

Coincidence Point Theorems for Ball Spaces and Their Applications

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ABSTRACT. We prove several coincidence point theorems for two functions in a minimal setting (“ball spaces”) which is flexible enough to allow applications in various areas. We illustrate their use by giving applications to ultrametric spaces, metric spaces, and ordered abelian groups and fields. Further, we present the general fixed point theorems that can be directly derived from our general coincidence point theorems.

1. Introduction

Coincidence point theorems (in short: “coincidence theorems”) consider two functions f and g from a set X into another set Y and give conditions for these functions to admit a *coincidence point*, that is, an element $x \in X$ such that

$$f(x) = g(x).$$

Fixed point theorems consider one function f from a set X into itself and give conditions for the existence of a *fixed point*, that is, an element $x \in X$ such that $f(x) = x$. We can view fixed point theorems as special cases of coincidence point theorems where we take $X = Y$ and the second function g to be the identity. The condition that most often makes classical coincidence point and fixed point theorems work is some sort of completeness of the sets under consideration, with respect to some structure like metric, ultrametric or topology. The proofs in many cases use some iteration method. A typical and well known example is the Newton algorithm for the (approximative) computation of zeros of functions (in particular, roots of polynomials); this can be seen as computing a coincidence point of the function f with the constant function $g = 0$. Apart from the case of real-valued functions, it is also known to work over certain valued fields, such as the field of p -adic numbers. It has been generalized to a much larger class of valued fields, containing all power

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series fields. But this originally came at the expense of replacing the simple iteration in the proofs by transfinite induction. However, in [7] S. Priß-Crampe showed that the transfinite induction can be replaced by the (more elegant) application of an ultrametric version of Banach's Fixed Point Theorem. Since then, an abundance of ultrametric fixed point, coincidence point and related theorems have been proven (see, e.g., [8], [9], [10], [2]).

The first and the second author have developed a simple approach that allows to extract common principles of proof for fixed point theorems in ultrametric spaces, metric spaces, ordered abelian groups and fields, topology, partially ordered sets, and lattices (see [3], [4], [5], and [6]). In this paper we will do the same for coincidence points and illustrate applications to the first three of the above mentioned areas. We will use our approach to give an alternative proof of a coincidence theorem by Priß-Crampe and Ribenboim ([10]) and a theorem by Goebel for metric spaces ([1]), as well as its analogues for ultrametric spaces and ordered abelian groups. Finally, we will present the general fixed point theorems that can be directly derived from our general coincidence theorems. We will give an application by proving the main fixed point theorem for functions that are contracting on orbits in ultrametric spaces, due to Priß-Crampe and Ribenboim.

We consider **ball spaces** (X, \mathcal{B}) , which are given by nonempty sets X with a nonempty collection \mathcal{B} of distinguished nonempty subsets B of X . The completeness property we need for our fixed point and coincidence theorems is inspired by the spherical completeness property of ultrametric spaces. A **nest of balls** in (X, \mathcal{B}) is a nonempty totally ordered subset of (\mathcal{B}, \subseteq) . A ball space (X, \mathcal{B}) is called **spherically complete** if every nest of balls has a nonempty intersection.

We will now introduce four coincidence theorems of various flavours. All of them are essentially more or less direct applications of Zorn's Lemma. However, they are not special cases of each other, and they offer quite distinct ways to prove coincidence theorems in applications.

Throughout, we will write fx in place of $f(x)$.

THEOREM 1.1 (Coincidence Theorem I). *Let (X, \mathcal{B}) be a spherically complete ball space and $f, g : X \rightarrow Y$ functions satisfying the following conditions:*

(CT1) *for every ball $B \in \mathcal{B}$, $f(B) \subseteq g(B)$,*

(CT2) *for every nest of balls \mathcal{N} , either $\bigcap_{B \in \mathcal{N}} g(B)$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneq \bigcap \mathcal{N}$.*

Then in every ball in \mathcal{B} there is some element z such that $fz = gz$.

The condition that (X, \mathcal{B}) be spherically complete can be dropped if for every $B \in \mathcal{B}$, $B = g^{-1}(g(B))$.

In [5], we introduce and study a hierarchy of spherical completeness properties. Apart from the basic property that we have defined above, we will only use one stronger property in the present paper. A ball space is said to be \mathbf{S}_2 if the intersection of each nest of balls contains a ball. By taking the ball space (X, \mathcal{B}) to be \mathbf{S}_2 , the conditions needed in our coincidence theorem can be made nicely symmetric, and the use of nests can be avoided in the formulation.

THEOREM 1.2 (Coincidence Theorem II). *Let (X, \mathcal{B}) be an \mathbf{S}_2 ball space and $f, g : X \rightarrow Y$ functions satisfying the following conditions:*

(CS1) *for every ball $B \in \mathcal{B}$, $f(B) \cap g(B) \neq \emptyset$,*

(CS2) for every ball $B \in \mathcal{B}$, either $f(B)$ is a singleton or $g(B)$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneq B$.

Then in every ball in \mathcal{B} there is some element z such that $fz = gz$.

REMARK 1.3. The above two theorems become trivial as soon as every ball in the ball space (X, \mathcal{B}) contains a singleton ball. This is for instance the case for the ultrametric ball spaces, which we will introduce in Section 3.1. However, for the ball spaces we work with in applications we do not know whether they contain any singleton balls; in certain cases the existence of coincidence points implies their existence. (See also Remark 1.9 below.)

The approach used in the following theorem was first developed in [3] to prove fixed point theorems. The version presented here is not itself a fixed point or coincidence theorem. But it allows high flexibility in applications. Below, we will derive from it coincidence theorems for two distinct cases, depending on whether the domain or the codomain of the functions under consideration is chosen to be a ball space.

In the following theorems we will not assume that the ball spaces we are working with are spherically complete. Instead, the spherical completeness appears in a modified form in our assumptions on specific nests of balls.

THEOREM 1.4 (Basic B_x -type Theorem). *Take an arbitrary set X and a ball space $(Z, \mathcal{B}(Z))$. Let $P(x)$ be any assertion about the element $x \in X$. Assume that*

$$X \ni x \longmapsto B_x \in \mathcal{B}(Z)$$

is a function such that the following conditions hold:

(*) *if $\mathcal{N} = (B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(Z)$ and $P(x_i)$ holds for all $i \in I$, then there exists some $y \in X$ such that $P(y)$ holds and $B_y \subseteq \bigcap \mathcal{N}$, with the inclusion being proper if $\bigcap \mathcal{N}$ is not a singleton.*

Then for every $x_0 \in X$ such that $P(x_0)$ holds, there is $z \in X$ such that $P(z)$ holds and B_z is a singleton contained in B_{x_0} .

The condition (*) can be broken down into two separate conditions, which in applications are often checked separately:

(*1) If B_x is not a singleton and $P(x)$ holds, then there exists $y \in X$ such that $B_y \subsetneq B_x$ and $P(y)$ holds.

(*2) If $\mathcal{N} = (B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(Z)$ and $P(x_i)$ holds for all $i \in I$, then there exists some $y \in X$ such that $P(y)$ holds and $B_y \subseteq \bigcap \mathcal{N}$.

Indeed, if condition (*) holds, then also the weaker condition (*2) holds, and (*1) is obtained by taking $\mathcal{N} = \{B_x\}$. Conversely, if $\mathcal{N} = (B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(Z)$ and $P(x_i)$ holds for all $i \in I$, then by (*2) there is some $y \in X$ such that $P(y)$ holds and $B_y \subseteq \bigcap \mathcal{N}$; if $B_y = \bigcap \mathcal{N}$ and this is not a singleton, then (*1) shows that B_y can be replaced by a smaller ball $B_y \subsetneq \bigcap \mathcal{N}$ for which $P(y)$ still holds.

Note that condition **(CT2)** of Theorem 1.1 can be broken into two separate conditions in a similar way.

From Theorem 1.4 we now derive two coincidence theorems for functions $f, g : X \rightarrow Y$. In the first theorem we consider the set X to be a ball space, and take the assertion $P(x)$ to state that $f(B_x) \cap g(B_x) \neq \emptyset$.

THEOREM 1.5 (B_x -type Coincidence Theorem A). *Take a ball space $(X, \mathcal{B}(X))$, a set Y , and functions $f, g : X \rightarrow Y$. Assume that there is a function*

$$X \ni x \mapsto B_x \in \mathcal{B}(X)$$

such that the following conditions hold:

(A1) *If B_x is not a singleton and $f(B_x) \cap g(B_x) \neq \emptyset$, then there is $y \in X$ such that $B_y \subsetneq B_x$ and $f(B_y) \cap g(B_y) \neq \emptyset$.*

(A2) *If $\mathcal{N} = (B_{x_i})_{i \in I}$ is a nest of balls in $\mathcal{B}(X)$ such that $f(B_{x_i}) \cap g(B_{x_i}) \neq \emptyset$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap \mathcal{N}$ and $f(B_y) \cap g(B_y) \neq \emptyset$.*

Then for every $x_0 \in X$ such that $f(B_{x_0}) \cap g(B_{x_0}) \neq \emptyset$, there is $z \in B_{x_0}$ such that $fz = gz$.

In applications we often find slightly stronger conditions to be satisfied which can make the formulation of the coincidence theorem more elegant:

COROLLARY 1.6. *Take a ball space $(X, \mathcal{B}(X))$, a set Y , and functions $f, g : X \rightarrow Y$. Assume that there is a function*

$$X \ni x \mapsto B_x \in \mathcal{B}(X)$$

such that $f(B_x) \cap g(B_x) \neq \emptyset$ for all $x \in X$ and the following conditions are satisfied:

(A1') *If B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneq B_x$.*

(A2') *The ball space $(X, \{B_x \mid x \in X\})$ is \mathbf{S}_2 .*

Then every ball B_x contains some $z \in X$ such that $fz = gz$.

In the second B_x -type coincidence theorem, we consider the set Y to be a ball space, and take the assertion $P(x)$ to state that $fx, gx \in B_x$.

THEOREM 1.7 (B_x -type Coincidence Theorem B). *Take a ball space $(Y, \mathcal{B}(Y))$, a set X , and functions $f, g : X \rightarrow Y$. Assume that there is a function*

$$X \ni x \mapsto B_x \in \mathcal{B}(Y)$$

such that the following conditions hold:

(B1) *If B_x is not a singleton and $fx, gx \in B_x$, then there is $y \in X$ such that $B_y \subsetneq B_x$ and $fy, gy \in B_y$.*

(B2) *If $\mathcal{N} = (B_{x_i})_{i \in I}$ is a nest of balls such that $fx_i, gx_i \in B_{x_i}$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap \mathcal{N}$ and $fy, gy \in B_y$.*

Then for every $x_0 \in X$ such that $fx_0, gx_0 \in B_{x_0}$ there is some $z \in X$ such that $fz = gz \in B_{x_0}$.

As before, a slightly stronger condition leads to a nicer formulation of the coincidence theorem. It will be used in Section 3.3 to derive an ultrametric B_x -type coincidence theorem.

COROLLARY 1.8. *Take a ball space $(Y, \mathcal{B}(Y))$, a set X , and functions $f, g : X \rightarrow Y$. Assume that there is a function*

$$X \ni x \mapsto B_x \in \mathcal{B}(Y)$$

such that $fx, gx \in B_x$ for all $x \in X$ and the following conditions are satisfied:

(B1') *If B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneq B_x$.*

(B2') *The ball space $(Y, \{B_x \mid x \in X\})$ is \mathbf{S}_2 .*

Then for every $x_0 \in X$ there is some $z \in X$ such that $fz = gz \in B_{x_0}$.

REMARK 1.9. The collections $\{B_x \mid x \in X\}$ induce ball space structures on the sets in which the balls B_x are taken. For the ball spaces we obtain in this way, we will in general not know whether they contain any singletons. But in certain cases, this is exactly what we want to prove. See also the proof of Theorem 4.1 for a similar situation.

2. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1.1: Take any ball $B_0 \in \mathcal{B}$. The set of all nests of balls containing B_0 is partially ordered by inclusion, and the union over any linearly ordered set of such nests is again a nest containing B_0 . Hence by Zorn's Lemma there is a maximal nest \mathcal{N}_0 containing B_0 . By (CT2), $\bigcap_{B \in \mathcal{N}_0} g(B)$ must be a singleton, say $\{y\}$ for some $y \in Y$, since otherwise there would exist a ball $B' \subsetneq \bigcap_{B \in \mathcal{N}_0} g(B)$ and $\mathcal{N}_0 \cup \{B'\}$ would be a nest of balls containing B_0 and larger than \mathcal{N}_0 , which contradicts its maximality.

Using (CT1),

$$f\left(\bigcap_{B \in \mathcal{N}_0} \mathcal{N}_0\right) \subseteq \bigcap_{B \in \mathcal{N}_0} f(B) \subseteq \bigcap_{B \in \mathcal{N}_0} g(B) = \{y\}.$$

Therefore, $fz = y = gz$ for every $z \in \bigcap \mathcal{N}_0 \subseteq B_0$. If (X, \mathcal{B}) is spherically complete, then $\bigcap \mathcal{N}_0 \neq \emptyset$ and there is at least one such z . If on the other hand $B = g^{-1}(g(B))$ for all $B \in \mathcal{B}$, then all preimages of y are contained in every $B \in \mathcal{N}_0$ and thus again, $\bigcap \mathcal{N}_0 \neq \emptyset$. \square

Proof of Theorem 1.2: As before, we choose a ball B_0 and find a maximal nest \mathcal{N}_0 containing B_0 . Since (X, \mathcal{B}) is an \mathbf{S}_2 ball space, the intersection of \mathcal{N}_0 contains a ball B . By (CS2) we have that $f(B)$ or $g(B)$ is a singleton $\{y\}$ for some $y \in Y$ because the existence of a ball $B' \in \mathcal{B}$ with $B' \subsetneq B$ is excluded by the maximality of the nest \mathcal{N}_0 . Since $f(B) \cap g(B) \neq \emptyset$ by (CS1), we see that $f(B) \cap g(B) = \{y\}$. Hence for each $z \in B \subseteq B_0$, we obtain that $fz = y = gz$ (note that B is nonempty as it is a ball). \square

Proof of Theorem 1.4: By (*2), the set $S = \{B_x \subseteq B_{x_0} \mid x \in X \text{ and } P(x) \text{ holds}\}$ is downward inductively ordered by inclusion. Hence by Zorn's Lemma, there is a minimal element B_z in S . Suppose that B_z is not a singleton. Then by (*1) there exists $y \in X$ such that $B_y \subsetneq B_z$ and $P(y)$ holds. Thus $B_y \in S$ which contradicts the minimality of B_z . Therefore B_z must be a singleton. Since $B_z \in S$, $P(z)$ must hold. \square

Proof of Theorem 1.5: Take any $x_0 \in X$ such that $f(B_{x_0}) \cap g(B_{x_0}) \neq \emptyset$. We apply Theorem 1.4 with $Z = X$ and $P(x)$ being the assertion that $f(B_x) \cap g(B_x) \neq \emptyset$, to obtain some $z_0 \in X$ such that B_{z_0} is a singleton contained in B_{x_0} and $f(B_{z_0}) \cap g(B_{z_0}) \neq \emptyset$. Since B_{z_0} is a singleton, say $B_{z_0} = \{z\}$, it follows that $\emptyset \neq f(B_{z_0}) \cap g(B_{z_0}) = \{fz\} \cap \{gz\}$, hence $fz = gz$. Since $B_{z_0} \subseteq B_{x_0}$, we have that $z \in B_{x_0}$. \square

Proof of Theorem 1.7: Take any $x_0 \in X$ such that $fx_0, gx_0 \in B_{x_0}$. We apply Theorem 1.4 with $Z = Y$ and $P(x)$ being the assertion that $fx, gx \in B_x$, to obtain some $z \in X$ such that B_z is a singleton contained in B_{x_0} and $fz, gz \in B_z$, so $fz = gz \in B_{x_0}$. \square

3. Applications to ultrametric spaces

In this section, we will use Coincidence Point Theorem II as well as Corollary 1.8 to prove various ultrametric coincidence theorems.

3.1. Preliminaries on ultrametric spaces. An ultrametric d on a set X is a function from $X \times X$ to a partially ordered set Γ with smallest element 0 , such that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $d(x, y) = 0$ if and only if $x = y$,

(U2) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$,

(U3) $d(x, y) = d(y, x)$ (symmetry).

(U2) is the ultrametric triangle law; if Γ is totally ordered, it is equivalent to:

(UT) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

A **closed ultrametric ball** is a set $B_\alpha(x) := \{y \in X \mid d(x, y) \leq \alpha\}$, where $x \in X$ and $\alpha \in \Gamma$. The problem with general ultrametric spaces is that closed balls $B_\alpha(x)$ are not necessarily **precise**, that is, there may not be any $y \in X$ such that $d(x, y) = \alpha$. Therefore, we prefer to work only with precise ultrametric balls, which we can write in the form

$$B(x, y) := \{z \in X \mid d(x, z) \leq d(x, y)\},$$

where $x, y \in X$. We obtain the **ultrametric ball space** (X, \mathcal{B}) from (X, d) by taking \mathcal{B} to be the set of all such balls $B(x, y)$.

It follows from symmetry and the ultrametric triangle law that

$$B(x, y) = B(y, x)$$

and that

$$(3.1) \quad B(t, z) \subseteq B(x, y) \text{ if and only if } t \in B(x, y) \text{ and } d(t, z) \leq d(x, y).$$

In particular,

$$B(t, z) \subseteq B(x, y) \text{ if } t, z \in B(x, y).$$

Two elements γ and δ of Γ are **comparable** if $\gamma \leq \delta$ or $\gamma \geq \delta$. Hence if $d(x, y)$ and $d(y, z)$ are comparable, then $B(x, y) \subseteq B(y, z)$ or $B(y, z) \subseteq B(x, y)$. If $d(y, z) < d(x, y)$, then in addition, $x \notin B(y, z)$ and thus, $B(y, z) \subsetneq B(x, y)$. We note:

$$(3.2) \quad d(y, z) < d(x, y) \implies B(y, z) \subsetneq B(x, y).$$

If Γ is totally ordered and B and B' are any two balls with nonempty intersection, then $B \subseteq B'$ or $B' \subseteq B$.

An ultrametric space is called **spherically complete** if its ultrametric ball space is spherically complete, and it is said to be \mathbf{S}_2 if its ultrametric ball space is \mathbf{S}_2 .

3.2. An application of Coincidence Theorem II. The following theorem was proved in [10].

THEOREM 3.1 (S. Priß-Crampe, P. Ribenboim). *Let (X, d) be an ultrametric space and take two functions $f, g : X \rightarrow X$. Assume that the following conditions hold:*

- a) $(g(X), d)$ is spherically complete,
- b) $f(X) \subseteq g(X)$,

- c) if $gx \neq fx = gy$, then $d(fx, fy) < d(fx, gx)$,
d) if $d(gx, gy) \leq d(gx, fx)$, then $d(gy, fy) \leq d(gx, fx)$.

Then there is some $z \in X$ such that $fz = gz$.

Analyzing this theorem, we see that there is no need of $g(X)$ being contained in X and X being itself an ultrametric space. On the other hand, the theorem shows some similarity with a theorem of K. Goebel for metric spaces (see Theorem 4.1 below). Learning from this, we formulate the following more general theorem:

THEOREM 3.2. *Take two sets X and Y , and two functions $f, g : X \rightarrow Y$. Assume that the following conditions hold:*

- (PR1) $(g(X), d)$ is a spherically complete ultrametric space,
(PR2) $f(X) \subseteq g(X)$,
(PR3) if $gx \neq fx = gy$, then $d(fx, fy) < d(fx, gx)$,
(PR4) if $d(gx, gy) \leq d(gx, fx)$, then $d(gy, fy) \leq d(gx, fx)$.

Then there is some $z \in X$ such that $fz = gz$.

PROOF. By condition (PR2), $fx \in g(X)$, so we can define the precise ball

$$B(gx, fx) = \{gw \mid w \in X \text{ and } d(gx, gw) \leq d(gx, fx)\}$$

in $(g(X), d)$. We define a ball space on the set X by taking $\mathcal{B}(X)$ to consist of the preimages

$$B_x := g^{-1}(B(gx, fx)) = \{w \in X \mid d(gx, gw) \leq d(gx, fx)\} \subseteq X,$$

for all $x \in X$. Note that $B_x \neq \emptyset$ since $x \in B_x$.

To prove that the conditions of Theorem 1.2 are satisfied, we will first show $B_y \subseteq B_x$ for every $y \in B_x$. Take $y \in B_x$ and $z \in B_y$. This means that $d(gx, gy) \leq d(gx, fx)$ and $d(gy, gz) \leq d(gy, fy)$. By condition (PR4) we have that $d(gy, fy) \leq d(gx, fx)$, whence $d(gy, gz) \leq d(gx, fx)$. By the ultrametric triangle law, we obtain that $d(gx, gz) \leq d(gx, fx)$, so $z \in B_x$. This proves that $B_y \subseteq B_x$.

Take a nest of balls $(B_{x_i})_{i \in I}$ in $(X, \mathcal{B}(X))$. Since $g(B_{x_i}) = B(gx_i, fx_i)$, it follows that $\mathcal{N} = (g(B_{x_i}))_{i \in I}$ is a nest of precise ultrametric balls in $(g(X), d)$. Since by (PR1), $g(X)$ is spherically complete, there is an element in $\bigcap \mathcal{N}$; in view of (PR2), we can write it as $g(z)$ for some $z \in X$. But then $z \in B_{x_i}$ and hence $B_z \subseteq B_{x_i}$ for every $i \in I$, which shows that B_z is contained in the intersection of the nest $(B_{x_i})_{i \in I}$. This proves that the ball space $(X, \mathcal{B}(X))$ is \mathbf{S}_2 .

By condition (PR2), for every $x \in X$ there is $y \in X$ such that $gy = fx \in B(gx, fx)$. Then also $y \in B_x$, so $fx = gy \in f(B_x) \cap g(B_x)$, showing that condition (CS1) is satisfied.

Take a ball $B_x \in \mathcal{B}(X)$. If $g(B_x) = B(gx, fx)$ is not a singleton, then $fx \neq gx$. By (PR2) there is some $y \in X$ such that $gy = fx \neq gx$. We have that $y \in B_x$, so $B_y \subseteq B_x$. Using condition (PR3), we obtain:

$$d(gy, gx) = d(fx, gx) > d(fx, fy) = d(gy, fy).$$

This shows that $gx \notin B(gy, fy)$, whence $x \notin B_y$ and consequently, $B_y \not\subseteq B_x$. Hence, condition (CS2) is satisfied.

Now we can apply Theorem 1.2 to obtain a coincidence point z for f and g . \square

REMARK 3.3. Analyzing the proof, we notice two things. First, we used preimages to obtain a spherically complete ball space on X from the spherically complete ultrametric ball space on $g(X)$. This technique will be discussed further in [5]. It

makes essential use of the fact that ball spaces are very flexible, as they carry only the minimal structure necessary for our purposes.

Second, we see that a function $X \ni x \mapsto B_x \in \mathcal{B}(X)$ was introduced. Both observations suggest that it takes less effort to deduce Theorem 3.2 from B_x -type Coincidence Theorem A. This will be done in the next section.

3.3. A B_x -type coincidence theorem for ultrametric spaces, and applications. In this section we will prove three different coincidence theorems for ultrametric spaces and study the relation between them. The following theorem is a direct application of Corollary 1.8 since by definition, $fx, gx \in B(fx, gx)$ for all $x \in X$.

THEOREM 3.4 (Ultrametric B_x -type Coincidence Theorem). *Let X be an arbitrary set and (Y, d) an ultrametric space. Take two functions $f, g : X \rightarrow Y$. For each $x \in X$, set*

$$B_x := B(fx, gx).$$

*Assume that conditions **(B1')** and **(B2')** of Corollary 1.8 hold. Then there is some $z \in X$ such that $fz = gz$.*

From this theorem we will now derive Theorem 3.2 as well as the following second theorem which is an ultrametric version of a theorem of K. Goebel for metric spaces (see Theorem 4.1 below).

THEOREM 3.5. *Let X be an arbitrary set and (Y, d) an ultrametric space. Take functions $f, g : X \rightarrow Y$ such that the following conditions hold:*

(GU1) $(g(X), d)$ is spherically complete,

(GU2) $f(X) \subseteq g(X)$, and

(GU3) $d(fx, fy) \leq d(gx, gy)$ for all $x, y \in X$, and if $gx \neq gy$, then $d(fx, fy) < d(gx, gy)$.

Then the following holds:

i) *there exists $z \in X$ such that $fz = gz$,*

ii) *if z is a coincidence point and $gz = gy$, then also y is a coincidence point,*

iii) *if z and y are coincidence points, then $gz = gy$.*

REMARK 3.6. Statements ii) and iii) are immediate consequences of the hypothesis, and only the existence of a coincidence point is nontrivial. Indeed, in order to derive ii), assume that z is a coincidence point and take $y \in X$ such that $gz = gy$. Then $d(fz, fy) \leq d(gz, gy) = 0$ by (GU3). Thus $fz = fy$, and since $fz = gz$, it follows that $fy = gz = gy$. In order to derive iii), assume that $y, z \in X$ are coincidence points. We have that $d(gz, gy) = d(fz, fy)$ which by (GU3) is only possible if $gz = gy$.

We have the following logical connection between the three theorems 3.4, 3.2 and 3.5:

PROPOSITION 3.7. *Theorem 3.4 implies Theorem 3.2, and Theorem 3.2 implies Theorem 3.5.*

This proposition is proved by the following lemma which exhibits the logical relations between the conditions of the three theorems.

LEMMA 3.8. Take a set X , an ultrametric space (Y, d) , and functions $f, g : X \rightarrow Y$. For each $x \in X$, set $B_x := B(fx, gx)$. Then:

1) Condition (GU3) of Theorem 3.5 implies conditions (PR3) and (PR4) of Theorem 3.2.

2) Condition (PR3) implies: if B_x is not a singleton, then

$$(3.3) \quad \forall x, y \in X : gy = fx \Rightarrow B_y \subsetneq B_x,$$

and condition (PR4) implies:

$$(3.4) \quad \forall x, y \in X : gy \in B_x \Rightarrow B_y \subseteq B_x.$$

3) Assume that (PR2) holds, so that we can set $Y = g(X)$ in Theorem 3.4. Then (3.3) implies condition **(B1')** of Corollary 1.8, and (3.4) together with (PR1) implies **(B2')**.

PROOF. 1): a) Assume that $gx \neq fx = gy$. Then we can apply (GU3) to obtain that

$$d(fx, fy) < d(gx, gy) = d(gx, fx) = d(fx, gx),$$

which proves (PR3).

b) Assume that $d(gx, gy) \leq d(gx, fx)$. Then we can apply (GU3) to obtain that

$$d(fx, fy) \leq d(gx, gy) \leq d(gx, fx).$$

By (U2), the two inequalities $d(fx, fy) \leq d(gx, fx)$ and $d(gx, fx) \leq d(gx, fx)$ yield:

$$d(gx, fy) \leq d(gx, fx).$$

Again by (U2), this together with $d(gx, gy) \leq d(gx, fx)$ gives

$$d(gy, fy) \leq d(gx, fx),$$

which proves (PR4).

2): a) Assume that B_x is not a singleton, i.e., $fx \neq gx$, and that $gy = fx$. Then by (PR3), $d(fx, fy) < d(fx, gx)$. Hence,

$$d(fy, gy) = d(gy, fy) = d(fx, fy) < d(fx, gx).$$

Since $gy = fx \in B_x$, we obtain from (3.2) that $B_y = B(fy, gy) \subsetneq B(fx, gx) = B_x$.

b) Assume that $gy \in B_x$. Then $d(gx, gy) \leq d(gx, fx)$ and (PR4) shows that $d(gy, fy) \leq d(gx, fx)$, whence by (3.1), $B_y = B(gy, fy) \subseteq B(gx, fx) = B_x$.

3): a) Assume that B_x is not a singleton. Since $Y = g(X)$, there is $y \in X$ such that $gy = fx$. From (3.3) it follows that $B_y \subsetneq B_x$. This proves **(B1')**.

b) Assume that (Y, d) is spherically complete. Take a nest $\mathcal{N} = (B_{x_i})_{i \in I}$. Then there is an element in $\bigcap \mathcal{N}$; since $Y = g(X)$, it can be written as gy for some $y \in X$. Then for all $i \in I$, $gy \in B_{x_i}$ and (3.4) yields that $B_y \subseteq B_{x_i}$. Thus $B_y \subseteq \bigcap \mathcal{N}$, which proves **(B2')**. \square

In the following, we will illustrate the use of Theorem 3.4 by deriving Theorem 3.5 directly from it.

Proof of Theorem 3.5 by direct application of Theorem 3.4:

We assume that the conditions of Theorem 3.5 are satisfied. Then as in part 3) of Lemma 3.8, we can take $Y = g(X)$ in Theorem 3.4, and we set $B_x := B(fx, gx) \in \mathcal{B}(Y)$ for every $x \in X$. Take $x \in X$ and assume that B_x is not a singleton, i.e.,

$fx \neq gx$. Since $f(X) \subseteq g(X)$, there is $y \in X$ such that $gy = fx \neq gx$. By condition (GU3), we obtain that

$$d(gy, fy) = d(fx, fy) < d(gx, gy),$$

which by (3.2) implies that $B_y = B(fy, gy) \subsetneq B(fx, gx) = B_x$. This shows that condition **(B1')** of Theorem 3.4 is satisfied.

Take a nest of balls $\mathcal{N} = (B_{x_i})_{i \in I}$. Since $g(X)$ is spherically complete, there is $y \in X$ such that $gy \in \bigcap \mathcal{N}$. We wish to show that $B_y \subseteq \bigcap \mathcal{N}$. By the ultrametric triangle inequality, we obtain for all $i \in I$:

$$d(fy, gy) \leq \max\{d(fy, fx_i), d(fx_i, gx_i), d(gx_i, gy)\}.$$

We have that

$$d(fy, fx_i) \leq d(gy, gx_i) \leq d(fx_i, gx_i),$$

where the first inequality follows from (GU3) and the second inequality holds since $gy \in B_{x_i} = B(fx_i, gx_i)$. Hence $d(fy, gy) \leq d(fx_i, gx_i)$, which by (3.1) implies that $B_y = B(fy, gy) \subseteq B(fx_i, gx_i) = B_{x_i}$. Thus $B_y \subseteq \bigcap \mathcal{N}$, which proves that condition **(B2')** of Theorem 3.4 holds.

Now by Theorem 3.4, there is some $z \in X$ such that $fz = gz$. \square

4. An application to complete metric spaces

Take a metric space (X, d) . The (closed) **metric balls** are defined as usual as

$$B_r(x) := \{y \in X \mid d(x, y) \leq r\}$$

for $r \in \mathbb{R}^{\geq 0}$ and $x \in X$. We can take these balls to form a ball space on X . The problem is only that even when (X, d) is a complete metric space, this ball space may not be spherically complete (see the discussion in [5]). But if we restrict the radii r to a subset of $\mathbb{R}^{\geq 0}$ which has 0 as its only accumulation point, then the so restricted ball space is spherically complete if and only if (X, d) is complete (we leave the easy proof to the reader). We will use this fact and the flexibility of the concept of ball spaces to prove a theorem by K. Goebel that appeared in [1].

It should be noted that in the case of metric spaces, the proofs of fixed point and coincidence theorems using ball spaces are in general not easier than the direct proofs using Cauchy sequences. However, it is worthwhile pointing out how it can be done by use of metric balls. More importantly, it is a warm-up to the case of ordered abelian groups and fields which we will discuss in the next section. There we will indeed need the idea how to define the balls B_x , which we will develop now.

THEOREM 4.1 (K. Goebel). *Let X be an arbitrary set and (Y, d) a metric space, and take functions $f, g : X \rightarrow Y$ which satisfy the following conditions:*

(GM1) $(g(X), d)$ is a complete metric space,

(GM2) $f(X) \subset g(X)$,

(GM3) there is a positive real number $c < 1$ such that $d(fx, fy) \leq cd(gx, gy)$ for all $x, y \in X$.

Then the following holds:

i) there exists $z \in X$ such that $fz = gz$,

ii) if z is a coincidence point and $gz = gy$, then also y is a coincidence point,

iii) if z and y are coincidence points, then $gz = gy$.

PROOF. In view of Remark 3.6, we only have to prove assertion i). We wish to apply Corollary 1.8. To this end, we define for each $x \in X$ a metric ball B_x in $(g(X), d)$ as follows. If $fx = gx$, we set $B_x = \{gx\}$. Otherwise, we set

$$n_x := \max\{n \in \mathbb{Z} \mid d(fx, gx) \leq c^n\} \quad \text{and} \quad r_x := \frac{c^{n_x}}{1-c}.$$

Then we define:

$$B_x := B_{r_x}(gx) = \left\{ gw \mid w \in X \text{ and } d(gx, gw) \leq \frac{c^{n_x}}{1-c} \right\}.$$

Note that all radii r_x lie in a subset of \mathbb{R} that has 0 as its only accumulation point. Since $fx = gw$ for some $w \in X$ by (GM2), and since $d(fx, gx) \leq c^{n_x} < r_x$ in the case where $fx \neq gx$, we always have that $fx, gx \in B_x$.

Assume that B_x is not a singleton. By (GM2) there is $y \in X$ such that $fx = gy$. By (GM3),

$$d(fy, gy) = d(fy, fx) \leq cd(gy, gx) = cd(fx, gx) \leq c^{n_x+1},$$

so we find that

$$(4.1) \quad n_y > n_x \quad \text{and} \quad r_y \leq cr_x < r_x.$$

We show that $B_y \subseteq B_x$. Take $gw \in B_y$ for some $w \in X$. Then

$$\begin{aligned} d(gw, gx) &\leq d(gw, gy) + d(gy, gx) = d(gw, gy) + d(fx, gx) \\ &\leq \frac{c^{n_y}}{1-c} + c^{n_x} \leq \frac{cc^{n_x}}{1-c} + c^{n_x} \\ &= \left(\frac{c}{1-c} + 1 \right) c^{n_x} = \frac{c^{n_x}}{1-c}. \end{aligned}$$

Thus $gw \in B_x$ and we have proved that $B_y \subseteq B_x$.

We know from (4.1) that B_y has a smaller radius than B_x . However, in a general metric space this does not automatically mean that $B_y \neq B_x$. To ensure inequality, we take $k \in \mathbb{N}$ so large that $c^k r_x < d(fx, gx)$ and iterate the above procedure k many times. In this way we will find some $y \in X$ such that $fx \notin B_y$ or $gx \notin B_y$ and consequently, $B_y \subsetneq B_x$. This shows that condition **(B1')** of Corollary 1.8 is satisfied.

Now we wish to show that also condition **(B2')** is satisfied. Take a nest $\mathcal{N} = (B_{x_i})_{i \in I}$. If this nest contains a smallest ball, then there is nothing to show. Otherwise the nest will contain balls of arbitrarily small radii. Since the ball space $(g(X), \{B_x \mid x \in X\})$ is spherically complete by (GM1) and our remark at the beginning of this section, there is an element in $\bigcap \mathcal{N}$; in view of (GM2), we can write it as $g(z)$ for some $z \in X$. We will show that $fz = gz$, so $B_z = \{gz\} \subseteq \bigcap \mathcal{N}$, as desired. Indeed, for each $i \in I$ we have that $gz \in B_{x_i}$, i.e., $d(gz, gx_i) \leq r_{x_i}$. Now we compute for x being any of the x_i , using (GM3):

$$\begin{aligned} d(gz, fz) &\leq d(gz, gx) + d(gx, fx) + d(fx, fz) \\ &\leq d(gz, gx) + d(gx, fx) + cd(gx, gz) \\ &\leq r_x + c^{n_x} + cr_x = r_x(1 + 1 - c + c) = 2r_x, \end{aligned}$$

which gets arbitrarily small with the radii r_x approaching 0. This shows that $d(gz, fz) = 0$, i.e., $fz = gz$.

Now we can apply Corollary 1.8 to obtain a coincidence point of f and g . \square

5. An application to ordered abelian groups and fields

In this section we will discuss an analogue of Goebel's Coincidence Theorem for the case of ordered abelian groups and fields. In [3] we have proved a fixed point theorem for this case that can be seen as an analogue of Banach's Fixed Point Theorem. Since the underlying additive group of an ordered field is an ordered abelian group, we will concentrate on discussing the case of ordered abelian groups.

The most natural idea to derive a ball space from the ordering of an ordered abelian group $(G, <)$ is to define the **order balls** in G to be the sets of the form

$$B_r(g) := \{h \in G \mid |g - h| \leq r\}$$

for arbitrary $g \in G$ and nonnegative $r \in G$. To obtain a ball space on G , we set

$$\mathcal{B} := \{B_r(g) \mid g \in G, 0 \leq r \in G\}.$$

Then (G, \mathcal{B}) is the **order ball space** associated with $(G, <)$. We say that $(G, <)$ is **symmetrically complete** if (G, \mathcal{B}) is spherically complete. In [6] we have characterized symmetrically complete ordered abelian groups and fields. We showed that every ordered abelian group (or field) can be extended to a symmetrically complete ordered abelian group (or field, respectively), and that all symmetrically complete ordered abelian groups are Hahn products with its archimedean components equal to \mathbb{R} ; it follows that they are divisible and therefore \mathbb{Q} -vector spaces.

Here is an analogue of Goebel's Coincidence Theorem 4.1. For its proof, we will generalize the idea used in the proof of Theorem 4.1, namely that we will work only with nests of balls of a certain type. But this time, this is not achieved by restricting the set of radii in general, but by imposing certain conditions on the nests we consider. This will mean that we cannot derive our theorem directly from one of our general coincidence theorems, but our proof will again use Zorn's Lemma. A similar approach was used in [3], where we introduced the notion of " f -nest".

THEOREM 5.1. *Take a set X , a symmetrically complete ordered abelian group $(G, <)$ and functions $f, g : X \rightarrow G$ such that g is surjective. Assume that there is a positive rational number $c < 1$ such that*

$$|fx - fy| \leq c|gx - gy| \quad \text{for all } x, y \in X.$$

Then there exists $z \in X$ such that $fz = gz$, and also assertions ii) and iii) of Theorem 4.1 hold.

PROOF. As before, we only have to show the existence of a coincidence point. We set $d(x, y) := |x - y|$ and choose any $q \in \mathbb{Q}$ such that

$$1 < \frac{1}{1-c} < q, \quad \text{so} \quad \frac{1}{q} + c < 1.$$

Now we set

$$\begin{aligned} r_x &:= qd(fx, gx) \\ B_x &:= B_{r_x}(gx) = \{gw \mid w \in X \text{ and } d(gx, gw) \leq qd(fx, gx)\}. \end{aligned}$$

Since $q > 1$ and g is surjective, we obtain that $fx, gx \in B_x$.

Assume that B_x is not a singleton, so $d(fx, gx) \neq 0$ and $r_x \neq 0$. Since g is surjective, there is $y \in X$ such that $gy = fx$. By assumption,

$$d(fy, gy) = d(fy, fx) \leq cd(gy, gx) = cd(fx, gx),$$

whence

$$r_y \leq cr_x < r_x.$$

We show that $B_y \subsetneq B_x$. Take $gw \in B_y$ for some $w \in X$. Then

$$\begin{aligned} d(gx, gw) &\leq d(gx, gy) + d(gy, gw) = d(gx, fx) + r_y \\ &\leq d(fx, gx) + cr_x = \left(\frac{1}{q} + c\right) r_x < r_x. \end{aligned}$$

Thus $gw \in B_x$ and we have proved that $B_y \subseteq B_x$. Since $r_x \in G$, the elements $gx \pm r_x$ lie in B_x . But since $r_y < r_x$, they cannot both lie in B_y , so $B_y \subsetneq B_x$.

Take a nest \mathcal{N} of non-singleton balls B_x with the property that for every $B_x \in \mathcal{N}$ there is some $y \in X$ with $gy = fx$ such that $B_y \in \mathcal{N}$. Starting with $x_0 = x$ and $x_1 = y$ we thus find a chain $(x_i)_{i \in \mathbb{N}}$ of elements in X such that $B_{x_i} \in \mathcal{N}$, $gx_{i+1} = fx_i$ and $r_{x_i} \leq c^i r_x$.

Since $g(G, <)$ is symmetrically complete, its order ball space is spherically complete and there is an element in $\bigcap \mathcal{N}$; in view of the surjectivity of g , we can write it as gz for some $z \in X$. We wish to show that $B_z \subseteq \bigcap \mathcal{N}$.

For each $B_x \in \mathcal{N}$ we have that $gz \in B_x$, i.e., $d(gx, gz) \leq r_x$. Note that $r_x \neq 0$ since by our assumption on \mathcal{N} the ball B_x is not a singleton. Now we compute, using the assumptions of our theorem:

$$\begin{aligned} d(fz, gz) &\leq d(fz, fx) + d(fx, gx) + d(gx, gz) \\ &\leq cd(gz, gx) + d(fx, gx) + d(gx, gz) \\ &\leq cr_x + d(fx, gx) + r_x \\ &= r_x \left(c + \frac{1}{q} + 1\right) < 2r_x. \end{aligned}$$

Substituting x_i for x , we find that $d(fz, gz) < 2c^i r_x$ for all $i \in \mathbb{N}$.

Take an arbitrary element $gw \in B_z$; then $d(gz, gw) \leq r_z = qd(fz, gz)$. For x_1 as defined above, we have that $B_{x_1} \in \mathcal{N}$ and thus $gz \in B_{x_1}$, that is, $d(gx_1, gz) \leq r_{x_1} \leq cr_x$. Using these facts, we compute:

$$\begin{aligned} d(gx, gw) &\leq d(gx, gx_1) + d(gx_1, gz) + d(gz, gw) \\ &\leq d(gx, fx) + cr_x + 2qc^i r_x = \left(\frac{1}{q} + c + 2qc^i\right) r_x \end{aligned}$$

for all $i \in \mathbb{N}$. Since $\frac{1}{q} + c < 1$, we can find some $i \in \mathbb{N}$ large enough so that the factor of r_x in the last expression is at most 1. This shows that $gw \in B_x$ and we have proved that $B_z \subseteq B_x$. As this holds for all $B_x \in \mathcal{N}$, we see that $B_z \subseteq \bigcap \mathcal{N}$.

Consider the set of all nests \mathcal{N} with the property that for every non-singleton $B_x \in \mathcal{N}$ there is some $y \in X$ with $gy = fx$ such that $B_y \in \mathcal{N}$. This set is inductively ordered by inclusion: the union over an ascending chain of nests with the prescribed property is again a nest with the prescribed property. By Zorn's Lemma, there is a maximal nest \mathcal{N} . Suppose that this nest does not contain a smallest ball. Then all of its balls are non-singletons and as we have shown above, there is an element $z \in X$ such that $B_z \subseteq \bigcap \mathcal{N}$. Since \mathcal{N} does not contain a smallest ball, we find that $B_z \notin \mathcal{N}$. We set $z_0 = z$, and choosing by induction a chain $(z_i)_{i \in \mathbb{N}}$ of elements in X such that $gz_{i+1} = fz_i$, we obtain a nest $\mathcal{N}_z = (B_{z_i})_{i \in \mathbb{N}}$ with the prescribed property. Then $\mathcal{N} \cup \mathcal{N}_z$ is a nest with the prescribed property, properly containing \mathcal{N} . As this contradicts the maximality of \mathcal{N} , we find that \mathcal{N} contains a

smallest ball B_z . This must be a singleton, because otherwise, taking $y \in X$ such that $gy = fz$, we would obtain smaller balls B_y , but none of them contained in \mathcal{N} , contradicting the prescribed property of \mathcal{N} . Since B_z is a singleton, we have that $fz = gz$, as desired. \square

6. Fixed point theorems

6.1. General fixed point theorems. In this section, we will derive general fixed point theorems for ball spaces from the main theorems and corollaries of the Introduction. Throughout, let (X, \mathcal{B}) be a ball space and consider a function

$$f : X \rightarrow X.$$

The following fixed point theorems are immediately obtained from the corresponding coincidence theorems by taking $Y = X$ and g to be the identity function.

THEOREM 6.1 (Fixed Point Theorem I). *Assume that f satisfies the following conditions:*

(FPT1) *for every $B \in \mathcal{B}$, $f(B) \subseteq B$,*

(FPT2) *for every nest of balls \mathcal{N} in \mathcal{B} , either $\bigcap \mathcal{N}$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneq \bigcap \mathcal{N}$.*

Then every ball in \mathcal{B} contains a fixed point of f .

THEOREM 6.2 (Fixed Point Theorem II). *Assume that (X, \mathcal{B}) is an \mathbf{S}_2 ball space and that f satisfies the following conditions:*

(FPS1) *for every $B \in \mathcal{B}$, $f(B) \cap B \neq \emptyset$,*

(FPS2) *for every $B \in \mathcal{B}$, $f(B)$ is a singleton or there is $B' \in \mathcal{B}$ such that $B' \subsetneq B$.*

Then every ball in \mathcal{B} contains a fixed point of f .

THEOREM 6.3 (B_x -type Fixed Point Theorem A). *Assume that*

$$(6.1) \quad X \ni x \mapsto B_x \in \mathcal{B}(X)$$

is a function such that the following conditions hold:

(FPA1) *If B_x is not a singleton and $f(B_x) \cap B_x \neq \emptyset$, then there is $y \in X$ such that $B_y \subsetneq B_x$ and $f(B_y) \cap B_y \neq \emptyset$.*

(FPA2) *If B_{x_i} , $i \in I$, is a nest of balls in $\mathcal{B}(X)$ such that $f(B_{x_i}) \cap B_{x_i} \neq \emptyset$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i \in I} B_{x_i}$ and $f(B_y) \cap B_y \neq \emptyset$.*

Then for each $x_0 \in X$ with $f(B_{x_0}) \cap B_{x_0} \neq \emptyset$, there is a fixed point for f in B_{x_0} .

COROLLARY 6.4. *Assume that there is a function (6.1) such that $f(B_x) \cap B_x \neq \emptyset$ for all $x \in X$ and the following conditions are satisfied:*

(FPA1') *If B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneq B_x$.*

(FPA2') *The ball space $(X, \{B_x \mid x \in X\})$ is \mathbf{S}_2 .*

Then X contains a fixed point of f .

THEOREM 6.5 (B_x -type Fixed Point Theorem B). *Assume that*

$$(6.2) \quad X \ni x \mapsto B_x \in \mathcal{B}(X)$$

is a function such that the following conditions hold:

(FPB1) *If B_x is not a singleton and $x, fx \in B_x$, then there is $y \in X$ such that $B_y \subsetneq B_x$ and $y, fy \in B_y$.*

(FPB2) If B_{x_i} , $i \in I$, is a nest of balls such that $x_i, fx_i \in B_{x_i}$ for all $i \in I$, then there is $y \in X$ such that $B_y \subseteq \bigcap_{i \in I} B_{x_i}$ and $y, fy \in B_y$.

If there is any $x_0 \in X$ such that $x_0, fx_0 \in B_{x_0}$, then X contains a fixed point of f .

COROLLARY 6.6. Assume that there is a function (6.2) such that $x, fx \in B_x$ for all $x \in X$ and the following conditions are satisfied:

(FPB1') If B_x is not a singleton, then there is $y \in X$ such that $B_y \subsetneq B_x$.

(FPB2') The ball space $(Y, \{B_x \mid x \in X\})$ is \mathbf{S}_2 .

Then X contains a fixed point of f .

We will illustrate the use of some of these results by proving one more fixed point theorem for ball spaces. If $f : X \rightarrow X$ is a function, then a subset $B \subseteq X$ will be called **f -contracting** if $f(B) \subset B$ and the inclusion is strict whenever B is not a singleton. The following is Theorem 1.1 of [3].

THEOREM 6.7. Take a ball space (X, \mathcal{B}) and a function $f : X \rightarrow X$ which satisfies the following conditions:

(FC1) there is at least one f -contracting ball,

(FC2) for every f -contracting ball $B \in \mathcal{B}$, the image $f(B)$ contains an f -contracting ball,

(FC3) the intersection of every nest of f -contracting balls contains an f -contracting ball.

Then f admits a fixed point.

PROOF. We take \mathcal{B}_f to be the collection of all f -contracting balls in \mathcal{B} . By **(FC1)**, \mathcal{B}_f is nonempty, so (X, \mathcal{B}_f) is itself a ball space. Since every $B \in \mathcal{B}_f$ is an f -contracting ball, we have that $f(B) \subset B$, hence condition **(FPT1)** of Theorem 6.1 is satisfied.

Now take a nest of balls \mathcal{N} in \mathcal{B}_f and assume that $\bigcap \mathcal{N}$ is not a singleton. By **(FC3)**, $\bigcap \mathcal{N}$ contains an f -contracting ball $B' \in \mathcal{B}_f$. If B' is a singleton, then $B' \subsetneq \bigcap \mathcal{N}$. If B' is not a singleton, then by definition, $f(B') \subsetneq B'$, and by **(FC2)**, $f(B')$ contains an f -contracting ball $B'' \in \mathcal{B}_f$. Then $B'' \subsetneq f(B') \subsetneq B'$, whence $B'' \subsetneq \bigcap \mathcal{N}$. This proves that condition **(FPT1)** is also satisfied.

Now Theorem 6.1 shows the existence of a fixed point for f . □

6.2. An ultrametric fixed point theorem. We show how to derive from Corollary 6.6 the main ultrametric fixed point theorem of Pri  -Crampe and Ribenboim ([9]).

THEOREM 6.8 (S. Pri  -Crampe, P. Ribenboim). Take a spherically complete ultrametric space (X, d) and a function $f : X \rightarrow X$ which is **contracting on orbits**, i.e., satisfies the following conditions for all $x, y \in X$:

(CO1) $d(fx, fy) \leq d(x, y)$,

(CO2) $d(fx, f^2x) < d(x, fx)$ if $x \neq fx$.

Then f has a fixed point in X .

PROOF. For every $x \in X$ we set

$$B_x := B(x, fx).$$

Then $x, fx \in B_x$.

If B_x is not a singleton, i.e., $x \neq fx$, then $d(fx, f^2x) < d(x, fx)$ by **(CO2)**. It follows from (3.2) that $B_{fx} = B(fx, f^2x) \subsetneq B(x, fx) = B_x$. This proves that condition **(FPB1')** of Corollary 6.6 is satisfied.

Take a nest $\mathcal{N} = (B_{x_i})_{i \in I}$. Since (X, d) is spherically complete, there is some $y \in \bigcap \mathcal{N}$. For all $i \in I$, $y \in B_{x_i}$ and therefore, $d(y, x_i) \leq d(x_i, fx_i)$. From **(CO1)** we infer that $d(fy, fx_i) \leq d(y, x_i)$, whence

$$d(fx_i, fy) = d(fy, fx_i) \leq d(y, x_i) \leq d(x_i, fx_i).$$

By (U2), the two inequalities $d(x_i, fx_i) \leq d(x_i, fx_i)$ and $d(y, x_i) \leq d(x_i, fx_i)$ yield:

$$d(y, fx_i) \leq d(x_i, fx_i).$$

Again by (U2), this together with $d(fx_i, fy) \leq d(x_i, fx_i)$ gives

$$d(y, fy) \leq d(x_i, fx_i).$$

Hence by (3.1), $B_y = B(y, fy) \subseteq B(x_i, fx_i) = B_{x_i}$. Consequently, $B_y \subseteq \bigcap \mathcal{N}$, hence also condition **(FPB2')** of Corollary 6.6 is satisfied. Now we can apply Corollary 6.6 to obtain a fixed point of f . \square

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