b_1^{-1}$ are $\overline{K}$-linearly independent.

"$\Leftarrow$": Let $b_1, \ldots, b_n \in \mathcal{B}$ and $d_1 \ldots, d_n \in K$. We have to show that $v(d_1 b_1 + \ldots + d_n b_n) = \min_i vd_i b_i$. If this holds under the assumption that all values $vd_i b_i$ are equal, then it also holds in the general case. But if all $vd_i b_i$ are equal, then the values of all b_i lie in the same coset modulo vK. So by assumption, there are $c_2 \ldots, c_n \in K$ such that the elements $1, c_2 b_1^{-1} b_2 \ldots, c_n b_n b_1^{-1}$ are of value 0 and their residues are $\overline{K}$-linearly independent. In particular, $1 + \overline{d_2 d_1^{-1} c_2^{-1}}\,\overline{c_2 b_2 b_1^{-1}} + \ldots + \overline{d_n d_1^{-1} c_n^{-1}}\,\overline{c_n b_n b_1^{-1}} \neq 0$ (note that $d_i d_1^{-1} c_i^{-1} \in K$ and that $v(d_i d_1^{-1} c_i^{-1}) = 0$ since $vd_1 b_1 = vd_i b_i$ and $vc_i b_i b_1^{-1} = 0$ imply that $vd_i = vd_1 b_1 b_i^{-1} = vd_1 c_i$). It follows that the value of $1 + d_2 b_2 d_1^{-1} b_1^{-1} + \ldots + d_n b_n d_1^{-1} b_1^{-1}$ is zero. In other words, $v(d_1 b_1 + \ldots + d_n b_n) = vd_1 b_1 = \min_i vd_i b_i$, as required. □

The proof of the next lemma is left to the reader as an exercise:

Lemma 6.20 *Let $(\Omega|K, v)$ an extension of valued fields with subextension $(L|K, v)$. Let a_i, $i \in I$, be a valuation basis of $(L|K, v)$. If $b_i \in \Omega$ are elements such that $v(a_i - b_i) > va_i$, then b_i, $i \in I$, are K-valuation independent. If in addition all b_i lie in L, then b_i, $i \in I$, is a valuation basis of $(L|K, v)$.*

How far is a finite valued field extension from having a valuation basis? This question probably makes no sense in general, but there is a situation where this question is the key to the proof of a nice result. Let us first observe (cf. Lemma 3.11 and Lemma 1.11):

Lemma 6.21 *Let (K, v) be a valued field. If the finite dimensional (K, v)-vector space (V, v) admits a K-valuation basis $\{b_1, \ldots, b_n\}$, then as an ultrametric space, (V, v) is the product $\prod_{i=1}^n (Kb_i, v)$ of the ultrametric spaces (Kb_i, v). Every (Kb_i, v) is isomorphic to (K, v) as a (K, v)-vector space. Hence if (K, v) is complete resp. spherically complete, then so is (V, v).*

We will later show that every finite extension of a spherically complete field admits a valuation basis and is thus again spherically complete. Unfortunately, not every extension of a complete field admits a valuation basis. Indeed, there are even complete fields of rank 1 which admit immediate extensions (cf. Example 11.50). Nevertheless, every finite extension of a complete field is again complete. This is because the extension of the valuation is "not too far" from having a valuation basis. We need the following observation whose proof is left to the reader.

Lemma 6.22 *Let (K, v) be a valued field, (V, v) a (K, v)-vector space of arbitrary dimension and $\mathcal{B}$ an arbitrary K-basis of V. Define*

$$v_{\mathcal{B}}(c_1 b_1 + \ldots c_n b_n) = \min_{1 \leq i \leq n} vc_i$$

for every choice of basis elements $b_1, \ldots, b_n \in \mathcal{B}$ and coefficients $c_1, \ldots, c_n \in K$. Then $(V, v_{\mathcal{B}})$ is a (K, v)-vector space with valuation basis $\mathcal{B}$.

Theorem 6.23 *Let (K, v) be a complete valued field, (V, v) a finite dimensional (K, v)-vector space and $\mathcal{B}$ any basis of V. Then there are elements $\beta, \gamma \in vV$ such that*

$$v_{\mathcal{B}} a + \beta \leq va \leq v_{\mathcal{B}} a + \gamma \tag{6.4}$$

for every $a \in V$. Furthermore, (V, v) is complete.

Proof: Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $a = c_1b_1 + \ldots + c_nb_n \in V$. Then $va \geq \min_{1\leq i\leq n} vc_ib_i \geq \min_{1\leq i\leq n} vc_i + \min_{1\leq i\leq n} vb_i = v_{\mathcal{B}}a + \min_{1\leq i\leq n} vb_i$. Hence $\beta := \min_{1\leq i\leq n} vb_i$ satisfies the lower inequality.

For the upper inequality, we proceed by induction on n. For $n = 1$, the assertion trivially follows with $\gamma := vb_1$. So let us assume that $n > 1$ and that the assertion of our theorem is already proved for dimension $n-1$. For every $i \in \{1, \ldots, n\}$ we let V_i denote the K-subvector space of V generated by $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$. Then $b_i \notin V_i$, hence at (b_i, V_i) is nontrivial by virtue of Lemma **??**. By induction hypothesis, (V_i, v) is complete. Hence by Lemma 1.38, every completion type over (V_i, v) is trivial. Consequently, at (b_i, V_i) is not a completion type. In view of Corollary **??** we conclude that $\mathrm{dist}\,(b_i, V_i) < \infty$ or vV_i is not cofinal in $\{v(b_i - d) \mid d \in V_i\} \cup vV_i$. If the former holds, then there is some $\gamma_i \in vV_i$ such that $v(b_i - d) \leq \gamma_i$ for every $d \in V_i$. If the latter holds, then there is some $d_i \in V_i$ such that $v(b_i - d_i) > vV_i$ and consequently, $\gamma_i := v(b_i - d_i) = \max\{v(b_i - d) \mid d \in V_i\}$. Hence in both cases, $v(b_i + \sum_{j\neq i} c_jb_j) \leq \gamma_i$ for every choice of $c_j \in K$. It follows that $v(\sum_{1\leq j\leq n} c_jb_j) \leq vc_i + \gamma_i \leq vc_i + \max_{1\leq \ell\leq n} \gamma_\ell$ for all i and all $c_j \in K$. Hence with $\gamma := \max_{1\leq \ell\leq n} \gamma_\ell$, we find that $v(\sum_{1\leq j\leq n} c_jb_j) \leq \min_{1\leq i\leq n} vc_i + \gamma = v_{\mathcal{B}}a + \gamma$.

Now we wish to prove that (V, v) is complete. Let $\mathbf{A}$ be a nontrivial completion type over (V, v). Hence $\Lambda(\mathbf{A}) = vV$. As usual, we let $\mathbf{A}_\alpha = B_\alpha(a_\alpha)$ for all $\alpha \in vV$ (the balls are taken with respect to v). We distinguish two cases:

<u>Case 1:</u> vV is included in the convex hull of vK. It follows by construction that also β and γ lie in this convex hull. This implies that for every $\alpha \in vV$ there is some $\alpha' \in vK$ such that $\alpha' + \beta > \alpha$.

Now we consider the balls $B'_\alpha(a_{\alpha+\gamma})$ in $(V, v_{\mathcal{B}})$ for all $\alpha \in v_{\mathcal{B}}V = vK$ (the prime indicates that the balls are taken with respect to $v_{\mathcal{B}}$). The inequalities of (6.4) imply that $B_{\alpha+\gamma}(a_{\alpha+\gamma}) \subset B'_\alpha(a_{\alpha+\gamma}) \subset B_{\alpha+\beta}(a_{\alpha+\gamma})$, and it follows that the latter ball is equal to $B_{\alpha+\beta}(a_{\alpha+\beta})$. That is,

$$\mathbf{A}_{\alpha+\gamma} \subset B'_\alpha(a_{\alpha+\gamma}) \subset \mathbf{A}_{\alpha+\beta}$$

for all $\alpha \in vK$. From this it follows that the balls $B'_\alpha(a_{\alpha+\gamma})$ form a nest (since the same is true for the balls $\mathbf{A}_\alpha$). Since α runs through all of vK, this nest is a completion nest. Its intersection is nonempty because $(V, v_{\mathcal{B}})$ is complete by virtue of the two preceeding lemmata. On the other hand, we know that for every $\alpha \in vK$ there is some $\alpha' \in vK$ such that $\alpha' + \beta > \alpha$. This means that for every $\alpha \in vV$ there is some $\alpha' \in vV$ such that $B'_{\alpha'}(a_{\alpha'+\gamma}) \subset \mathbf{A}_{\alpha'+\beta} \subset \mathbf{A}_\alpha$. This shows that the nonempty intersection of the balls $B'_\alpha(a_{\alpha+\gamma})$ must lie in the intersection of the balls $\mathbf{A}_\alpha$. Hence the approximation type $\mathbf{A}$ is trivial, showing that (V, v) is complete.

<u>Case 2:</u> vV is not included in the convex hull of vK. Without loss of generality, we can assume that vb_1 is the minimal value among the values $vb_1, \ldots, vb_n$. Since completeness is preserved under isomorphisms (as the reader may show), we can replace (V, v) by the isomorphic (K, v)-vector space $(b_1^{-1}V, v)$. So we can assume that $vb_1 = 0$ and that no value vb_i is smaller than all values in vK. So also no K-linear combination of the b_i (that is, no element of V) has a value which is smaller than all values in vK. If our replacement has produced a vector space V whose value set lies in the convex hull of vK, then we proceed as in Case 1. Otherwise, there exist elements in V whose value is bigger than vK. Now we choose a basis $\{b'_1, \ldots, b'_n\}$ of V containing a maximal number of elements of value bigger than vK. This number can not be n since otherwise all elements of V would have a value bigger than vK, but by construction, V contains the element b_1 of value 0. Let us assume

that $vb'_1, \ldots, vb'_m$ are bigger than vK, and that $vb'_{m+1}, \ldots, vb'_n$ lie in the convex hull of vK. Then $1 \leq m < n$ by what we have shown. If a K-linear combination of the b'_i has a value bigger than vK, then the coefficients of all b'_i with $i > m$ must be zero since otherwise, some b'_i with $i > m$ could be replaced by this linear combination, in contradiction to the maximality of m.

Now for every $\alpha \in vV$, $\alpha > vK$, consider the representation of the elements of the ball $\mathbf{A}_\alpha$ as linear combinations of the basis elements b'_i. Since $v(a-b) \geq \alpha > vK$ for every two elements $a, b \in \mathbf{A}_\alpha$, we find that if $a = c_1 b'_1 + \ldots + c_n b'_n$ and $a = \tilde{c}_1 b'_1 + \ldots + \tilde{c}_n b'_n$, then $c_i = \tilde{c}_i$ for $i = m+1, \ldots, n$. Hence, subtraction of $d := c_{m+1} b'_{m+1} + \ldots + c_n b'_n$ transfers the nest $\{\mathbf{A}_\alpha \mid \alpha > vK\}$ onto the nest $\{B_\alpha(a_\alpha - d) \mid \alpha > vK\}$. The latter is a nest of balls in the (K, v)-subvector space of V which is generated by $b'_1, \ldots, b'_m$. Since this is of dimension less than n, it is complete by our induction hypothesis. Consequently, the nest $\{B_\alpha(a_\alpha - d) \mid \alpha > vK\}$ and thus also the nest $\{\mathbf{A}_\alpha \mid \alpha > vK\}$ has a nonempty intersection. But this means that $\mathbf{A}$ is realized in (V, v), showing that (V, v) is complete. □

The proof (with the exception of Case 2) is due to P. Roquette ([ROQ1], Lemma 2) and is based on lectures given by E. Artin. Since for an algebraic extension $(L|K, v)$ the value group vL lies in the divisible hull of vK by virtue of Corollary 6.15, the first case of the proof suffices to deduce:

Corollary 6.24 *Let (K, v) be a complete valued field and $(L|K, v)$ a finite extension of valued fields. Then also (L, v) is complete.*

Lemma 6.25 *Let $(L|K, v)$ be a finite extension of valued fields and $(K, v)^c = (K^c, v)$ the completion of (K, v). Then there is a unique extension of v from K^c to $L.K^c$ which is also an extension of v from L to $L.K^c$. With this extension, the valued field $(L.K^c, v)$ is the completion of (L, v). So we can write*

$$L^c = L.K^c \, .$$

Proof: Since $L|K$ is finite, so is $L.K^c|K^c$. Hence by the foregoing corollary, $(L.K^c, v)$ is complete for every extension of v from K^c to $L.K^c$. Let $a \in L.K^c$ and $va < \alpha \in v(L.K^c)$. Choose any K-basis $\{b_1, \ldots, b_n\}$ of L. Since these elements also generate $L.K^c$ over K^c, we can write $a = c_1 b_1 + \ldots + c_n b_n$ with $c_i \in K^c$. Since $L.K^c|K^c$ is finite, Corollary 6.15 shows that $v(L.K^c)$ lies in the divisible hull of vK. Hence, there is some $\beta \in vK$ such that $\beta \geq \alpha - \min_i vb_i$. Since (K, v) is dense in its completion $(K, v)^c$, we can find elements $c'_i \in K$ such that $v(c'_i - c_i) \geq \beta$. We set $a' := c'_1 b_1 + \ldots + c'_n b_n \in L$. Then it follows that $v(a - a') \geq \min_i v(c_i b_i - c'_i b_i) \geq \beta + \min_i vb_i \geq \alpha$. This proves that $va = va'$, so v is uniquely determined by its restriction to L. Moreover, since α was arbitrarily large, we have proved that (L, v) is dense in $(L.K^c, v)$. □

Corollary 6.26 *Let $(K, v) = (k((t)), v_t)$ and $(K(s)|K, v)$ an algebraic extension such that $s^n = t$ for some natural number n. Then $(K(s), v) = (k((s)), v_s)$.*

Proof: By the foregoing lemma, $K(s)$ is the completion of $(k(t, s), v) = (k(s), v_s)$. But this completion is just $(k((s)), v_s)$. □

The main notions introduced in this and the last section are invariant under isomorphisms. We leave it to the reader to prove the following facts:

Lemma 6.27 *Let $(L|K,v)$ an extension of valued fields and $\iota : L \to \iota L$ a field isomorphism. Then ι is an isomorphism from (L,v) onto $(\iota L, v\iota^{-1})$ and $\mathrm{res}_K(\iota)$ is an isomorphism from (K,v) onto $(\iota K, v\iota^{-1})$. For the valued field extension $(\iota L|\iota K, v\iota^{-1})$, the following facts hold:*
a) $(\iota L|\iota K, v\iota^{-1})$ is vs-defectless if and only if $(L|K,v)$ is,
b) $(\iota L|\iota K, v\iota^{-1})$ is immediate if and only if $(L|K,v)$ is,
c) $(\iota K, v\iota^{-1})$ is dense in $(\iota L, v\iota^{-1})$ if and only if (K,v) is dense in (L,v),
d) If $L|K$ is finite, then so is $\iota L|\iota K$, and

$$e(\iota L|\iota K, v\iota^{-1}) = e(L|K,v) \quad \textit{and} \quad (\iota L|\iota K, v\iota^{-1}) = f(L|K,v)\;,$$

e) $\mathcal{B}$ is a set of K-valuation independent elements in (L,v) if and only if $\iota\mathcal{B} = \{\iota b \mid b \in \mathcal{B}\}$ is a set of ιK-valuation independent elements in $(\iota L, v\iota^{-1})$,
f) $\mathcal{B}$ is a valuation basis of $(L|K,v)$ if and only if $\iota\mathcal{B}$ is a valuation basis of $(\iota L|\iota K, v\iota^{-1})$.

Exercise 6.1 *Show how a finite valuation basis can be transformed into a standard valuation basis.*

Exercise 6.2 *Show that isomorphisms of valued fields preserve the properties of being a spherically complete or a maximal field.*

Exercise 6.3 *Define the notion of a* **direct sum of ultrametric spaces** *and apply it to infinite valued field extensions with valuation bases. What can be said about completeness and spherical completeness?*

6.4 Algebraic valuation independence

In field theory, we have the notions of linear independence and algebraic independence. In valuation theory, the notion of valuation independence is the analogue to the first. We shall now introduce the notion of algebraic valuation independence, which is the analogue to algebraic independence. Let $(L|K,v)$ be an extension of valued fields. A subset $\mathcal{T} \subset L$ is called **algebraically valuation independent over** (K,v), if for every choice of finitely many distinct elements $t_1,\ldots,t_n \in \mathcal{T}$, the value of every polynomial f in $K[t_1,\ldots,t_n]$ is equal to the value of a summand of f of minimal value. That means,

$$v\left(\sum_{\underline{\nu}} c_{\underline{\nu}} t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}\right) = \min_{\underline{\nu}}\,(vc_{\underline{\nu}} t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}) \tag{6.5}$$

where $\underline{\nu} = (\nu_1,\ldots,\nu_n)$ runs over all n-tuples of integers ≥ 0 and only finitely many $c_{\underline{\nu}}$ are nonzero. Observe that if at least one coefficient is nonzero, then the value of the polynomial is less or equal to the value of this coefficient and thus $< \infty$, that is, the polynomial does not equate to zero. This proves:

Lemma 6.28 *Let $(L|K,v)$ be a valued field extension. If $\mathcal{T} \subset L$ is algebraically valuation independent over (K,v), then the elements of $\mathcal{T}$ are algebraically independent over K.*

If a transcendence basis of $L|K$ is algebraically valuation independent over (K,v), then we call it a **valuation transcendence basis of** $(L|K,v)$. Note that the elements $t_1,\ldots,t_n$ are algebraically valuation independent over (K,v) if and only if the elements $t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}$, $\nu_1,\ldots,\nu_n\in\mathbb{N}$, are linearly valuation independent over (K,v). From this, we deduce:

Lemma 6.29 *Let $(L|K,v)$ be a valued field extension. If $\mathcal{T}\subset L$ is algebraically valuation independent over (K,v), then the subextension $(K(\mathcal{T})|K,v)$ of $(L|K,v)$ is vs-defectless.*

Proof: Let $V\subset K(\mathcal{T})$ be a K-vector space of finite dimension and containing K. We have to show that (V,v) admits a valuation basis over the subspace (K,v). Every K-basis of V over K consists of finitely many elements of $K(\mathcal{T})$, which are rational functions in the elements of $\mathcal{T}$. If we take $a\in K[\mathcal{T}]$ to be the common denominator of these rational functions, we find that $aV\subset K[\mathcal{T}]$. On the other hand, if $a\neq 0$ is any element of $K(\mathcal{T})$, then $\mathcal{B}$ is a valuation basis of (V,v) over the subspace (K,v) if and only if $a\mathcal{B}$ is a valuation basis of (aV,v). So we can assume from the start that $V\subset K[\mathcal{T}]$. We choose again a K-basis of V over K and collect the finitely many products of the form $t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}$ with $n,\nu_1,\ldots,\nu_n\in\mathbb{N}$ and $t_i\in\mathcal{T}$ that appear in the elements of that basis. These products form a valuation basis of a valued K-vector space $(V',v)\subset(K(\mathcal{T}),v)$ which contains (V,v). By Lemma 3.59 it follows that also (V,v) admits a valuation basis. □

If the elements $t_1,\ldots,t_n$ have values which are rationally independent over vK, then no two summands of the sum appearing on the left hand side of (6.5) will have equal value. This yields that the equality in (6.5) holds. Indeed, if two distinct summands $c_{\underline{\nu}}t_1^{\nu_1}\cdot\ldots\cdot t_n^{\nu_n}$ and $c_{\underline{\nu}'}t_1^{\nu_1'}\cdot\ldots\cdot t_n^{\nu_n'}$ would have equal value, we would obtain that $vc_{\underline{\nu}}-vc_{\underline{\nu}'}+(\nu_1-\nu_1')vt_1+\ldots+(\nu_n-\nu_n')vt_n=0$, contradicting our assumption that the values $vt_1,\ldots,vt_n$ are rationally independent over vK.

Now assume that $n=r+s$, that $t_i=x_i$ have values which are rationally independent over vK, $1\leq i\leq r$, and that $t_{r+j}=y_j$ have residues which are algebraically independent over $\overline{K}$, $1\leq j\leq s$. Suppose that we have a polynomial $f\in K[t_1,\ldots,t_n]$ which is a counterexample to (6.5). Then it will still be a counterexample if we omit all summands that are not of minimal value; hence without loss of generality, we may assume that all summands in f have equal value. But then it follows from our above arguments that every element x_i has to appear to the same power ν_i in every of these summands. So dividing f by $x_i^{\nu_i}$ for $1\leq i\leq r$, we obtain a polynomial $g\in K[y_1,\ldots,y_s]$ with summands of equal value, which is still a counterexample to (6.5). Since the elements y_j all have value zero, we find that all nonzero coefficients must have equal value. After dividing by one of them, we can assume that all summands have value zero, but that the value of g is >0. Passing to the residue field through the residue map, we obtain a nontrivial polynomial $\overline{g}(X_1,\ldots,X_s)\in\overline{K}[X_1,\ldots,X_s]$ such that $\overline{g}(\overline{y_1},\ldots,\overline{y_s})=0$. But this contradicts our assumption that the residues $\overline{y_1},\ldots,\overline{y_s}$ be algebraically independent over $\overline{K}$. We have proved:

Lemma 6.30 *Let $(L|K,v)$ be an extension of valued fields. Assume that $x_i\in L$, $i\in I$, are elements whose values vx_i, $i\in I$, are rationally independent over vK. Assume further that $y_j\in L$, $j\in J$, are elements whose residues $\overline{y}_j$, $j\in J$, are algebraically independent over $\overline{K}$. Then $\mathcal{T}=\{x_i\,,\,y_j\mid i\in I\,,\,j\in J\}$ is algebraically valuation independent over (K,v) and thus also algebraically independent over K.*

Given any extension $(L|K,v)$ of valued fields, we can choose a special set of algebraically valuation independent elements as follows. We choose elements $y_j \in L$, $j \in J$, such that their residues $\overline{y}_j$ form a transcendence basis of $\overline{L}|\overline{K}$. For the extension $vL|vK$ of abelian groups, one can show by means of Zorn's Lemma that there exist maximal sets of elements in vL which are rationally independent over vK. This is analogous to the existence of transcendence bases for arbitrary field extensions. In view of this analogy, such a maximal subset of vL will be called a **transcendence basis of** $vL|vK$. Note that elements $\alpha_i \in vL$ are rationally independent over vK if and only if their cosets $\alpha_i + vK$ are rationally independent. That is, the cardinality of a transcendence basis of $vL|vK$ is equal to the rational rank $\mathrm{rr}\, vL/vK$. We choose elements $x_i \in L$, $i \in I$, such that their values vx_i form a transcendence basis of $vL|vK$. By the foregoing lemma, $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ is algebraically valuation independent over (K,v) and thus also algebraically independent over K. A set $\mathcal{T}$ of this form will be called a **standard algebraically valuation independent set of** $(L|K,v)$. We obtain:

Lemma 6.31 *Let $(L|K,v)$ be an extension of valued fields. Then*

$$\mathrm{trdeg}\, L|K \geq \mathrm{rr}\, vL/vK \,+\, \mathrm{trdeg}\, \overline{L}|\overline{K}\,. \tag{6.6}$$

(If one of these numbers is infinite, the inequality is to be understood as an inequality of cardinal numbers.)

If $\mathrm{trdeg}\, L|K$ is finite, then the lemma shows that both $\mathrm{rr}\, vL/vK$ and $\mathrm{trdeg}\, \overline{L}|\overline{K}$ are finite, and that

$$\mathrm{trdef}\,(L|K,v) \;:=\; \mathrm{trdeg}\, L|K - \mathrm{rr}\, vL/vK - \mathrm{trdeg}\, \overline{L}|\overline{K}$$

is a nonnegative integer. We will call it the **transcendence defect of** $(L|K,v)$. If $\mathrm{trdef}\,(L|K,v) = 0$, then we call $(L|K,v)$ an **extension without transcendence defect**.

The transcendence degree of field extensions is additive, as we have stated in Section 24.5. The same holds for extensions of abelian groups, that is, $\mathrm{rr}\, C/A = \mathrm{rr}\, C/B + \mathrm{rr}\, B/A$ if $A \subset B \subset C$. Hence it follows that also the transcendence defect is additive:

$$\mathrm{trdef}\,(L|K,v) = \mathrm{trdef}\,(L|E,v) \,+\, \mathrm{trdef}\,(E|K,v)$$

if E is an intermediate field of $L|K$. Consequently:

Lemma 6.32 *Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. If $E|K$ is any subextension of $L|K$, then $(L|K,v)$ is an extension without transcendence defect if and only if both $(L|E,v)$ and $(E|K,v)$ are extensions without transcendence defect.*

In view of this lemma, we can generalize our definition to extensions of infinite transcendence degree as follows. We will say that an arbitrary extension $(L|K,v)$ is an **extension without transcendence defect** if every subextension $(E|K,v)$ of finite transcendence degree is an extension without transcendence defect.

If a standard algebraically valuation independent set of $(L|K,v)$ is a transcendence basis of $L|K$, then we call it a **standard valuation transcendence basis of** $(L|K,v)$. Now we can prove:

Lemma 6.33 *Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following assertions are equivalent:*

1) $(L|K,v)$ is an extension without transcendence defect.
2) $(L|K,v)$ admits a standard valuation transcendence basis,
3) $(L|K,v)$ admits a valuation transcendence basis.

Proof: 1)⇒2): As we have done above, we choose a standard algebraically valuation independent set of $(L|K,v)$. Its elements are algebraically independent over K, and its cardinality is equal to $\mathrm{rr}\, vL/vK + \mathrm{trdeg}\, \overline{L}|\overline{K}$. Hence if $(L|K,v)$ has transcendence defect 0, then its cardinality is equal to $\mathrm{trdeg}\, L|K$, which yields that it is a transcendence basis of $L|K$.

2)⇒3) is trivial.

3)⇒1): Let $\mathcal{T} = \{t_1,\ldots,t_n\}$ be a valuation transcendence basis of $(L|K,v)$. Hence $n = \mathrm{trdeg}\, L|K$. We can assume that the numeration is such that for some $r \geq 0$, the values $vt_1,\ldots,vt_r$ are rationally independent over vK and the values of every $r+1$ elements in $\mathcal{T}$ are rationally dependent over vK. That is, for every j such that $0 < j \leq s := n-r$, there are integers $\nu_j > 0$ and ν_{ij} , $1 \leq i \leq r$, and a constant $c_j \in K$ such that the element

$$t'_j \;:=\; c_j t_{r+j}^{\nu_j} \prod_{i=1}^{r} t_i^{\nu_{ij}}$$

has value 0. Observe that $r \leq \mathrm{rr}\, vL/vK$ and that $r + s = \mathrm{trdeg}\, L|K$.

Now assume that $(L|K,v)$ has nontrivial transcendence defect. Then

$$s = \mathrm{trdeg}\, L|K - r \geq \mathrm{trdeg}\, L|K - \mathrm{rr}\, vL|vK > \mathrm{trdeg}\, \overline{L}|\overline{K}\ .$$

This yields that the residues $\overline{t'_1},\ldots,\overline{t'_s}$ are not $\overline{K}$-algebraically independent. Hence, there is a nontrivial polynomial $g(X_1,\ldots,X_s) \in \mathcal{O}_{\mathbf{K}}^{\times}[X_1,\ldots,X_s]$ such that $\overline{g}(\overline{t'_1},\ldots,\overline{t'_s}) = 0$. Hence, $vg(t'_1,\ldots,t'_s) > 0$. After multiplying with sufficiently high powers of every element t_i , $1 \leq i \leq r$, we obtain a polynomial f in $t_1,\ldots,t_n$ which violates (6.5). But this contradicts our assumption that $\mathcal{T}$ be a valuation basis. Consequently, $(L|K,v)$ can not have a nontrivial transcendence defect. □

Corollary 6.34 *If a valued field extension admits a valuation transcendence basis, then it is an extension without transcendence defect.*

Proof: Let $(L|K,v)$ be an extension with valuation transcendence basis $\mathcal{T}$. Given a subextension $(E|K,v)$ of finite transcendence degree, we have to show that it is without transcendence defect. Let $\mathcal{T}'$ be a transcendence basis of $E|K$. Since $\mathcal{T}'$ is finite, there exists a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ such that all elements of $\mathcal{T}'$ are algebraic over $K(\mathcal{T}_0)$. Now let F be the relative algebraic closure of $K(\mathcal{T}_0)$ in L. Then $E \subset F$. The extension $(F|K,v)$ has valuation transcendence basis $\mathcal{T}_0$, because F it is algebraic over $K(\mathcal{T}_0)$. By the foregoing lemmy, it is an extension without transcendence defect. In view of Lemma 6.32 we conclude that also $(E|K,v)$ is an extension without transcendence defect. □

Equality (6.5) allows us to determine the value group and residue field of extensions which are generated by algebraically valuation independent elements. When we write

"$vK \oplus \mathbb{Z}vx$" then we mean the direct sum of vK with the group generated by vx as abelian groups; the symbol gives no infromation about the ordering on this direct sum. The ordering will usually be given by the ordering on the value group of a valued field extension of (K,v) which contains x. Extensions of the form $vK \oplus \bigoplus_{i=1}^n \mathbb{Z}\alpha_i \,|\, vK$ are the abelian group analogues of rational function fields $K(x_1 \ldots, x_n)|K$ of transcendence degree n (resp. of similar extensions of $\overline{K}$).

Lemma 6.35 *Let $(L|K,v)$ be an extension of valued fields containing a standard valuation independent set $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\} \subset L$ such that the values $vx_i\,,\ i \in I$, are rationally independent over vK, and that the residues $\overline{y}_j\,,\ j \in J$, are algebraically independent over $\overline{K}$. Then*

$$v\left(\sum_{\underline{\nu}} c_{\underline{\nu}} x_1^{\nu_1} \cdot \ldots \cdot x_r^{\nu_r} \cdot y_1^{\nu_{r+1}} \cdot \ldots \cdot y_s^{\nu_{r+s}}\right) = \min_{\underline{\nu}} vc_{\underline{\nu}} + \nu_1 vx_1 + \ldots \nu_r vx_r\ ,$$

and

$$\begin{aligned} vK(\mathcal{T}) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i \\ \overline{K(\mathcal{T})} &= \overline{K}(\overline{y}_j \mid j \in J)\ . \end{aligned}$$

If J is finite, then $\overline{K(\mathcal{T})}$ is a rational function field over $\overline{K}$. If P is the place associated with v, then we can write $K(\mathcal{T})P = KP(\mathcal{T}P)$, where $\mathcal{T}P = \{tP \mid t \in \mathcal{T}\}$.

Further, v is uniquely determined on $K(\mathcal{T})$ by its restriction to K, the prescription of the values vx_i, and the condition that the residues $\overline{y}_j$ be algebraically independent over $\overline{K}$. If in addition, the residues $\overline{y}_j$ are prescribed, then also the residue map is uniquely determined on $K(\mathcal{T})$.

The following corollary shows that the value group of a rational place of a function field in one variable is isomorphic to $\mathbb{Z}$. Note that if a valuation v is trivial on a field K, then we can assume that the residue map is the identity on K, so that we can write $\overline{K} = K$.

Corollary 6.36 *If $(L|K,v)$ is a finitely generated extension without transcendence defect, then also the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are finitely generated. In particular, if $(F|K,v)$ is a valued algebraic function field of transcendence degree 1 such that v is trivial on K but nontrivial on F, then $vF \cong \mathbb{Z}$ and $\overline{F}|K$ is finite.*

Proof: By Lemma 24.16 the finitely generated extension $L|K$ has finite transcendence degree. Hence by virtue of Lemma 6.33, we can choose a (finite) standard valuation transcendence basis $\mathcal{T}$ of $(L|K,v)$. The foregoing lemma shows that the extensions $vK(\mathcal{T})|vK$ and $\overline{K(\mathcal{T})}|\overline{K}$ are finitely generated. Since $L|K$ is finitely generated and $\mathcal{T}$ is a transcendence basis of $L|K$, it follows that $L|K(\mathcal{T})$ is a finite extension. From Lemma 6.13 we can thus infer that $vL|vK(\mathcal{T})$ and $\overline{L}|\overline{K(\mathcal{T})}$ are finite. Consequently, also the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are finitely generated.

Assume that $(F|K,v)$ is a valued algebraic function field such that v is trivial on K. Then $vK = \{0\}$ and $\overline{K} = K$. If v is nontrivial on F, then $vF \neq \{0\}$, and there is some $t \in F$ such that $vt \neq 0$. Now $\{t\}$ is algebraically valuation independent and thus also algebraically independent over K. If $\mathrm{trdeg}\, F|K = 1$, then $\{t\}$ is a transcendence basis and thus also a valuation transcendence basis over K. In particular, $F|K(t)$ is finite. By Lemma 6.13, $vF|vK(t)$ and $\overline{F}|\overline{K(t)}$ are finite. On the other hand, the foregoing lemma shows that $vK(t) = \mathbb{Z}vt \cong \mathbb{Z}$ and $\overline{K(t)} = \overline{K}$. The latter implies that $\overline{F}|K$ is finite. Finally, since $vF|vK(t)$ is finite, we conclude that also vF is isomorphic to $\mathbb{Z}$. □

As a further consequence of Lemma 6.35, we prove the following embedding lemma:

Lemma 6.37 *Let the assumptions be as in Lemma 6.35. Further, let $(K,v) \subset (F,w)$ be a valued field extension and assume that there are embeddings*

$$\rho:\ vK(\mathcal{T}) \longrightarrow wF$$

over vK and

$$\sigma:\ \overline{K(\mathcal{T})} \longrightarrow Fw$$

over $\overline{K}$. For every $i \in I$, choose some $x'_i \in F$ such that $wx'_i = \rho vx_i$, and for every $j \in J$, choose some $y'_j \in F$ such that $y'_j w = \sigma\overline{y}_j$. Then $\mathcal{T}' = \{x'_i\,,\, y'_j \mid i \in I\,,\, j \in J\}$ is a standard valuation independent set in (F,w) such that the values wx'_i, $i \in I$, are rationally independent over vK, and that the residues $y'_j w$, $j \in J$, are algebraically independent over $\overline{K}$. Further, the assignment $x_i \mapsto x'_i$, $y_j \mapsto y'_j$ induces an embedding of $(K(\mathcal{T},v)$ in (F,w) over K which respects ρ and σ.

Proof: The assertions about the values wx'_i, $i \in I$ and the residues $y'_j w$, $j \in J$ follow from the assumption that ρ is an embedding (over vK) and σ is an embedding (over $\overline{K}$). Now it follows from Lemma 6.30 that both $\mathcal{T}$ and $\mathcal{T}'$ are algebraically independent over K. Hence, the assignment $x_i \mapsto x'_i$, $y_j \mapsto y'_j$ induces an embedding ι of $K(\mathcal{T}$ in F over K of fields. Through this embedding, a valuation $w\iota$ and a residue map $a \mapsto (\iota^{-1}a)w$ are induced on $K(\mathcal{T})$. By the choice of the elements x'_i and y'_j, the induced valuation satisfies $w\iota x_i = wx'_i = \rho vx_i$, and the induced residue map satisfies $(\iota y_j)w = y'_j w = \sigma\overline{y_j}$. By the uniqueness assertion of Lemma 6.35, we now obtain that $w\iota a = \rho va$ holds for all $a \in K(\mathcal{T})$, and $(\iota a)w = \sigma\overline{a}$ holds for all $a \in K(\mathcal{T})$ of value ≥ 0. That is, ι respects ρ and σ. □

If t is an element of a standard algebraically valuation independent set of $(L|K,v)$, then
– either its value vt is 0 and its residue $\overline{t}$ is transcendental over $\overline{K}$,
– or its value vt is rationally independent over vK.
In the first case, we call t **residue-transcendental**, in the second case **value-transcendental** (**over** K). If one of these cases holds for t, we call t **valuation-transcendental**.

In the residue-transcendental case, the valuation obtained on $K(t)$ is called the **functional valuation** or **Gauß valuation associated with** t. For a polynomial

$$f(x) = c_0 + c_1 t + \ldots + c_n t^n \in K[t]$$

we have

$$vf(t) = \min_{0\leq i\leq n} vc_i$$

and consequently,

$$vK(t) = vK \quad\text{and}\quad \overline{K(t)} = \overline{K}(\overline{t})\ , \tag{6.7}$$

with $\overline{K}(\overline{t})$ a rational function field in one variable over $\overline{K}$. The valuation v on $K(t)$ is uniquely determined by its restriction to K and the prescription that t be valuation-transcendental.

In the value-transcendental case, given a polynomial $f(t)$ like above, we have that

$$v(f(t)) = \min_{0\leq i\leq n} (vc_i + ivt) \;,$$

and consequently,

$$vK(t) = vK \oplus \mathbb{Z}vt \;\text{ and }\; \overline{K(t)} = \overline{K} \;. \tag{6.8}$$

The valuation v on $K(t)$ is uniquely determined by its restriction to K and the rationally independent value of t (which in turn is described by the cut induced by vt in the divisible hull $\widetilde{vK}$ of the value group vK).

Lemma 6.38 *An element x in a valued field extension of (K,v) is algebraically valuation independent over (K,v) if and only if*
– either there is an integer $e>0$ and some $c\in K$ such that cx^e is residue-transcendental over (K,v),
– or x is value-transcendental over (K,v).
If the first case holds, then cx^e is residue-transcendental for every e such that $evx\in vK$ and every $c\in K$ such that $vc=-evx$.

Proof: Assume that x is algebraically valuation independent over (K,v). If it is not value-transcendental, then its value is a torsion element over vK. Let $e>0$ be any integer such that $evx\in vK$ and choose any $c\in K$ with $vc=-evx$. Then cx^e has value 0. If its residue were algebraic over $\overline{K}$, then there would exist a polynomial $g(X)\in\mathcal{O}_K[X]$ such that $\overline{g}$ is nonzero and $0=\overline{g}(\overline{cx^e})=\overline{g(cx^e)}$. But then $vg(cx^e)>0$ although at least one of its monomials has value 0. Since this contradicts our assumption that x be algebraically valuation independent, we conclude that cx^e is residue-transcendental.

Now we wish to prove the converse. If x is value-transcendental, then it is algebraically valuation independent by Lemma 6.30. So it remains to discuss the case where cx^e is residue-transcendental for some e and c. Take $e_0>0$ minimal with $e_0vx\in vK$. Since $evx=-vc\in vK$, e is a multiple of e_0, say, $e=me_0$. Take any $c_0\in K$ such that $vc_0=-e_0vx$. Then

$$\overline{cx^e} \;=\; \overline{c/c_0^m}\,\overline{c_0x^{e_0}}^m \;.$$

Since $\overline{cx^e}$ is transcendental over $\overline{K}$, the same must hold for $\zeta:=\overline{c_0x^{e_0}}$. Now we apply the criterion of Lemma 6.19 to $\mathcal{B}:=\{x^\ell\mid \ell=0,1,\ldots\}$. The values of elements $x^{\ell_1},\ldots,x^{\ell_n}$ belong to the same coset modulo vK if and only $\ell_1,\ldots,\ell_n$ are equivalent modulo e_0. Hence, there are distinct nonzero integers $k_2,\ldots,k_n$ such that $k_ie_0=\ell_i-\ell_1$. It follows that the elements $1,c_0^{k_2}x^{\ell_2-\ell_1},\ldots,c_0^{k_n}x^{\ell_n-\ell_1}$ have the residues $1,\zeta^{k_2},\ldots,\zeta^{k_n}$, which are $\overline{K}$-linearly independent since ζ is transcendental over $\overline{K}$. This proves that $\mathcal{B}$ is valuation independent over (K,v), that is, x is algebraically valuation independent. □

Exercise 6.4 *Let $t_1,\ldots,t_n$ be algebraically valuation independent.*
a) Show that (6.5) also holds if $\nu_1,\ldots,\nu_n$ are arbitrary integers.
b) Let $e_1,\ldots,e_n$ be nonzero integers. Prove that also $t_1^{e_1},\ldots,t_n^{e_n}$ are algebraically valuation independent. What happens if we replace t_i by an arbitrary polynomial in t_i?

Exercise 6.5 *Following the arguments of the proof of implication 3)⇒1) of Lemma 6.33, give criteria for algebraic valuation independence, by looking at the values and residues of given elements.*

Exercise 6.6 *Let $(L|K,v)$ be an extension which admits a valuation transcendence basis. Prove that $|L| \leq \max\{\aleph_0\,,\,|K|\,,\,|vL|\,,\,|\overline{L}|\}$ and that equality holds if L is infinite.*

6.5 Construction of valued fields

In this section, we will deal with the following problem. Suppose that (K,v) is a valued field and that $\Gamma|vK$ is an extension of ordered abelian groups and $k|\overline{K}$ is a field extension. Does there exist an extension $(L|K,v)$ of valued fields such that $vL=\Gamma$ and $Lv=k$? We include the case of (K,v) being trivially valued (like the finite field $\mathbb{F}_p$); this amounts to the construction of a valued field with given value group and residue field. We know from Section 4.3 that there are extensions of the valuation from (K,v) to every extension field, but there was nothing said about their value groups and residue fields.

If Γ/vK is a torsion group, then set $I=\emptyset$. Otherwise, pick a maximal set $\{\alpha_i \mid i\in I\}$ of elements in Γ rationally independent over vK. If $k|Kv$ is algebraic, then set $J=\emptyset$. Otherwise, pick a transcendence basis $\{\zeta_j \mid j\in J\}$ of $k|Kv$. By Theorem **??** there is a purely transcendental field extension $L_0|K$ and an extension of v to L_0 such that $vL_0 = vK\oplus\bigoplus_{i\in I}\mathbb{Z}\alpha_i$ and $L_0v = Kv\,(\zeta_j \mid j\in J)$. Now Γ/vL_0 is a torsion group and $k|L_0v$ is algebraic. Therefore, we may from now on assume that already Γ/vK is a torsion group and $k|Kv$ is algebraic, and we will construct an algebraic extension of (K,v) having value group Γ and residue field k. (Under certain conditions, such value group and residue field extensions can also be realized by purely transcendental field extensions, see [] and Section **??**.

Before we continue, let us adjust the following notion to our purposes. Usually, when one speaks of an **Artin-Schreier extension** then one means an extension of a field K generated by a root of an irreducible polynomial of the form X^p-X-c, provided that $p=\mathrm{char}\,K$. When working with a valued field, we will replace the condition "$p=\mathrm{char}\,K$" by the weaker condition "$p=\mathrm{char}\,\overline{K}$". In fact, such extensions also play an important role in the mixed characteristic case, where $\mathrm{char}\,K=0$. This will be seen when we deal with the defect of valued field extensions. See also R. E.MacKenzie and G. Whaples [MCK–WHA].

Every Artin-Schreier polynomial X^p-X-c is separable since its derivative does not vanish. The following is a simple but very useful observation:

Lemma 6.39 *Let (K,v) be a valued field and $c\in K$ such that $vc<0$. If an element a in some valued field extension (L,v) of (K,v) satisfies $v(a^p-a-c)\geq 0$, then $v(a^p-c) = va > pva = vc$. In particular, if $a\in\tilde{K}$ such that $a^p-a=c$, then this holds for every extension of v from K to $K(a)$.*

Proof: Necessarily, $va<0$ since otherwise, $0\leq v(a^p-a-c)=\min\{pva,va,vc\}=vc$, a contradiction. Since $va<0$, it follows that $va^p=pva<va$ and thus, $vc=v(a^p-a)=\min\{pva,va\}=pva$. Moreover, $v(a^p-c)=\min\{v(a^p-a-c),va\}=va$. □

Lemma 6.40 *Let (K,v) be a valued field, p a prime and α an element of the divisible hull of vK such that $p\alpha\in vK$, $\alpha\notin vK$. Choose an element $a\in\tilde{K}$ such that $a^p\in K$ and $va^p=p\alpha$. Then v extends in a unique way from K to $K(a)$. It satisfies $va=\alpha$ and*

$$vK(a)=vK+\mathbb{Z}\alpha \quad \textit{and} \quad \overline{K(a)}=\overline{K}\;.$$

If $\mathrm{char}\,K = \mathrm{char}\,\overline{K} = p > 0$*, then this extension* $K(a)|K$ *is purely inseparable. On the other hand, if* $\mathrm{char}\,\overline{K} = p > 0$*, then there is always a separable Artin-Schreier extension* $K(a)|K$ *with the same properties (if* $\alpha < 0$*, then* a *itself can be chosen to be an Artin-Schreier root).*

Proof: Choose some $c \in K$ such that $vc = p\alpha$. Further, choose some $a \in \tilde{K}$ such that $a^p = c$. By Corollary 4.11, there is an extension of v from K to $K(a)$. For this extension, we must have that $pva = vc = p\alpha$ by virtue of Lemma 6.39. Consequently, $va = \alpha$ and $(vK(a) : vK) \geq (vK + \mathbb{Z}\alpha : vK) = p$. On the other hand, the fundamental inequality (6.2) shows that

$$p = [K(a) : K] \geq (vK(a) : vK) \cdot [\overline{K(a)} : \overline{K}] \geq (vK(a) : vK) \geq p\,.$$

Hence, equality holds everywhere, and we find that $(vK(a) : vK) = p$ and $[\overline{K(a)} : \overline{K}] = 1$. That is, $vK(a) = vK + \mathbb{Z}\alpha$ and $\overline{K(a)} = \overline{K}$. Further, $1, a, \ldots, a^{p-1}$ is a valuation basis of $(K(a)|K, v)$ since the values of these elements belong to distinct cosets modulo vK. The uniqueness of v on $K(a)$ thus follows from Lemma 6.18.

Now suppose that p is equal to the characteristic of K. Then the polynomial $X^p - c$ is purely inseparable. To construct a separable extension, assume that $vc = p\alpha < 0$; otherwise, just replace c by $1/c$ in the following. By the foregoing lemma, every root a of the Artin-Schreier polynomial $X^p - X - c$ must satisfy $pva = vc$. Now we replace a by $1/a$ if $\alpha > 0$ (but note that $1/a$ is not an Artin-Schreier root). □

Lemma 6.41 *Let* (K, v) *be a valued field and* ζ *an element of the algebraic closure of* $\overline{K}$*. Choose a monic polynomial* $f \in \mathcal{O}_{\mathbf{K}}[X]$ *whose reduction* $\overline{f}$ *is the minimal polynomial of* ζ *over* $\overline{K}$*. Further, choose a root* $b \in \tilde{K}$ *of* f*. Then* v *extends in a unique way from* K *to* $K(b)$*. The associated residue map can be extended such that* $\overline{b} = \zeta$*. Further,* $[K(b) : K] = [\overline{K}(\zeta) : \overline{K}]$*, and*

$$vK(b) = vK \quad \textit{and} \quad \overline{K(b)} = \overline{K}(\zeta)\,.$$

In all cases, f *can be chosen to be separable, provided that the valuation* v *is nontrivial. On the other hand, if* $\mathrm{char}\,K = \mathrm{char}\,\overline{K} = p > 0$ *and* ζ *is purely inseparable over* $\overline{K}$*, then* b *can be chosen to be purely inseparable over* K*.*

If v *is not trivial on* K*,* $\mathrm{char}\,\overline{K} = p > 0$ *and* $\zeta^p \in \overline{K}$*,* $\zeta \notin \overline{K}$*, then there is also an Artin-Schreier extension* $K(b)|K$ *such that* $[K(b) : K] = [\overline{K}(\zeta) : \overline{K}]$*,* $vK(b) = vK$ *and* $\overline{K(b)} = \overline{K}(\zeta)$*.*

Proof: By Corollary 4.11, there is an extension of v from K to $K(b)$. Since f has integral coefficients, b must also be integral for this extension, and $\overline{b}$ must be a root of $\overline{f}$. We may compose the residue map with an isomorphism in $\mathrm{Gal}\,\overline{K}$ which sends this root to ζ. Doing so, we obtain a residue map (still associated with v) that satisfies $\overline{b} = \zeta$. Now $\zeta \in \overline{K(b)}$ and consequently, $[\overline{K(b)} : \overline{K}] \geq [\overline{K}(\zeta) : \overline{K}] = \deg \overline{f} = \deg f$. On the other hand, the fundamental inequality (6.2) shows that

$$\deg f = [K(b) : K] \geq (vK(b) : vK) \cdot [\overline{K(b)} : \overline{K}] \geq [\overline{K(b)} : \overline{K}] \geq \deg f\,.$$

Hence, equality holds everywhere, and we find that $[\overline{K(b)} : \overline{K}] = [\overline{K}(\zeta) : \overline{K}] = [K(b) : K]$ and $(vK(b) : vK) = 1$. That is, $vK(b) = vK$ and $\overline{K(b)} = \overline{K}(\zeta)$. Further, $1, b, \ldots, b^{\deg f - 1}$

is a valuation basis of $(K(b)|K,v)$ since the residues of these elements are $\overline{K}$-linearly independent. The uniqueness of v on $K(a)$ thus follows from Lemma 6.18.

If $\overline{f}$ is separable, then so is f. If $\overline{f}$ is not separable and v is nontrivial on K, then f can always chosen to be separable since we can add a summand cX with $0 \neq c \in \mathcal{M}_{\mathbf{K}}$ without changing the reduction of f. On the other hand, if $\overline{f} = X^{p^\nu} - \overline{c}$ is purely inseparable, then we can choose $f = X^{p^\nu} - c$ which also is purely inseparable if $\mathrm{char}\, K = p$.

Now suppose that $\mathrm{char}\,\overline{K} = p > 0$ and $\zeta^p \in \overline{K}$, $\zeta \notin \overline{K}$. Choose $c \in K$ such that $\overline{c} = \zeta^p$. To construct an Artin-Schreier extension, choose any $d \in K$ with $vd < 0$, and let b be a root of the Artin-Schreier polynomial $X^p - X - d^pc$. Since $vd^pc = pvd < 0$, Lemma 6.39 shows that $v(b^p - d^pc) = vb > vb^p$. Consequently, $v((b/d)^p - c) > v(b/d)^p = vc = 0$, and $\overline{b/d} = \overline{c}^{1/p} = \zeta$. As before, it follows that $vK(b) = vK$ and $\overline{K(b)} = \overline{K}(\zeta)$. □

Theorem 6.42 *Let (K,v) be an arbitrary valued field. For every extension $\Gamma|vK$ of ordered abelian groups and every field extension $k|\overline{K}$, there is an extension (L,v) of (K,v) such that $vL = \Gamma$ and $\overline{L} = k$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then $L|K$ can be chosen to be algebraic.*

In general, L can be chosen to be a separable extension of K (provided that $\Gamma \neq \{0\}$) and such that $(L|K,v)$ admits a standard valuation transcendence basis. On the other hand, if Γ/vK is a p-group with p the characteristic exponent of $\overline{K}$, and if $k|\overline{K}$ is a purely inseparable extension, then L can be chosen to be a purely inseparable extension of K.

Proof: For the proof, we assume that $\Gamma \neq \{0\}$ (the other case is trivial). Let α_i, $i \in I$, be a transcendence basis of $\Gamma|vK$. Then by Lemma **??** there is an extension $(K_1,v) := (K(x_i \mid i \in I), v)$ of (K,v) such that $vK_1 = vK \oplus \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\overline{K_1} = \overline{K}$. Next, choose a transcendence basis ζ_j, $j \in J$, of $k|\overline{K}$. Then by Lemma 5.5 there is an extension $(K_2,v) := (K_1(y_j \mid j \in J), v)$ of (K_1,v) such that $vK_2 = vK_1$ and $\overline{K_2} = \overline{K}(\zeta_j \mid j \in J)$. Note that $\{x_i, y_j \mid i \in I, j \in J\}$ is a standard valuation transcendence basis of $(K_2|K,v)$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then $K_2 = K$.

If we are given an ascending chain of valued fields whose value groups are subgroups of Γ and whose residue fields are subfields of k, then the union over this chain is again a valued field whose value group is a subgroup of Γ and whose residue field is a subfield of k. So a standard argument using Zorn's Lemma together with the transitivity of separable extensions shows that there are maximal separable algebraic extension fields of (K_2,v) with these properties. Choose one of them and call it (L,v). We have to show that $vL = \Gamma$ and $\overline{L} = k$. Since already Γ/vK_2 is a torsion group and $k|\overline{K_2}$ is algebraic, the same holds for Γ/vL and $k|\overline{L}$. If vL is a proper subgroup of Γ, then there is some prime p and some element $\alpha \in \Gamma \setminus vL$ such that $p\alpha \in vL$. But then, Lemma 6.40 shows that there exists a proper separable algebraic extension (L',v) of (L,v) such that $vL' = vL + \mathbb{Z}\alpha \subset \Gamma$ and $\overline{L'} = \overline{L} \subset k$, which contradicts the maximality of L. If $\overline{L}$ is a proper subfield of k, then there is some element $\zeta \in k \setminus \overline{L}$, and ζ is algebraic over $\overline{L}$. But then, Lemma 6.41 shows that there exists a proper separable algebraic extension (L',v) of (L,v) such that $vL' = vL \subset \Gamma$ and $\overline{L'} = \overline{L}(\zeta) \subset k$, which again contradicts the maximality of L (here we have used $\Gamma \neq \{0\}$, which implies that v is not trivial on L). This proves that $vL = \Gamma$ and $\overline{L} = k$, and (L,v) is the required extension of (K,v). Since K_2 is generated over K by a set of elements which are algebraically independent over K, we know from Lemma 24.40 that $K_2|K$ is separable. Since also $L|K_2$ is separable, we find that (L,v) is a separable extension of (K,v). Since $L|K_2$ is algebraic, $\{x_i\,,\, y_j \mid i\in I\,,\, j\in J\}$ is a valuation transcendence basis of $(K_2|K,v)$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then L is an algebraic extension of $K=K_2$.

In the case of Γ/vK being a p-group and $k|\overline{K}$ being a purely inseparable extension, we set $K_2=K$ and consider the purely inseparable extensions of (K,v) whose value group is a subgroup of Γ and whose residue field is a subfield of k. The same arguments as above work, with "purely inseparable" in the place of "separable". We then obtain a purely inseparable extension of (K,v) with the required properties. □

Every ordered abelian group is an extension of the trivial group $\{0\}$ as well as of the ordered abelian group $\mathbb{Z}$. Every field of characteristic 0 is an extension of $\mathbb{Q}$, and every field of characteristic $p>0$ is an extension of $\mathbb{F}_p$. Let Γ be an ordered abelian group and k a field. If $\operatorname{char} k=0$, then $\mathbb{Q}$ endowed with the trivial valuation v will satisfy $v\mathbb{Q}=\{0\}\subset\Gamma$ and $\overline{\mathbb{Q}}=\mathbb{Q}\subset k$. If $\operatorname{char} k=p>0$, then we can choose v to be the p-adic valuation on $\mathbb{Q}$ (see Section 4.2) to obtain that $v\mathbb{Q}=\mathbb{Z}\subset\Gamma$ and $\overline{\mathbb{Q}}=\mathbb{F}_p\subset k$. But also $\mathbb{F}_p$ endowed with the trivial valuation v will satisfy $v\mathbb{F}_p=\{0\}\subset\Gamma$ and $\overline{\mathbb{F}_p}=\mathbb{F}_p\subset k$. An application of the foregoing theorem now proves:

Corollary 6.43 *Let Γ be an ordered abelian group and k a field. Then there is a valued field (L,v) with $vL=\Gamma$ and $\overline{L}=k$. If $\operatorname{char} k=p>0$, then L can be chosen to be of characteristic 0 (mixed characteristic case) or of characteristic p (equal characteristic case).*

We can also derive information about extensions of the form $(\tilde{K}|K,v)$ and $(K^{\rm sep}|K,v)$. From Corollary 6.15 we know that $v\tilde{K}|vK$, $vK^{\rm sep}|vK$, $\overline{\tilde{K}}|\overline{K}$ and $\overline{K^{\rm sep}}|\overline{K}$ are algebraic extensions. On the other hand, Lemma 6.40 shows that the value group of a separable-algebraically closed field must be divisible. Similarly, it follows from Lemma 6.41 that the residue field of a separable-algebraically closed field must be algebraically closed. This proves:

Lemma 6.44 *Let (K,v) be a valued field and extend v to $\tilde{K}$. Then the value groups $v\tilde{K}$ and $vK^{\rm sep}$ are equal to the divisible hull of vK, and the residue fields $\overline{\tilde{K}}$ and $\overline{K^{\rm sep}}$ are equal to the algebraic closure of $\overline{K}$.*

Corollary 6.45 *Let (K,v) be a valued field with divisible value group and algebraically closed residue field. Then every maximal immediate extension and every maximal immediate algebraic extension is algebraically closed. In particular, every maximal immediate extension of an algebraically closed valued field is again algebraically closed.*

Proof: If K is algebraically closed, then by the foregoing lemma, vK is divisible and $\overline{K}$ is algebraically closed. Hence it suffices to prove the first assertion. So let vK be divisible and $\overline{K}$ be algebraically closed, and let (L,v) be an immediate extension of (K,v). Then by the foregoing lemma, the extension $(\tilde{L}|L,v)$ must be immediate for every extension of v from L to $\tilde{L}$. Hence if (L,v) admits no immediate algebraic extension, then this extension is trivial, that is, L is algebraically closed. □

A valued field (K,v) of residue characteristic $p>0$ will be called **Artin-Schreier closed** if every Artin-Schreier polynomial with coefficients in K admits a root in K. Recall that if $\operatorname{char} K=p$, then this means that every Artin-Schreier polynomial with coefficients in K splits into linear factors over K. As a corollary to Lemma 6.41 and Lemma 6.40, we obtain:

Corollary 6.46 *Every Artin-Schreier closed valued field of residue characteristic $p > 0$ has p-divisible value group and perfect residue field.*

Similarly, Lemma 6.41 and Lemma 6.40 show that the value group of the perfect hull of a valued field (K, v) of characteristic $p > 0$ is p-divisible, and that its residue field is perfect. On the other hand, if $\alpha \in vK^{1/p^\infty}$ and $a \in K^{1/p^\infty}$ such that $va = \alpha$, then $a^{p^m} \in K$ and thus $p^m\alpha \in vK$ for some integer $m \geq 0$. This shows that vK^{1/p^∞} is contained in the p-divisible hull of vK. Similarly, if ζ is an element of $\overline{K^{1/p^\infty}}$ and $b \in K^{1/p^\infty}$ is such that $\overline{b} = \zeta$, then $b^{p^m} \in K$ and thus $\zeta^{p^m} \in \overline{K}$ for some integer $m \geq 0$. This shows that $\overline{K^{1/p^\infty}}$ is contained in $\overline{K}^{1/p^\infty}$. This proves:

Lemma 6.47 *Let (K, v) be a valued field of characteristic $p > 0$ and extend v to K^{1/p^∞}. Then vK^{1/p^∞} is the p-divisible hull of vK, and $\overline{K^{1/p^\infty}}$ is the perfect hull of $\overline{K}$.*

We leave it to the reader to prove the following equalities along the lines of our earlier arguments:

$$vK^{1/p} = \frac{1}{p}vK \quad \text{and} \quad \overline{K^{1/p}} = \overline{K}^{1/p}$$

and

$$vK^p = pvK \quad \text{and} \quad \overline{K^p} = \overline{K}^p \,.$$

6.6 Extensions of a valuation to an algebraic field extension

In this section, we let $L|K$ be an algebraic field extension. We want to determine all possible extensions of a given valuation v from K to L. If $\tilde{v}$ is an arbitrary extension of v from K to $\tilde{K}$, then for every embedding ι of L in $\tilde{K}$ over K, the valuation $\tilde{v}\iota$ (which shall denote the more correct expression $\tilde{v}|_{\iota L}\iota$) is an extension of v from K to L. We shall show that these are all possible extensions (see Theorem 6.53 below).

We need a basic approximation theorem:

Theorem 6.48 (Chinese Remainder Theorem)
Let $\mathcal{R}$ be a commutative ring with 1 and $\mathcal{I}_1, \ldots, \mathcal{I}_n$ ideals of $\mathcal{R}$. Assume that these ideals are **pairwise comaximal**, *that is, $\mathcal{I}_i + \mathcal{I}_j = \mathcal{R}$ whenever $1 \leq i < j \leq n$. Then for every choice of $r_1, \ldots, r_n \in \mathcal{R}$ there is some $r \in \mathcal{R}$ such that $r \equiv r_i$ modulo $\mathcal{I}_i$ for $1 \leq i \leq n$.*

Proof: In the case $n = 2$, there is an equation $1 = s_1 + s_2$ with $s_i \in \mathcal{I}_i$, and $r = r_2s_1 + r_1s_2$ is the required element.

Now assume that we have proved our theorem for $n - 1$ ideals. For every $i \geq 2$ we choose elements $s_i' \in \mathcal{I}_1$ and $s_i \in \mathcal{I}_i$ such that $1 = s_i' + s_i$. The product

$$\prod_{i=2}^{n}(s_i' + s_i) \in \mathcal{I}_1 + \prod_{i=2}^{n}\mathcal{I}_i$$

is equal to 1. That is, the ideals $\mathcal{I}_1$ and $\prod_{i=2}^n \mathcal{I}_i$ of $\mathcal{R}$ are comaximal. By the first part of our proof, we may choose an element $t_1 \in \mathcal{R}$ such that

$$t_1 \equiv 1 \pmod{\mathcal{I}_1} \quad \text{and} \quad t_1 \equiv 0 \pmod{\prod_{i=2}^n \mathcal{I}_i} .$$

By a similar procedure we find elements $t_1, \ldots, t_n$ such that we obtain

$$t_i \equiv 1 \pmod{\mathcal{I}_i} \quad \text{and} \quad t_j \equiv 0 \pmod{\mathcal{I}_j} \quad \text{for } j \neq i .$$

Now $r = r_1 t_1 + \ldots + r_n t_n$ is the required element. □

Now let us fix the following situation:

$$\left.\begin{array}{rl} \mathbf{K} & = (K, v) \text{ a valued field with} \\ \mathcal{O}_{\mathbf{K}} & \text{its valuation ring, which has} \\ \mathcal{M}_{\mathbf{K}} & \text{as its maximal ideal,} \\ L|K & \text{an algebraic field extension,} \\ \mathcal{R} & \text{the integral closure of } \mathcal{O}_{\mathbf{K}} \text{ in } L. \end{array}\right\} \tag{6.9}$$

We need a first statement about the prime ideals of $\mathcal{R}$.

Lemma 6.49 *Let $\mathcal{R} \supset \mathcal{R}'$ be an extension of commutative rings with* 1 *and assume that $\mathcal{R}$ is integral over $\mathcal{R}'$. Let $\mathcal{I}$ be a prime ideal of $\mathcal{R}$ and set $\mathcal{I}' = \mathcal{I} \cap \mathcal{R}'$. Then $\mathcal{I}'$ is a maximal ideal if and only if $\mathcal{I}$ is.*

Proof: Assume that $\mathcal{I}'$ is maximal. Then $\mathcal{R}'/\mathcal{I}'$ is a field. Since $\mathcal{I}$ is a prime ideal satisfying $\mathcal{I}' = \mathcal{I} \cap \mathcal{R}'$, we have that $\mathcal{R}/\mathcal{I}$ is an entire ring extending $\mathcal{R}'/\mathcal{I}'$. Further, $\mathcal{R}/\mathcal{I}$ is integral over $\mathcal{R}'/\mathcal{I}'$ because $\mathcal{R}$ is integral over $\mathcal{R}'$. Hence, every element $a \in \mathcal{R}/\mathcal{I}$ is algebraic over the field $\mathcal{R}'/\mathcal{I}'$, which yields that $\mathcal{R}'/\mathcal{I}'[a]$ is a field. Consequently, every element of $\mathcal{R}/\mathcal{I}$ is invertible, that is, $\mathcal{R}/\mathcal{I}$ is a field and $\mathcal{I}$ is a maximal ideal.

For the converse, assume that $\mathcal{I}$ is a maximal ideal. Then $\mathcal{R}/\mathcal{I}$ is field, and as before, it is integral over the subring $\mathcal{R}'/\mathcal{I}'$. Suppose that $\mathcal{I}'$ is not maximal, that is, $R' := \mathcal{R}'/\mathcal{I}'$ is not a field. Then it admits a nonzero maximal ideal I'. By Theorem 4.7 there exists a valuation ring R of $\mathcal{R}/\mathcal{I}$ containing R' such that the maximal ideal of R contains I' and is thus nonzero. But by virtue of Theorem 4.14, R must contain the integral closure of R' in $\mathcal{R}/\mathcal{I}$ which is $\mathcal{R}/\mathcal{I}$ itself. Hence, R is a field and can not have a proper nonzero ideal. This contradiction shows that $\mathcal{I}'$ must be maximal. □

Lemma 6.50 *Assume that $\mathcal{O}$ is a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}$, and let $\mathcal{I}$ be the prime ideal $\mathcal{M} \cap \mathcal{R}$ of $\mathcal{R}$. Then $\mathcal{I}$ is a maximal ideal and $\mathcal{O}$ is the localization $\mathcal{R}_{\mathcal{I}}$.*

Proof: Since $\mathcal{O}$ lies above $\mathcal{O}_{\mathbf{K}}$, we have $\mathcal{I} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M} \cap \mathcal{R} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M}_{\mathbf{K}}$. Hence by Lemma 6.49, $\mathcal{I}$ is a maximal ideal. From $\mathcal{I} = \mathcal{M} \cap \mathcal{R}$ it follows that $\mathcal{R}_{\mathcal{I}} \subset \mathcal{O}$. Now let $x \in \mathcal{O}$; we want to show that $x \in \mathcal{R}_{\mathcal{I}}$. Since $L|K$ is assumed to be algebraic, there is an equation $a_n x^n + \ldots + a_0 = 0$ with coefficients $a_i \in K$, not all 0. Let m be the maximal index for which va_m is the minimum of the values of these coefficients. Hence, the elements $b_i := a_i/a_m$ are all contained in $\mathcal{O}_\mathbf{K}$. Dividing our equation by $a_m x^m$, and defining

$$y := b_n x^{n-m} + \ldots + b_{m+1}x + 1 \in \mathcal{O}_\mathbf{K}[x]\,, \quad z = b_{m-1} + \ldots + b_1 \left(\frac{1}{x}\right)^{m-2} + b_0 \left(\frac{1}{x}\right)^{m-1} \in \mathcal{O}_\mathbf{K}\left[\frac{1}{x}\right]\,,$$

we obtain $y + \frac{1}{x}z = 0$. Now let $\mathcal{O}'$ be any valuation ring of L containing $\mathcal{O}_\mathbf{K}$. If $\mathcal{O}'$ contains x, then $y \in \mathcal{O}_\mathbf{K}[x] \subset \mathcal{O}'$, and $z = -yx \in \mathcal{O}'$. If $\mathcal{O}'$ contains $\frac{1}{x}$, then $z \in \mathcal{O}_\mathbf{K}[\frac{1}{x}] \subset \mathcal{O}'$, and $y = -\frac{1}{x}z \in \mathcal{O}'$. Since $\mathcal{O}'$ satisfies (VR), we see that it will always contain y and z. From Theorem 4.14 we deduce $y, z \in \mathcal{R}$.

By the minimality of m, we have $b_n, \ldots, b_{m+1} \in \mathcal{M}_\mathbf{K} \subset \mathcal{I}$. This shows $y \in 1 + \mathcal{I}$, yielding that $y \notin \mathcal{I}$. Consequently, $x = -z/y \in \mathcal{R}_\mathcal{I}$. □

We will now study how the automorphisms of $L|K$ act on the maximal ideals of $\mathcal{R}$. Note that every automorphism of $L|K$ maps $\mathcal{R}$ into $\mathcal{R}$. Indeed, if $\sigma \in \mathrm{Gal}\, L|K$ and $b \in \mathcal{R}$ then σb satisfies the same equation with coefficients from $\mathcal{O}_\mathbf{K}$ as b and is consequently also integral over $\mathcal{O}_\mathbf{K}$. Further, σ sends prime ideals of $\mathcal{R}$ onto prime ideals of $\mathcal{R}$, and maximal ideals onto maximal ideals.

Lemma 6.51 *Let $L|K$ be a finite normal extension. Assume that $\mathcal{I}$ and $\mathcal{J}$ are maximal ideals of $\mathcal{R}$ and that $\mathcal{I}$ contains $\mathcal{M}_\mathbf{K}$. Then there exists $\sigma \in \mathrm{Gal}\, L|K$ such that $\sigma\mathcal{J} = \mathcal{I}$.*

Proof: Suppose that $\mathcal{I} \neq \sigma\mathcal{J}$ for all $\sigma \in \mathrm{Gal}\, L|K$. Then $\tau\mathcal{I} \neq \sigma\mathcal{J}$ for all $\sigma, \tau \in \mathrm{Gal}\, L|K$. Since $\mathcal{J}$ and $\mathcal{I}$ are maximal ideals of $\mathcal{R}$, the same holds for all $\sigma\mathcal{J}$ and $\sigma\mathcal{I}$, $\sigma \in \mathrm{Gal}\, L|K$. Hence, the sum of any two distinct of them is equal to $\mathcal{R}$. So we may apply the Chinese Remainder Theorem 6.48. According to this theorem, there exists $b \in \mathcal{R}$ such that for all $\sigma \in \mathrm{Gal}\, L|K$, $b \equiv 0$ modulo $\sigma\mathcal{J}$ and $b \equiv 1$ modulo $\sigma\mathcal{I}$. Let $[L : K]_{\mathrm{insep}} = p^\nu$ where p is the characteristic exponent of K. The norm $a := \mathrm{N}_{L|K}(b) = \prod_{\sigma \in \mathrm{Gal}\, L|K} \sigma b^{p^\nu}$ lies in $\mathcal{R} \cap K$. Since $\mathcal{O}_\mathbf{K}$ is integrally closed by Lemma 4.1, we have $\mathcal{R} \cap K = \mathcal{O}_\mathbf{K}$, showing that $\mathrm{N}_{L|K}(b) \in \mathcal{O}_\mathbf{K}$.

Since $b \in \mathcal{J}$ by our choice of b, it follows that $a \in \mathcal{J} \cap \mathcal{O}_\mathbf{K}$. Since $\mathcal{J}$ is a proper ideal of $\mathcal{R}$, we know that $\mathcal{J} \cap \mathcal{O}_\mathbf{K}$ is a proper ideal of $\mathcal{O}_\mathbf{K}$ and thus contained in $\mathcal{M}_\mathbf{K}$. By assumption, $\mathcal{I} \cap \mathcal{O}_\mathbf{K} = \mathcal{M}_\mathbf{K}$, hence $\mathrm{N}_{L|K}(b) \in \mathcal{I}$. But $b \notin \sigma\mathcal{I}$ for all $\sigma \in \mathrm{Gal}\, L|K$ by our choice of b, that is, $\sigma b \notin \mathcal{I}$ for all $\sigma \in \mathrm{Gal}\, L|K$. This contradicts our assumption that $\mathcal{I}$ be a prime ideal. This contradiction completes the proof of our lemma. □

Theorem 6.52 *Let $L|K$ be a normal algebraic field extension and $\mathcal{O}_\mathbf{K}$ a valuation ring of K with maximal ideal $\mathcal{M}_\mathbf{K}$. Let $\mathcal{R}$ be the integral closure of $\mathcal{O}_\mathbf{K}$ in L. For every two valuation rings $\mathcal{O}, \mathcal{O}'$ of L lying above $\mathcal{O}_\mathbf{K}$, with maximal ideals $\mathcal{M}, \mathcal{M}'$ respectively, there is an automorphism $\sigma \in \mathrm{Gal}\, L|K$ such that $\sigma\mathcal{O}' = \mathcal{O}$ and $\sigma\mathcal{M}' = \mathcal{M}$. The valuation rings of L lying above $\mathcal{O}_\mathbf{K}$ are precisely all the localizations of $\mathcal{R}$ with respect to its maximal ideals. Moreover, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{O}_\mathbf{K}$.*

Proof: According to Lemma 6.50, with the maximal ideals $\mathcal{I} = \mathcal{M} \cap \mathcal{R}$ and $\mathcal{I}' = \mathcal{M}' \cap \mathcal{R}$, we have $\mathcal{O} = \mathcal{R}_\mathcal{I}$ and $\mathcal{O}' = \mathcal{R}_{\mathcal{I}'}$. Suppose that we are able to show for every maximal ideal $\mathcal{J}$ the existence of some $\sigma \in \mathrm{Gal}\, L|K$ such that $\sigma\mathcal{J} = \mathcal{I}$. Then because of $\sigma\mathcal{R} = \mathcal{R}$, we obtain $\sigma\mathcal{R}_\mathcal{J} = \mathcal{R}_{\sigma\mathcal{J}} = \mathcal{R}_\mathcal{I} = \mathcal{O}$. Applying this to $\mathcal{J} = \mathcal{I}'$, we obtain $\sigma\mathcal{O}' = \mathcal{O}$. But then, σ

will also send the unique maximal ideal $\mathcal{M}'$ of $\mathcal{O}'$ onto the unique maximal ideal $\mathcal{M}$ of $\mathcal{O}$. With $\mathcal{J}$ running through all maximal ideals of $\mathcal{R}$, also the last assertion of the theorem will follow: By Theorem 4.7 there is at least one valuation ring $\mathcal{O}$ of L lying above $\mathcal{O}_{\mathbf{K}}$; every localization $\mathcal{R}_{\mathcal{J}}$ will thus be sent onto $\mathcal{O}$ by some $\sigma \in \operatorname{Gal} L|K$ and will therefore be itself a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. On the other hand, we have seen at the beginning that every valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$ is the localization $\mathcal{R}_{\mathcal{J}}$ with respect to some maximal $\mathcal{J}$ of $\mathcal{R}$.

In the case of a finite normal extension $L|K$, the existence of σ is assured by Lemma 6.51. We have to generalize to the infinite case. Since $L|K$ is a normal algebraic extension, L is the union over all finite normal subextensions $L_i|K$, $i \in I$. Let $\mathcal{R}_i := \mathcal{R} \cap L_i$ which is the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L_i. Further, let $\mathcal{I}_i := \mathcal{I} \cap \mathcal{R}_i$ and $\mathcal{J}_i := \mathcal{J} \cap \mathcal{R}_i$. Since $\mathcal{I}$ and $\mathcal{J}$ are maximal ideals, so are $\mathcal{I}_i$ and $\mathcal{J}_i$ by virtue of Lemma 6.49; moreover, $\mathcal{I}_i$ contains $\mathcal{M}_{\mathbf{K}}$. Hence by Lemma 6.51 there exists $\sigma_i \in \operatorname{Gal} L_i|K$ such that $\sigma_i \mathcal{J}_i = \mathcal{I}_i$. By the Compactness Principle for Algebraic Extensions (cf. Lemma 24.5 where we take $F = \tilde{L}$), there exists $\sigma \in \operatorname{Gal} L|K$ and a subset $J \subset I$ such that $L = \bigcup_{j\in J} L_j$ and $\operatorname{res}_{L_i}(\sigma) = \sigma_i$. Since $\mathcal{I} = \bigcup_{j\in J} \mathcal{I}_i$ and $\mathcal{J} = \bigcup_{j\in J} \mathcal{J}_i$, it follows that $\sigma\mathcal{J} = \mathcal{I}$.

It remains to prove the last assertion of the theorem. By Theorem 4.14, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{R}$. In view of what we have proved so far, it suffices to show that every valuation ring $\mathcal{O}_0$ of L lying above $\mathcal{R}$ contains a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. Let $\mathcal{M}_0$ be the maximal ideal of $\mathcal{O}_0$. Choose some maximal ideal $\mathcal{I}$ of $\mathcal{R}$ containing the prime ideal $\mathcal{I}_0 = \mathcal{O}_0 \cap \mathcal{R}$. Since the latter contains precisely all elements of $\mathcal{R}$ which are non-units in $\mathcal{O}_0$ we find that $\mathcal{O}_0$ must contain the localization $\mathcal{R}_{\mathcal{I}}$. But this is a valuation ring of L which lies above $\mathcal{O}_{\mathbf{K}}$. □

For $L|K$ an arbitrary algebraic extension, we are now able to determine all extensions of v from K to L. We have $\tilde{L} = \tilde{K}$, and $\tilde{L}|K$ is a normal algebraic extension containing $L|K$. Given two extensions w, w' of v from K to L, by virtue of Theorem 4.10 we may choose extensions $\tilde{w}, \tilde{w}'$ to $\tilde{L}$. Then by the preceding theorem, there is $\sigma \in \operatorname{Gal} \tilde{L}|K$ such that $\sigma\mathcal{O}_{\tilde{w}'} = \mathcal{O}_{\tilde{w}}$ and $\sigma\mathcal{M}_{\tilde{w}'} = \mathcal{M}_{\tilde{w}}$. Consequently, $\tilde{w}' = \tilde{w}\sigma$ showing that also $w' = \tilde{w}\operatorname{res}_L(\sigma)$. If $L|K$ is normal, then the embedding $\iota := \operatorname{res}_L(\sigma)$ of L in $\tilde{L}$ over K is an automorphism of L and we may write $w' = w\iota$. Even if ι is not an automorphism of L, we shall call w and $w' = \tilde{w}\iota$ **conjugate extensions of** v. One also says that the valuations w and w' resp. their associated places P_w and $P_{w'} = P^\iota$ are **conjugate over** K. Fixing an arbitrary extension $\tilde{v}$ of v from K to $\tilde{K}$, we may write $w = \tilde{v}|_L$ and $\tilde{w} = \tilde{v}$. Then we find that every extension w' of v from K to L is of the form $\tilde{v}\iota$. We have proved:

Theorem 6.53 *If $L|K$ is algebraic, then every two extensions of v from K to L are conjugate. Given an arbitrary extension $\tilde{v}$ of v from K to $\tilde{K}$, the set of all extensions of v from K to L is*

$$\{(\tilde{v}\iota)|_L \mid \iota \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\}\ .$$

As a corollary to this theorem and Corollary 4.11, we can note for the special case of the algebraic closure of a valued field:

Corollary 6.54 *Let (K, v) be an arbitrary valued field. Then there exists an extension $\tilde{v}$ of v from K to $\tilde{K}$, and the set of all such extensions is given by*

$$\{\tilde{v}\iota \mid \iota \in \operatorname{Gal} K\}\ .$$

In other words, every two algebraic closures of (K,v) are isomorphic over K.

If the algebraic extension $L|K$ is normal and we pick one extension of v to L and denote it again by v, then the set of all extensions of v from K to L can be written as

$$\{v\iota \mid \iota \in \mathrm{Gal}\, L|K\} .$$

In view of part c) of Lemma 6.27, this yields:

Corollary 6.55 *If $(L|K,v)$ is a finite normal extension, then for every extension w of v from K to L,*

$$e(L|K,w) = e(L|K,v) \quad \textit{and} \quad f(L|K,w) = f(L|K,v) .$$

For a finite extension $L|K$, the number of distinct embeddings of L in $\tilde{L}$ over K is equal to the separable degree $[L:K]_{\mathrm{sep}}$. Hence, we may deduce from the foregoing theorem:

Corollary 6.56 *If $L|K$ is finite, then the number of distinct extensions of v from K to L is not greater than the separable degree $[L:K]_{\mathrm{sep}}$.*

As a special case, we obtain the following corollary:

Corollary 6.57 *If $L|K$ is a purely inseparable algebraic extension, then the extension w of v from K to L is uniquely determined.*

Let us give another proof of this corollary. Let $p = \mathrm{char}\, K$. For every $b \in L$ there is some nonzero $n \in \mathbb{N}$ such that $b^{p^n} = a \in K$. Hence, $wb = \frac{1}{p^n} va$ for every extension w.

We generalize the last two assertions of Theorem 6.52 to algebraic extensions which are not necessarily normal.

Lemma 6.58 *Let $L|K$ be an algebraic field extension and $\mathcal{O}_{\mathbf{K}}$ a valuation ring of K with maximal ideal $\mathcal{M}_{\mathbf{K}}$. Let $\mathcal{R}$ be the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L. Then the valuation rings of L lying above $\mathcal{O}_{\mathbf{K}}$ are precisely all the localizations of $\mathcal{R}$ with respect to its maximal ideals. Moreover, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{O}_{\mathbf{K}}$.*

Proof: It follows already from Lemma 6.50 that every valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$ is the localization of $\mathcal{R}$ with respect to some maximal ideal. For the converse, assume that $\mathcal{I}$ is a maximal ideal of $\mathcal{R}$. Since $\mathcal{R}$ is integral over $\mathcal{O}_{\mathbf{K}}$, Lemma 6.49 shows that $\mathcal{I} \cap \mathcal{O}_{\mathbf{K}}$ is a maximal ideal and thus equal to $\mathcal{M}_{\mathbf{K}}$. From Theorem 4.7 we infer the existence of a valuation ring $\mathcal{O}$ of L containing $\mathcal{R}$ whose maximal ideal $\mathcal{M}$ contains $\mathcal{I}$. Consequently, $\mathcal{O}$ contains $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{M}$ contains $\mathcal{M}_{\mathbf{K}}$ which shows that $\mathcal{O}$ lies over $\mathcal{O}_{\mathbf{K}}$. By Lemma 6.50, $\mathcal{O}$ is the localization of $\mathcal{R}$ with respect to the maximal ideal $\mathcal{M} \cap \mathcal{R}$. Since this maximal ideal contains $\mathcal{I}$, it must be equal to $\mathcal{I}$.

The last assertion is shown as in the proof of Theorem 6.52. □

Two valuation rings $\mathcal{O}_1$, $\mathcal{O}_2$ resp. their associated valuations are called **comparable** if $\mathcal{O}_1 \subset \mathcal{O}_2$ or $\mathcal{O}_1 \supset \mathcal{O}_2$; otherwise, they are called **incomparable**. From the foregoing lemma, we obtain:

Corollary 6.59 *Every two distinct extensions of a valuation to an algebraic extension field are incomparable.*

Proof: Let the notation be as in the foregoing lemma. Given two extensions of v from K to L with valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, the foregoing lemma tells us that $\mathcal{O}_i = \mathcal{R}_{\mathcal{I}_i}$ with maximal ideals $\mathcal{I}_i$ of $\mathcal{R}$, $i = 1, 2$. Let us assume that $\mathcal{O}_1 \subset \mathcal{O}_2$. Then $\mathcal{I}_1 \subset \mathcal{I}_2$ which yields $\mathcal{I}_1 = \mathcal{I}_2$ because of their maximality. Consequently, $\mathcal{O}_1 = \mathcal{O}_2$. □

Let us exploit once more the Chinese Remainder Theorem 6.48:

Lemma 6.60 *Let (K, v) be a valued field, $L|K$ an algebraic extension and $v_1, \ldots, v_n$ distinct extensions of v from K to L. Given any elements $\zeta_1 \in L/v_1$, ..., $\zeta_n \in L/v_n$, there is some $a \in L$ such that $a/v_1 = \zeta_1$, ..., $a/v_n = \zeta_n$.*

Proof: Let $\mathcal{O}_v$ be the valuation ring of v in K and let $\mathcal{R}$ be its integral closure in L. Denote the valuation ring of v_i in L by $\mathcal{O}_i$, its maximal ideal by $\mathcal{M}_i$ and set $\mathcal{I}_i := \mathcal{M}_i \cap \mathcal{R}$. By Lemma 6.50, $\mathcal{O}_i = \mathcal{R}_{\mathcal{I}_i}$. Since $L/v_i = \mathcal{O}_i/\mathcal{M}_i = \mathcal{R}_{\mathcal{I}_i}/\mathcal{R}_{\mathcal{I}_i}\mathcal{I}_i = \mathcal{R}/\mathcal{I}_i$, we may choose $a_i \in \mathcal{R}$ such that $a_i/v_i = a_i/\mathcal{I}_i = \zeta_i$. Since the $\mathcal{I}_i$ are maximal ideals, we have $\mathcal{I}_i + \mathcal{I}_j = \mathcal{R}$ for $i \neq j$. So we may use the Chinese Remainder Theorem to find $a \in \mathcal{R}$ which satisfies $a \equiv a_i$ modulo $\mathcal{I}_i$ for $1 \leq i \leq n$. That is, $a/v_i = a/\mathcal{I}_i = a_i/\mathcal{I}_i = \zeta_i$. □

With Lemma 5.6 as a tool, we prove:

Lemma 6.61 *Let $(L|K, v)$ be a normal extension of valued fields. Then for every $\zeta \in \overline{L}$ and every $\xi \in \widetilde{\overline{K}}$ which is conjugate to ζ over $\overline{K}$, there exists some $a \in L$ and an automorphism $\sigma \in \operatorname{Gal} L|K$ such that $\overline{a} = \zeta$ and $\overline{\sigma a} = \xi$. In particular, $\overline{L}|\overline{K}$ is normal.*

Proof: Let g be the minimal polynomial of ζ over $\overline{K}$, so ζ, ξ are roots of g. By Lemma 6.60, there exists some $a \in L$ such that $\overline{a} = \zeta$ and $v_i a > 0$ for all extensions $v_i \neq v$ of v from K to L. In view of Theorem 6.53, this yields that all conjugates of a, hence all roots of f, are integral. Consequently, the minimal polynomial f of a over K has integral coefficients. Since a is integral and a root of f, we know that $\overline{a} = \zeta$ is a root of $\overline{f} \in \overline{K}[X]$. Consequently, g must divide $\overline{f}$, showing that also ξ is a root of $\overline{f}$. Then by part b) of Lemma 5.6, ξ is the residue of some root $b \in L$ of f. By our choice of f, there is some $\sigma \in \operatorname{Gal} L|K$ such that $b = \sigma a$. So $\xi = \overline{\sigma a}$, as contended. In particular, $\xi \in \overline{L}$. Since ζ was an arbitrary element of $\overline{L}$ and ξ was an arbitrary conjugate of ζ over $\overline{K}$, this proves that $\overline{L}|\overline{K}$ is normal. □

Exercise 6.7 *Let $\mathbf{K} = (K, v)$ be an arbitrary valued field and let a be algebraic with minimal polynomial f over K. Prove that $f \in \mathcal{O}_{\mathbf{K}}[X]$ if and only if a is integral for all extensions of v from K to $\tilde{K}$ (or to $K(a)$).*

6.7 Valuation disjoint extensions

Let $(\Omega|K, v)$ be an extension of valued fields and $F|K$ and $L|K$ two subextensions of $\Omega|K$. We say that $(F|K, v)$ is **valuation disjoint from** $(L|K, v)$ (**in** (Ω, v)) if every standard valuation independent set $\mathcal{B}$ of $(F|K, v)$ is also a standard valuation independent set of $(F.L|L, v)$. Let us observe that it is possible that a standard valuation independent set $\mathcal{B}$ of $(F|K, v)$ remains valuation independent over (L, v) without remaining a standard valuation independent set of $(F.L|L, v)$:

Example 6.62 Let k be a non-perfect field of characteristic p and $c \in k \setminus k^p$. Let t be transcendental over k and consider the valued field $(k(t), v_t)$ with the t-adic valuation. Extend v_t to $\widetilde{k(t)}$ and choose $a, b \in \widetilde{k(t)}$ such that $a^p = ct$ and $b^p = t$. Define $K := k(t)$. Then the standard valuation basis $\{1, a, \ldots, a^{p-1}\}$ of $(K(a)|K, v_t)$ will remain valuation independent over $(K(b), v_t)$ since $a^i/b^i = c^{i/p}$ and $\{1, c^{1/p}, \ldots, c^{p-1/p}\}$ is valuation independent over $(K(b), v_t)$. But $\{1, a, \ldots, a^{p-1}\}$ is not a standard valuation basis of $(K(a,b)|K(b), v_t)$ since the values of its elements are all in the same coset modulo $vK(b)$ without being equal.

With this example, we can observe also another phenomenon which will play a role later in this section. We have that $vK(a) = vK(b) = \mathbb{Z}\frac{vt}{p}$, hence $\overline{K(a)} = \overline{K(b)} = \overline{K}$ by the fundamental inequality. Consequently, $\overline{K(a).K(b)} = \overline{K(a,b)} = \overline{K}(\bar{c}^{1/p}) \neq \overline{K} = \overline{K(a)}.\overline{K(b)}$. To obtain a similar phenomenon for the value groups, it suffices to change the definition of a slightly, requiring now that $a^p = c + t$. Then we have that $\overline{K(a)} = \overline{K(b)} = \overline{K}(\bar{c}^{1/p})$, hence $vK(a) = vK(b) = vK$ by the fundamental inequality. Then $K(a,b) = K(t^{1/p}, c^{1/p})$ and consequently, $vK(a,b) = \mathbb{Z}\frac{vt}{p} \neq \mathbb{Z}vt = vK(a) + vK(b)$. $\diamondsuit$

We have chosen the strong condition using *standard* valuation independent sets since then, valuation disjoint extensions can be characterized by means of their value groups and residue fields. We need some preparation.

Let G and G' be two subgroups of some group $\mathcal{G}$, and H be a common subgroup of G and G'. The elements $\alpha_1, \ldots, \alpha_n \in G$ are said to be H**-independent** if they belong to distinct cosets modulo H. We will say that the group extension $G|H$ is **disjoint from** the group extension $G'|H$ **(in** $\mathcal{G}$**)** if for every $n \in \mathbb{N}$ and every choice of H-independent elements $\alpha_1, \ldots, \alpha_n \in G$, these elements will also be G'-independent. Contrary to the field case, this is already the case if every two H-independent elements α, α' remain G'-independent. Observe that α, α' are H-independent if and only if $\alpha - \alpha' \notin H$. So if α, α' do not remain G'-independent, then $\alpha - \alpha' \in G \cap G'$, showing that $H \neq G \cap G'$. Conversely, if there exists $\alpha \in G \cap G'$, $\alpha \notin H$, then α and 0 are H-independent, but not G'-independent. We have proved:

$G|H$ is disjoint from $G'|H$ if and only if $G \cap G' = H$.

Hence, the notion "disjoint from" is symmetrical, like "linearly disjoint from" in the field case.

Now valuation disjoint extensions can be characterized as follows:

Lemma 6.63 *Let $(\Omega|K, v)$ be an extension of valued fields and $F|K$ and $L|K$ subextensions of $\Omega|K$. Then $(F|K, v)$ is valuation disjoint from $(L|K, v)$ in (Ω, v) if and only if*

1) $vF|vK$ is disjoint from $vL|vK$ in $v\Omega$, and

2) $\overline{F}|\overline{K}$ is linearly disjoint from $\overline{L}|\overline{K}$ in $\overline{\Omega}$.

Consequently, valuation disjointness is symmetrical: if $(F|K, v)$ is valuation disjoint from $(L|K, v)$, then $(L|K, v)$ is valuation disjoint from $(F|K, v)$.

Proof: From the characterization of valuation independence given in Lemma 6.19 we infer that $(F|K, v)$ is valuation disjoint from $(L|K, v)$ if the following two conditions are satisfied:

1) if $b, b' \in F$ have values which belong to distinct cosets modulo vK, then these values also belong to distinct cosets modulo vL, and

2) if $b_1, \ldots, b_n \in \mathcal{O}_{\mathbf{F}}$ have residues which are $\overline{K}$-linearly independent, then these residues are also $\overline{L}$-linearly independent.

Each of these conditions is equivalent to the corresponding condition in the assertion of our lemma.

For the converse, assume that condition 1) or 2) is not satisfied. If 1) is not satisfied, then there are two elements $b, b' \in F$ whose values belong to distinct cosets modulo vK but to the same coset modulo vL. From the former it follows that $\{b, b'\}$ is a standard valuation independent set in $(F|K, v)$, and from the latter it follows that $\{b, b'\}$ is not a standard valuation independent set in $(F.L|L, v)$. If 2) is not satisfied, then there are elements $b_1, \ldots, b_n \in \mathcal{O}_{\mathbf{F}}$ such that their residues are $\overline{K}$-linearly independent but not $\overline{L}$-linearly independent. Now $\{b_1, \ldots, b_n\}$ is a standard valuation independent set in $(F|K, v)$, but not in $(F.L|L, v)$. This shows that $(F|K, v)$ is not valuation disjoint from $(L|K, v)$. $\square$

The following is a partial analogue to Lemma 24.12; we leave its straightforward proof to the reader:

Lemma 6.64 *Let $(\Omega|K, v)$ be an extension of valued fields, and let $L|K$ and $F \supset E \supset K$ be subextensions of $\Omega|K$. If $(L|K, v)$ is valuation disjoint from $(E|K, v)$ and $(E.L|E, v)$ is valuation disjoint from $(F|E, v)$, then $(L|K, v)$ is valuation disjoint from $(F|K, v)$. Conversely, if $(L|K, v)$ is valuation disjoint from $(F|K, v)$, then it is also valuation disjoint from $(E|K, v)$, and if in addition $v(E.L) = vE + vL$ and $\overline{E.L} = \overline{E}.\overline{L}$, then also $(E.L|E, v)$ is valuation disjoint from $(F|E, v)$.*

An extension $(F|K, v)$ will be called **valuation separable** if it is valuation disjoint from $(K^{1/p^\infty}|K, v)$. Note that this does not depend on the embedding of (F, v) and $(K^{1/p^\infty}, v)$ in a common valued extension field, since K^{1/p^∞} admits a unique embedding in F^{1/p^∞} and the extension of v from F to F^{1/p^∞} is unique. From Lemma 6.63 we infer that $(F|K, v)$ is valuation separable if and only if $vF \cap vK^{1/p^\infty} = vK$ and $\overline{F}|\overline{K}$ is linearly disjoint from $\overline{K^{1/p^\infty}}|\overline{K}$. But $\overline{K^{1/p^\infty}} = \overline{K}^{1/p^\infty}$, so the latter is equivalent to the condition that $\overline{F}|\overline{K}$ is separable. Similarly, vK^{1/p^∞} is the p-divisible hull of vK, so the former is equivalent to the condition that every torsion element in vF/vK has order prime to p. We have proved:

Lemma 6.65 *An extension $(F|K, v)$ is valuation separable if and only if*
1) the torsion subgroup of vF/vK is a p'-group,
2) $\overline{F}|\overline{K}$ is separable.

An extension $(F|K, v)$ will be called **valuation regular** if it is valuation disjoint from $(\tilde{K}|K, v)$ in $(\tilde{F}, \tilde{v})$ for some extension $\tilde{v}$ of the valuation v from F to $\tilde{F}$. As it was done for the notion "valuation separable", one deduces the following characterization for "valuation regular", using that $v\tilde{K}$ is the divisible hull of K and that $\overline{\tilde{K}} = \tilde{\overline{K}}$:

Lemma 6.66 *An extension $(F|K, v)$ is valuation regular if and only if*
1) vF/vK is torsionfree,
2) $\overline{F}|\overline{K}$ is regular.
Consequently, $(F|K, v)$ is valuation regular if and only if it is valuation disjoint from $(\tilde{K}|K, v)$ for every *extension $\tilde{v}$ of the valuation v from F to $\tilde{F}$.*

By this lemma, every extension of an algebraically closed valued field is valuation regular. For the scope of this book, the most important examples of valuation regular extensions are the valued field extensions which are generated by algebraically valuation independent sets. Indeed, it follows from Lemma 6.35 that they satisfy the conditions of the above lemma. Using also Lemma 6.63, we obtain:

Lemma 6.67 *Let $(\Omega|K,v)$ be an extension of valued fields containing a standard algebraically valuation independent set $\mathcal{T}$. Then $(K(\mathcal{T})|K,v)$ is a valuation regular extension. More generally, if $(L|K,v)$ is a subextension of $(\Omega|K,v)$ such that $\mathcal{T}$ remains algebraically valuation independent over (L,v), then $(K(\mathcal{T})|K,v)$ is valuation disjoint from $(L|K,v)$.*

Lemma 6.68 *Assume that $(F|K,v)$ is a valuation regular subextension of a valued field extension $(\Omega|K,v)$. If $\mathcal{T}$ is a standard algebraically valuation independent set in $(\Omega|F,v)$, then also $(F(\mathcal{T})|K(\mathcal{T}),v)$ and $(F(\mathcal{T})|K,v)$ are valuation regular extensions.*

Proof: We write $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\} \subset F$ such that the values $vx_i\,,\ i \in I$, are rationally independent over vF, and that the residues $\overline{y}_i\,,\ j \in J$, are algebraically independent over $\overline{F}$. Since $(F|K,v)$ is assumed to be valuation regular we know from Lemma 6.66 that vF/vK is torsion free and that $\overline{F}|\overline{K}$ is regular. The former implies that also $vF \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$ is torsion free modulo $vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$. Similarly, the latter implies by Corollary 24.51 that also the extension $\overline{F}(\overline{y}_j \mid j \in J) \,|\, \overline{K}(\overline{y}_j \mid j \in J)$ is regular. Again by Lemma 6.66, we conclude that $(F(\mathcal{T})|K(\mathcal{T}),v)$ is a valuation regular extension. In view of Lemma 6.67 and Lemma 6.64, the same now follows for the extension $(F(\mathcal{T})|K,v)$. $\square$