Chapter 7

Ramification theory

7.1 Basic definitions

Let $(L|K,v)$ be a normal algebraic extension of valued fields, not necessarily finite. We shall investigate three distinguished subgroups of the Galois group $\mathrm{Gal}\,L|K$. The subgroup

$$G^d(L|K,v) := \{\sigma \in \mathrm{Gal}\,L|K \mid v\sigma = v \text{ on } L\} \tag{7.1}$$

of $\mathrm{Gal}\,L|K$ is called the **decomposition group of** $(L|K,v)$. The condition "$v\sigma = v$ on L" means that $\forall x \in L : v\sigma x = vx$.

Remark 7.1 In the literature, one finds a definition of the decomposition group which appears to be different from ours. In the place of $v\sigma = v$, the condition is used that $v\sigma$ and v be equivalent, that is, that they have the same valuation ring. This holds if and only if there is an isomorphism ρ of vL onto $v\sigma L$ over vK such that $v\sigma = \rho \circ v$. Since we are dealing with algebraic extensions $L|K$, Lemma 6.15 shows that vL lies in the divisible hull of vK. But then, ρ can only be the identity.

From infinite Galois Theory (Section 24.4), we know that $\sigma \in \mathrm{Gal}\,L|K$ lies in the closure of $G^d(L|K,v)$ if and only if $\mathrm{res}_{L_i}(\sigma) \in \mathrm{res}_{L_i}(G^d(L|K,v))$ for every finite normal subextension $L_i|K$ of $L|K$. But then, $v\sigma = v$ on every L_i, and since L is the union over all L_i, this yields that $v\sigma = v$ on L, that is, $\sigma \in G^d(L|K,v)$. This proves that $G^d(L|K,v)$ is a closed subgroup of $\mathrm{Gal}\,L|K$.

Let $\mathcal{O}_{\mathbf{L}}$ and $\mathcal{M}_{\mathbf{L}}$ be the valuation ring and valuation ideal of $\mathbf{L} = (L,v)$. Then for every $\sigma \in G^d(L|K,v)$ we have $\sigma\mathcal{O}_{\mathbf{L}} = \mathcal{O}_{\mathbf{L}}$ and consequently also $\sigma\mathcal{M}_{\mathbf{L}} = \mathcal{M}_{\mathbf{L}}$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_{\mathbf{L}}/\mathcal{M}_{\mathbf{L}} = \overline{L}$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. We will call it the **reduction of** σ. Since σ fixes K, it follows that $\overline{\sigma}$ fixes $\overline{K}$. Moreover, the map

$$G^d(L|K,v) \ni \sigma \;\mapsto\; \overline{\sigma} \in \mathrm{Gal}\,\overline{L}|\overline{K} \tag{7.2}$$

is a group homomorphism. Note that we have written "$\mathrm{Gal}\,\overline{L}|\overline{K}$" since by virtue of Lemma 6.61, our general hypothesis that $L|K$ be normal yields that also $\overline{L}|\overline{K}$ is normal.

The homomorphism (7.2) is continuous. Indeed, we only have to show that for every open subgroup of $\mathrm{Gal}\,\overline{L}|\overline{K}$ there is an open subgroup of $G^d(L|K,v)$ that is mapped into it. Now an open subgroup of $\mathrm{Gal}\,\overline{L}|\overline{K}$ is of the form $\mathrm{Gal}\,\overline{L}|\overline{K}_1$ where $\overline{K}_1$ is a finite extension of $\overline{K}$. Let $a_1,\ldots,a_n \in L$ such that $\overline{a_1},\ldots,\overline{a_n} \in \overline{K}_1$ generate $\overline{K}_1$ over $\overline{K}$. Then $a_1,\ldots,a_n$ generate a finite extension K_2 of $(L|K,v)^d$ such that $\overline{K}_1 \subset \overline{K_2}$, and the open subgroup $\mathrm{Gal}\,L|K_2$ of $G^d(L|K,v)$ is mapped into $\mathrm{Gal}\,\overline{L}|\overline{K_2} \subset \mathrm{Gal}\,\overline{L}|\overline{K}_1$. Consequently, the kernel of the homomorphism (7.2) is a closed normal subgroup of $G^d(L|K,v)$; it is called the **inertia group of** $(L|K,v)$ and denoted by $G^i(L|K,v)$. We shall show that

$$\begin{aligned} G^i(L|K,v) &= \{\sigma\in \operatorname{Gal} L|K \mid \forall x\in \mathcal{O}_{\mathbf{L}} : \sigma x - x\in \mathcal{M}_{\mathbf{L}}\} \\ &= \{\sigma\in \operatorname{Gal} L|K \mid \forall x\in \mathcal{O}_{\mathbf{L}} : v(\sigma x - x) > 0\}\,. \end{aligned} \tag{7.3}$$

Let us abbreviate

$$G_{\mathrm{T}} := G^i(L|K,v) \quad\text{and}\quad G_{\mathrm{Z}} := G^d(L|K,v)$$

as long as we are working with our fixed extension $(L|K,v)$. Let $\sigma\in\operatorname{Gal} L|K$ such that $\sigma a - a\in\mathcal{M}_{\mathbf{L}}$ for all $a\in\mathcal{O}_{\mathbf{L}}$. Then $\sigma a\in\mathcal{O}_{\mathbf{L}}$ for every $a\in\mathcal{O}_{\mathbf{L}}$, because otherwise we would have $v(\sigma a-a)<0$. This gives $\sigma\mathcal{O}_{\mathbf{L}}\subset\mathcal{O}_{\mathbf{L}}$. Since both $\sigma\mathcal{O}_{\mathbf{L}}$, $\mathcal{O}_{\mathbf{L}}$ are valuation rings lying above $\mathcal{O}_{\mathbf{K}}$, Corollary 6.59 shows that both must be equal. Hence, $v\sigma = v$ which proves that $\sigma\in G_{\mathrm{Z}}$. Now for $\sigma\in G_{\mathrm{Z}}$, we find that $\overline{\sigma}$ is the identity if and only if $\overline{\sigma a}=\overline{a}$ for all $a\in\mathcal{O}_{\mathbf{L}}$. But $\overline{\sigma a}=\overline{a}$ is equivalent to $\sigma a - a\in\mathcal{M}_{\mathbf{L}}$ and to $v(\sigma a-a)>0$. This gives (7.3).

We will now consider a **pairing**, that is, a bilinear map

$$(\,.\,,.\,):\ L^\times\times G_{\mathrm{T}}\ \longrightarrow\ \overline{L}^\times \quad\text{which sends } a\in L^\times \text{ and } \sigma\in G_{\mathrm{T}} \text{ to}\quad (a,\sigma):=\overline{\left(\frac{\sigma a}{a}\right)} \tag{7.4}$$

where $L^\times$ and $\overline{L}^\times$ denote the multiplicative groups of L and $\overline{L}$. Note that $\sigma\in G_{\mathrm{T}}\subset G_{\mathrm{Z}}$ implies $v\sigma a = va$ and thus, $\sigma a\,/\,a\in\mathcal{O}_{\mathbf{L}}$ and $\overline{\sigma a\,/\,a}\neq 0$. For $\sigma\in G_{\mathrm{T}}$, every $a\in L$ with $va=0$ will satisfy $v(\sigma a-a)>0$ and hence, $v(\frac{\sigma a}{a}-1)>0$. This shows

$$(a,\sigma)=1 \quad\text{for all } a\in\mathcal{O}_{\mathbf{L}}^\times\,,\ \sigma\in G_{\mathrm{T}}\,. \tag{7.5}$$

For fixed $\sigma\in G_{\mathrm{T}}$, the map $(\,.\,,\sigma)$ is a homomorphism from $L^\times$ into $\overline{L}^\times$ since $\frac{\sigma ab}{ab}=\frac{\sigma a}{a}\frac{\sigma b}{b}$. For fixed $a\in L^\times$, the map $(a,\,.\,)$ is a homomorphism from G_{T} into $\overline{L}^\times$. To show this, let also $\tau\in G_{\mathrm{T}}$. Then $\frac{\sigma\tau a}{a}=\frac{\sigma\tau a}{\sigma a}\frac{\sigma a}{a}=\sigma\left(\frac{\tau a}{a}\right)\frac{\sigma a}{a}=\left(\sigma\left(\frac{\tau a}{a}\right)/\frac{\tau a}{a}\right)\frac{\sigma a}{a}\frac{\tau a}{a}$ showing that $(a,\sigma\tau)=(\frac{\tau a}{a},\sigma)(a,\sigma)(a,\tau)$. But $v\frac{\tau a}{a}=0$, so (7.5) shows that $(\frac{\tau a}{a},\sigma)=1$, and we have proved that $(a,\,.\,)$ is linear.

If G and H are groups, then $\operatorname{Hom}(G,H)$ denotes the set of all group homomorphisms from G to H. For $\varphi,\psi\in\operatorname{Hom}(G,H)$, define $\varphi\cdot\psi$ by $(\varphi\cdot\psi)(g)=\varphi(g)\psi(g)$ (using the operation in H) for all $g\in G$. Then $\operatorname{Hom}(G,H)$ is a group under this operation.

The bilinearity of the pairing can be restated as follows: the maps

$$G_{\mathrm{T}}\ \longrightarrow\ \operatorname{Hom}(L^\times,\overline{L}^\times)\qquad \sigma\ \mapsto\ (\,.\,,\sigma) \tag{7.6}$$

$$L^\times\ \longrightarrow\ \operatorname{Hom}(G_{\mathrm{T}},\overline{L}^\times)\qquad a\ \mapsto\ (a,\,.\,) \tag{7.7}$$

are group homomorphisms. The kernel of (7.6) is a normal subgroup of G_{T}, called the **ramification group of** $(L|K,v)$. It is denoted by $G^r(L|K,v)$, and for the present discussion we will abbreviate it by G_{V} . It consists of all $\sigma\in G_{\mathrm{T}}$ for which $(\,.\,,\sigma)=1$, or in other words, $\overline{\sigma a\,/\,a}=1$ for all $a\in L^\times$. But $\overline{\sigma a\,/\,a}=1$ is equivalent to $v(\frac{\sigma a}{a}-1)>0$, which is the same as $v(\sigma a-a)>va$. Note that if $\sigma\in\operatorname{Gal} L|K$ satisfies $v(\sigma a-a)>va$ and hence $v(\sigma a-a)>0$ for all $a\in\mathcal{O}_{\mathbf{L}}$, then $\sigma\in G_{\mathrm{T}}$. We have proved that

$$\begin{aligned} G^r(L|K,v) &= \{\sigma\in \operatorname{Gal} L|K \mid \forall x\in \mathcal{O}_{\mathbf{L}} : \tfrac{\sigma x}{x}-1\in \mathcal{M}_{\mathbf{L}}\} \\ &= \{\sigma\in \operatorname{Gal} L|K \mid \forall x\in \mathcal{O}_{\mathbf{L}} : v(\sigma x - x) > vx\}\,. \end{aligned} \tag{7.8}$$

Now let $\sigma \in G_{\rm V}$, $\tau \in G_{\rm Z}$ and $a \in L^\times$. Since $G_{\rm T} \lhd G_{\rm Z}$, we have $\tau\sigma\tau^{-1} \in G_{\rm T}$. Setting $b := \tau^{-1}a$, we compute: $\overline{\tau\sigma\tau^{-1}a\,/\,a} = \overline{\tau\sigma b\,/\,\tau b} = \overline{\tau}\,\overline{\sigma b\,/\,b} = \overline{\tau}\,1 = 1$. This shows that $\tau\sigma\tau^{-1} \in G_{\rm V}$, proving that $G_{\rm V}$ is a normal subgroup also of $G_{\rm Z}$. Further, $G_{\rm V}$ is closed in $\mathrm{Gal}\, L|K$; the proof of this fact is similar to that for the decomposition group.

We extend the homomorphism introduced in (7.6) to a crossed homomorphism from $G_{\rm Z}$ to $\mathrm{Hom}(L^\times, \overline{L}^\times)$. For the definition and basic properties of crossed homomorphisms, see Section 24.9.

For $\sigma \in G_{\rm Z}$ and $d \in L^\times$, we set

$$\chi_\sigma(d) \;:=\; \overline{\frac{\sigma(d)}{d}v}\,.$$

Since $\sigma \in G_{\rm Z}$, we know that $v\sigma(d) = vd$, and as above is seen that $chi_\sigma \in \mathrm{Hom}(L^\times, \overline{L}^\times)$. This group is a right $\mathrm{Gal}\, L|K$-module under the scalar multiplication

$$\chi^\rho \;:=\; \chi \circ \rho\,.$$

We have $\chi_{\sigma\tau}(d) = \frac{\sigma\tau(d)}{d} = \frac{\sigma\tau(d)}{\tau(d)}\frac{\tau(d)}{d} = (\chi_\sigma \circ \tau)(d) \cdot \chi_{\sigma\tau}(d)$. Thus,

$$\chi_{\sigma\tau} \;=\; \chi_\sigma^\tau \cdot \chi_\tau\,.$$

In other words, the map

$$G_{\rm Z} \ni \sigma \;\mapsto\; \chi_\sigma \in \mathrm{Hom}(L^\times, \overline{L}^\times) \tag{7.9}$$

is a crossed homomorphism. Hence, it is injective if and only if its kernel is trivial. This kernel consists of all $\sigma \in G_{\rm Z}$ for which $\overline{\sigma a\,/\,a} = 1$ for all $a \in L^\times$. By what we have shown above, this is $G_{\rm V}$.

We summarize our results in the following theorem.

Theorem 7.2 *Take any normal algebraic extension $(L|K, v)$ of valued fields.*

a) The decomposition group $G^d(L|K, v)$, defined in (7.1), is a closed subgroup of $\mathrm{Gal}\, L|K$.

b) The inertia group $G^i(L|K, v)$, defined as the kernel of the homomorphism (7.2), is a closed normal subgroup of $G^d(L|K, v)$, and (7.3) holds.

c) The ramification group $G^r(L|K, v)$, defined as the kernel of the homomorphism (7.6), is a closed normal subgroup of $G^d(L|K, v)$ and of $G^i(L|K, v)$, and (7.8) holds. It is also the kernel of the crossed homomorphism (7.9).

The fixed field of $G^d(L|K, v)$ in $K_s := (L|K)^{\rm sep}$ is called the **decomposition field of** $(L|K, v)$ and will be denoted by $(L|K, v)^d$ or by $(L|K)^{d(v)}$. Similarly, the fixed field of $G^i(L|K, v)$ in K_s is called the **inertia field of** $(L|K, v)$ and will be denoted by $(L|K, v)^i$ or by $(L|K)^{i(v)}$. Finally, the fixed field of $G^r(L|K, v)$ in K_s is called the **ramification field of** $(L|K, v)$ and will be denoted by $(L|K, v)^r$ or by $(L|K)^{r(v)}$. By definition, these fields are separable subextensions of $L|K$. Note that in contrast to the common use in the literature, we define the decomposition field to be the fixed field of the decomposition group in the maximal separable subextension $K_s|K$ of $L|K$, and similarly we do for the inertia field and the ramification field. This has the consequence that the ramification field is a separable extension of K and that all inseparability is shifted "to the top", that is, to the extension $L \,|\, (L|K, v)^d$ (cf. the table on page 185). This version has significant advantages for the formulation of certain facts (see for instance Lemma 11.22 and the definition of tame extensions in Section 13.1).

Since decomposition group, inertia group and ramification group are closed subgroups of $\mathrm{Gal}\, L|K$, Theorem 24.10 shows that they are equal to $\mathrm{Gal}\, L\,|\,(L|K,v)^d$, $\mathrm{Gal}\, L\,|\,(L|K,v)^i$ and $\mathrm{Gal}\, L\,|\,(L|K,v)^r$ respectively.

7.2 Functorial properties of the groups in ramification theory

We take over from J. Neukirch [NEU] the following description of the functorial properties of G^d, G^i and G^r. Let us assume that $(L'|K',v')$ is an arbitrary normal algebraic valued field extension and that τ is an embedding of (L,v) in (L',v') such that $\tau K \subset K'$. This embedding induces a homomorphism

$$\tau_* : \mathrm{Gal}\, L'|K' \to \mathrm{Gal}\, L|K\,, \quad \tau_*(\sigma') = \tau^{-1}\sigma'\tau\,.$$

Note that $\tau L|\tau K$ is normal since $L|K$ is; hence $\sigma'\tau L \subset \tau L$ for $\sigma' \in \mathrm{Gal}\, L'|K'$ in view of $\tau K \subset K'$. This shows that the expression $\tau^{-1}\sigma'\tau$ makes sense. Furthermore, τ_* is continuous and open since it is the composition of the continuous open restriction map $\mathrm{res}_{\tau L}$ with the topological isomorphism $\mathrm{Gal}\,\tau L|\tau K \to \mathrm{Gal}\, L|K$ which is induced by the isomorphism $\tau : L \to \tau L$.

Lemma 7.3 *The continuous homomorphism τ_* induces continuous homomorphisms*

$$\begin{array}{rcl} G^d(L'|K',v') & \longrightarrow & G^d(L|K,v) \\ G^i(L'|K',v') & \longrightarrow & G^i(L|K,v) \\ G^r(L'|K',v') & \longrightarrow & G^r(L|K,v)\,. \end{array}$$

Proof: By assumption, $v = v'\tau$ on L or equivalently, $v\tau^{-1} = v'$ on τL; we also have $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$. Let $\sigma' \in \mathrm{Gal}\, L'|K'$ and $\sigma = \tau_*(\sigma')$. If $\sigma' \in G^d(L'|K',v')$, then $v'\sigma' = v'$, hence

$$v\sigma = v\tau^{-1}\sigma'\tau = v'\sigma'\tau = v'\tau = v$$

on L, showing that $v \in G^d(L|K,v)$. If $\sigma' \in G^i(L'|K',v')$, then in view of $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$,

$$v(\sigma a - a) = v\tau^{-1}(\sigma'\tau a - \tau a) = v'(\sigma'\tau a - \tau a) > 0$$

for every $a \in \mathcal{O}_{\mathbf{L}}$, showing that $v \in G^i(L|K,v)$. If $\sigma' \in G^r(L'|K',v')$, then again in view of $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$,

$$v(\sigma a - a) = v\tau^{-1}(\sigma'\tau a - \tau a) = v'(\sigma'\tau a - \tau a) > v'\tau a = va\,,$$

for every $a \in \mathcal{O}_{\mathbf{L}}$, showing that $v \in G^r(L|K,v)$. □

If $\tau : L \to L'$ and $\tau : K \to K'$ are isomorphisms, then so are τ_* and the above homomorphisms induced by τ_*. For $L = L'$ and $K = K'$, this yields the following corollary. In view of the fact that all extensions of a valuation to an algebraic extension field are conjugate (Theorem 6.53), it gives information about the decomposition field, inertia field and ramification field with respect to the other extensions of v from K to L:

Corollary 7.4 *Let $\iota \in \mathrm{Gal}\, L|K$. Then*

$$\begin{array}{rclcrcl} G^d(L|K,v\iota) &=& \iota^{-1}\, G^d(L|K,v)\,\iota & \text{and} & (L|K,v\iota)^d &=& \iota^{-1}(L|K,v)^d \\ G^i(L|K,v\iota) &=& \iota^{-1}\, G^i(L|K,v)\,\iota & \text{and} & (L|K,v\iota)^i &=& \iota^{-1}(L|K,v)^i \\ G^r(L|K,v\iota) &=& \iota^{-1}\, G^r(L|K,v)\,\iota & \text{and} & (L|K,v\iota)^r &=& \iota^{-1}(L|K,v)^r\,. \end{array}$$

The left hand sides of our assertions follow directly from Lemma 7.3, where we set $v' = v\iota$ and $\tau = \iota^{-1}$. The right hand sides follow from the left by Theorem 24.10.

For our present discussion of the given extension $(L|K,v)$, let us abbreviate $\mathrm{Z} = (L|K,v)^d$, $\mathrm{T} = (L|K,v)^i$ and $\mathrm{V} = (L|K,v)^r$. So we are studying the following situation:

$$\left.\begin{array}{l} (L|K,v) \;\text{ a normal algebraic extension of valued fields with} \\ G_{\mathrm{Z}} \;\text{ its decomposition group and } (\mathrm{Z},v) \text{ its decomposition field,} \\ G_{\mathrm{T}} \;\text{ its inertia group and } (\mathrm{T},v) \text{ its inertia field,} \\ G_{\mathrm{V}} \;\text{ its ramification group and } (\mathrm{V},v) \text{ its ramification field,} \\ K_s|K \;\text{ the maximal separable subextension of } L|K, \\ p \;\text{ the characteristic exponent of the residue field } \overline{K}. \end{array}\right\} \tag{7.10}$$

Remark 7.5 The abbreviations Z, T, V are a reference to the german words “Zerlegungskörper”, “Trägheitskörper” and “Verzweigungskörper”.

Let us summarize what we have already shown for Z, T and V:

Lemma 7.6 *In the situation (7.10),*

$$G_{\mathrm{V}} \subset G_{\mathrm{T}} \subset G_{\mathrm{Z}} \qquad \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z}$$

where the groups are the Galois groups in L of the respective fields. Moreover, the inertia group G_{T} and the ramification group G_{V} are normal subgroups of the decomposition group G_{Z} and thus, the inertia field T *and the ramification field* V *are Galois extensions of the decomposition field* Z.

Assume that $E|K$ is any subextension of $L|K$. With $L' = L$, $K' = E$ and $\tau = \mathrm{id}_L$, this constitutes another special case of Lemma 7.3; in this case, τ_* is just the inclusion of $\mathrm{Gal}\, L|E$ in $\mathrm{Gal}\, L|K$.

Lemma 7.7 *Assume (7.10). If $E|K$ is any arbitrary subextension of $L|K$, then*

$$\begin{array}{rclcrcl} G^d(L|E,v) &=& G_{\mathrm{Z}} \cap \mathrm{Gal}\, L|E & \text{and} & (L|E,v)^d &=& (E.\mathrm{Z},v) \\ G^i(L|E,v) &=& G_{\mathrm{T}} \cap \mathrm{Gal}\, L|E & \text{and} & (L|E,v)^i &=& (E.\mathrm{T},v) \\ G^r(L|E,v) &=& G_{\mathrm{V}} \cap \mathrm{Gal}\, L|E & \text{and} & (L|E,v)^r &=& (E.\mathrm{V},v)\,. \end{array}$$

Proof: The inclusions “$\subset$” on the left hand side follow directly from Lemma 7.3. But actually, we do not need to employ this lemma, since already the equalities follow immediately from (7.1), (7.3) and (7.8).

If $L|K$ is Galois, then the right hand side follows from the left hand side by (Gal2′). If $L|K$ is not separable, then we have to proceed as follows. In that case, we have to take the fixed fields in $(L|E)^{\mathrm{sep}} = E.K_s$. Since $L|E.K_s$ is purely inseparable, we may view all subgroups of $\mathrm{Gal}\, L|K$ as subgroups of $\mathrm{Gal}\, E.K_s|K$. The extension $E.K_s|E$ is separable. Hence, the right hand side now follows from Lemma 24.39, applied to the extension $E.K_s|K$ with $F = \mathrm{Z}$, $F = \mathrm{T}$ and $F = \mathrm{V}$ respectively. □

A further special case of Lemma 7.3 is given when $E|K$ is a normal subextension of $L|K$. In the lemma, we have to replace L by $E_s = (E|K)^{\rm sep} = E \cap K_s$ and L' by K_s, and we set $K' = K$ and $\tau = {\rm id}_{E_s}$. In this case, τ_* is the restriction ${\rm res}_{E_s} : {\rm Gal}\, K_s|K \rightarrow {\rm Gal}\, E_s|K$. Using Lemma 7.3 and (Gal7), we obtain:

Corollary 7.8 *Assume (7.10). If $E|K$ is a normal subextension of $L|K$, then*

$$\begin{array}{lclcrcl} {\rm res}_E(G_{\rm Z}) & \subset & G^d(E|K,v) & \textit{and} & (E\cap {\rm Z},v) & \supset & (E|K,v)^d \\ {\rm res}_E(G_{\rm T}) & \subset & G^i(E|K,v) & \textit{and} & (E\cap {\rm T},v) & \supset & (E|K,v)^i \\ {\rm res}_E(G_{\rm V}) & \subset & G^r(E|K,v) & \textit{and} & (E\cap {\rm V},v) & \supset & (E|K,v)^r \,. \end{array}$$

The inclusions are in fact equalities, as we will show later. Note that the fixed fields of the restricted groups are $E_s \cap {\rm Z}$, $E_s \cap {\rm T}$, $E_s \cap {\rm V}$ respectively, by virtue of (Gal7). But since the extensions ${\rm Z}|K$, ${\rm T}|K$, ${\rm V}|K$ are separable, these intersections are equal to $E\cap{\rm Z}$, $E\cap{\rm T}$, $E \cap {\rm V}$ respectively.

7.3 The decomposition field

Let us now study the properties of the decomposition field.

Lemma 7.9 *Assume (7.10). Then:*

a) The extension of v from ${\rm Z}$ to L is unique.

b) For $\sigma, \tau \in {\rm Gal}\, L|K$, the following assertions are equivalent:
1) $v\sigma = v\tau$ on $\tau^{-1}{\rm Z}$,
2) $v\sigma = v\tau$ on L,
3) $\sigma\tau^{-1} \in G_{\rm Z}$.

c) If the automorphism $\sigma \in {\rm Gal}\, L|K$ does not fix ${\rm Z}$, then $v\sigma \neq v$ on ${\rm Z}$.

d) If $E|K$ is a subextension of $L|K$, then the extension of v from E to L is unique if and only if ${\rm Z} \subset E$. In particular, the extension of v from K to L is unique if and only if ${\rm Z} = K$, that is, if and only if $G_{\rm Z} = {\rm Gal}\, L|K$.

e) If ${\rm Z}|K$ is finite, then the number $g(L|K,v)$ of distinct extensions of v from K to L is finite and equal to $({\rm Gal}\, L|K : G_{\rm Z}) = [{\rm Z} : K]$.

f) For every $\iota \in {\rm Gal}\, L|K$, the restriction ${\rm res}_{\rm Z}(\iota^{-1})$ is the unique isomorphism over K sending $({\rm Z}, v)$ onto $(L|K, v\iota)^d$.

Proof: a): From Theorem 6.53 we know that every two extensions of v from ${\rm Z}$ to L are conjugate. But by definition, every automorphism of $G_{\rm Z} = {\rm Gal}\, L|{\rm Z}$ fixes v.

b): If $\sigma, \tau \in {\rm Gal}\, L|K$ such that $v\sigma = v\tau$ on $\tau^{-1}{\rm Z}$ and hence $v\sigma\tau^{-1} = v$ on ${\rm Z}$, then also $v\sigma\tau^{-1} = v$ on L by virtue of part a). Since $v\sigma\tau^{-1} = v$ on L if and only if $v\sigma = v\tau$ on L, this proves 1)⇒2). The converse is trivial. By definition of $G_{\rm Z}$, $v\sigma\tau^{-1} = v$ holds on L if and only if $\sigma\tau^{-1} \in G_{\rm Z}$. This proves the equivalence of 2) and 3).

c): This is the implication 1)⇒3) of part b) with $\tau = {\rm id}_L$.

d): Every extension of v from E to L is an extension of v from K to L. Hence if ${\rm Z} \subset E$, then by part a), v admits a unique extension from E to L. For the conversely, assume that ${\rm Z} \not\subset E$. Then there is some $a \in {\rm Z} \setminus E$. Since ${\rm Z}|K$ is separable, a is separable algebraic over

E. Thus, there is some $\sigma \in \mathrm{Gal}\, L|E$ which moves a. Hence $\sigma \notin G_{\mathrm{Z}}$, so the implication 2)⇒3) of part b) shows that $v \neq v\sigma$ on L. On the other hand, $v = v\sigma$ on E. This shows that v and $v\sigma$ are two distinct extensions of v from E to L.

e): By the equivalence 2)⇔3) of part b), $g(L|K,v)$ is equal to the number of cosets of $\mathrm{Gal}\, L|K$ modulo $G_{\mathrm{Z}} = \mathrm{Gal}\, L|\mathrm{Z}$. This in turn is equal to $[\mathrm{Z} : K]$.

f): It follows from Corollary 7.4 that the restriction of ι^{-1} is the required isomorphism. If there would be a second isomorphism, say σ^{-1}, then $v\sigma = v\iota$ on $\iota^{-1}\mathrm{Z}$, so by the implication 1)⇒3) of part b), ι^{-1} and σ^{-1} must coincide on Z. □

Lemma 7.10 *Assume (7.10) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\mathrm{res}_E(G_{\mathrm{Z}}) = G^d(E|K,v) \quad \textit{and} \quad (E \cap \mathrm{Z}, v) = (E|K,v)^d .$$

Proof: In view of Corollary 7.8, we have to show that $\mathrm{res}_E : G_{\mathrm{Z}} \to G^d(E|K,v)$ is surjective. Let $\rho \in G^d(E|K,v)$ and $\sigma \in \mathrm{Gal}\, L|K$ such that $\rho = \mathrm{res}_E(\sigma)$. By assumption on ρ, we have $v = v\rho = v\sigma$ on E. Hence, by Theorem 6.53 there is some $\tau \in \mathrm{Gal}\, L|E$ such that $v\tau = v\sigma$ on L. That is, $\sigma\tau^{-1} \in G_{\mathrm{Z}}$ with $\rho = \mathrm{res}_E(\sigma\tau^{-1})$. □

The reader may prove that the number $g(L|K,v)$ of distinct extensions of v from K to L is multiplicative, for arbitrary finite extensions:

Lemma 7.11 *Let $(L|K,v)$ be a finite extension of valued fields and $E|K$ a subextension of $L|K$. Then $g(L|K,v) = g(L|E,v) \cdot g(E|K,v)$.*

We the help of Theorem 7.9, we prove:

Lemma 7.12 *Assume (7.10). Then $(\mathrm{Z}|K,v)$ is an immediate extension.*

Proof: Assume first that $L|K$ is finite. Then also $\mathrm{Z}|K$ is finite. Let $\zeta \neq 0$ be an element of the residue field of Z. By Lemma 6.60, there exists some $c \in \mathrm{Z}$ such that $\bar{c} = \zeta$ and $v'c > 0$ for all extensions $v' \neq v$ of v from K to Z. By Theorem 7.9 we know that $v\sigma \neq v$ on Z for every $\sigma \in \mathrm{Gal}\, L|K \setminus G_{\mathrm{Z}}$. Hence for all conjugates $\sigma c \neq c$ we have $v\sigma c > 0$. Now $\mathrm{Tr}_{\mathrm{Z}|K}(c) = c + \sum_{\sigma \in H} \sigma c$ where $H \subset \mathrm{Gal}\, L|K$ is a set of representatives of the cosets $\mathrm{Gal}\, L|K$ modulo G_{Z} which are different from G_{Z}. Hence $v\sigma c > 0$ for all $\sigma \in H$. Consequently, $\zeta = \bar{c} = \overline{\mathrm{Tr}_{\mathrm{Z}|K}(c)} \in \overline{K}$. We have proved that $\overline{\mathrm{Z}} = \overline{K}$.

Now let α be an element of the value group of Z. Choose $b \in \mathrm{Z}$ with $vb = \alpha$. Let c and H be as before, that is, $vc = 0$ and $v\sigma c > 0$ for all $\sigma \in H$. We choose some $m \in \mathbb{N}$ such that $vc^m b = vb \neq m \cdot v\sigma c + v\sigma b = v\sigma c^m b$ for all $\sigma \in H$. Set $a := c^m b$ and observe that $va = \alpha$. The conjugates of a over K are precisely the roots of the minimal polynomial f of a over K. By construction of a, it is the only root of value α. Thus, an application of part d) of Lemma 5.6 shows that $\alpha = va \in vK$. We have proved that $v\mathrm{Z} = vK$. This concludes the proof of our assertion for the case of finite $L|K$.

In the case of an infinite extension $L|K$, it suffices to show that every finite subextension $(\mathrm{Z}_1|K,v)$ of $(\mathrm{Z}|K,v)$ is immediate. Let L_1 be the normal hull of $\mathrm{Z}_1|K$; it lies in L since $L|K$ is assumed to be normal. Moreover, $L_1|K$ is finite. By the previous lemma, $(L_1|K,v)^d = L_1 \cap (L|K,v)^d \supset \mathrm{Z}_1$. By what we have already shown, $(L_1|K,v)^d$ is an immediate extension of (K,v), and the same is consequently true for (Z_1, v). □

7.4 The inertia field

Next, we investigate the extension $\mathrm{T}|\mathrm{Z}$.

Theorem 7.13 *Assume (7.10). The value group of the inertia field* T *is equal to* $v\mathrm{Z} = vK$, *and its residue field is equal to the relative separable-algebraic closure of* $\overline{K}$ *in* $\overline{L}$ *(and hence a Galois extension of* $\overline{K}$*). The homomorphism (7.2) (which sends* σ *to* $\overline{\sigma}$*) is onto and consequently,* $\mathrm{T}|\mathrm{Z}$ *is a Galois extension with Galois group*

$$\operatorname{Gal}\mathrm{T}|\mathrm{Z} \;\cong\; G_{\mathrm{Z}}\,/\,G_{\mathrm{T}} \;\cong\; \operatorname{Gal}\overline{\mathrm{T}}|\overline{\mathrm{Z}} \;\cong\; \operatorname{Gal}\overline{L}|\overline{K}$$

(these are topological isomorphisms).

For every subextension $K'|\mathrm{Z}$ *of* $\mathrm{T}|\mathrm{Z}$*, the image of* $\operatorname{Gal}\mathrm{T}|K'$ *under this isomorphism is equal to* $\operatorname{Gal}\overline{\mathrm{T}}|\overline{K'}$*. Hence if* $K'|\mathrm{Z}$ *is normal, the isomorphism induces an isomorphism of* $\operatorname{Gal}K'|\mathrm{Z}$ *onto* $\operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}$*. Moreover,* $[K_1 : K_0] = [\overline{K_1} : \overline{K_0}]$ *for every finite extension* $(K_1|K_0, v)$ *such that* $\mathrm{Z} \subset K_0 \subset K_1 \subset \mathrm{T}$.

Proof: By Lemma 7.12, $\overline{\mathrm{Z}} = \overline{K}$. Since $L|\mathrm{Z}$ is a Galois extension, Lemma 6.61 shows that $\overline{L}|\overline{\mathrm{Z}}$ is a normal extension. Now let $k|\overline{\mathrm{Z}}$ be a finite Galois subextension of $\overline{L}|\overline{\mathrm{Z}}$ and let ζ be a primitive element of it. Let $\tau \in \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}}$; its restriction to k is uniquely determined by the conjugate $\tau\zeta$. From Lemma 6.61 we know that there is some $a \in L$ and an automorphism $\sigma \in \operatorname{Gal}L|\mathrm{Z}$ such that $\overline{a} = \zeta$ and $\overline{\sigma a} = \tau\zeta$. That is, $\overline{\sigma}$ coincides with τ on k.

If $L|K$ is finite, then also $\overline{L}|\overline{\mathrm{Z}}$ is finite by virtue of the fundamental inequality (6.2), and we may choose $k = (\overline{L}|\overline{\mathrm{Z}})^{\mathrm{sep}}$; then our argument shows that (7.2) is surjective. For the infinite case, we proceed as follows. Let $\tau \in \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}}$ be given. We know that the maximal separable subextension of $\overline{L}|\overline{\mathrm{Z}}$ is the union of finite Galois subextensions $k_i|\overline{\mathrm{Z}}$, $i \in I$, and we have shown that for every $i \in I$, the restriction τ_i of τ to k_i coincides with that of $\overline{\sigma_i}$ for some $\sigma_i \in \operatorname{Gal}L|\mathrm{Z}$. From the Compactness Principle for Algebraic Extensions (Lemma 24.5), it now follows that there is $\sigma \in \operatorname{Gal}L|\mathrm{Z} = G_{\mathrm{Z}}$ such that $\overline{\sigma} = \tau$. This proves the surjectivity of (7.2) in the general case. Since G_{T} is defined to be the kernel of the continuous homomorphism (7.2), we thus obtain a topological isomorphism $G_{\mathrm{Z}}/G_{\mathrm{T}} \cong \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}} = \operatorname{Gal}\overline{L}|\overline{K}$. The topological isomorphism $\operatorname{Gal}\mathrm{T}|\mathrm{Z} \cong G_{\mathrm{Z}}\,/\,G_{\mathrm{T}}$ is inferred from infinite Galois theory.

Now let $K'|\mathrm{Z}$ be any subextension of $\mathrm{T}|\mathrm{Z}$. Then by Lemma 7.7, K' is the decomposition field and T is the inertia field of the normal extension $(L|K', v)$. What we have just proved may be applied to K' in the place of K, showing that the restriction of (7.2) to $\operatorname{Gal}L|K'$ induces an isomorphism from $\operatorname{Gal}L|K'\,/\,G_{\mathrm{T}} \cong \operatorname{Gal}\mathrm{T}|K'$ onto $\operatorname{Gal}\overline{L}|\overline{K'}$. On the one hand, this result yields that if $K'|\mathrm{Z}$ is normal, then the isomorphism induces an isomorphism of $\operatorname{Gal}\mathrm{T}|\mathrm{Z}\,/\operatorname{Gal}\mathrm{T}|K' \cong \operatorname{Gal}K'|\mathrm{Z}$ onto $\operatorname{Gal}\overline{L}|\overline{\mathrm{Z}}\,/\operatorname{Gal}\overline{L}|\overline{K'} \cong \operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}$. On the other hand, we apply this result to $K' = \mathrm{T}$ to find that $\operatorname{Gal}\overline{L}|\overline{\mathrm{T}}$ is trivial. That is, $\overline{L}|\overline{\mathrm{T}}$ must be a purely inseparable extension.

Let $K'|\mathrm{Z}$ be a finite normal subextension of the normal extension $\mathrm{T}|\mathrm{Z}$. We have already shown that $\operatorname{Gal}K'|\mathrm{Z} \cong \operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}$. Since $\mathrm{T}|\mathrm{Z}$ and thus also $K'|\mathrm{Z}$ is separable, we obtain $[K' : \mathrm{Z}] = |\operatorname{Gal}K'|\mathrm{Z}\,| = |\operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}\,| \leq [\overline{K'} : \overline{\mathrm{Z}}] \leq [K' : \mathrm{Z}]$, where the last inequality follows from the fundamental inequality (6.2). Thus, equality holds everywhere, showing that $[K' : \mathrm{Z}] = [\overline{K'} : \overline{\mathrm{Z}}]$ and that $\overline{K'}|\overline{\mathrm{Z}}$ is Galois. Moreover, from the fundamental inequality we infer that $vK' = v\mathrm{Z}$, which in view of Lemma 7.12 tells us that $vK' = vK$. Since every value of $v\mathrm{T}$ is already contained in the value group of a finite normal subextension $K'|\mathrm{Z}$, we find that $v\mathrm{T} = vK$. Similarly, since every element of $\overline{\mathrm{T}}$ is already contained in the residue field of a finite normal subextension $K'|\mathrm{Z}$, we see that $\overline{\mathrm{T}}|\overline{K}$ is separable. Since $\overline{L}|\overline{\mathrm{T}}$ is purely inseparable, we may conclude that $\overline{\mathrm{T}}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. From this, we also obtain the topological isomorphism of $\mathrm{Gal}\,\overline{L}|\overline{K} = \mathrm{Gal}\,\overline{L}|\overline{\mathrm{Z}}$ with $\mathrm{Gal}\,\overline{\mathrm{T}}|\overline{\mathrm{Z}}$.

Finally, let $\mathrm{Z} \subset K_0 \subset K_1 \subset \mathrm{T}$ with $K_1|K_0$ finite. We choose a finite normal subextension $K'|K_0$ of $\mathrm{T}|K_0$ such that $K_1 \subset K'$. From Lemma 7.7 we know that the decomposition field of $(L|K_0, v)$ is $K_0.\mathrm{Z} = K_0$ and that its inertia field is $K_0.\mathrm{T} = \mathrm{T}$, and similarly for K_1 in the place of K_0. What we have already shown may thus be applied to K_0 and K_1 in the place of Z to obtain that $[K' : K_0] = [\overline{K'} : \overline{K_0}]$ and $[K' : K_1] = [\overline{K'} : \overline{K_1}]$. This proves that $[K_1 : K_0] = [\overline{K_1} : \overline{K_0}]$. □

Lemma 7.14 *Assume (7.10) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\mathrm{res}_E(G_\mathrm{T}) = G^i(E|K, v) \quad \textit{and} \quad (E \cap \mathrm{T}, v) = (E|K, v)^i .$$

Proof: In view of Corollary 7.8, we have to show that $\mathrm{res}_E : G_\mathrm{T} \to G^i(E|K, v)$ is surjective. Let $\rho \in G^i(E|K, v)$, which means that $\overline{\rho}$ is the identity on $\overline{E}$. It also implies that $\rho \in G^d(E|K, v)$, hence by Lemma 7.10 there exists $\sigma \in G_\mathrm{Z}$ such that $\rho = \mathrm{res}_E(\sigma)$. We see that $\overline{\sigma}$ fixes $\overline{E}$, so the surjectivity of (7.2) proved in Theorem 7.13 (applied to the extension $(L|E, v)$) shows that there is some $\tau \in G^d(L|E, v)$ such that $\overline{\tau} = \overline{\sigma}$. Consequently, $\sigma\tau^{-1} \in G_\mathrm{T}$ with $\rho = \mathrm{res}_E(\sigma\tau^{-1})$. □

7.5 The ramification field

Proceeding not too systematically may sometimes turn out to be of advantage. In this spirit, let us jump to the extension $L|\mathrm{V}$. The next theorem will describe its main properties. Beforehand, we need a lemma.

Lemma 7.15 *Let $(K_1|K_0, v)$ be an algebraic extension of valued fields. If $K_1|K_0$ is a p-extension, then vK_1/vK_0 is a p-group. The same is true if $K_1|K_0$ is purely inseparable.*

Proof: Assume that $K_1|K_0$ is a p-extension. Let Z_0 be the decomposition field of $(K_1|K_0, v)$. From Lemma 7.12 we know that $vK_0 = v\mathrm{Z}_0$. Since $K_1|K_0$ is a p-extension, the same is true for $K_1|\mathrm{Z}_0$. Hence, it suffices to show our assertion for the extension $K_1|\mathrm{Z}_0$.

Let $a \in K_1$. Since $\mathrm{Gal}\,K_1|\mathrm{Z}_0$ is a pro-p-group, the number of conjugates of a over Z_0 is a power of p, say p^n. Since the extension of v from Z_0 to K_1 is unique, all conjugates have the same value. The norm $\mathrm{N}_{\mathrm{Z}_0(a)|\mathrm{Z}_0}(a) \in \mathrm{Z}_0$, being the product of these conjugates, thus has value $p^n va$. This shows that $p^n va \in v\mathrm{Z}_0$. We have proved that $vK_1/v\mathrm{Z}_0 = vK_1/vK_0$ is a p-group.

Now assume that $K_1|K_0$ is purely inseparable. Then for every $a \in K_1$ there is some $n \in \mathbb{N}$ such that $a^{p^n} \in K_0$ and thus again, $p^n va \in vK_0$. As before we find that vK_1/vK_0 is a p-group. □

Theorem 7.16 *Assume (7.10). The ramification group* G_{V} *is a pro-p-group. Hence* $K_s|\mathrm{V}$ *is a p-extension, and* $[L:\mathrm{V}]$ *is a (possibly infinite) power of p. In particular,* $G_{\mathrm{V}}=1$ *and* $\mathrm{V}=L$ *if* $\mathrm{char}\,\overline{K}=0$.

The factor group $vL/v\mathrm{V}$ *is a p-group, and the residue field extension* $\overline{L}|\overline{\mathrm{V}}$ *is purely inseparable.*

Proof: Assume that G_{V} is not a pro-p-group, that is, that there exists a prime $q\neq p$ and a finite normal subextension $E|V$ of $K_s|V$ whose Galois group contains an element σ of order q. Let K_0 be the fixed field in E of the cyclic group generated by σ. Then $E|K_0$ is a Galois extension of degree q. Let a be a primitive element for $E|K_0$ and $f=X^q+c_{q-1}X^{q-1}+\ldots+c_0$ its minimal polynomial over K_0. Note that $-c_{q-1}$ is equal to the trace $\mathrm{Tr}_{E|K_0}(a)$. Replacing a by $a+c_{q-1}/q$ (which is possible since $q\neq 0$ in E), we may assume from the start that this trace of a is zero. On the other hand, let $\sigma_i a$ be all conjugates of a, with a suitable choice of $\sigma_1,\ldots,\sigma_q\in\mathrm{Gal}\,K_s|K_0\subset G_{\mathrm{V}}$. Then $\overline{\sigma_i a/a}=1$ for all i, and the element $0=a^{-1}\mathrm{Tr}_{E|K_0}(a)=\sum_{1\leq i\leq q}\sigma_i a/a$ has residue q, but $q\neq p$ is not zero in $\overline{K_1}$. This contradiction shows that G_{V} must be a pro-p-group. Since it is the Galois group of the separable extension $K_s|\mathrm{V}$, this extension is a p-extension. On the other hand, $L|K_s$ is purely inseparable by definition of K_s, so both degrees $[L:K_s]$ and $[K_s:\mathrm{V}]$ and consequently also $[L:\mathrm{V}]$ are powers of p. (Recall that $\mathrm{char}\,K=\mathrm{char}\,\overline{K}=p$ if $\mathrm{char}\,K\neq 0$.)

From the preceding lemma it now follows that vL/vK_s and $vK_s/v\mathrm{V}$ are p-groups. Hence also $vL/v\mathrm{V}$ is a p-group. Finally, the residue field extension $\overline{L}|\overline{\mathrm{V}}$ must be purely inseparable since it was asserted by Theorem 7.13 that $\overline{\mathrm{T}}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. □

In view of Corollary 24.29 and Corollary 24.56, the foregoing theorem yields:

Corollary 7.17 *Assume (7.10) with* $p>1$ *and let* $\mathrm{V}\subset K_0\subset K_1\subset L$. *Then* $K_1|K_0$ *is a tower of normal extensions of degree p, the separable ones being Artin-Schreier extensions.*

We will now examine the extension $\mathrm{V}|\mathrm{T}$. To this end, we return to our pairing (7.4) and ask for the kernel of the homomorphism (7.7). It contains $\mathrm{T}^\times$ since for all $a\in\mathrm{T}$ and all $\sigma\in G_{\mathrm{T}}$ we have $\sigma a=a$ and consequently, $(a,\sigma)=1$. By (7.5), the kernel also contains $\mathcal{O}_{\mathbf{L}}^\times$. It follows that every element $a\in L$ with $va\in vK$ lies in the kernel since it can be written as $a=bc$ with $b\in K$, $vb=va$, so that $c\in\mathcal{O}_{\mathbf{L}}^\times$.

Now we see that the value of (a,σ) only depends on the coset of va modulo vK and on the coset of σ modulo G_{V} (since the latter was defined to be the kernel of (7.6)). But $G_{\mathrm{T}}/G_{\mathrm{V}}\cong\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. So the pairing (7.4) is in fact a pairing

$$(.\,,.)':\,vL/vK\times\mathrm{Gal}\,\mathrm{V}|\mathrm{T}\;\longrightarrow\;\overline{L}^\times \tag{7.11}$$

between the additive group vL/vK and the Galois group $\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. For $\overline{\alpha}:=\alpha+vK\in vL/vK$ and $\sigma\in\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$, it satisfies

$$(\overline{\alpha},\sigma)'=(a,\sigma_L) \tag{7.12}$$

where $a\in L$ is an arbitrary element such that $va=\alpha$, and $\sigma_L\in G_{\mathrm{T}}$ is an arbitrary automorphism such that $\mathrm{res}_{\mathrm{V}}(\sigma_L)=\sigma$. If $\sigma\in G_{\mathrm{T}}$, then we will simply write "$(\overline{\alpha},\sigma)'$" instead of "$(\overline{\alpha},\mathrm{res}_{\mathrm{V}}(\sigma))'$". Since every element in vL/vK has finite order, the same is true for every element $(\overline{\alpha},\sigma)'\in\overline{L}^{\times}$. That is, the range of the pairing $(\,.\,,.)'$ (which is equal to that of $(\,.\,,.)$) lies in the subgroup of all torsion elements in $\overline{L}^{\times}$. This is in fact the subgroup of all roots of unity in $\overline{L}$. It is an abelian torsion group.

Since char $\overline{L}=p$, the group of roots of unity in $\overline{L}$ is a p'-group. Consequently, $(\overline{\alpha},\sigma)'=1$ for arbitrary $\sigma\in\operatorname{Gal}\mathrm{V}|\mathrm{T}$ if $\alpha\in vL$ is an element whose order modulo vK is a power of p. We write $vL/vK=(vL/vK)_p\oplus(vL/vK)_{p'}$ where $(vL/vK)_p$ is an abelian p-group and $(vL/vK)_{p'}$ is an abelian p'-group. We find that $((vL/vK)_p\,,\operatorname{Gal}\mathrm{V}|\mathrm{T})'=\{1\}$, so the above pairing can be rewritten as

$$(vL/vK)_{p'}\times\operatorname{Gal}\mathrm{V}|\mathrm{T}\;\longrightarrow\;\overline{L}^{\times}\;.$$

Given any extension $\Delta\subset\Delta'$ of abelian groups, the **(relative) p'-divisible closure of Δ in Δ'** is defined to be the subgroup $\{\alpha\in\Delta'\mid\exists n:\ (p,n)=1\wedge n\alpha\in\Delta\}$ of all elements in Δ' whose order modulo Δ is prime to p. It is equal to the preimage of $(\Delta'/\Delta)_{p'}$ under the canonical epimorphism $\Delta'\to\Delta'/\Delta$. Since $vL/v\mathrm{V}$ is a p-group by virtue of Lemma 7.16, the p'-divisible closure of vK in vL lies already in $v\mathrm{V}$. Hence, $(vL/vK)_{p'}=(v\mathrm{V}/vK)_{p'}$. In view of Theorem 7.13 we may also replace vK by $v\mathrm{T}$. Furthermore, observe that we may replace $\overline{L}^{\times}$ by $\overline{\mathrm{V}}^{\times}$. Indeed, since the group of all roots of unity in a field of characteristic exponent p is a p'-group, it remains unchanged under purely inseparable extensions. On the other hand, Theorem 7.16 shows that $\overline{L}|\overline{\mathrm{V}}$ is purely inseparable. So the groups of roots of unity in $\overline{L}$ and in $\overline{\mathrm{V}}$ are equal, and the above pairing can be rewritten as

$$(v\mathrm{V}/v\mathrm{T})_{p'}\times\operatorname{Gal}\mathrm{V}|\mathrm{T}\;\longrightarrow\;\overline{\mathrm{V}}^{\times}\;. \tag{7.13}$$

The homomorphism (7.6) with kernel G_{V} turns into an embedding

$$\operatorname{Gal}\mathrm{V}|\mathrm{T}\;\longrightarrow\;\operatorname{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right) \tag{7.14}$$

of $\operatorname{Gal}\mathrm{V}|\mathrm{T}$ in the p-character group $\operatorname{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)$. Since the latter is an abelian group, this shows that also the Galois group $\operatorname{Gal}\mathrm{V}|\mathrm{T}$ is abelian.

Now let H be any pro-p-Sylow group of G_{T} containing the pro-p-subgroup G_{V}, and let L_0 be its fixed field in K_s. Then $\mathrm{T}\subset L_0\subset\mathrm{V}$, and $V|L_0$ is a p-extension. From Lemma 7.7 we infer that L_0 itself is the inertia field of $(L|L_0,v)$ and that V is its ramification field. Hence, in (7.14) we can just replace T by L_0 to obtain an embedding

$$H/G_{\mathrm{V}}\cong\operatorname{Gal}\mathrm{V}|L_0\;\longrightarrow\;\operatorname{Hom}\left((v\mathrm{V}/vL_0)_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)\;.$$

But Lemma 7.15 shows that $(v\mathrm{V}/vL_0)_{p'}=0$ since $V|L_0$ is a p-extension. Consequently, $\operatorname{Hom}\left((v\mathrm{V}/vL_0)_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)$ and thus also H/G_{V} are trivial, showing that $H=G_{\mathrm{V}}$. Hence, G_{V} is a pro-p-Sylow group in G_{T}; it is the only one since it is a normal subgroup of G_{T}. We have proved:

Lemma 7.18 *Assume (7.10). The ramification group G_{V} is the unique pro-p-Sylow group in the inertia group G_{T}, and* $\operatorname{Gal}\mathrm{V}|\mathrm{T}\cong G_{\mathrm{T}}/G_{\mathrm{V}}$ *is an abelian pro-p'-group.*

Now we can state the main properties of the extension V|T:

Theorem 7.19 *Assume (7.10). Then* $\mathrm{V}|\mathrm{T}$ *is an abelian* p'*-extension. The value group of the ramification field* V *is the* p'*-divisible closure of* vK *in* vL*, that is,* $v\mathrm{V}/vK = (vL/vK)_{p'}$ *. Its residue field is equal to the residue field of the inertia field* T*. The homomorphism (7.14) is onto and consequently, the Galois group of* V *over* T *is*

$$G_{\mathrm{T}}\,/\,G_{\mathrm{V}} \;\cong\; \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\cong\; \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}\right)\;.$$

This is isomorphic to $v\mathrm{V}/v\mathrm{T}$ *if* $\mathrm{V}|\mathrm{T}$ *is finite. The isomorphisms are topological isomorphisms, and* $\mathrm{Hom}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times})$ *is the full* p*-character group of* $v\mathrm{V}/v\mathrm{T}$*.*

For every subextension $K'|\mathrm{T}$ *of* $\mathrm{V}|\mathrm{T}$*, the image of* $\mathrm{Gal}\,\mathrm{V}|K'$ *under this isomorphism is equal to* $\mathrm{Hom}_{vK'/v\mathrm{T}}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}) \cong \mathrm{Hom}(v\mathrm{V}/vK'\,,\,\overline{\mathrm{T}}^{\times})$*. If* $K'|\mathrm{T}$ *is normal, then this isomorphism induces an isomorphism of* $\mathrm{Gal}\,K'|\mathrm{T}$ *onto* $\mathrm{Hom}(vK'/v\mathrm{T}\,,\,\overline{\mathrm{V}}^{\times})$*. If* $K'|\mathrm{T}$ *is also finite, then* $\mathrm{Gal}\,K'|\mathrm{T}$ *is isomorphic to* $vK'/v\mathrm{T} = vK'/vK$*. Moreover,* $[K_1 : K_0] = (vK_1 : vK_0)$ *for every finite extension* $(K_1|K_0, v)$ *such that* $\mathrm{T} \subset K_0 \subset K_1 \subset \mathrm{V}$*.*

Proof: According to the previous lemma, the Galois group of the Galois extension $\mathrm{V}|\mathrm{T}$ is a p'-group, that is, $\mathrm{V}|\mathrm{T}$ is a p'-extension. Assume that $K'|\mathrm{T}$ is a finite normal subextension of $\mathrm{V}|\mathrm{T}$. Since $\mathrm{V}|\mathrm{T}$ is a p'-extension, the same is true for the subextension $K'|\mathrm{T}$. Theorem 7.16 shows that the ramification group of a p'-extension must be trivial; hence, the ramification field of $(K'|\mathrm{T}, v)$ must be equal to K'. On the other hand, Lemma 7.7 shows that the inertia field of $(L|\mathrm{T}, v)$ is T. Together with Lemma 7.14, this in turn yields that the inertia field of $(K'|\mathrm{T}, v)$ is again T. In (7.14), we can thus replace V to obtain an embedding of $\mathrm{Gal}\,K'|\mathrm{T}$ in the p-character group $\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\overline{K'}^{\times})$. This in turn is a subgroup of the full p-character group $\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\widetilde{k}^{\times})$, where $k = \overline{K'}$. For k a field of characteristic exponent p and a finite abelian p'-group Δ, the full p-character group $\mathrm{Hom}(\Delta, \tilde{k})$ is isomorphic to Δ (cf. Lemma 24.58). Using also the inequality $|vK'/v\mathrm{T}| \leq [K' : \mathrm{T}]$ which we infer from the fundamental inequality (6.2), we compute

$$\begin{aligned}[K' : \mathrm{T}] &= \left|\,\mathrm{Gal}\,K'|\mathrm{T}\,\right| \leq \left|\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\overline{K'}^{\times})\right| \\ &\leq \left|\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\widetilde{k}^{\times})\right| = \left|(vK'/v\mathrm{T})_{p'}\right| \\ &\leq |vK'/v\mathrm{T}| \leq [K' : \mathrm{T}]\;.\end{aligned}$$

Hence, equality holds everywhere. In particular, $(vK'/v\mathrm{T})_{p'} = vK'/v\mathrm{T}$ which means that $vK'/v\mathrm{T}$ is a p'-group, and

$$\mathrm{Gal}\,K'|\mathrm{T} \;\cong\; \mathrm{Hom}(vK'/v\mathrm{T}\,,\,\overline{K'}^{\times}) \;\cong\; vK'/v\mathrm{T}\;.$$

Note that we also obtain that $\mathrm{Hom}(vK'/v\mathrm{T}\,,\,\overline{K'}^{\times})$ is already the full character group of $vK'/v\mathrm{T}$. Furthermore, we see that $[K' : \mathrm{T}] = (vK' : v\mathrm{T})$, and in view of the fundamental inequality (6.2), we may conclude that $\overline{K'} = \overline{\mathrm{T}}$. Since $K'|\mathrm{T}$ was an arbitrary finite normal subextension of $\mathrm{V}|\mathrm{T}$, we have now proved that $v\mathrm{V}/v\mathrm{T}$ is a p'-group, and that $\overline{\mathrm{V}} = \overline{\mathrm{T}}$. Consequently,

$$\mathrm{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right) = \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}\right)\;.$$

If $\mathrm{V}|\mathrm{T}$ is finite, we may choose $K' = \mathrm{V}$; then our argument shows that the embedding (7.14) is surjective, and in this case, it follows that $\mathrm{Gal}\,\mathrm{V}|\mathrm{T} \cong v\mathrm{V}/v\mathrm{T}$. To show the surjectivity of (7.14) in the infinite case, we proceed as follows. Let $\chi \in \mathrm{Hom}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times})$ be given. The extension $\mathrm{V}|\mathrm{T}$ is the union of finite Galois extensions $K_i|\mathrm{T}$, $i \in I$ and consequently, $v\mathrm{V}$ is the union of the groups vK_i. By the surjectivity in the finite case that we have proved above, we know that for every $i \in I$, the restriction χ_i of χ to vK_i is of the form $(\,.\,,\sigma_i)'$ for some $\sigma_i \in \mathrm{Gal}\, K_i|\mathrm{T}$. From the Compactness Principle for Algebraic Extensions (Lemma 24.5) it now follows that there is $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$ such that $(\,.\,,\sigma)' = \chi$. This proves the surjectivity of (7.14) in the general case. In this argument, $\mathrm{V}|\mathrm{T}$ can be replaced by an arbitrary normal subextension $K'|\mathrm{T}$, showing that the induced embedding $\mathrm{Gal}\, K'|\mathrm{T} \to \mathrm{Hom}(vK'/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times})$ is onto.

Now let $K'|\mathrm{T}$ be an arbitrary subextension of $\mathrm{V}|\mathrm{T}$. By Lemma 7.7, the ramification field of $(L|K',v)$ is V, and its inertia field is K'. The pairing associated with this extension is obtained by restricting (7.4) to the group $G^i(L|K',v) = G_{\mathrm{T}} \cap \mathrm{Gal}\, L|K'$. By what we have shown already, the restricted pairing yields an isomorphism of $\mathrm{Gal}\,\mathrm{V}|K'$ onto $\mathrm{Hom}_{vK'/v\mathrm{T}}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}) \cong \mathrm{Hom}(v\mathrm{V}/vK'\,,\,\overline{\mathrm{T}}^{\times})$ (note that $\overline{\mathrm{T}} = \overline{K'}$).

Finally, let $\mathrm{T} \subset K_0 \subset K_1 \subset \mathrm{V}$ with $K_1|K_0$ finite. We choose a finite normal subextension $K'|K_0$ of $\mathrm{T}|K_0$ such that $K_1 \subset K'$. From Lemma 7.7 we know that the inertia field of $(L|K_0,v)$ is $K_0.\mathrm{T} = K_0$ and that its ramification field is $K_0.\mathrm{V} = \mathrm{V}$, and similarly for K_1 in the place of K_0. What we have already shown may thus be applied to K_0 and K_1 in the place of T to obtain that $[K' : K_0] = (vK' : vK_0)$ and $[K' : K_1] = (vK' : vK_1)$. This proves that $[K_1 : K_0] = (vK_1 : vK_0)$. □

Lemma 7.20 *Assume (7.10) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\mathrm{res}_E(G_{\mathrm{V}}) = G^r(E|K,v) \quad \textit{and} \quad (E \cap \mathrm{V}, v) = (E|K,v)^r\;.$$

Proof: In view of Corollary 7.8, we have to show that $\mathrm{res}_E : G_{\mathrm{V}} \to G^r(E|K,v)$ is surjective. Let $(\,.\,,.\,)'_{L|K}$, $(\,.\,,.\,)'_{E|K}$ and $(\,.\,,.\,)'_{L|E}$ denote the pairing (7.11) for the extension $L|K$, $E|K$ and $L|E$ respectively. Assume that $\rho \in G^r(E|K,v)$, which means that $(\,.\,,\rho)'_{E|K}$ is the trivial character of vE/vK. It also implies that $\rho \in G^i(E|K,v)$, hence by Lemma 7.14 there exists $\sigma \in G_{\mathrm{T}}$ such that $\rho = \mathrm{res}_E(\sigma)$. It follows that $(vE/vK,\sigma)'_{L|K} = (vE/vK,\rho)'_{E|K} = 1$, so $(\,.\,,\sigma)'_{L|K}$ is in fact a character of vL/vE. Now the surjectivity proved in Theorem 7.19 shows that there is some $\tau \in G^i(L|E,v)$ such that $(\,.\,,\tau)'_{L|E} = (\,.\,,\sigma)'_{L|K}$. It follows that also $(\,.\,,\tau)'_{L|K} = (\,.\,,\sigma)'_{L|K}$. Consequently, $(\,.\,,\sigma\tau^{-1})'_{L|K} = 1$, showing that $\sigma\tau^{-1} \in G_{\mathrm{V}}$ with $\rho = \mathrm{res}_E(\sigma\tau^{-1})$. □

7.6 Synopsis

From the foregoing theorem together with Theorem 7.13, we deduce:

Corollary 7.21 *Assume (7.10). If the maximal separable subextension $\overline{\mathrm{T}}|\overline{K}$ of $\overline{L}|\overline{K}$ is a p'-extension, then also* $\mathrm{T}|\mathrm{Z}$ *and* $\mathrm{V}|\mathrm{Z}$ *are p'-extensions.*

Corollary 7.22 *Assume (7.10). Let $(K_1|K_0,v)$ be a finite normal extension such that $K \subset K_0 \subset K_1 \subset \mathrm{V}$. Then*

$$[K_1 : K_0] = e \cdot f \cdot g$$

where $e = (vK_1 : vK_0)$*,* $f = [\overline{K_1} : \overline{K_0}]$ *and* g *is the number of distinct extensions of* v *from* K_0 *to* K_1 *. If* $\mathrm{Z} \subset K_0$ *, then* $g = 1$*.*

Proof: From Lemma 7.20 it follows that the ramification field of $(K_1|K, v)$ is K_1 . Together with Lemma 7.7, this in turn yields that the ramification field of $(K_1|K_0, v)$ is again K_1 . In view of

$$[K_1 : K_0] = [(K_1|K,v)^r : (K_1|K,v)^i] \cdot [(K_1|K,v)^i : (K_1|K,v)^d] \cdot [(K_1|K,v)^d : K_0] ,$$

our first assertion follows from Theorem 7.19, Theorem 7.13 and Theorem 7.9. If $\mathrm{Z} \subset K_0$, then by Lemma 7.20 and Lemma 7.7, K_0 is the decomposition field of $(K_1|K_0, v)$, and Theorem 7.9 shows that $g = 1$. □

Remark 7.23 The isomorphism of Theorem 7.19 induces a topology on the group $\mathrm{Hom}(v\mathrm{V}/v\mathrm{T}\,,\, \overline{\mathrm{T}}^{\times})$ which shows that it is in fact a profinite group. Further, Theorem 7.19 shows that the subgroups $\mathrm{Hom}(v\mathrm{V}/vK'\,,\, \overline{\mathrm{T}}^{\times})$ are closed for every subextension $K'|\mathrm{T}$ of $\mathrm{V}|\mathrm{T}$, and that they are open if and only if $K'|\mathrm{T}$ is finite. See O. Endler [END8], §20 for further details.

We summarize our main results of this section in the following table:

Galois group	**field**		**value group**	**residue field**
	(L,v)		vL	$\overline{L}$
	purely inseparable			
1	$(L\mid K,v)^{\rm sep}$	maximal separable subextension	$(vL/vK)_p$	purely inseparable
	p-extension			
$G^r(L\mid K,v)$	$\mathrm{V} = (L\mid K,v)^r$	ramification field	$\frac{1}{p'^\infty}vK \cap vL$	$(\overline{L}\mid\overline{K})^{\rm sep}$
Char	p'-extension		$(vL/vK)_{p'}$	$\Vert$
$G^i(L\mid K,v)$	$\mathrm{T} = (L\mid K,v)^i$	inertia field	vK	$(\overline{L}\mid\overline{K})^{\rm sep}$
$\mathrm{Gal}\,\overline{L}\mid\overline{K}$	unramified		$\Vert$	Galois
$G^d(L\mid K,v)$	$\mathrm{Z} = (L\mid K,v)^d$	decomposition field	vK	$\overline{K}$
	immediate		$\Vert$	$\Vert$
$\mathrm{Gal}\,L\mid K$	(K,v)		vK	$\overline{K}$

where $\frac{1}{p'^\infty}vK \cap vL$ is the relative p'-divisible closure of vK in vL, and Char denotes the character group

$$\mathrm{Hom}\left(vL/vK\,,\,\overline{L}^\times\right) \cong \mathrm{Hom}\left((vL/vK)_{p'}\,,\,((\overline{L}|\overline{K})^{\rm sep})^\times\right)\ ,$$

which is also isomorphic to the character group appearing in Theorem 7.19.

Let us return to the pairing (7.13). Lemma 7.19 has shown that $v\mathrm{V}/v\mathrm{T}$ is a p'-group, hence $(v\mathrm{V}/v\mathrm{T})_{p'} = v\mathrm{V}/v\mathrm{T}$. Further, $\overline{\mathrm{V}} = \overline{\mathrm{T}}$ by Theorem 7.19. Therefore, the pairing (7.13) induces a homomorphism

$$v\mathrm{V}/v\mathrm{T} \longrightarrow \mathrm{Hom}\left(\mathrm{Gal}\,\mathrm{V}|\mathrm{T}\,,\,\overline{\mathrm{T}}^\times\right) \tag{7.15}$$

of $v\mathrm{V}/v\mathrm{T}$ into the character group $\mathrm{Hom}\left(\mathrm{Gal}\,\mathrm{V}|\mathrm{T}\,,\,\overline{\mathrm{T}}^\times\right)$. We prove:

Lemma 7.24 *The homomorphism (7.15) is an embedding.*

Proof: Assume that $b \in \mathrm{V}^\times$ is such that $\overline{\sigma b\,/\,b} = 1$ for every $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. Then in particular, all conjugates $\sigma_i b$ of b over T can be written as $\sigma_i b = b(1+c_i)$ with $c_i \in \mathcal{M}_{(\mathrm{V},v)}$ $(1 \le i \le n := [\mathrm{T}(b) : \mathrm{T}])$. Hence,

$$\mathrm{Tr}_{\mathrm{T}(b)|\mathrm{T}}(b) = b(n+c) \qquad \text{with } c = \sum_{1\le i\le n} c_i \in \mathcal{M}_{(\mathrm{V},v)}\ .$$

Since V|T is a p'-extension according to Lemma 7.19, n is prime to p. That is, $vn = 0$ showing that $vb = vb(n+c) = v\mathrm{Tr}_{\mathrm{T}(b)|\mathrm{T}}(b) \in v\mathrm{T}$. This yields our assertion. □

Corollary 7.25 *The pairing*

$$(\,.\,,.)' :\ v\mathrm{V}/v\mathrm{T} \times \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\longrightarrow\; \overline{\mathrm{T}}^\times \tag{7.16}$$

is faithful: If $1 \ne \sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$*, then* $(\,.\,,\sigma)'$ *is a non-trivial character of* $v\mathrm{V}/v\mathrm{T}$*. If* $0 \ne \overline{\alpha} \in v\mathrm{V}/v\mathrm{T}$*, then* $(\overline{\alpha},\,.\,)'$ *is a non-trivial character of* Gal V|T*.*

Finally, let us return once more to the "functorial discussion" which was initiated by Lemma 7.3. Let us now consider the following case. Let $(K'|K,v)$ be an arbitrary extension, $(L'|K',v)$ a normal algebraic extension and $(L|K,v)$ a normal algebraic subextension of $(L'|K,v)$. In this case, τ_* is the restriction map res_L , which is a topological epimorphism from $\mathrm{Gal}\,L.K'|K'$ onto $\mathrm{Gal}\,L\,|\,L\cap K'$.

For the purposes of ramification theory, we take fixed fields in the maximal separable subextension of a given normal algebraic extension. So let $L_s := (L\,|\,L\cap K')^{\mathrm{sep}}$. Then also $L_s.K'|K'$ is separable, so $L_s.K'$ and thus also L_s are contained in $L'_s := (L'|K')^{\mathrm{sep}}$. Further, $L_s\cap K' = L\cap K'$ by definition of L_s. Now τ_* is the restriction map res_{L_s} , which again gives the topological epimorphism from $\mathrm{Gal}\,L'|K'$ onto $\mathrm{Gal}\,L\,|\,L\cap K'$ since $\mathrm{Gal}\,L'|K' = \mathrm{Gal}\,L'_s|K'$ and $\mathrm{Gal}\,L\,|\,L\cap K' = \mathrm{Gal}\,L_s\,|\,L_s\cap K'$. From Lemma 7.3, we obtain:

Lemma 7.26 *Let* $(K'|K,v)$ *be an arbitrary extension,* $(L'|K',v)$ *a normal algebraic extension and* $(L|K,v)$ *a normal algebraic subextension of* $(L'|K,v)$*. Then*

$$\begin{array}{lcl}
\mathrm{res}_L(G^d(L'|K',v)) \subset G^d(L\,|\,L\cap K',v) & \textit{and} & (L\cap(L'|K',v)^d,v) \supset (L\,|\,L\cap K',v)^d\\
\mathrm{res}_L(G^i(L'|K',v)) \subset G^i(L\,|\,L\cap K',v) & \textit{and} & (L\cap(L'|K',v)^i,v) \supset (L\,|\,L\cap K',v)^i\\
\mathrm{res}_L(G^r(L'|K',v)) \subset G^r(L\,|\,L\cap K',v) & \textit{and} & (L\cap(L'|K',v)^r,v) \supset (L\,|\,L\cap K',v)^r\ .
\end{array}$$

Later, we will determine criteria for the above inclusions to be equalities. One important case will be met in the next section.

Exercise 7.1 *Show that* $\mathrm{Hom}((v\mathrm{V}/v\mathrm{T})_{p'}\,,\,\overline{\mathrm{V}}^\times)$ *is a pro-*p'*-group, without using that it is isomorphic to some Galois group.*

7.7 Absolute ramification theory

Let (K,v) be a henselian field. The inertia field of the normal extension $(\tilde{K}|K,v)$ (or equivalently, of the normal extension $(K^{\mathrm{sep}}|K,v)$) will be called the **absolute inertia field** or just **inertia field of** (K,v) and denoted by $(K,v)^i$ or by (K^i,v^i). Similarly, the ramification field of $(\tilde{K}|K,v)$ will be called the **absolute ramification field** or just **ramification field of** (K,v) and denoted by $(K,v)^r$ or by (K^r,v^r). Observe that both fields are uniquely determined by the henselian field (K,v) since the extension of v from K to L is unique. In the following, we will always refer to the valuation v and its unique extension to $\tilde{K}$. For instance, instead of writing (K^i,v^i), we will just write K^i.

Theorem 7.27 *Let (K,v) be a henselian field and p the characteristic exponent of its residue field. Then the following assertions hold:*

a) $K^i|K$ and $K^r|K$ are Galois extensions,

b) $vK^i = vK$, $\overline{K^i} = \overline{K}^{\mathrm{sep}}$ and $\mathrm{Gal}\, K^i|K \cong \mathrm{Gal}\, \overline{K}$,

c) vK^r is the p'-divisible hull of vK, $\overline{K^r} = \overline{K^i}$ and $\mathrm{Gal}\, K^r|K^i$ is an abelian pro-p'-group,

d) $K^{\mathrm{sep}}|K^r$ is a p-extension.

Proof: Recall first that vK^{sep} is the divisible hull of vK and that $\overline{K^{\mathrm{sep}}} = \overline{K}^{\mathrm{sep}}$ (cf. Lemma 6.44). Since (K,v) is henselian, the decomposition field of the Galois extension $(K^{\mathrm{sep}}|K,v)$ is equal to K. Now the assertions follow from Lemma 7.6, Theorem 7.13 and Theorem 7.19. □

Table 7.6 now takes the following form:

Galois group	**field**		**value group**	**residue field**
	$\tilde{K}$		$\widetilde{vK}$	$\widetilde{Kv}$
	purely inseparable		‖	‖
1	$K^{\rm sep}$	separable-algebraic closure	$\widetilde{vK}$	$\widetilde{Kv}$
	Galois p-extension		division by p	purely inseparable
G^r	K^r	absolute ramification field	$\frac{1}{p'^\infty}vK$	$(Kv)^{\rm sep}$
Char	abelian Galois p'-extension, defectless		division prime to p	‖
G^i	K^i	absolute inertia field	vK	$(Kv)^{\rm sep}$
$\mathrm{Gal}\, Kv$	Galois, defectless		‖	Galois
G^d	K^h	absolute decomposition field	vK	Kv
	immediate		‖	‖
$\mathrm{Gal}\, K$	K		vK	Kv

where $\frac{1}{p'^\infty}vK$ denotes the p'-divisible hull of vK and Char denotes the character group

$$\mathrm{Hom}\left(vK^r/vK^i\,,\,(K^iv)^\times\right) \;\cong\; \mathrm{Hom}\left(v\tilde{K}/vK\,,\,(\tilde{K}v)^\times\right)\;. \tag{7.17}$$

From Lemma 7.7 we obtain:

Lemma 7.28 *Let $(L|K,v)$ be an algebraic extension of henselian fields. Then $L^i = L.K^i$ and $L^r = L.K^r$.*

As a consequence of Corollary 7.22, we have

Lemma 7.29 *Let (K,v) be a henselian field and $K\subset K_0\subset K_1\subset K^r$ such that $K_1|K_0$ is finite. Then $(K_1|K_0,v)$ is defectless. Further, $e(K_1|K_0,v)$ is not divisible by the characteristic of $\overline{K}$, and $\overline{K_1}|\overline{K_0}$ is a separable-algebraic extension.*

Proof: Since (K,v) is henselian, the same is true for (K_0,v), and we thus have $g(K_1|K_0,v) = 1$. Hence, our assertion follows directly from Corollary 7.22 if $K_1|K_0$ is normal. If this is not the case, we choose N to be the normal hull of K_1 over K_0. Since

$K^r|K_0$ is normal, N lies in K^r. Now Corollary 7.22 applied to the finite normal extension $(N|K_0, v)$ gives the corresponding assertion for this extension. From this, the assertion also follows for the subextension $(K_1|K_0, v)$ of $(N|K_0, v)$. □

By use of this lemma, we can add the following fact to the assertion of Lemma 7.28:

Lemma 7.30 *Let $(L|K, v)$ be an immediate extension of henselian fields. Then $L^i = L.K^i$ and $L^r = L.K^r$.*

Proof: Since $vK = vL$, we have by Theorem 7.27 that $vK^i = vK = vL = vL^i$. It also follows that the p'-divisible hull of vK is the same as that of vL. Hence in view of Theorem 7.27, $vK^r = vL^r$. Since $\overline{K} = \overline{L}$, the separable-algebraic closure of $\overline{K}$ is the same as that of $\overline{L}$. Hence in view of Theorem 7.27, $\overline{K^i} = \overline{L^i}$ and $\overline{K^r} = \overline{L^r}$. This shows that

$$(L^i|K^i, v) \quad \text{and} \quad (L^i|K^i, v)$$

are immediate extensions. Consequently, also

$$(L^i|L.K^i, v) \quad \text{and} \quad (L^i|L.K^i, v)$$

are immediate algebraic extensions. Now Lemma 7.29 shows that they must be trivial. □

The absolute inertia field and the absolute ramification field of (K, v) are universal in the sense that their intersection with an arbitrary normal extension of (K, v) produces the inertia and ramification field of that extension. Indeed, this follows from Lemma 7.14 and Lemma 7.20. So let us note:

Lemma 7.31 *Let $(L|K, v)$ be a normal extension of henselian fields. Then $(L|K, v)^i = (L \cap K^i, v)$ and $(L|K, v)^r = (L \cap K^r, v)$. In particular, the inertia field of every normal extension of (K, v) is contained in K^i, and the ramification field of every normal extension of (K, v) is contained in K^r.*

The following lemma shows that even if the extension $L|K$ is not normal, then the intersections have the main properties of the inertia and ramification field:

Lemma 7.32 *Let $(L|K, v)$ be an algebraic extension of henselian fields, and let p denote the characteristic exponent of $\overline{K}$. Then the following holds:*

a) $\overline{L \cap K^i}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$, and $v(L \cap K^i) = vK$. If $L|K$ is finite, then $[L \cap K^i : K] = [\overline{L} : \overline{K}]_{\mathrm{sep}}$.

b) $v(L \cap K^r)$ is the p'-divisible hull of vK in vL, and $\overline{L \cap K^r} = \overline{L \cap K^i}$. If $L|K$ is finite and $e(L|K, v) = p^\mu \cdot e'$ with e' prime to p, then $[L \cap K^r : L \cap K^i] = e'$.

Consequently, if $\overline{L}|\overline{K}$ is not purely inseparable, then $L|K$ is not linearly disjoint from $K^i|K$. Similarly, if $(vL : vK)$ is not a power of p, then $L|K$ is not linearly disjoint from $K^r|K$.

Proof: We prove our assertions for the case of a finite extension $L|K$. The deduction of the assertions for arbitrary algebraic extensions is left to the reader as a straightforward exercise. We set $\mathrm{V}_0 := L \cap K^r$ and $\mathrm{T}_0 := L \cap K^i \subset \mathrm{V}_0$. Let N be the normal hull of $L|K$. Since (K, v) is henselian, we have $g = 1$ for every extension of valued fields which are algebraic extensions of (K, v).

By Lemma 7.14, $\mathrm{T} := N \cap K^i$ is the inertia field of $(N|K,v)$ and of $(N|\mathrm{T}_0,v)$. Hence, $\overline{\mathrm{T}}|\overline{\mathrm{T}_0}$ is separable with $[\mathrm{T}:\mathrm{T}_0] = [\overline{\mathrm{T}}:\overline{\mathrm{T}_0}]$, and $\overline{N}|\overline{\mathrm{T}}$ is purely inseparable. This shows that

$$[\overline{L.\mathrm{T}}:\overline{\mathrm{T}_0}]_{\mathrm{sep}} = [\overline{\mathrm{T}}:\overline{\mathrm{T}_0}] = [\mathrm{T}:\mathrm{T}_0]\;. \tag{7.18}$$

By Lemma 24.14 in the Appendix, $\mathrm{T}|\mathrm{T}_0$ is linearly disjoint from $L|\mathrm{T}_0$, hence we have $[\mathrm{T}:\mathrm{T}_0] = [L.\mathrm{T}:L]$. By Lemma 7.7, $L.\mathrm{T}$ is the inertia field of $(N|L,v)$. Thus, the extension $\overline{L.\mathrm{T}}|\overline{L}$ is separable, and $[L.\mathrm{T}:L] = [\overline{L.\mathrm{T}}:\overline{L}]$. Altogether, we find that

$$[\overline{L.\mathrm{T}}:\overline{\mathrm{T}_0}]_{\mathrm{sep}} = [\overline{L.\mathrm{T}}:\overline{L}]\;, \tag{7.19}$$

which proves that the extension $\overline{L}|\overline{\mathrm{T}_0}$ must be purely inseparable. Consequently, $\overline{\mathrm{T}_0}$ contains the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. But $\overline{\mathrm{T}_0}|\overline{K}$ is separable, being a subextension of the separable extension $\overline{K^i}|\overline{K}$. This proves that $\overline{\mathrm{T}_0} = (\overline{L}|\overline{K})^{\mathrm{sep}}$. By Theorem 7.13, $v\mathrm{T}_0 = vK$ and $[\mathrm{T}_0:K] = [\overline{\mathrm{T}_0}:\overline{K}] = [\overline{L}:\overline{K}]_{\mathrm{sep}}$.

Similarly, we treat the field V_0. By Lemma 7.20 and Lemma 7.7, $\mathrm{V} := N \cap K^r$ is the ramification field of $(N|K,v)$, of $(N|\mathrm{T}_0,v)$ and of $(N|\mathrm{V}_0,v)$. Hence by Theorem 7.19, $\overline{\mathrm{V}}|\overline{\mathrm{T}_0}$ is separable and $v\mathrm{V}/v\mathrm{V}_0$ is a p'-group. The former proves that $\overline{\mathrm{V}_0} = \overline{\mathrm{T}_0}$. Indeed, we have already seen that $\overline{\mathrm{T}_0} \subset \overline{\mathrm{V}_0}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$, hence the separable subextension $\overline{\mathrm{V}_0}|\overline{\mathrm{T}_0}$ of $\overline{\mathrm{V}}|\overline{\mathrm{T}_0}$ must be trivial.

By Theorem 7.19, we know that $\overline{\mathrm{V}} = \overline{\mathrm{T}}$. Since also $\overline{\mathrm{V}_0} = \overline{\mathrm{T}_0}$, we have that $[\overline{\mathrm{V}}:\overline{\mathrm{V}_0}] = [\overline{\mathrm{T}}:\overline{\mathrm{T}_0}] = [\mathrm{T}:\mathrm{T}_0]$. In view of Lemma 7.29, it follows that

$$[\mathrm{V}:\mathrm{V}_0] = (v\mathrm{V}:v\mathrm{V}_0)[\overline{\mathrm{V}}:\overline{\mathrm{V}_0}] = (v\mathrm{V}:v\mathrm{V}_0)[\mathrm{T}:\mathrm{T}_0]\;. \tag{7.20}$$

By Lemma 7.7, $L.\mathrm{V}$ is the ramification field of $(N|L.\mathrm{T},v)$. Hence by Theorem 7.19, $\overline{L.\mathrm{V}} = \overline{L.\mathrm{T}}$. Hence from (7.20) and (7.21), we obtain that

$$[\overline{L.\mathrm{V}}:\overline{L}] = [\overline{L.\mathrm{T}}:\overline{L}] = [\mathrm{T}:\mathrm{T}_0]\;.$$

Again in view of Lemma 7.29, we conclude that

$$[L.\mathrm{V}:L] = (v(L.\mathrm{V}):vL)[\overline{L.\mathrm{V}}:\overline{L}] = (v(L.\mathrm{V}):vL)[\mathrm{T}:\mathrm{T}_0]\;. \tag{7.21}$$

By Lemma 24.34, $\mathrm{V}|\mathrm{V}_0$ is linearly disjoint from $L|\mathrm{V}_0$, hence we have that $[\mathrm{V}:\mathrm{V}_0] = [L.\mathrm{V}:L]$. From this together with (7.20) and (7.21), we find that $(v\mathrm{V}:v\mathrm{V}_0) = (v(L.\mathrm{V}):vL)$. Consequently, $(v(L.\mathrm{V}):v\mathrm{V}) = (vL:v\mathrm{V}_0)$. From Theorem 7.16 we know that $vN/v\mathrm{V}$ is a p-group. Hence also its subgroup $v(L.\mathrm{V})/v\mathrm{V}$ is a p-group, which yields that also $vL/v\mathrm{V}_0$ is a p-group. Consequently, $v\mathrm{V}_0$ contains the p'-divisible hull of vK in vL. But $v\mathrm{V}_0/vK$ is a p'-group, being a subgroup of the p'-group vK^r/vK. This proves that $v\mathrm{V}_0$ is the p'-divisible hull of vK in vL. So if $e(L|K,v) = p^\mu \cdot e'$ with e' prime to p, then $e' = (v\mathrm{V}_0:vK)$. By Corollary 7.22, $[\mathrm{V}_0:K] = (v\mathrm{V}_0:vK)\cdot[\overline{\mathrm{V}_0}:\overline{K}]$. As we have seen already, $\overline{\mathrm{V}_0} = \overline{\mathrm{T}_0}$ and $[\mathrm{T}_0:K] = [\overline{\mathrm{T}_0}:\overline{K}]$. Dividing by $[\mathrm{T}_0:K]$, we thus find that $[\mathrm{V}_0:\mathrm{T}_0] = (v\mathrm{V}_0:vK) = e'$. □

In view of this lemma, one may define the inertia and ramification field of an arbitrary algebraic extension of a henselian field (K,v) to be the respective intersections with K^i and K^r. We leave it to the reader to work out the details of such a generalized ramification theory.

7.8 Henselian fields and henselizations

To the lazy mathematician it may appear uncomfortable to work with too many valuations, so we take the occasion to define a very handy (and important) class of valued fields: (K, v) will be called **henselian** if it admits a unique extension of v to *every* algebraic extension. If (K, v) is henselian and if it is clear that the symbol "v" refers to the valuation v on K, then we will also say that v is a **henselian valuation** and that its associated place P_v is a **henselian place**. Henceforth, if we are working with a henselian field (K, v), we will automatically assume the valuation extended to every algebraic field extension and to be called v again; since the extension is unique (and since it is again henselian, as we will see below), this can't cause confusion.

Let $L|K$ be an arbitrary algebraic extension of fields and $E|K$ a subextension of $L|K$. Let v be a valuation on K. Suppose that v_1 and v_2 are two distinct extensions of v to E. By Corollary 4.11, v_1 and v_2 can be extended to valuations v_1 and v_2 of L, and we will have $v_1 \neq v_2$ on L since already $v_1 \neq v_2$ on E. Hence K admits a unique extension of v to L if and only if it admits a unique extension of v to every intermediate field E.

Now let v be extended to E and call this extension again v. Every extension of v from E to L is also an extension of v from K to L. Hence if K admits a unique extensions of v to L, then so does E.

If K admits two distinct extensions v_1, v_2 of v to $\tilde{K}$, then in view of Corollary 6.57, already their restrictions to the separable-algebraic closure $(L|K)^{\mathrm{sep}}$ will be distinct. But then, there is some a separable over K such that $v_1 a \neq v_2 a$. So we have found a finite separable subextension $K(b)|K$ of $L|K$ such that v_1 and v_2 are distinct on $K(b)$. If in addition, $L|K$ is normal, then we can pass to the normal hull N of $K(a)$ over K, and we have found a finite Galois extension of $N|K$ such that v_1 and v_2 are distinct on N.

With $L = \tilde{K}$, these considerations prove:

Lemma 7.33 *A valued field (K, v) is henselian if and only if it admits a unique extension of v to $\tilde{K}$, and this holds if and only if (K, v) admits a unique extension of v to every finite Galois extension. In particular, every separable-algebraically closed valued field is henselian. Every algebraic extension of a henselian field is again henselian.*

This lemma gives rise to a slightly different characterization of henselian fields:

Lemma 7.34 *Take a field (K, v) and a valuation preserving embedding φ of (K, v) in an algebraically closed valued field (F, w). Extend v to a valuation $\tilde{v}$ of $\tilde{K}$. Then (K, v) is henselian if and only if every field embedding ψ of an algebraic extension L of K in F that extends φ is already a valuation preserving embedding of (L, v) in (F, w).*

Proof: Take L and ψ as in the lemma. Then by **??**, $w \circ \psi$ is a valuation on L. Since φ is valuation preserving, $w \circ \psi$ extends v. Now if (K, v) is henselian, then $w \circ \psi = v$ by the foregoing lemma, which means that ψ is valuation preserving.

If on the other hand (K, v) is not henselian, then there are two distinct extensions $\tilde{v}$ and w of v from K to $\tilde{K}$, and the identity is not a valuation preserving embedding of $(\tilde{K}, \tilde{v}$ in $(\tilde{K}, w)$, although it extends the identity on K which is a valuation preserving embedding of (K, v) in $(\tilde{K}, w)$. □

The property of being henselian is preserved under isomorphisms of valued fields. To see this, let $\iota : (K, v) \to (K', v')$ be such an isomorphism. In particular, $\iota\mathcal{O}_K = \mathcal{O}_{K'}$ (cf.

Exercise **??**). The field isomorphism ι can be extended to an isomorphism $\tilde{\iota}: \tilde{K} \to \widetilde{K'}$. If (K', v') is not henselian, then there are two distinct valuation rings $\mathcal{O}'_1$ and $\mathcal{O}'_2$ of $\widetilde{K'}$ lying above $\mathcal{O}_{K'}$. Then the valuation rings $\tilde{\iota}^{-1}(\mathcal{O}'_1)$ and $\tilde{\iota}^{-1}(\mathcal{O}'_2)$ of $\tilde{K}$ are distinct and lie above $\mathcal{O}_K$. This shows that also (K, v) is not henselian.

Setting $L = \tilde{K}$ in Theorem 7.9, we infer that

$$(K, v)^h := (\tilde{K}|K, v)^d$$

is henselian. This valued field is called the **henselization of** (K, v) **in** $(\tilde{K}, v)$. By definition, it is a separable algebraic extension of K. To denote the underlying field of $(K, v)^h$, we will also write K^h or $K^{h(v)}$. It should be pointed out that our definition depends on the extension of v from K to $\tilde{K}$ that we have chosen. A different extension leads to a different henselization, but all of these henselizations are isomorphic over (K, v), as we will see now.

Lemma 7.35 *Choose an extension of the valuation v from K to $\tilde{K}$ and call it again v. Denote by (K^h, v) the henselization of (K, v) in $(\tilde{K}, v)$. Then for every $\iota \in \mathrm{Gal}\, K$, the field $(\iota^{-1}K^h, v\iota)$ is the henselization $(K^{h(v\iota)}, v\iota)$ of (K, v) in $(\tilde{K}, v\iota)$, and (K^h, v) is isomorphic over K to $(\iota^{-1}K^h, v\iota)$ via the uniquely determined isomorphism $\mathrm{res}_{K^h}(\iota^{-1})$.*

Proof: The assertion follows from the definition of the henselization together with Corollary 7.4; the uniqueness of $\mathrm{res}_{K^h}(\iota^{-1})$ is stated in part f) of Theorem 7.9. □

Lemma 7.36 *Let w be any extension of v from K to $\tilde{K}$, and let E be an algebraic extension of K such that (E, w) is henselian. Then (E, w) contains the henselization $(K^{h(w)}, w)$ of (K, v) in $(\tilde{K}, w)$. Further, there is a unique embedding of (K^h, v) over K in (E, w), and its image is $(K^{h(w)}, w)$.*

Proof: Since (E, w) is assumed to be henselian, the extension of w to $\tilde{E} = \tilde{K}$ is unique. Applying part d) of Theorem 7.9 to the extension $(\tilde{K}|K, w)$, we find that (E, w) contains $(\tilde{K}|K, w)^d$. But the latter is the henselization $(K^{h(w)}, w)$ of (K, v) in $(\tilde{K}, w)$. By virtue of Theorem 6.53, $w = v\iota$ for some $\iota \in \mathrm{Gal}\, K$. By the foregoing lemma, $\mathrm{res}_{K^h}(\iota^{-1})$ is the unique isomorphism of (K^h, v) onto $(K^{h(w)}, w)$ over K.

If σ is any embedding of (K^h, v) over K in (E, w), then its image $\sigma(K^h, v)$ is henselian. So by what we have just proved, it must contain $(K^{h(w)}, w)$. Conversely, $\sigma^{-1}(K^{h(w)}, w)$ is a henselian subfield of $(\tilde{K}, v)$, and thus, it contains (K^h, v). This proves that $\sigma(K^h, v) = (K^{h(w)}, w)$ and that $\mathrm{res}_{K^h}(\iota^{-1}) = \mathrm{res}_{K^h}(\sigma^{-1})$ is the unique embedding of (K^h, v) in (E, w) over K. □

As we will later work with a universal extension that is fixed once and for all it is convenient to choose the notation $(K, v)^h$ as introduced above. As the henselization is a henselian field, the following is a direct consequence of the foregoing lemma:

Corollary 7.37 *(K, v) is henselian if and only if $(K, v)^h = (K, v)$.*

For the generalization of Lemma 7.36 to the case of arbitrary (not necessarily algebraic) extensions, we need the following corollary to Lemma 7.26:

Corollary 7.38 *The relative separable-algebraic closure of any subfield in a henselian field is again henselian.*

Proof: Let (F, v) be a henselian field and (K, v) a subfield. We apply Lemma 7.26 with $K' = F$, $L' = F^{\rm sep}$ and $L = K^{\rm sep}$ to find that $K^{\rm sep} \cap (F^{\rm sep}|F, v)^d \supset (K^{\rm sep}|K^{\rm sep} \cap F, v)^d$. But $(F^{\rm sep}|F, v)^d = (F, v)^h = (F, v)$ by the foregoing corollary because (F, v) is henselian by assumption. Consequently, $(K^{\rm sep} \cap F, v)^h = (K^{\rm sep}|K^{\rm sep} \cap F, v)^d \subset (K^{\rm sep} \cap F, v)$, showing that the latter is henselian. But $K^{\rm sep} \cap F$ is just the separable-algebraic closure of K in F. □

We are now able to prove the following universal property of the henselization:

Theorem 7.39 *For every henselian extension field (F, w) of (K, v), there is a unique embedding of $(K, v)^h$ into (F, w) over K. Consequently, there is a henselization of (K, v) in every henselian extension field of (K, v).*

Proof: If (F, w) is henselian, then by the preceding corollary, the same holds for the relative separable-algebraic closure of (K, v) in (F, w). Since the henselization is an algebraic extension, every embedded image of $(K, v)^h$ must be contained in this relative algebraic closure. Hence, it suffices to prove the theorem for the case of F being algebraic over K. But for this case, it is already asserted in Lemma 7.36. □

The following property of the henselization is a consequence of Lemma 7.7, applied with $L = \tilde{K}$.

Corollary 7.40 *Let $(K, v) \subset (E, v) \subset (\tilde{K}, v)$. Then $(E, v)^h = (E.K^h, v)$.*

This assertion can also be proved as follows. Since $(E.K^h, v)$ is henselian according to Lemma 7.33, it contains $(E, v)^h$ by virtue of Lemma 7.36. Conversely, E^h must contain K^h and E, so we have $(E, v)^h = (E.K^h, v)$.

If there are more than one extension of the valuation in a normal field extension $L|K$, then the henselization K^h allows us to shift the setting to a normal extension $L.K^h|K^h$ which admits a unique extension of the valuation (this is the so-called **local case**). The ramification theory of both extensions is connected as follows.

Theorem 7.41 *Let $(L|K, v)$ be a normal subextension of $(\tilde{K}|K, v)$ and (K^h, v) the henselization of (K, v) in $(\tilde{K}, v)$. Then*

$$\begin{array}{lll} G^d(L|K, v) \cong {\rm Gal}\, L.K^h|K^h & \textit{and} & (L|K, v)^d = (L \cap K^h, v) \\ G^i(L|K, v) \cong G^i(L.K^h|K^h, v) & \textit{and} & (L|K, v)^i = (L \cap (L.K^h|K^h, v)^i, v) \\ G^r(L|K, v) \cong G^r(L.K^h|K^h, v) & \textit{and} & (L|K, v)^r = (L \cap (L.K^h|K^h, v)^r, v)\ . \end{array}$$

(the isomorphisms being induced by the restriction map ${\rm res}_L$, which is a topological isomorphism from ${\rm Gal}\, L.K^h|K^h$ onto ${\rm Gal}\, L \,|\, L \cap K^h$). Further, $L \,|\, L \cap K^h$ is linearly disjoint from $K^h \,|\, L \cap K^h$. In particular, the extension of v from K to L is unique if and only if $L|K$ is linearly disjoint from $K^h|K$.

Proof: Since $K^h|K$ is separable and since $L|K$ is normal by assumption, it follows from Lemma 24.34 that $L \mid L \cap K^h$ is linearly disjoint from $K^h \mid L \cap K^h$, and that $L \cap K^h = K$ if and only if $L|K$ is linearly disjoint from $K^h|K$. It is asserted by (Gal8) that res_L is a topological isomorphism from $\mathrm{Gal}\, L.K^h|K^h$ onto $\mathrm{Gal}\, L \mid L \cap K^h$. From Lemma 7.10 we know that $\mathrm{res}_L(G^d(\tilde{K}|K, v)) = G^d(L|K, v)$. By the definition of K^h, $G^d(\tilde{K}|K, v) = \mathrm{Gal}\, K^h$, so we have $G^d(L|K, v) = \mathrm{res}_L(G^d(\tilde{K}|K, v)) = \mathrm{res}_L(\mathrm{Gal}\, K^h) = \mathrm{res}_L(\mathrm{res}_{L.K^h}(\mathrm{Gal}\, K^h)) = \mathrm{res}_L(\mathrm{Gal}\, L.K^h|K^h)$.

From Lemma 7.14 we know that $\mathrm{res}_L(G^i(\tilde{K}|K, v)) = G^i(L|K, v)$. Since $G^i(\tilde{K}|K, v) \subset G^d(\tilde{K}|K, v) = \mathrm{Gal}\, K^h$, Lemma 7.7 shows that $G^i(\tilde{K}|K^h, v) = G^i(\tilde{K}|K, v) \cap \mathrm{Gal}\, K^h = G^i(\tilde{K}|K, v)$. Consequently, we have $G^i(L|K, v) = \mathrm{res}_L(G^i(\tilde{K}|K, v)) = \mathrm{res}_L(G^i(\tilde{K}|K^h, v)) = \mathrm{res}_L(\mathrm{res}_{L.K^h}(G^i(\tilde{K}|K^h, v))) = \mathrm{res}_L(G^i(L.K^h|K^h, v))$, where the first and the last equality are inferred from Lemma 7.14. An analogous argument works for the ramification groups, by use of Lemma 7.20. The equalities on the right hand side follow by (Gal8), applied to the maximal separable algebraic subextension as explained preceding to Lemma 7.26.

The last assertion of our theorem is a consequence of part d) of Theorem 7.9 and Lemma 24.34. □

The next important property of the henselization follows from its definition together with Lemma 7.12:

Theorem 7.42 *The henselization is an immediate separable algebraic extension.*

Remark 7.43 The method we have used here to prove that the henselization is an immediate extension does not seem to be widely known. Most authors and lecturers have a hard time proving that the value group of the henselization (resp. of the decomposition field of an extension $(L|K, v)$) is equal to vK. The main ingredient in many proofs is some version of the Strong Approximation Theorem (Theorem **??**). The strikingly simple method used in the proof of Lemma 7.12 appears in the appendix of James Ax' paper [AX3]. This appendix gives a remarkably concise introduction to the valuation theory which is needed for the deduction of the Ax–Kochen–Ershov Theorem (Theorem 21.30).

In view of Corollary 7.37, the foregoing theorem enables us to provide important examples for henselian fields. Recall that by Lemma **??**, $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are spherically complete.

Theorem 7.44 *Every maximal and thus also every spherically complete valued field is henselian. In particular, power series fields are henselian. For every prime p, the valued fields $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are henselian.*

Finally, we want to state an easy but important lemma about approximation types over henselian fields.

Lemma 7.45 *Let $(L|K, v)$ be a normal extension of valued fields such that the extension of v from K to L is unique. Then for every $a \in L$ and every $\iota \in \mathrm{Gal}\, L|K$,*

$$v(a - \iota a) \geq \Lambda(a, K) \ .$$

Consequently, if (K, v) is henselian and f is an irreducible polynomial over K, then all roots of f have the same approximation type over K.

Proof: Let $(L|K,v)$, $a \in L$ and $\iota \in \operatorname{Gal} L|K$ be as in the assumption. Let $\alpha \in \Lambda(a,K)$ and $c \in K$ such that $v(a-c) \geq \alpha$. Since $v\iota = v$ on L, it follows that $v(a-c) = v\iota(a-c) = v(\iota a - c)$, showing that $v(a-\iota a) \geq \min\{v(a-c), v(\iota a - c)\} = \alpha$. This proves that $v(a-\iota a) \geq \Lambda(a,K)$.

If f is an irreducible polynomial over K, then there is a normal extension $L|K$ which contains all roots of f, and all these roots are conjugate over K. If (K,v) is henselian, then the extension of v from K to L is unique. Let a_1 and a_2 be two roots of f, and choose $\iota \in \operatorname{Gal} L|K$ such that $a_2 = \iota a_1$. As we have just shown, for every $c \in K$ we will have that $v(a_1 - c) = v(\iota a_1 - c) = v(a_2 - c)$. Hence $c \in \operatorname{at}(a_1,K)_\alpha$ if and only if $c \in \operatorname{at}(a_2,K)_\alpha$, and $c \in \operatorname{at}(a_1,K)^\circ_\alpha$ if and only if $c \in \operatorname{at}(a_2,K)^\circ_\alpha$. That is, $\operatorname{at}(a_1,K) = \operatorname{at}(a_2,K)$. □

Exercise 7.2 *Let (K,v) be henselian, and let a be algebraic over K. Show that $v\mathrm{Tr}_{K(a)|K}(a) \geq va$ and $v\mathrm{N}_{K(b)|K}(a) = [K(b):K] \cdot va$. Given an algebraic extension $(L|K,v)$ of degree n, conclude that $va = \frac{1}{n} v\mathrm{N}_{L|K}(a)$ for every $a \in L$.*

7.9 The fundamental inequality

In this section, we will relate the degree of a finite valued field extension $(L|K,v)$ with the ramification indices and the inertia degrees of all extensions of the valuation. We know already from Corollary 6.56 that the number of distinct extensions does not exceed $[L:K]$. If v admits more than one extension from K to L, then the estimate given in Lemma 6.13 will never be sharp. To obtain a better inequality involving also the other extensions, we will lift the extension $L|K$ to the henselizations of L with respect to these extensions. We need two lemmata. Recall the following. If H_1 and H_2 are subgroups of a group G, then for $g \in G$, the set $H_1 g H_2 = \{h_1 g h_2 \mid h_1 \in H_1 \wedge h_2 \in H_2\}$ is called a **double coset of** G. Further, $g \sim h :\Leftrightarrow H_1 g H_2 = H_1 h H_2$ is an equivalence relation, and its equivalence classes are double cosets of the form $H_1 g H_2$. If $(G:H_1)$ is finite, then there are at most $(G:H_1)$ distinct such double cosets.

Lemma 7.46 *Let $L|K$ be a finite and $K'|K$ an arbitrary algebraic extension. Let $g \in \mathbb{N}$ and $\iota_1,\ldots,\iota_g$ be representatives of the double cosets $\{\operatorname{Gal} K' \,\iota \operatorname{Gal} L \mid \iota \in \operatorname{Gal} K\}$ (it follows that $g \leq (\operatorname{Gal} K : \operatorname{Gal} L) \leq [L:K] < \infty$).*

a) An automorphism $\iota \in \operatorname{Gal} K$ lies in $\operatorname{Gal} K' \,\iota_i \operatorname{Gal} L$ if and only if the isomorphism $\mathrm{res}_L(\iota\,\iota_i^{-1}) : \iota_i L \to \iota L$ can be extended to an isomorphism of $\iota_i L.K'$ onto $\iota L.K'$ over K'.

b) For $1 \leq i \leq g$, let q_i denote the quotient $[L:K]_{\mathrm{ins}}/[\iota_i L.K' : K']_{\mathrm{ins}}$, which is a power of charexp K. *Then*

$$[L:K] = \sum_{1 \leq i \leq g} [\iota_i L.K' : K'] \cdot q_i \ . \tag{7.22}$$

c) If $L|K$ or $K'|K$ is separable, then $q_i = 1$ for $1 \leq i \leq g$.

d) Assume that $K'|K$ is separable, $f \in K[X]$ is an arbitrary polynomial and $L = K(a)$ for some root $a \in \tilde{K}$ of f. Then $f = f_1 \cdot \ldots \cdot f_g$ with f_i irreducible polynomials over K' and $\deg f_i = [\iota_i L.K' : K']$.

Proof: a): Let $\iota \in \operatorname{Gal} K$. Then an automorphism in $\operatorname{Gal} K$ extends $\mathrm{res}_{\iota_i L}(\iota\,\iota_i^{-1})$ if and only if it lies in the coset $\iota\,\iota_i^{-1} \operatorname{Gal} \iota_i L$. This coset is equal to $\iota\,\iota_i^{-1} \iota_i \operatorname{Gal} L\, \iota_i^{-1} = \iota \operatorname{Gal} L\, \iota_i^{-1}$.

Hence, there is an extension of $\mathrm{res}_{\iota_i L}(\iota\,\iota_i^{-1})$ to an isomorphism over K' if and only if $\iota\,\mathrm{Gal}\,L\,\iota_i^{-1}\cap\mathrm{Gal}\,K'\neq\emptyset$. But this is equivalent to $\iota\in\mathrm{Gal}\,K'\,\iota_i\,\mathrm{Gal}\,L$.

b): Let $K_s=(L|K)^{\mathrm{sep}}$ be the maximal separable subextension of K in L. Then $L|K_s$ is purely inseparable and thus, $\mathrm{Gal}\,L=\mathrm{Gal}\,K_s$. As a finite separable extension, $K_s|K$ is simple. Let $f\in K[X]$ be the minimal polynomial of some generator b of this extension. Let $\prod_i f_i$ be the decomposition of f into irreducible factors over K'. Then $\mathrm{res}_{\iota_i K_s}(\iota\,\iota_i^{-1}):\iota_i K_s\to\iota K_s$ can be extended to an isomorphism of $\iota_i K_s.K'$ onto $\iota K_s.K'$ over K' if and only if $\iota_i b$ and ιb are roots of the same irreducible factor. By virtue of part a), applied to K_s in the place of L, there are g such factors, and we may enumerate them such that $\iota_i b$ is a root of f_i. Then $[\iota_i K_s.K':K']$ is equal to the degree of f_i. Hence,

$$[L:K]_{\mathrm{sep}}=[K_s:K]=\deg f=\sum_{1\le i\le g}\deg f_i=\sum_{1\le i\le g}[\iota_i K_s.K':K']\,. \tag{7.23}$$

Like $K_s|K$, every extension $\iota_i K_s|K$ is separable. It follows from Lemma 24.43 that also every extension $\iota_i K_s.K'|K'$ is separable. On the other hand, $\iota_i L\,|\,\iota_i K_s$ is purely inseparable like $L|K_s$. Consequently, $\iota_i L.K'\,|\,\iota_i K_s.K'$ is purely inseparable, showing that

$$[\iota_i K_s.K':K']=[\iota_i L.K':K']_{\mathrm{sep}}=[\iota_i L.K':K']\cdot[\iota_i L.K':K']_{\mathrm{ins}}^{-1}\,.$$

Thus, multiplying equation (7.23) with $[L:K]_{\mathrm{ins}}$ gives equation (7.22).

c): If $L|K$ is separable, then so is $\iota_i L|K$ and hence also $\iota_i L.K'|K'$, for $1\le i\le g$. Hence $[L:K]_{\mathrm{ins}}=1=[\iota_i L.K':K']_{\mathrm{ins}}^{-1}$, which yields that $q_i=1$.

Now assume that $K'|K$ is separable. Since we have already shown that $\iota_i K_s.K'|K'$ is separable, it follows that $\iota_i K_s.K'|K$ and hence also $\iota_i K_s.K'\,|\,\iota_i K_s$, are separable. Since $\iota_i L\,|\,\iota_i K_s$ is purely inseparable, it is linearly disjoint from $\iota_i K_s.K'\,|\,\iota_i K_s$, and $\iota_i L.K'\,|\,\iota_i K_s.K'$ is purely inseparable. This yields that $[\iota_i L:\iota_i K_s]=[\iota_i L.K':\iota_i K_s.K']$ and that $\iota_i K_s.K'|K'$ is the maximal separable subextension of $\iota_i L.K'|K'$. Hence,

$$[L:K]_{\mathrm{ins}}=[L:K_s]=[\iota_i L:\iota_i K_s]=[\iota_i L.K':\iota_i K_s.K']=[\iota_i L.K':K']_{\mathrm{ins}}$$

which yields that $q_i=1$.

d): Assume the hypothesis of d) and let ν be the maximal natural number such that $f(X)=\tilde f(X^{p^\nu})$ with $\tilde f\in K[X]$. Then $\tilde f$ is separable and irreducible over K, and $K_s=K(a^{p^\nu})$. By what we have shown already, $\tilde f$ splits into irreducible factors $\tilde f_1,\ldots,\tilde f_g$ over K' such that $\deg\tilde f_i=[\iota_i K_s.K':K']$. We define $f_i:=\tilde f_i(X^{p^\nu})$ and note that $f=f_1\cdot\ldots\cdot f_g$. We observe that $\iota_i L=\iota_i K(a)=K(\iota_i a)$ and $\iota_i K_s=K((\iota_i a)^{p^\nu})$. Since $K'|K$ is assumed to be separable, also $K'((\iota_i a)^{p^\nu})|K((\iota_i a)^{p^\nu})$ is separable and thus linearly disjoint from $K(\iota_i a)|K((\iota_i a)^{p^\nu})$. Consequently, $[\iota_i L.K':K']=p^\nu[\iota_i K_s.K':K']=\deg f_i$, showing that the f_i are irreducible over K'. □

Lemma 7.47 *Let (K,v) be a valued field and $L|K$ a finite extension. Choose an extension of the valuation v from K to $\tilde K$ and call it again v. Denote by (K^h,v) the henselization of (K,v). Let $\iota_1,\ldots,\iota_g\in\mathrm{Gal}\,K$ be representatives of the double cosets*

$$\{\mathrm{Gal}\,K^h\,\iota\,\mathrm{Gal}\,L\mid\iota\in\mathrm{Gal}\,K\}\,.$$

Then the distinct extensions of v from K to L are given by the restrictions of the valuations $v_i=v\iota_i$ to L, $1\le i\le g$. Further, $(L.\iota_i^{-1}K^h,v\iota_i)$ is the henselization of (L,v_i) in $(\tilde K,v\iota_i)$, and it is isomorphic over K to $(\iota_i L.K^h,v)$ via ι_i.

Proof: By virtue of Lemma 7.35, $(\iota_i^{-1}K^h, v_i)$ is the henselization of (K,v) in $(\tilde{K}, v_i)$. From Corollary 7.40 it follows that $(L.\iota_i^{-1}K^h, v\iota_i)$ is the henselization of (L,v) in $(\tilde{K}, v_i)$. The restriction of ι_i is an isomorphism from $(L.\iota_i^{-1}K^h, v\iota_i)$ onto $(\iota_i L.K^h, v)$ over K.

Assume that $v\iota = v\iota_i$ on L. Then $v\iota$ and $v\iota_i$ are both extensions of the same valuation from L to $\tilde{K}$. From Theorem 6.53 we infer the existence of $\tau \in \operatorname{Gal} L$ such that $v\iota_i\tau = v\iota$ on $\tilde{K}$. Consequently, the restrictions of both ι^{-1} and $\tau^{-1}\iota_i^{-1}$ are embeddings of (K^h, v) in the henselian field $(\tilde{K}, v\iota) = (\tilde{K}, v\iota_i\tau)$. By Lemma 7.36, they must be equal, that is, $\sigma := \iota\tau^{-1}\iota_i^{-1}$ must be an element of $\operatorname{Gal} K^h$. So we find $\iota = \sigma\iota_i\tau \in \operatorname{Gal} K^h \cdot \iota_i \cdot \operatorname{Gal} L$.

For the converse, assume that ι lies in the double coset represented by ι_i, say $\iota = \sigma\iota_i\tau$ with $\sigma \in \operatorname{Gal} K^h = G^d(\tilde{K}|K,v)$ and $\tau \in \operatorname{Gal} L$. Then we have $v\sigma = v$ on $\tilde{K}$ and $\tau a = a$ for all $a \in L$. This yields that $v\iota a = v\sigma\iota_i\tau a = v\iota_i a$ for all $a \in L$, that is, $v\iota = v\iota_i$ on L. □

From this lemma together with part a) of Lemma 7.46 we can also deduce: *If $\iota, \iota' \in \operatorname{Gal} K$, then $v\iota = v\iota'$ on L if and only if $\iota L.K^h$ and $\iota' L.K^h$ are isomorphic over K^h.*

With $K^{h(v_i)}$ denoting the henselization of (K,v) in $(\tilde{K}, v_i)$, we have $K^{h(v_i)} = \iota_i^{-1}K^h$. This field lies in the henselization $L^{h(v_i)}$ (cf. Lemma 7.36), which is equal to $L.\iota_i^{-1}K^h$ (cf. Corollary 7.40). Since ι_i sends $\iota_i^{-1}K^h$ onto K^h and $L.\iota_i^{-1}K^h$ onto $\iota_i L.K^h$, we find that $[L^{h(v_i)} : K^{h(v_i)}] = [\iota_i L.K^h : K^h]$. Since the henselization is a separable extension, we can apply equation (7.22) of Lemma 7.46 with $q_i = 1$ to obtain:

Corollary 7.48 *Let (K,v) be a valued field and L a finite extension of K. Let $v_1, \ldots, v_g$ be the distinct extensions of v from K to L. Then*

$$[L : K] = \sum_{1 \le i \le g} [L^{h(v_i)} : K^{h(v_i)}] \,. \tag{7.24}$$

The degree $[L^{h(v_i)} : K^{h(v_i)}]$ is called the **local degree** of the extension $(L, v_i)|(K,v)$. Actually, in the literature this name is mainly used for the degree $[L^{c(v_i)} : K^c]$ of the extension of the respective completions. But to work with the latter degree only makes sense in this connection if the valuations are of rank 1, that is, their value groups are archimedean.

By Lemma 6.13 we know that $[L^{h(v_i)} : K^{h(v_i)}] \ge (v_i L^{h(v_i)} : vK^{h(v_i)}) \cdot [L^{h(v_i)}v_i : K^{h(v_i)}v]$ for $1 \le i \le g$. Since the henselization is an immediate extension by Theorem 7.42, we have $v_i K^{h(v_i)} = vK$, $v_i L^{h(v_i)} = v_i L$, $K^{h(v_i)}v_i = Kv$ and $L^{h(v_i)}v_i = Lv_i$. It follows that

$$[L^{h(v_i)} : K^{h(v_i)}] \ge (v_i L^{h(v_i)} : v_i K^{h(v_i)}) \cdot [L^{h(v_i)}v_i : K^{h(v_i)}v_i] = (v_i L : vK) \cdot [Lv_i : Kv] \,. \tag{7.25}$$

Together with equation (7.24), this proves:

Theorem 7.49 *Let (K,v) be a valued field and L a finite extension of K. Let $v_1, \ldots, v_g$ be the distinct extensions of v to L. Then we have the* **fundamental inequality**

$$[L : K] \ge \sum_{1 \le i \le g} (v_i L : vK) \cdot [Lv_i : Kv] \,. \tag{7.26}$$

Writing shortly $n = [L : K]$, $e_i = e(L|K, v_i) = (v_i L : vK)$ and $f_i = f(L|K, v_i) = [Lv_i : Kv]$, equation (7.26) can be expressed in the following mnemonic form:

$$n \ge \sum_{1 \le i \le g} e_i \cdot f_i \,.$$

The fundamental inequality can also be written in the form of an equality, see (11.2) below. If $L|K$ is a finite normal extension, then by Corollary 6.55, all e_i are equal and all f_i are equal. For this case, we obtain:

Corollary 7.50 *Let $(L|K,v)$ be a finite normal extension. Let g be the number of extensions of v from K to L, and set $n=[L:K]$, $e=e(L|K,v)$ and $f=f(L|K,v)$. Then*

$$n \;\geq\; e\cdot f\cdot g\;. \tag{7.27}$$

If (K,v) is henselian, then $g=1$ and

$$n \;\geq\; e\cdot f\;. \tag{7.28}$$