THE VALUATION THEORY OF DEEPLY RAMIFIED FIELDS AND ITS CONNECTION WITH DEFECT EXTENSIONS

FRANZ-VIKTOR KUHLMANN AND ANNA RZEPKA

Abstract. We study in detail the valuation theory of deeply ramified fields and introduce and investigate several other related classes of valued fields. Further, a classification of defect extensions of prime degree of valued fields that was earlier given only for the equicharacteristic case is generalized to the case of mixed characteristic by a unified definition that works simultaneously for both cases. It is shown that deeply ramified fields and the other valued fields we introduce only admit one of the two types of defect extensions, namely the ones that appear to be more harmless in open problems such as local uniformization and the model theory of valued fields in positive characteristic. We use our knowledge about such defect extensions to give a new, valuation theoretic proof of the fact that algebraic extensions of deeply ramified fields are again deeply ramified. We also prove finite descent, and under certain conditions even infinite descent, for deeply ramified fields. These results are also proved for two other related classes of valued fields. The classes of valued fields under consideration can be seen as generalizations of the class of tame valued fields. Our paper supports the hope that it will be possible to generalize to deeply ramified fields several important results that have been proven for tame fields and were at the core of partial solutions of the two open problems mentioned above.

1. Introduction

The main topics of this paper are the defect of valued field extensions, which lies at the heart of longstanding open problems in algebraic geometry and model theoretic algebra, and the valuation theory of deeply ramified fields. By studying the latter in depth, we will exhibit the connection with the former. On the one hand, this enables us to better understand deeply ramified fields, and on the other hand, it shows us a possible direction in our attempt to tame the defect.

Our interest in the defect owes its existence to the following well known deep open problems in positive characteristic:

1) resolution of singularities in arbitrary dimension,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, and of its perfect hull.

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Both problems are connected with the structure theory of valued function fields of positive characteristic p. The main obstruction here is the phenomenon of the **defect**, which we will define now.

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the valuation ideals of L and K. The value group of the valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av.

We will say that a valued field extension $(L|K,v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski,

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{1}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p}=\operatorname{char} Kv$ if it is positive and $\tilde{p}=1$ otherwise. The factor $d(L|K,v):=\tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. We call $(L|K,v)$ a **defect extension** if $d(L|K,v)>1$, and a **defectless extension** if $d(L|K,v)=1$. Nontrivial defect only appears when $\operatorname{char} Kv=p>0$, in which case $\tilde{p}=p$. A henselian field (K,v) is called a **defectless field** if all of its finite extensions are defectless.

Throughout this paper, when we talk of a **defect extension** $(L|K,v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. Then it follows from (1) that $[L:K]=p=\operatorname{char} Kv$ and that $(vL:vK)=1=[Lv:Kv]$; the latter means that $(L|K,v)$ is an **immediate extension**, i.e., the canonical embeddings $vK\hookrightarrow vL$ and $Kv\hookrightarrow Lv$ are onto.

Via ramification theory, the study of defect extensions can be reduced to the study of purely inseparable extensions and of Galois extensions of degree $p=\operatorname{char} Kv$. To this end, we fix an extension of v from K to its algebraic closure $\tilde{K}$. We denote the separable-algbraic closure of K by $K^{\rm sep}$. The **absolute ramification field of** (K,v) (with respect to the chosen extension of v), denoted by (K^r,v), is the ramification field of the normal extension $(K^{\rm sep}|K,v)$. If $a\in\tilde{K}$ such that $(K(a)|K,v)$ is a defect extension, then $(K^r(a)|K^r,v)$ is a defect extension with the same defect (see Proposition 2.12). On the other hand, $K^{\rm sep}|K^r$ is a p-extension, so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p.

Galois defect extensions of degree p of valued fields of characteristic $p>0$ (valued fields of **equal characteristic**) have been classified by the first author in [19]. There the extension is said to have **dependent** defect if it is related to a purely inseparable defect extension of degree p in a way that we will explain in Section 3.3, and to have **independent** defect otherwise. Note that the condition for the defect to be dependent implies that the purely inseparable defect extension does not lie in the completion of (K,v), hence if (K,v) lies dense in its perfect hull (with respect to the topology induced by the valuation), then it cannot have Galois defect extensions of prime degree with dependent defect.

The classification of defect extensions is important because work by M. Temkin (see e.g. [32]) and by the first author indicates that dependent defect appears to be more harmful to the above cited problems than independent defect. Also results in the present paper point in this direction; see the discussion in Remark 1.11.

An analogous classification of Galois defect extensions of degree p of valued fields of characteristic 0 with residue fields of characteristic $p > 0$ (valued fields of **mixed characteristic**) has so far not been given. But such a classification is important for instance for the study of infinite algebraic extensions of the field $\mathbb{Q}_p$ of p-adic numbers, which in contrast to $\mathbb{Q}_p$ itself may well admit defect extensions. Indeed, $\mathbb{Q}_p^{ab}$, the maximal abelian extension of $\mathbb{Q}_p$, is such a field. Other examples will be given in a subsequent paper [27]. Moreover, we wish to study the valuation theory of deeply ramified fields (such as $\mathbb{Q}_p^{ab}$), which will be introduced below, in full generality without restriction to the equal characteristic case. For these fields in particular it is important to work out the similarities between the equal and the mixed characteristic cases.

The obvious problem for the definition of "dependent defect" in the mixed characteristic case is that a field of characteristic 0 has no nontrivial inseparable extensions. However, there is a characterization of independent defect equivalent to the one given in [19] that readily works also in the mixed characteristic case, and we use it to give a unified definition, as follows. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. For every σ in its Galois group $\mathrm{Gal}\,(L|K)$, with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \middle| \; f \in L^\times \right\} \tag{2}$$

This set is a final segment of vK and independent of the choice of σ (see Theorems 3.4 and 3.5); we denote it by $\Sigma_{\mathcal{E}}$. We will show that it is the unique ramification jump of $\mathcal{E}$ and that $I_{\mathcal{E}} := \{a \in L \mid va \in \Sigma_{\mathcal{E}}\}$ is the unique ramification ideal of $\mathcal{E}$ (for definitions, see Section 2.4). We will explicitly compute $\Sigma_{\mathcal{E}}$ and $I_{\mathcal{E}}$ in Section 3.

We say that $\mathcal{E}$ has **independent defect** if

$$\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\} \text{ for some proper convex subgroup } H_{\mathcal{E}} \text{ of } vK; \tag{3}$$

otherwise we will say that $\mathcal{E}$ has **dependent defect**. If (K, v) has rank 1 (i.e., its value group is order isomorphic to a subgroup of $\mathbb{R}$), then condition (3) just means that $\Sigma_{\mathcal{E}}$ consists of all positive elements in vK.

That our definition of "independent defect" in mixed characteristic is the right one is supported by the following observation. Take a valued field of positive characteristic. If it lies dense in its perfect hull, then by what we have said before, all Galois defect extensions must have independent defect. If in addition the field is complete and of rank 1, then it is a perfectoid field. What about perfectoid fields of mixed characteristic? They share with their tilts, which are perfectoid fields of positive characteristic, isomorphic absolute Galois groups. Hence we expect that also perfectoid fields in mixed characteristic admit only independent defect extensions. This indeed holds with our definition. Similarly, the Fontaine-Wintenberger Theorem states that the fields $\mathbb{Q}_p(p^{1/p^n} \mid n \in \mathbb{N})$ and $\mathbb{F}_p((t))(t^{1/p^n} \mid n \in \mathbb{N})$ have isomorphic absolute Galois groups. Both are deeply ramified (and even semitame) fields (definitions are given below), and as such are independent defect fields, as we will show in Theorem 1.10.

For our purposes, the properties of completeness and rank 1 are irrelevant, and we prefer to work with a more flexible (and first order axiomatizable) notion. In fact, all perfectoid fields are deeply ramified, in the sense of [11]. Take a valued field (K,v) with valuation ring $\mathcal{O}_K$. Choose any extension of v to $K^{\rm sep}$ and denote the valuation ring of $K^{\rm sep}$ with respect to this extension by $\mathcal{O}_{K^{\rm sep}}$. Then (K,v) is a **deeply ramified field** if

$$\Omega_{\mathcal{O}_{K^{\rm sep}}|\mathcal{O}_K} \;=\; 0\,, \tag{4}$$

where $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra. This definition does not depend on the chosen extension of the valuation from K to $K^{\rm sep}$.

According to [11, Theorem 6.6.12 (vi)], a nontrivially valued field (K,v) is deeply ramified if and only if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneqq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) if $\operatorname{char} Kv = p > 0$, then the homomorphism

$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \tag{5}$$

is surjective, where $\mathcal{O}_{\hat{K}}$ denotes the valuation ring of the completion of (K,v).

Axiom (DRvr) means that modulo $p\mathcal{O}_{\hat{K}}$ every element in $\mathcal{O}_{\hat{K}}$ is a p-th power.

By altering axiom (DRvg) we will now introduce new classes of valued fields, one of them containing the class of deeply ramified fields, and one contained in it in the case of positive residue characteristic. We will call (K,v) a **roughly deeply ramified field**, or in short an **rdr field**, if it satisfies axiom (DRvr) together with:

(DRvp) if $\operatorname{char} Kv = p > 0$, then vp is not the smallest positive element in the value group vK.

The reason for the choice of this notion will become visible in Proposition 1.3 and will be further discussed in Remark 1.11. Note that (DRvg) implies (DRvp).

If $\operatorname{char} Kv = p > 0$, then (DRvg) certainly holds whenever vK is divisible by p. We will call (K,v) a **semitame field** if it satisfies axiom (DRvr) together with:

(DRst) if $\operatorname{char} Kv = p > 0$, then the value group vK is p-divisible.

We note:

Proposition 1.1. *The properties (DRvg), (DRvp) and (DRst) are first order axiomatizable in the language of valued fields, and so are the classes of semitame, deeply ramified and rdr fields of fixed characteristic.*

We will give the proof of this proposition and of almost all results that we will describe now in Section 4.

Let us mention at this point that it has been conjectured that the elementary theory of the perfect hull of $\mathbb{F}_p((t))$ is decidable, but no proof has been given so far. As a perfect valued field of positive characteristic, it is semitame, and understanding its valuation theory and in particular its defects may lay the basis for a future proof. Mastering the defect has already shown to be an efficient tool to prove results on local uniformization and the model theory of valued fields, as demonstrated in [15, 16, 17, 22].

The notion of "semitame field" is reminiscent of that of "tame field". Let us recall the definition of "tame". For the purpose of this paper we will slightly generalize the notion of "tame extension" as defined in [22] (there, tame extensions were only defined over henselian fields). A unibranched extension $(L|K,v)$ will be called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(**TE1**) The ramification index $(vE : vK)$ is not divisible by $\mathrm{char}\, Kv$.
(**TE2**) The residue field extension $Ev|Kv$ is separable.
(**TE3**) The extension $(E|K, v)$ is defectless.

A valued field (K, v) is called a **tame field** if it is henselian and its algebraic closure with the unique extension of the valuation is a tame extension, and a **separably tame field** if it is henselian and its separable-algebraic closure is a tame extension. The absolute ramification field (K^r, v) is the unique maximal tame extension of the henselian field (K, v) by [9, Theorem (22.7)] (see also [26, Proposition 4.1]). Hence a henselian field is tame if and only if its absolute ramification field is already algebraically closed; in particular, every tame field is perfect.

In contrast to tame and separably tame fields, we do not require semitame fields to be henselian; in this way they become closer to deeply ramified fields. The other fundamental difference to tame fields is that semitame fields may admit defect extensions, but as we will see in Theorem 1.10 below, only those with independent defect. This justifies the hope that many of the results that have been proved for tame fields and applied to the problems we have cited in the beginning (see [22, 23]) can be generalized (at least) to the case of henselian semitame fields.

All valued fields of residue characteristic 0 are semitame and rdr fields, and they are deeply ramified fields if and only if (DRvg) holds. Likewise, all henselian valued fields of residue characteristic 0 are tame fields. In the present paper, we are not interested in the case of residue characteristic 0, so we will always assume that $\mathrm{char}\, Kv = p > 0$. We will now summarize the basic facts about the connections between the properties we have introduced. The proofs will be provided in Section 4.

Theorem 1.2. *1) If (K, v) is a nontrivially valued field with $\mathrm{char}\, Kv = p > 0$, then the following logical relations between its properties hold:*

tame field $\Rightarrow$ separably tame field $\Rightarrow$ semitame field $\Rightarrow$
deeply ramified field $\Rightarrow$ rdr field.

2) For a valued field (K, v) of rank 1 with $\mathrm{char}\, Kv = p > 0$, the three properties "semitame field", "deeply ramified field" and "rdr field" are equivalent.

3) For a nontrivially valued field (K, v) of characteristic $p > 0$, the following properties are equivalent:
a) (K, v) is a semitame field,
b) (K, v) is a deeply ramified field,
c) (K, v) is an rdr field,
d) (K, v) satisfies axiom (DRvr),
e) the completion of (K, v) is perfect,
f) (K, v) is dense in its perfect hull,
g) (K^p, v) is dense in (K, v).

4) Every perfect valued field of positive characteristic is a semitame field.

We note that for valued fields of mixed characteristic, axiom (DRvr) can be substituted by a version where $\hat{K}$ is replaced by K (see Lemma 4.1), and even p can be replaced by elements of certain lower or higher values (see Propositions 4.4 and 4.11).

In [24] the equivalence of assertions a) and f) of part 3) of this theorem is used to show that every valued field of positive characteristic that has only finitely many

Artin-Schreier extensions is a semitame field. This proves that a nontrivially valued field of positive characteristic that is definable in an NTP_2 theory is a semitame field, as it is shown in [6] that such a field has only finitely many Artin-Schreier extensions.

Take a valued field (K,v) of characteristic 0 with residue characteristic $p>0$. Decompose $v=v_0\circ v_p\circ\overline{v}$, where v_0 is the finest coarsening of v that has residue characteristic 0, v_p is a rank 1 valuation on Kv_0, and $\overline{v}$ is the valuation induced by v on the residue field of v_p (which is of characteristic $p>0$). The valuations v_0 and $\overline{v}$ may be trivial. Further, following [13], we call the value group vK **roughly p-divisible** if $v_p\circ\overline{v}\,(Kv_0)$ (the value group of $v_p\circ\overline{v}$ on Kv_0) is p-divisible. Then we have:

Proposition 1.3. *Under the above assumptions, the following assertions are equivalent:*

a) (K,v) *is a roughly deeply ramified field,*

b) $(Kv_0,v_p\circ\overline{v})$ *is a deeply ramified field,*

c) (Kv_0,v_p) *is a deeply ramified field,*

d) (K,v) *satisfies (DRvr) and* vK *is roughly* p*-divisible.*

Note that by part 2) of Theorem 1.2, the properties "semitame field", "deeply ramified field" and "rdr field" are equivalent for (Kv_0,v_p).

From Theorem 1.2 and Proposition 1.3 it can be deduced that the three properties "semitame", "deeply ramified" and "rdr" behave well for composite valuations.

Proposition 1.4. *Take an arbitrary valued field* (K,v) *and assume that* $v=w\circ\overline{w}$ *with* w *and* $\overline{w}$ *nontrivial. Then* (K,v) *is an rdr field if and only if* (K,w) *and* $(Kw,\overline{w})$ *are. If* $\mathrm{char}\,Kw>0$*, then for* (K,v) *to be an rdr field it suffices that* (K,w) *is an rdr field. The same holds for "semitame" and "deeply ramified" in place of "rdr".*

If $\mathrm{char}\,Kw=0$*, then for* (K,v) *to be an rdr field it suffices that* $(Kw,\overline{w})$ *is an rdr field.*

For deeply ramified fields, the first assertion of the next theorem has been proved before (see [11, Corollary 6.6.16 (i)]), based on their definition given in (4).

Theorem 1.5. *Every algebraic extension of a deeply ramified field is again deeply ramified. The same holds for semitame fields and for rdr fields.*

We will give the easy proof for the equal characteristic case in Proposition 4.7. The proof for the mixed characteristic case can be reduced to the study of Galois defect extensions of prime degree via the following theorem:

Theorem 1.6. *Take a valued field* (K,v)*, fix any extension of* v *to* $\tilde{K}$*, and let* (K^r,v) *be the corresponding absolute ramification field of* (K,v)*. Then* (K^r,v) *is an rdr field if and only if* (K,v) *is, and* (K^r,v) *is a semitame field if and only if* (K,v) *is. If* (K,v) *is an rdr field, then* (K^r,v) *is a deeply ramified field.*

Note that the last assertion holds since if (K^r,v) is an rdr field, then it is already a deeply ramified field because vK^r is divisible by every prime distinct from the residue characteristic. However, it is not true in general that this implies that (K,v) is deeply ramified, since (DRvg) always holds in (K^r,v) (as long as v is nontrivial), while it may not hold in (K,v).

Corollary 1.7. *1) Take an algebraic (not necessarily finite) extension $(L|K,v)$ of valued fields. If $K^r = L^r$ with respect to some extension of v from L to $\tilde{L}$, then (L,v) is an rdr field if and only if (K,v) is, and the same holds for "semitame" in place of "rdr".*

2) Take a valued field (K,v), fix any extension of v to $\tilde{K}$, and let (K^h,v) be the henselization of (K,v) in $(\tilde{K},v)$. Then (K^h,v) is a deeply ramified field if and only if (K,v) is, and the same holds for "rdr" and "semitame" in place of "deeply ramified".

Note that the assumption of part 1) holds in particular if $(L|K,v)$ is a tame extension. We see that we have infinite descent of the properties "rdr" and "semitame" through extensions in the absolute ramification field and in particular through tame extensions. If the lower field already satisfies (DRvg), then the descent also works for "deeply ramified". For all of the properties, we have finite descent in general:

Theorem 1.8. *Take a finite extension $(L|K,v)$. If (L,v) is a deeply ramified field, then so is (K,v). The same holds for "rdr" and "semitame" in place of "deeply ramified".*

The next theorem addresses the connection of the properties we have defined with the classification of the defect. Take a valued field (K,v) of residue characteristic $p>0$. If $\operatorname{char} K = 0$, then we denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp; it is equal to the value group $v_p \circ \overline{v}\,(Kv_0)$ which we defined earlier. If $\operatorname{char} K > 0$, then we set $(vK)_{vp} = vK$. If (K,v) is of mixed characteristic, then we set $K' := K(\zeta_p)$, where ζ_p is a primitive p-th roots of unity; otherwise, we set $K' := K$. Then we call (K,v) an **independent defect field** if for some extension of v to K', all Galois defect extensions of (K',v) of degree p have independent defect. (This definition does not depend on the chosen extension of v as all extensions are conjugate.) We will show in Theorem 1.10 that all rdr fields, and hence all deeply ramified and semitame fields, are independent defect fields.

Remark 1.9. If (K,v) is an rdr field of mixed characteristic, then it does not necessarily contain a primitive p-th root of unity. In this case, a condition on Galois defect extensions may not contain enough information. We also need information on extensions by p-th roots which will then not be Galois. This is why we pass to the field K' in our definition. #

From Proposition 1.3 we see that in the case of fields (K,v) of mixed characteristic, (Kv_0, v_p) is essential for the rdr property. In the theory of formally p-adic fields (cf. [30]), Kv_0 is called the **core field**, and it is usually considered with the valuation $v_p \circ \overline{v}$. However, as we have just noted, the valuation v_p itself is important, and we will have to work with its residue field $(Kv_0)v_p = Kv_0v_p$. In the decomposition $v = v_0 \circ v_p \circ \overline{v}$ the valuation v_p is at the center, and so we define $\operatorname{crf}(K,v) := Kv_0v_p$ as one may call it the **central residue field**. If (K,v) is of equal characteristic, we set $\operatorname{crf}(K,v) := Kv$.

Theorem 1.10. *1) Take a valued field (K,v) with $\operatorname{char} Kv = p > 0$. Then (K,v) is an rdr field if and only if $(vK)_{vp}$ is p-divisible, $\operatorname{crf}(K,v)$ is perfect, and (K,v) is an independent defect field.*

2) A nontrivially valued field (K,v) is semitame if and only if every unibranched Galois extension of (K',v) of prime degree is either tame or an extension with independent defect.

Remark 1.11. Let us consider a field (K,v) of mixed characteristic. Then (K,v_0) is a field of residue characteristic 0. If (K,v) is henselian, then so is (K,v_0), and it is a defectless field and satisfies strong model theoretic principles (e.g. completeness, model completeness and decidability relative to their value groups and residue fields). Recently, the idea appeared in the literature that in order for (K,v) to have such good properties, one only has to ask that the core field satisfies suitable conditions. For example, [12] deals with the generalization of model theoretic properties from the class of algebraically maximal Kaplansky fields to the more general class of henselian fields whose core fields are algebraically maximal Kaplansky fields. A valued field is called **algebraically maximal** (respectively, **separable-algebraically maximal**) if it admits no nontrivial immediate algebraic (respectively, separable-algebraic) extensions. Since henselizations are immediate separable-algebraic extensions, every separable-algebraically maximal field is henselian. If the core field of (K,v) is a Kaplansky field, then we may call (K,v) a **roughly Kaplansky field**, as the passage from Kaplansky field to roughly Kaplansky field is nothing but the passage from p-divisible value group to roughly p-divisible value group.

Similarly, (K,v) is a defectless field if and only if its core field is. Let us call (K,v) a roughly tame field if it is henselian and its core field is a tame field; again the generalization consists in replacing "p-divisible value group" by "roughly p-divisible value group". It is shown in [31] that a henselian field (K,v) is roughly tame if and only if all of its algebraic extensions are defectless fields. The reader may note that in general, infinite algebraic extensions of defectless fields may not again be defectless fields.

We have chosen the name "roughly deeply ramified field" since in this case, the generalization of the notion "deeply ramified field" consists in replacing condition (DRvg) by condition "roughly p-divisible value group", which removes any condition on the value group v_0K. It should be noted that the value group of every rdr and hence also every deeply ramified field is roughly p-divisible (see Lemma 4.3). From Theorem 1.5 together with part 1) of Theorem 1.10 we know that for every rdr field, in analogy to the case of roughly tame fields, every algebraic extension is an independent defect field. At this point, we do not know whether the latter property characterizes the rdr fields. In any case, it appears to be unlikely that a similar result can be shown with "independent" replaced by "dependent"; if this is indeed impossible, then it is another indication that independent defect is more harmless than dependent defect. #

The classification of Galois defect extensions of prime degree in the equal characteristic case is also an important tool in the proof of Theorem 1.2 of [19], which we will state now.

Theorem 1.12. *A valued field of positive characteristic is a henselian and defectless field if and only if it is separable-algebraically maximal and each finite purely inseparable extension is defectless.*

This theorem in turn is used in [18] for the construction of an example showing that a certain natural axiom system for the elementary theory of $\mathbb{F}_p((t))$ ("henselian defectless valued field of characteristic p with residue field $\mathbb{F}_p$ and value group a $\mathbb{Z}$-group") is not complete.

A full analogue of Theorem 1.12 in mixed characteristic is not presently known. However, in a subsequent paper [27] we will show:

Theorem 1.13. *Every algebraically maximal rdr field is a perfect, henselian and defectless field.*

The study of mixed characteristic independent defect fields that are not rdr fields is only at its infancy. We hope that the valuation theoretic proof of Theorem 1.5 will be a basis for further insight. At this point, we are able to prove:

Proposition 1.14. *1) If (K^r,v) is an independent defect field, then so is (K,v).*
2) A valued field (K,v) of equal positive characteristic is an independent defect field if and only if every immediate purely inseparable extension of (K,v) lies in its completion.

It is an important fact that the properties of valued fields of being henselian, tame, semitame, deeply ramified or rdr all are preserved under infinite algebraic extensions. In contrast to this, the properties of being a defectless or an independent defect field are not necessarily preserved, as will be shown in [27] by the construction of a suitable algebraic extension of $\mathbb{Q}_p$. However, we conjecture that if (K,v) is an independent defect field, then so is (K^r,v).

Continuing the work presented in [7], the idea has been suggested to employ higher ramification groups for the study of the ramification theory of 2-dimensional valued function fields. When working over valued fields with arbitrary value groups, the classical ramification numbers have to be replaced by **ramification jumps** which can be understood as cuts (or equivalently, final segments) in the value group (cf. Section 2.4). We will characterize Galois defect extensions $\mathcal{E}$ of prime degree having independent defect via their ramification jumps $\Sigma_{\mathcal{E}}$ and their associated ramification ideals $I_{\mathcal{E}}$. In Section 3.2 we will prove:

Theorem 1.15. *Take a Galois defect extension $\mathcal{E}=(K(a)|K,v)$ of prime degree. Then $\Sigma_{\mathcal{E}}$ is the unique ramification jump of $\mathcal{E}$, $I_{\mathcal{E}}$ is the unique ramification ideal of $\mathcal{E}$, and the following assertions are equivalent:*

a) $\mathcal{E}$ has independent defect,

b) the ramification jump of $\mathcal{E}$ is equal to $\{\alpha\in vK\mid \alpha>H_{\mathcal{E}}\}$ for some proper convex subgroup $H_{\mathcal{E}}$ of vK,

c) the ramification ideal of $\mathcal{E}$ is a nontrivial prime ideal of $\mathcal{O}_L$.

If assertion c) holds, then the localization of $\mathcal{O}_L$ with respect to the ramification ideal is a valuation ring on L containing $\mathcal{O}_L$, i.e., the associated valuation is a coarsening of v.

If the rank of (K,v) is 1, then $H_{\mathcal{E}}$ can only be equal to $\{0\}$ and if the ramification ideal is a prime ideal, then it can only be equal to $\mathcal{M}_L$.

2. Preliminaries

2.1. Cuts, distances and defect.

We recall basic notions and facts connected with cuts in ordered abelian groups and distances of elements of valued field extensions. For the details and proofs see Section 2.3 of [19] and Section 3 of [28].

Take a totally ordered set $(T,<)$. For a nonempty subset S of T and an element $t\in T$ we will write $S<t$ if $s<t$ for every $s\in S$. A set $S\subseteq T$ is called an **initial segment** of T if for each $s\in S$ every $t<s$ also lies in S. Similarly, $S\subseteq T$ is called

a **final segment** of T if for each $s \in S$ every $t > s$ also lies in S. A pair (Λ^L, Λ^R) of subsets of T is called a **cut** in T if Λ^L is an initial segment of T and $\Lambda^R = T \setminus \Lambda^L$; it then follows that Λ^R is a final segment of T. To compare cuts in $(T, <)$ we will use the lower cut sets comparison. That is, for two cuts $\Lambda_1 = (\Lambda_1^L, \Lambda_1^R)$, $\Lambda_2 = (\Lambda_2^L, \Lambda_2^R)$ in T we will write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$, and $\Lambda_1 \leq \Lambda_2$ if $\Lambda_1^L \subseteq \Lambda_2^L$.

For any $s \in T$ define the following **principal cuts**:

$$s^- := (\{t \in T \mid t < s\}, \{t \in T \mid t \geq s\}) \,,\; s^+ := (\{t \in T \mid t \leq s\}, \{t \in T \mid t > s\}) \,.$$

We identify the element s with s^+. Therefore, for a cut $\Lambda = (\Lambda^L, \Lambda^R)$ in T and an element $s \in T$ the inequality $\Lambda < s$ means that for every element $t \in \Lambda^L$ we have $t < s$. Similarly, for any subset M of T we define M^+ to be a cut (Λ^L, Λ^R) in T such that Λ^L is the smallest initial segment containing M, that is,

$$M^+ \;=\; (\{t \in T \mid \exists\, m \in M \;\, t \leq m\}, \{t \in T \mid t > M\}) \,.$$

Likewise, we denote by M^- the cut (Λ^L, Λ^R) in T such that Λ^L is the largest initial segment disjoint from M, i.e.,

$$M^- \;=\; (\{t \in T \mid t < M\}, \{t \in T \mid \exists\, m \in M \;\, t \geq m\}) \,.$$

For every extension $(L|K, v)$ of valued fields and $z \in L$ define

$$v(z - K) \;:=\; \{v(z - c) \mid c \in K\} \,.$$

The set $v(z - K) \cap vK$ is an initial segment of vK and thus the lower cut set of a cut in vK. However, it is more convenient to work with the cut

$$\mathrm{dist}\,(z, K) \;:=\; (v(z - K) \cap vK)^+ \quad \text{in the divisible hull } \widetilde{vK} \text{ of } vK \,.$$

We call this cut the **distance of z from K**. The lower cut set of $\mathrm{dist}\,(z, K)$ is the smallest initial segment of $\widetilde{vK}$ containing $v(z - K) \cap vK$. If $(F|K, v)$ is an algebraic subextension of $(L|K, v)$ then $\widetilde{vF} = \widetilde{vK}$. Thus $\mathrm{dist}\,(z, K)$ and $\mathrm{dist}\,(z, F)$ are cuts in the same group and we can compare these cuts by set inclusion of the lower cut sets. Since $v(z - K) \subseteq v(z - F)$ we deduce that

$$\mathrm{dist}\,(z, K) \;\leq\; \mathrm{dist}\,(z, F) \,.$$

If $\mathrm{char}\, K = p > 0$ and $z \in K$, then K^p is a subfield of K, and the expressions $v(z - K^p)$ and $\mathrm{dist}\,(z, K^p)$ are covered by our above definitions. We generalize this to the case where $\mathrm{char}\, K = 0$ with the same definitions but note that $v(z - K^p) \cap vK$ is not necessarily an initial segment of vK.

If y is another element of L, then we define:

$$z \sim_K y \;:\Leftrightarrow\; v(z - y) \;>\; \mathrm{dist}\,(z, K) \,.$$

The next lemma shows, among other things, that the relation $\sim_K$ is symmetrical.

Lemma 2.1. *Take a valued field extension $(L|K, v)$ and elements $z, y \in L$.*

1) If $z \sim_K y$, then $v(z - c) = v(y - c)$ for all $c \in K$ such that $v(z - c) \in vK$, $v(z - K) = v(y - K)$, $\mathrm{dist}\,(z, K) = \mathrm{dist}\,(y, K)$, and $y \sim_K z$.

2) If $(K(z)|K, v)$ is immediate, then $v(z - K)$ is a subset of vK without largest element.

Proof. 1): This is part (1) of Lemma 2.17 in [19].

2): It follows from [14, Theorem 1] that $v(z - K)$ has no largest element. To prove that it is a subset of vK, take $c \in K$; we wish to show that $v(z - c) \in vK$. Choose

$d \in K$ such that $v(z-d) > v(z-c)$. Then $v(z-c) = \min\{v(z-c), v(z-d)\} = v(c-d) \in vK$. □

For any $\alpha \in vK$ and each cut Λ in vK we set $\alpha + \Lambda := (\alpha + \Lambda^L, \alpha + \Lambda^R)$. An immediate consequence of the above definitions is the following lemma:

Lemma 2.2. *Take an extension $(L|K, v)$ of valued fields. Then for every element $c \in K$ and $y, z \in L$,*
1) $\mathrm{dist}\,(z + c, K) = \mathrm{dist}\,(z, K)$*,*
2) $\mathrm{dist}\,(cz, K) = vc + \mathrm{dist}\,(z, K)$*.*

Here are some important properties of distances in valued field extensions. For the proof of the next lemma see [3, Lemma 7] and [19, Lemma 2.5].

Lemma 2.3. *Take any immediate extension $(F|K, v)$ and a finite defectless unibranched extension $(L|K, v)$. Then the extension of v from F to $F.L$ is unique, $(F.L|F, v)$ is defectless, $(F.L|L, v)$ is immediate, and for every $a \in F \setminus K$ we have:*

$$\mathrm{dist}\,(a, K) = \mathrm{dist}\,(a, L).$$

Moreover, $F|K$ and $L|K$ are linearly disjoint, i.e.,

$$[F.L : F] \;=\; [L : K]\,.$$

For the proof of the following results see [3, Lemmas 5 and 9].

Lemma 2.4. *Take a unibranched extension $(F|K, v)$ and an extension of v to the algebraic closure of F. Take K^h to be the henselization of K with respect to this fixed extension of v. Then for every $a \in F$ we have that $[K(a) : K] = [K^h(a) : K^h]$ as well as*

$$d(K(a)|K, v) \;=\; d(K^h(a)|K^h, v) \quad \text{and} \quad \mathrm{dist}\,(a, K) \;=\; \mathrm{dist}\,(a, K^h)\,.$$

For the proof of the next proposition, see [19], Proposition 2.8.

Proposition 2.5. *Take a henselian field (K, v) and a tame extension N of K. Then for any finite extension $L|K$,*

$$d(L|K, v) \;=\; d(L.N|N, v)\,.$$

In particular, (K, v) is defectless if and only if (K^r, v) is defectless.

For the following theorem, see [14, Theorem 1] and [19, Theorem 2.19].

Theorem 2.6. *If $(L|K, v)$ is an immediate extension of valued fields, then for every element $a \in L \setminus K$ the set $v(a - K)$ is an initial segment of vK without maximal element. In particular, $va < \mathrm{dist}\,(a, K)$.*

The following partial converse of this theorem also holds (see [1, Lemma 4.1], cf. also [19, Lemma 2.21]):

Lemma 2.7. *Assume that $(K(a)|K, v)$ is a unibranched extension of prime degree such that $v(a - K)$ has no maximal element. Then the extension $(K(a)|K, v)$ is immediate and hence a defect extension.*

The property that the set $v(a - K)$ has no maximal element does not in general imply that $(K(a)|K, v)$ is immediate. However, the next lemma (see e.g. [28, Lemma 2.1]) shows that if in addition (K, v) is henselian and a is algebraic over K, then $(K(a)|K, v)$ is a defect extension.

Lemma 2.8. *If (K,v) is henselian and $(L|K,v)$ is a finite defectless extension, then for every element $a\in L$ the set $v(a-K)$ admits a maximal element.*

We will need a version of Lemma 2.7 that also works for extensions that are not assumed to be unibranched.

Lemma 2.9. *Assume that $(K(a)|K,v)$ is an extension of degree at most $p=$ char Kv of rank 1 valued fields such that $v(a-K)$ has no maximal element but is bounded from above in vK. Then the extension $(K(a)|K,v)$ is a unibranched defect extension.*

Proof. Take a henselization (K^h,v) and consider the extension $(K^h(a)|K^h,v)$ which again is of degree at most p. Take any $b\in K^h$. Since $K^h(a)|K$ is algebraic, we know that $vK^h(a)$ lies in the divisible hull $\widetilde{vK}$ of vK and thus there is some $\alpha\in vK$ such that $\alpha>v(a-b)$. Since (K,v) is of rank 1 by assumption, (K,v) lies dense in (K^h,v) (cf. [33, 32.11 and 32.18]). Therefore, there is some $c\in K$ such that $v(b-c)\geq\alpha>v(a-b)$, so that $v(a-b)=v(a-c)\in v(a-K)$. This shows that $v(a-K^h)=v(a-K)$. Hence also $v(a-K^h)$ has no maximal element and is bounded from above in $vK=vK^h$. Thus in particular $(K^h(a)|K^h,v)$ is a nontrivial extension.

Suppose the extension $(K^h(a)|K^h,v)$ were defectless. Then by Lemma 2.8 the set $v(a-K^h)$ would admit a maximal element, a contradiction. This shows that $(K^h(a)|K^h,v)$ has nontrivial defect $d(K^h(a)|K^h,v)$. By the Lemma of Ostrowski,

$$[K^h(a):K^h]=d(K^h(a)|K^h,v)(vK^h(a):vK^h)[K^h(a)v:K^hv],$$

where $d(K^h(a)|K^h,v)=p^n$ for some $n\geq 1$. Since $[K^h(a):K^h]\leq p$, we deduce that

$$[K^h(a):K^h]=p=d(K^h(a)|K^h,v).$$

Hence $K(a)|K$ is linearly disjoint from $K^h|K$ and thus it is a unibranched extension (cf. Lemma 2.1 of [2]). Moreover, from Lemma 2.4 we deduce that $(K(a)|K,v)$ is a defect extension.

□

The next lemma follows from [14, Lemma 8] and [28, Lemma 5.2]. Note that if $(K(a)|K,v)$ is an extension such that $v(a-K)$ has no maximal element, then by the proof of [14, Theorem 1], a is limit of a pseudo Cauchy sequence in (K,v) without limit in K, and by [28, part a) of Lemma 4.1] its approximation type over (K,v) is immediate. We use the Taylor expansion

$$f(X)=\sum_{i=0}^{n}\partial_i f(c)(X-c)^i \tag{6}$$

where $\partial_i f$ denotes the i-th **Hasse-Schmidt derivative** of f.

Lemma 2.10. *Take a nontrivial extension $(K(a)|K,v)$ of degree p. Assume that $v(a-K)$ has no maximal element. Then for every nonconstant polynomial $f\in K[X]$ of degree $<p$ there is some $\gamma\in v(a-K)$ such that for all $c\in K$ with $v(a-c)\geq\gamma$ and all i with $1\leq i\leq\deg f$, we have:*
the values $v\partial_i f(c)$ are fixed, equal to $v\partial_i f(a)$,

the values $v\partial_i f(c) + i\cdot v(a-c)$ *are pairwise distinct,*

$$v\partial_1 f(c) + v(a-c) \;<\; v\partial_i f(c) + i\cdot v(a-c) \;\text{ whenever } i\neq 1, \tag{7}$$

$$v(f(a)-f(c)) \;=\; v\partial_1 f(c) + v(a-c)\,, \;\text{ and} \tag{8}$$

$$\mathrm{dist}\,(f(a),K) \;=\; v\partial_1 f(c) + \mathrm{dist}\,(a,K)\,. \tag{9}$$

The following is Lemma 2.4 of [19].

Lemma 2.11. *Take a valued field* (K,v)*, a finite extension* $(L|K,v)$ *and a coarsening* w *of* v *on* L*. If* (K,v) *is henselian, then so is* (K,w)*. If* $(L|K,v)$ *is defectless, then so is* $(L|K,w)$*.*

2.2. **The absolute ramification field.**

Proposition 2.12. *Take an immediate unibranched extension* $(K(a)|K,v)$*. Extend* v *to the algebraic closure of* K *and let* (K^h,v) *be the henselization and* (K^r,v) *the absolute ramification field of* (K,v) *with respect to this extension. Then* $(K^r(a)|K^r,v)$ *is an immediate extension with*

$$[K^r(a):K^r] \;=\; [K^h(a):K^h] \;=\; [K(a):K]\,, \tag{10}$$

$$d(K^r(a)|K^r,v) \;=\; d(K^h(a)|K^h,v) \;=\; d(K(a)|K,v)\,, \tag{11}$$

$$\mathrm{dist}\,(a,K^r) \;=\; \mathrm{dist}\,(a,K^h) \;=\; \mathrm{dist}\,(a,K)\,. \tag{12}$$

If $(N|K,v)$ *is any subextension of* $(K^r|K,v)$*, then* $[N(a):N]=[K(a):K]$ *and*

$$d(N(a)|N,v) \;=\; d(K(a)|K,v) \;\text{ and }\; \mathrm{dist}\,(a,N) \;=\; \mathrm{dist}\,(a,K)\,. \tag{13}$$

Proof. Since $(K(a)|K,v)$ is a unibranched extension, we know from Lemma 2.4 that $[K^h(a):K^h]=[K(a):K]$ as well as $d(K^h(a)|K^h,v)=d(K(a)|K,v)$ and $\mathrm{dist}\,(a,K^h)=\mathrm{dist}\,(a,K)$. Since $(K(a)|K,v)$ is also immediate,

$$[K^h(a):K^h] \;=\; [K(a):K] \;=\; d(K(a)|K,v) \;=\; d(K^h(a)|K^h,v)\,,$$

showing that also $(K^h(a)|K^h,v)$ is immediate.

Further, $(K^r|K^h,v)$ is tame and hence the union of its finite defectless subextensions. Thus by Lemma 2.3, $(K^r(a)|K^r,v)$ is immediate with $[K^r(a):K^r]=[K^h(a):K^h]$ and $\mathrm{dist}\,(a,K^r)=\mathrm{dist}\,(a,K^h)$. By Proposition 2.5, $d(K^r(a)|K^r,v)=d(K^h(a)|K^h,v)$.

Finally, if $(N|K,v)$ is a subextension of $(K^r|K,v)$, then $N^r=K^r$. Hence by (10), $[N(a):N]=[N^r(a):N^r]=[K^r(a):K^r]=[K(a):K]$, by (11), $d(N(a)|N,v)=d(N^r(a)|N^r,v)=d(K^r(a)|K^r,v)=d(K(a)|K,v)$, and by (12), $\mathrm{dist}\,(a,N)=\mathrm{dist}\,(a,N^r)=\mathrm{dist}\,(a,K^r)=\mathrm{dist}\,(a,K)$. □

For the proof of the following results, see Lemma 2.9 of [19].

Lemma 2.13. *Take any valued field* (K,v) *and let* K^h *and* K^r *be its henselization and its absolute ramification field with respect to any extension of* v *to the algebraic closure of* K*. If* $\mathrm{char}\,Kv=0$*, then* K^r *is algebraically closed. If* $\mathrm{char}\,Kv=p>0$*, then every finite extension of* K^r *is a tower of normal extensions of degree* p*. Further, if* $L|K$ *is a finite extension, then there is already a finite tame extension* N *of* K^h *such that* $L.N|N$ *is such a tower.*

The proof of this lemma uses the fact that if $\mathrm{char}\,Kv=p>0$, then $K^{\mathrm{sep}}|K^r$ is a p-extension. From this we can also conclude:

Corollary 2.14. *Every absolute ramification field contains all* p*-th roots of unity.*

Finally, we will need the following fact:

Lemma 2.15. *Let (K^r, v) be the absolute ramification field of (K, v), and assume that $v = w \circ \overline{w}$. Then the extension $K^r w | Kw$ is separable.*

Proof. For a detailed proof, see the chapter on ramification theory in [25]. For the convenience of the reader, we give here a sketch of the proof. We use several facts from ramification theory. The assertion is trivial if $\mathrm{char}\, Kv = 0$, so we let $\mathrm{char}\, Kv = p > 0$.

i) If (K^h, v) is the henselization of (K, v), then $(K^h w, \overline{w})$ is the henselization of $(Kw, \overline{w})$. In particular, $K^h w | Kw$ is separable.

ii) If $(L_1 | L_2, w \circ \overline{w})$ is a finite defectless extension, then so are $(L_1 | L_2, w)$ and $(L_1 w | L_2 w, \overline{w})$.

iii) Since $(K^r | K^h, v)$ is a tame extension, also $(K^r w | K^h w, \overline{w})$ is a tame extension. Indeed, if $(L | K^h, v)$ is a finite subextension, then p does not divide $(vL : vK^h)$ and hence also not $(\overline{w}(Lw) : \overline{w}(K^h w))$, the extension $Lv | K^h v = (Lw)\overline{w} | (K^h w)\overline{w}$ is separable, and $(L | K^h, v)$ is defectless, which by ii) implies that $(Lw | K^h w, \overline{w})$ is defectless.

Now as $(K^r w | K^h w, \overline{w})$ is a tame extension, $K^r w | K^h w$ is separable, and in view of i) we obtain that $K^r w | Kw$ is separable. □

2.3. **1-units and p-th roots in valued fields of mixed characteristic.**
Throughout this section, (K, v) will be a valued field of characteristic zero and residue characteristic $p > 0$, with valuation ring $\mathcal{O}$ and valuation ideal $\mathcal{M}$. We assume that v is extended to the algebraic closure $\tilde{K}$ of K. We will need a few easy observations about the relation of congruences and powers of elements.

Lemma 2.16. *1) If $b_1, \ldots, b_n \in \mathcal{O}$, then*

$$(b_1 + \ldots + b_n)^p \;\equiv\; b_1^p + \ldots + b_n^p \mod p\mathcal{O}\,. \tag{14}$$

2) Take elements $b_1, \ldots, b_n \in K$ of values $\geq -\frac{vp}{p}$. Then

$$(b_1 + \cdots + b_n)^p \;\equiv\; b_1^p + \cdots + b_n^p \mod \mathcal{O}\,.$$

3) Take $\eta \in \tilde{K}$ such that $\eta^p \in \mathcal{O}$. Then for every $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$ we have that $\eta^p \equiv c^p \mod p\mathcal{O}$.

Proof. 1): We have:

$$(b_1 + b_2)^p \;=\; b_1^p + \sum_{i=1}^{p-1} \binom{p}{i} b_1^{p-i} b_2^i + b_2^p\,. \tag{15}$$

Since the binomial coefficients under the sum are all divisible by p and since $b_1, b_2 \in \mathcal{O}$, all summands on the right hand side for $1 \leq i \leq p-1$ lie in $p\mathcal{O}$, which proves our assertion in the case of $n = 2$. The general case follows by induction on n.

2): If $vb_1 \geq -\frac{vp}{p}$ and $vb_2 \geq -\frac{vp}{p}$, then $vb_1^{p-i} b_2^i \geq -vp$ for $1 \leq i \leq p-1$, so all summands in the sum on the right hand side of (15) have non-negative value. As for part 1), the assertion now follows by induction on n.

3): For $c \in K$ with $v(\eta - c) > 0$ we have that $vc \geq 0$ and, by part 1):

$$(\eta - c)^p \;\equiv\; \eta^p - c^p \mod p\mathcal{O}_{K(\eta)}\,.$$

If $v(\eta - c) \geq \frac{vp}{p}$, then $v(\eta - c)^p \geq vp$, i.e., $\eta^p - c^p \equiv (\eta - c)^p \equiv 0$ modulo $p\mathcal{O}_{K(\eta)} \cap K = p\mathcal{O}$. □

Lemma 2.17. *Take $\eta \in \tilde{K}$ such that $\eta^p \in K$ and $v\eta = 0$. Then for $c \in K$ such that $v(\eta - c) > 0$, $v(\eta - c) < \frac{1}{p-1}vp$ holds if and only if $v(\eta^p - c^p) < \frac{p}{p-1}vp$, and if this is the case, then $v(\eta^p - c^p) = pv(\eta - c)$. If $v(\eta - c) > \frac{1}{p-1}vp$, then $v(\eta^p - c^p) = vp + v(\eta - c)$.*

Proof. Take any $c \in K$ such that $0 < v(\eta - c)$. Then $vc = v\eta = 0$. We have:

$$\eta^p \;=\; (\eta - c + c)^p \;=\; (\eta - c)^p + \sum_{i=1}^{p-1} \binom{p}{i} (\eta - c)^i c^{p-i} \;+\; c^p \,.$$

Since $vc = 0$ and the binomial coefficients under the sum all have value vp, the unique summand with the smallest value is $p(\eta - c)c^{p-1}$. Therefore,

$$v(\eta^p - c^p) \;\geq\; \min\{v(\eta - c)^p, vp(\eta - c)\} \;=\; \min\{pv(\eta - c), vp + v(\eta - c)\}\,, \tag{16}$$

with equality holding if $pv(\eta - c) \neq vp + v(\eta - c)$. We observe that

$$v(\eta - c) < \frac{vp}{p-1} \iff pv(\eta - c) < vp + v(\eta - c)\,, \tag{17}$$

and the same holds for “$>$” in place of “$<$”. Assume that $v(\eta - c) < \frac{vp}{p-1}$. Then

$$v(\eta^p - c^p) \;=\; pv(\eta - c) \;<\; \frac{p}{p-1}vp$$

by (17) and (16). Now assume that $v(\eta - c) \geq \frac{1}{p-1}vp$. Then by (17), $pv(\eta - c) \geq vp + v(\eta - c)$, and (16) yields that

$$v(\eta^p - c^p) \;\geq\; vp + v(\eta - c) \;\geq\; vp + \frac{1}{p-1}vp \;=\; \frac{p}{p-1}vp\,.$$

Finally, if $v(\eta - c) > \frac{1}{p-1}vp$, then from (17) and (16) we conclude:

$$v(\eta^p - c^p) \;=\; vp + v(\eta - c)\,.$$

□

A **1-unit** in (K, v) is an element of the form $u = 1 + b$ with $b \in \mathcal{M}$; in other words, u is a unit in $\mathcal{O}$ with residue 1. We will call the value $v(u - 1)$ the **level** of the 1-unit u. Taking η to be a 1-unit u in Lemma 2.17, we obtain:

Corollary 2.18. *Assume that u is a 1-unit. Then the level of u is smaller than $\frac{1}{p-1}vp$ if and only if the level of u^p is smaller than $\frac{p}{p-1}vp$, and if this is the case, then $v(u^p - 1) = pv(u - 1)$.*

Lemma 2.19. *Take $\eta \in \tilde{K}$ such that $\eta^p \in K$. If there is some $c \in K$ such that*

$$v(\eta - c) \;>\; v\eta + \frac{vp}{p-1}\,, \tag{18}$$

then η lies in the henselization of (K, v) within $(\tilde{K}, v)$.

Proof. If $\eta \in K$, then there is nothing to show, so let us assume that $\eta \notin K$. We fix a primitive p-th root of unity ζ_p. Every root of $X^p - \eta^p$ is of the form $\eta\zeta_p^i$ with $0 \leq i \leq p - 1$. For $0 \leq i \neq j \leq p - 1$,

$$v(\eta\zeta_p^i - \eta\zeta_p^j) \;=\; v\eta + jv\zeta_p + v(\zeta_p^{i-j} - 1) \;=\; v\eta + \frac{vp}{p-1}\,,$$

where the last equality holds since $v\zeta_p = 0$ and

$$v(\zeta - 1) \;=\; \frac{vp}{p-1} \tag{19}$$

for every primitive p-th root of unity ζ (see e.g. the proof of [21, Lemma 2.9]). Hence if (18) holds, then it follows from Krasner's Lemma that $\eta \in K(c)^h = K^h$, where K^h denotes the henselization of (K, v) within $(\tilde{K}, v)$. $\square$

For our work with 1-units, we will need a result that is Lemma 2.9 of [21].

Lemma 2.20. *A henselian field of characteristic 0 and residue characteristic $p > 0$ contains an element C such that $C^{p-1} = -p$ if and only if it contains a primitive p-th root ζ_p of unity.*

The element C satisfies:

$$C^p = -pC \quad \text{and} \quad vC = \frac{vp}{p-1}\,. \tag{20}$$

The following construction will play an important role in the proof of Theorem 1.5. Take a 1-unit $\eta \in \tilde{K}$ such that $\eta^p \in K$. Then also η^p is a 1-unit. Assume that K contains an element C as in Lemma 2.20. Consider the substitution $X = CY + 1$ for the polynomial $X^p - \eta^p$. We then obtain the polynomial $(CY+1)^p - \eta^p$. Dividing this polynomial by C^p and using the fact that $C^p = -pC$, we obtain the polynomial

$$f_\eta(Y) = Y^p + g(Y) - Y - \frac{\eta^p - 1}{C^p}, \tag{21}$$

where

$$g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i\,. \tag{22}$$

Note that $g(Y) \in \mathcal{M}_K[Y]$ since $C \in K$ and $vC = \frac{vp}{p-1}$. We see that an element $\tilde{\eta}$ is a root of $X^p - \eta^p$ if and only if the element $\frac{\tilde{\eta}-1}{C}$ is a root of f_η. Thus the roots of f_η are of the form $\frac{\zeta_p^i \eta - 1}{C}$ with $0 \le i \le p-1$. Set

$$\vartheta_\eta \;:=\; \frac{\eta - 1}{C}\,. \tag{23}$$

Then $K(\eta) = K(\vartheta_\eta)$, with f_η the minimal polynomial of ϑ_η over K. Modulo $\mathcal{M}_K[Y]$, the polynomial $f_\eta(Y)$ has the form of an Artin-Schreier polynomial (see Section 3.2). We note that by Lemma 2.2,

$$\operatorname{dist}(\vartheta_\eta, K) \;=\; \operatorname{dist}(\eta, K) - \frac{vp}{p-1}\,.$$

The following is an easy consequence of the above.

Lemma 2.21. *In a henselian field (K, v) of mixed characteristic with residue characteristic p which contains a primitive p-th root of unity, every 1-unit of level strictly greater than $\frac{p}{p-1} vp$ is a p-th power.*

Proof. By Lemma 2.20, K contains an element C as in that lemma. Take a 1-unit $u \in K$ of level strictly greater than $\frac{p}{p-1} vp$. Apply the above transformation to the polynomial $X^p - u$ with $\eta^p = u$. By our assumption on u we have that $\frac{\eta^p - 1}{C^p} \in \mathcal{M}_K$. Hence $f_\eta(Y)$ is equivalent modulo $\mathcal{M}_K[Y]$ to $Y^p - Y$, which splits in the henselian field K. Therefore, $\vartheta_\eta \in K$, hence $\eta \in K$. $\square$

2.4. **Higher ramification groups.**

Take a henselian field (K,v). Assume that $L|K$ is a Galois extension, and let $G=\mathrm{Gal}\,(L|K)$ denote its Galois group. For ideals I of $\mathcal{O}_L$ we consider the (upper series of) **higher ramification groups**

$$G_I \;:=\; \left\{\sigma\in G \;\middle|\; \frac{\sigma b-b}{b}\in I \;\text{ for all } b\in L^\times\right\} \tag{24}$$

(see [34], §12). Note that $G_{\mathcal{M}_L}$ is the ramification group of $(L|K,v)$. For every ideal I of $\mathcal{O}_L$, G_I is a normal subgroup of G ([34] (d) on p.79). The function

$$\varphi:\; I\;\mapsto\; G_I \tag{25}$$

preserves $\subseteq$, that is, if $I\subseteq J$, then $G_I\subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the higher ramification groups are linearly ordered by inclusion. Note that in general, φ will neither be injective, nor surjective as a function to the set of normal subgroups of G.

The function

$$v:\; I\;\mapsto\;\Sigma_I\;:=\;\{vb\mid 0\neq b\in I\} \tag{26}$$

is an order preserving bijection from the set of all ideals of $\mathcal{O}_L$ onto the set of all final segments of the positive part $(vL)^{>0}$ of the value group vL (including the final segment $\emptyset$). The set of these final segments is again linearly ordered by inclusion, and the function (26) is order preserving: $J\subseteq I$ holds if and only if $\Sigma_J\subseteq\Sigma_I$ holds. The inverse of the above function is the order preserving function

$$\Sigma\;\mapsto\; I_\Sigma\;:=\;\{a\in L\mid va\in\Sigma\}\cup\{0\}\,. \tag{27}$$

Now the higher ramification groups can be represented as

$$G_\Sigma\;:=\;G_{I_\Sigma}\;=\;\left\{\sigma\in G\;\middle|\; v\,\frac{\sigma b-b}{b}\in\Sigma\cup\{\infty\}\;\text{ for all } b\in L^\times\right\}\;,$$

where Σ runs through all final segments of $(vL)^{>0}$.

Like the function (25), also the function $\Sigma\mapsto G_\Sigma$ is in general not injective. We call Σ a **ramification jump** if

$$\Sigma'\subsetneq\Sigma\;\Rightarrow\;G_{\Sigma'}\subsetneq G_\Sigma\,.$$

If Σ is a ramification jump, then I_Σ is called a **ramification ideal**.

Given any ramification group $H\subseteq G$, we define

$$\Sigma_-(H)\;:=\;\bigcap_{G_\Sigma=H}\Sigma\qquad\text{and}\qquad\Sigma_+(H)\;:=\;\bigcup_{G_\Sigma=H}\Sigma\,. \tag{28}$$

and note that arbitrary unions and intersections of final segments of $(vL)^{>0}$ are again final segments of $(vL)^{>0}$. From its definition it is obvious that $\Sigma_-(H)$ is a ramification jump, $G_{\Sigma_-(H)}=H$, and that

$$I_-(H)\;:=\;I_{\Sigma_-(H)}$$

is a ramification ideal. It is generated by the set

$$\left\{\frac{\sigma b-b}{b}\;\mid\;\sigma\in H\,,\;b\in L^\times\right\}\,. \tag{29}$$

In this paper we are particularly interested in the case where $(L|K,v)$ is a Galois extension of prime degree p. Then $G = \mathrm{Gal}\,(L|K)$ is a cyclic group of order p and thus has only one proper subgroup, namely $\{\mathrm{id}\}$, and this subgroup is equal to G_Σ for $\Sigma = \emptyset$. If in this case G itself is the ramification group of the extension, then there must be a unique ramification jump. As we will show in the next section, this ramification jump carries important information about the extension $(L|K,v)$.

3. Defect extensions of prime degree

We will investigate defect extensions $(L|K,v)$ of prime degree p. By what we have already stated in the Introduction, such extensions are immediate unibranched extensions; moreover, $p = \mathrm{char}\,Kv > 0$. By Theorem 2.6, for every $a \in L\setminus K$ the set $v(a-K)$ is an initial segment of vK without maximal element, and $\mathrm{dist}\,(a,K) > va$.

In the following, we distinguish two cases:

- the equal characteristic case where $\mathrm{char}\,K = p$,
- the mixed characteristic case where $\mathrm{char}\,K = 0$ and $\mathrm{char}\,Kv = p$.

We fix an extension of v from L to the algebraic closure $\tilde{K}$ of K.

Note: to shorten expressions, we will often write "independent defect extension" in place of "defect extension with independent defect".

In a first section, we investigate the set Σ_σ defined in (2) for σ in the absolute Galois group $\mathrm{Gal}\,(K) := \mathrm{Gal}\,(K^{\mathrm{sep}}|K)$.

3.1. The set Σ_σ.

We start with the following easy but helpful observations. The first is obvious.

Lemma 3.1. *Let $(K(a)|K,v)$ be any algebraic extension of valued fields. If $\sigma \in \mathrm{Gal}\,(K)$ is such that $\sigma a \neq a$, then*

$$\left\{ v\,\frac{\sigma(a-c)-(a-c)}{a-c} \;\middle|\; c \in K \right\} = \left\{ v\,\frac{\sigma a - a}{a-c} \;\middle|\; c \in K \right\} = -v(a-K)+v(\sigma a-a).$$

Lemma 3.2. *Take a nontrivial immediate unibranched extension $(K(a)|K,v)$. Then the following assertions hold.*

1) For each $\sigma \in \mathrm{Gal}\,(K)$ and $c \in K$,

$$v(a-c) \;<\; v(\sigma a - a)\;.$$

2) For each $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma a \neq a$,

$$\mathrm{dist}\,(a,K) \;\leq\; v(\sigma a - a)^-.$$

Proof. 1): Since the extension is immediate and $a \notin K$, the set $v(a-K)$ has no maximal element. Thus it suffices to prove that $v(a-c) \leq v(\sigma a - a)$. If this were not true, then for some $\sigma \in \mathrm{Gal}\,(K)$ and $c \in K$, $v(a-c) > v(\sigma a - a)$. But then,

$$v\sigma(a-c) \;=\; v(\sigma a - c) \;=\; \min\{v(\sigma a - a)\,,\, v(a-c)\} \;=\; v(\sigma a - a) \;<\; v(a-c)\;,$$

which contradicts our assumption that $K(a)|K$ is a unibranched extension, as $v\sigma$ is also an extension of v from K to $K(a)$.

2): This is an immediate consequence of part 1). □

With the help of Lemma 2.10, we prove:

Lemma 3.3. *Take a defect extension $(K(a)|K,v)$ of prime degree and any $b \in K(a)^\times$. Then for all $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma a \neq a$ there is some $c \in K$ such that*

$$v\,\frac{\sigma b - b}{b} \;>\; -v(a-c) \,+\, v(\sigma a - a)\;. \tag{30}$$

Proof. As stated already, $(K(a)|K,v)$ is immediate with $[K(a):K] = p = \mathrm{char}\, Kv$. The element $b \in K(a)^\times$ can be written as $f(a)$ for $f(X) \in K[X]$ of degree smaller than p. By Theorem 2.6, $v(a-K)$ has no maximal element. Hence by Lemma 2.10, we can choose $\gamma \in v(a-K)$ so large that for all $c \in K$ with $v(a-c) \geq \gamma$, all values $v\partial_i f(c)$ are fixed and equal to $v\partial_i f(a)$ whenever $0 \leq i < p$, and that (7) and (8) hold. It suffices to restrict our attention to those $c \in K$ for which $v(a-c) \geq \gamma$. Then we have that

$$v\partial_1 f(a)(a-c) \;=\; v\partial_1 f(c)(a-c) \;<\; v\partial_i f(c)(a-c)^i \;=\; v\partial_i f(a)(a-c)^i \tag{31}$$

for all $i > 1$. From part 1) of Lemma 3.2 we infer that

$$0 \;<\; v\left(\frac{\sigma a - a}{a-c}\right) \;<\; v\left(\frac{\sigma a - a}{a-c}\right)^i$$

for all $i > 1$. Using this together with (31), we obtain:

$$\begin{aligned} v\partial_1 f(a)(\sigma a - a) \;&=\; v\partial_1 f(a)(a-c)\left(\frac{\sigma a - a}{a-c}\right) \\ &<\; v\partial_i f(a)(a-c)^i\left(\frac{\sigma a - a}{a-c}\right)^i \;=\; v\partial_i f(a)(\sigma a - a)^i\;. \end{aligned}$$

It follows that

$$\begin{aligned} v(\sigma f(a) - f(a)) \;&=\; v(f(\sigma a) - f(a)) \;=\; v\left(\sum_{i=1}^{\deg f} \partial_i f(a)(\sigma a - a)^i\right) \\ &=\; v\partial_1 f(a)(\sigma a - a) \;=\; v\partial_1 f(c) \,+\, v(\sigma a - a)\;. \end{aligned}$$

Now (8) shows that

$$v\partial_1 f(c) + v(a-c) \;=\; v(f(a) - f(c)) \;\geq\; \min\{vf(a), vf(c)\}\;.$$

The value on the right hand side is fixed, but the value of the left hand side increases with $v(a-c)$. Since $v(a-K)$ has no maximal element, we can choose γ so large that the value on the left hand side is larger than the one on the right hand side, which can only be the case if $vf(a) = vf(c)$, whence $vf(a) < v\partial_1 f(c) + v(a-c)$. Consequently,

$$v\,\frac{\sigma f(a) - f(a)}{f(a)} \;=\; \partial_1 f(c) \,+\, v(\sigma a - a) \,-\, vf(a) \;>\; -v(a-c) \,+\, v(\sigma a - a)\;.$$

□

Theorem 3.4. *Take a defect extension $\mathcal{E} = (L|K,v)$ of prime degree. Then for every generator $a \in L$ of the extension and every $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma a \neq a$,*

$$\Sigma_\sigma \;=\; -v(a-K) + v(\sigma a - a)\,, \tag{32}$$

and this set is a final segment of $vK^{>0} = \{\alpha \in vK \mid \alpha > 0\}$.

Proof. The inclusion "$\supseteq$" in (32) follows from Lemma 3.1. To show the reverse inclusion, we use Lemma 3.3. Since $v(a-K)$ is an initial segment of vK, $-v(a-K)$ is a final segment of vK. Thus we can infer from (30) that

$$v\,\frac{\sigma b-b}{b}\;\in\;-v(a-K)\,+\,v(\sigma a-a)\,.$$

This proves the inclusion "$\subseteq$".

Since $\mathcal{E}$ is an immediate unibranched extension, taking $c=0$ in part 1) of Lemma 3.2 yields that $v(\sigma b-b)>vb$ for all $b\in L^{\times}$, showing that $v\frac{\sigma b-b}{b}\in vL^{>0}=vK^{>0}$. Since $-v(a-K)$ is a final segment of vK, the same holds for $\Sigma_\sigma=-v(a-K)+v(\sigma a-a)$. □

3.2. Galois defect extensions of prime degree.

A Galois extension of degree p of a field K of characteristic $p>0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** X^p-X-c with $c\in K$. A Galois extension of degree p of a field K of characteristic 0 which contains all p-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^p\in K$. For these facts, see [29, Chapter VIII, §8].

If $(L|K,v)$ is a Galois defect extension of degree p of fields of characteristic 0, then a Kummer generator of $L|K$ can be chosen to be a 1-unit. Indeed, choose any Kummer generator η. Since $(L|K,v)$ is immediate, we have that $v\eta\in vK(\eta)=vK$, so there is $c\in K$ such that $vc=-v\eta$. Then $v\eta c=0$, and since $\eta cv\in K(\eta)v=Kv$, there is $d\in K$ such that $dv=(\eta cv)^{-1}$. Then $v(\eta cd)=0$ and $(\eta cd)v=1$. Hence ηcd is a 1-unit. Furthermore, $K(\eta cd)=K(\eta)$ and $(\eta cd)^p=\eta^p c^p d^p\in K$. Thus we can replace η by ηcd and assume from the start that η is a 1-unit. It follows that also $\eta^p\in K$ is a 1-unit.

Throughout this article, whenever we speak of "Artin-Schreier extension" we refer to fields of positive characteristic, and with "Kummer extension" we refer to fields of characteristic 0.

Theorem 3.5. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree with Galois group G. Then G is the ramification group of $\mathcal{E}$. The set Σ_σ does not depend on the choice of the generator σ of G. Writing $\Sigma_{\mathcal{E}}$ for Σ_σ, we have that $\Sigma_{\mathcal{E}}$ is a final segment of $vK^{>0}$ and satisfies*

$$\Sigma_{\mathcal{E}}\;=\;\Sigma_-(G)\;=\;\Sigma_+(\{\mathrm{id}\})\,,$$

showing that $\Sigma_{\mathcal{E}}$ is the unique ramification jump of the extension $\mathcal{E}$. Further, the ramification ideal $I_{\mathcal{E}}=I_-(G)$ is equal to the ideal of $\mathcal{O}_L$ generated by the set

$$\left\{\frac{\sigma b-b}{b}\;\middle|\;b\in L^{\times}\right\}\,, \tag{33}$$

for any generator σ of G.

If $(L|K,v)$ is an Artin-Schreier defect extension with any Artin-Schreier generator ϑ, then

$$\Sigma_{\mathcal{E}}\;=\;-v(\vartheta-K)\,. \tag{34}$$

If K contains a primitive root of unity and $(L|K,v)$ is a Kummer extension with Kummer generator η of value 0, then

$$\Sigma_{\mathcal{E}}\;=\;\frac{vp}{p-1}\;-\;v(\eta-K)\,. \tag{35}$$

Proof. It follows from Theorem 3.4 that G is the ramification group of $\mathcal{E}$ and that Σ_σ is a final segment of $vK^{>0}$.

Assume first that $(L|K,v)$ is an Artin-Schreier defect extension with Artin-Schreier generator ϑ. Then for every generator σ of G, we have that $\sigma\vartheta = \vartheta + i$ for some $i \in \mathbb{F}_p$ and thus, $v(\sigma\vartheta - \vartheta) = vi = 0$. Hence equation (32) shows that Σ_σ does not depend on the choice of σ and that (34) holds.

Now assume that K contains a primitive root of unity and $(L|K,v)$ is a Kummer extension with Kummer generator η which is a 1-unit. Then $\sigma\eta - \eta = (\zeta_p - 1)\eta$ for some primitive root of unity ζ_p , and by equation (19),

$$v(\sigma\eta - \eta) \;=\; v(\zeta_p - 1) + v\eta \;=\; \frac{vp}{p-1}\,. \tag{36}$$

Hence by equation (32), Σ_σ does not depend on the choice of σ, and (35) holds.

If $\Sigma \subsetneq \Sigma_\sigma$, then $\sigma \notin G_\Sigma$ and hence $G_\Sigma = \{\mathrm{id}\}$. If $\Sigma_\sigma \subseteq \Sigma$, then $\sigma \in G_\Sigma$ and hence $G_\Sigma = G$. As $\Sigma_{\mathcal{E}}$ is the intersection of all final segments that contain it,

$$\Sigma_{\mathcal{E}} \;=\; \bigcap_{G_\Sigma = G} \Sigma \;=\; \Sigma_-(G)\,.$$

Since the sets $-v(\vartheta - K)$ in equation (34) and $-v(\eta - K)$ in equation (35) have no smallest element, the same is true for $\Sigma_{\mathcal{E}}$. Therefore, $\Sigma_{\mathcal{E}}$ is the union of all final segments properly contained in it, whence

$$\Sigma_{\mathcal{E}} \;=\; \bigcup_{G_\Sigma = \{\mathrm{id}\}} \Sigma \;=\; \Sigma_+(\{\mathrm{id}\})\,.$$

Finally, from Section 2.4 we know that $I_-(G)$ is generated by the set (29). However, as $\Sigma_{\mathcal{E}} = \Sigma_\sigma$ for every generator σ of G, it is also generated by the set (33). □

We define the **distance of $\mathcal{E}$** to be the cut

$$\mathrm{dist}\,\mathcal{E} \;:=\; (-\Sigma_{\mathcal{E}})^+$$

in $\widetilde{vK}$. By applying the distance operator to the right hand sides of equations (34) and (35), we obtain:

Corollary 3.6. *If $\mathcal{E}$ is an Artin-Schreier defect extension, then*

$$\mathrm{dist}\,\mathcal{E} \;=\; \mathrm{dist}\,(\vartheta, K)$$

for every Artin-Schreier generator ϑ of $\mathcal{E}$. Consequently, all Artin-Schreier generators of $\mathcal{E}$ have the same distance.

If $\mathcal{E}$ is a Kummer extension, then

$$\mathrm{dist}\,\mathcal{E} \;=\; -\frac{vp}{p-1} \;+\; \mathrm{dist}\,(\eta, K) \;=\; \mathrm{dist}\,(\vartheta_\eta, K)$$

for every Kummer generator η of value 0. Consequently, all Kummer generators of $\mathcal{E}$ of value 0 have the same distance.

Proposition 3.7. *Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p.*
1) We have that

$$\mathrm{dist}\,\mathcal{E} \;\leq\; 0^-\,. \tag{37}$$

If $\mathcal{E}$ is an Artin-Schreier defect extension, then

$$\mathrm{dist}\,(\vartheta, K) \;\leq\; 0^- \tag{38}$$

for every Artin-Schreier generator ϑ. If $\mathcal{E}$ is a Kummer defect extension, then

$$0 \;<\; \mathrm{dist}\,(\eta, K) \;\leq\; \left(\frac{vp}{p-1}\right)^{-} \tag{39}$$

for every Kummer generator η of value 0.

2) The extension $\mathcal{E}$ has independent defect if and only if

$$\mathrm{dist}\,\mathcal{E} \;=\; H^{-} \tag{40}$$

for some proper convex subgroup H of vK. In particular, if the value group of (K,v) is archimedean, then $\mathcal{E}$ has independent defect if and only if $\mathrm{dist}\,\mathcal{E} = 0^{-}$.

3) If $\mathcal{E}$ is an Artin-Schreier defect extension with Artin-Schreier generator ϑ, then it has independent defect if and only if

$$\mathrm{dist}\,(\vartheta, K) \;=\; H^{-} \tag{41}$$

for some proper convex subgroup H of vK.

A Kummer defect extension of prime degree with Kummer generator η of value 0 *has independent defect if and only if*

$$\mathrm{dist}\,(\eta, K) \;=\; \frac{vp}{p-1} + H^{-}\,, \tag{42}$$

or equivalently,

$$\mathrm{dist}\,(\vartheta_{\eta}, K) \;=\; H^{-}\,, \tag{43}$$

for some convex subgroup H of vK that does not contain vp.

Proof. 1): Inequality (37) follows from Theorem 3.4 together with the definition of $\Sigma_{\mathcal{E}}$ in Theorem 3.5. From inequality (37) we obtain inequality (38) and the second inequality in (39) by an application of Corollary 3.6. The first inequality in (39) follows from Theorem 2.6 since $v\eta = 0$.

2): By definition, the lower cut set of $\mathrm{dist}\,\mathcal{E}$ is the smallest initial segment of $\widetilde{vK}$ containing $-\Sigma_{\mathcal{E}}$. Since $-\Sigma_{\mathcal{E}}$ is an initial segment of vK, $\mathrm{dist}\,\mathcal{E} = H^{-}$ is equivalent to $\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H\}$.

The final assertion of part 2) follows from the fact that the only proper convex subgroup in an archimedean ordered abelian group is $\{0\}$.

3): This follows from part 2) together with Corollary 3.6. In the case of a Kummer extension we have that $\mathrm{dist}\,(\eta, K) > v\eta = 0$, so H cannot contain vp. □

Proposition 3.8. *Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p with an Artin-Schreier or Kummer generator a. Further, choose any extension of v from $K(a)$ to $\tilde{K}$, take (K^r, v) to be the absolute ramification field of (K, v), and N to be an intermediate field of $K^r|K$. Then also $\mathcal{E}_N := (L.N|N, v)$ is a Galois defect extension of degree p,*

$$\mathrm{dist}\,\mathcal{E}_N \;=\; \mathrm{dist}\,\mathcal{E}\,,$$

and $\mathcal{E}_N$ has independent defect if and only $\mathcal{E}$ has. Further, if (N, v) is an independent defect field, then so is (K, v).

Proof. We may assume that a is a generator of $\mathcal{E}$ as in Theorem 3.5. By equation (13) of Proposition 2.12, also $\mathcal{E}_N$ is a Galois defect extension of prime degree p, and $\mathrm{dist}\,(a, N) = \mathrm{dist}\,(a, K)$. In view of Corollary 3.6, we obtain that $\mathrm{dist}\,\mathcal{E}_N = \mathrm{dist}\,\mathcal{E}$. From this, the third assertion follows by part 2) of Proposition 3.7.

In order to prove the final assertion, assume that (N,v) is an independent defect field. Take a p-th root of unity ζ_p. Then by definition, also $N(\zeta_p) \subseteq K^r = K(\zeta_p)^r$ is an independent defect field with respect to v. Take any Galois defect extension of degree p of $K(\zeta_p)$ with a generator a as above. Then $(N(\zeta_p)(a)|N(\zeta_p),v)$ has independent defect, and by what we have proved already, the same is true for $(K(\zeta_p)(a)|K(\zeta_p),v)$. This shows that (K,v) is an independent defect field. □

Proof of Theorem 1.15. We have shown in Theorem 3.5 that $\Sigma_{\mathcal{E}}$ is the unique ramification jump of $\mathcal{E}$, and it follows that $I_{\mathcal{E}}$ is the unique ramification ideal of $\mathcal{E}$. Thus the equivalence of assertions a) and b) follows from the definition of independent defect. The equivalence of assertions b) and c) in Theorem 1.15 holds because $vL = vK$ and an ideal I_Σ of $\mathcal{O}_L$ is prime if and only if $\Sigma = \{\alpha \in vL \mid \alpha > H\}$ for some proper convex subgroup H of vL.

The remaining assertions follow from basic facts of valuation theory. □

For Artin-Schreier defect extensions, a different definition was given for dependent and independent defect in [19]. We will show in the next section that our new definition is consistent with the previous one.

3.3. **Artin-Schreier defect extensions.**

In this section, we consider the case of a valued field (K,v) of positive characteristic p and an Artin-Schreier defect extension $(L|K,v)$ with Artin-Schreier generator ϑ, that is, $\vartheta^p - \vartheta \in K$. The following definition was introduced in [19]: if there is an immediate purely inseparable extension $(K(\eta)|K,v)$ of degree p such that

$$\vartheta \sim_K \eta\,, \tag{44}$$

then we say that the Artin-Schreier defect extension has **dependent defect**; otherwise it has **independent defect**. Note that (44) implies that $\operatorname{dist}(\eta,K) < \infty$, that is, η does not lie in the completion of (K,v), since otherwise it would follow that $\vartheta = \eta$.

The above definition does not depend on the Artin-Schreier generator of the extension $L|K$. Indeed, by [19, Lemma 2.26], $\vartheta' \in L$ is another Artin-Schreier generator of $L|K$ if and only if $\vartheta' = i\vartheta + c$ for some $i \in \mathbb{F}_p^\times$ and $c \in K$. If we set $\eta' = i\eta + c$, then $K(\eta) = K(\eta')$ and $v(\vartheta' - \eta') = v(i(\vartheta - \eta)) = v(\vartheta - \eta) > \operatorname{dist}(\vartheta, K)$, that is, $\vartheta' \sim_K \eta'$.

Proposition 3.9. *An Artin-Schreier defect extension is dependent in the sense as defined in* [19] *if and only if it is dependent in the sense as defined in the introduction.*

Proof. Take an Artin-Schreier defect extension $(L|K,v)$ and write $p = \operatorname{char} K$. First assume that $\vartheta \sim_K \eta$ holds for some Artin-Schreier generator ϑ of $(L|K,v)$ and an element η such that $\eta^p \in K$. Then $v(\vartheta - \eta) > \operatorname{dist}(\vartheta, K)$. On the other hand, $pv(\vartheta - \eta) = v(\vartheta^p - \eta^p) = v(\vartheta + \vartheta^p - \vartheta - \eta^p) \in v(\vartheta - K)$. Suppose that (41) holds for some proper convex subgroup H of vK. Then $v(\vartheta - \eta) \in H$, but $pv(\vartheta - \eta) \notin H$, a contradiction. Now Proposition 3.7 shows that $(L|K,v)$ has a defect that is dependent in the sense as defined in the introduction.

Now assume that (41) does not hold for any proper convex subgroup H of vK. This means that the set $\{\alpha \in vK \mid v(\vartheta - K) < \alpha \leq 0\}$ is not closed under addition; more specifically, there is some α in this set and some $c \in K$ such that $p\alpha \leq v(\vartheta - c)$. Set $b := (\vartheta - c)^p - (\vartheta - c) \in K$ so that the Artin-Schreier generator

$\vartheta - c$ becomes a root of the Artin-Schreier polynomial $X^p - X - b$. Then by [19, Theorem 4.5 (c)], the root η of the polynomial $X^p - b$ generates an immediate extension which does not lie in the completion of (K, v), and $\vartheta - c \sim_K \eta$ holds. □

The name "dependent defect" was chosen in [19] because the existence of Artin-Schreier defect extensions with a defect of (K, v) that is dependent according to the definition given in [19] depends on the existence of purely inseparable defect extensions of degree p that do not lie in the completion; [19, Proposition 4.3] shows how the former are constructed from the latter. If (K, v) admits any purely inseparable defect extension not contained in its completion, then it also admits one of degree p. This is proved in the beginning of Section 4.3 of [19].

The reverse construction is given in the foregoing proof. Hence if (K, v) is an independent defect field in the sense as defined in the introduction, then it admits an immediate purely inseparable extension of degree p that does not lie in its completion. This proves part 2) of Proposition 1.14.

4. Semitame, deeply ramified and rdr fields

Throughout this section, we will consider a valued field (K, v) **of residue characteristic** $p > 0$**.** All statements we will prove are trivial for valued fields of residue characteristic 0.

When we deal with valued fields (K, v) of mixed characteristic with residue characteristic p, we will write $v = v_0 \circ v_p \circ \overline{v}$ as in the paragraph before Proposition 1.3, set $\mathrm{crf}\,(K, v) := Kv_0v_p$ and denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp. Further, $\frac{1}{p^\infty}\mathbb{Z}vp$ will denote the p-divisible hull of the subgroup $\mathbb{Z}vp$ of vK generated by vp. If K has positive characteristic p, then we set $\mathrm{crf}\,(K, v) := Kv$ and $(vK)_{vp} = Kv$.

4.1. Some basic results.

To start with, we state a few simple observations.

Lemma 4.1. *1) If* $\mathrm{char}\, K = p > 0$*, then*

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K \tag{45}$$

is surjective if and only if K *is perfect; in particular, (DRvr) holds if and only if* $\hat{K}$ *is perfect.*

2) If (45) is surjective, then (DRvr) holds.

3) If $\mathrm{char}\, K = 0$*, then the following assertions are equivalent:*

a) (45) is surjective,

b) for every $a \in \mathcal{O}_K$ *there is* $c \in \mathcal{O}_K$ *such that* $a \equiv c^p \mod p\mathcal{O}_K$*,*

c) for every $\hat{a} \in \mathcal{O}_{\hat{K}}$ *there is* $c \in \mathcal{O}_K$ *such that* $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$*,*

d) (DRvr) holds.

4) If (K, v) *satisfies (DRvr), then so does every extension of* (K, v) *within its completion.*

Proof. 1): From $\mathrm{char}\, K = p > 0$ it follows that $p\mathcal{O}_K = \{0\}$, hence the surjectivity of the homomorphism in (5) means that every element in $\mathcal{O}_K$ is a p-th power. Hence the same is true for every element in K, i.e., K is perfect. Replacing K by $\hat{K}$ in (45), we thus obtain that $\hat{K}$ is perfect.

2): Assume first that $\operatorname{char} K = p > 0$. Then by part 1) the surjectivity of (45) implies that K is perfect. Since the completion of a perfect field is again perfect, it follows that $\hat{K}$ is perfect. Hence again by part 1), (DRvr) holds.

Now assume that $\operatorname{char} K = 0$. Take $\hat{a} \in \mathcal{O}_{\hat{K}}$. Then there exists $a \in K$ such that $\hat{a} \equiv a \mod p\mathcal{O}_{\hat{K}}$. By assumption, there is some $c \in \mathcal{O}_K$ such that $a \equiv c^p \mod p\mathcal{O}_K$. It follows that $\hat{a} \equiv a \equiv c^p \mod p\mathcal{O}_{\hat{K}}$, showing that (DRvr) also holds in this case.

3): Assume that $\operatorname{char} K = 0$. The proof of the equivalence of a) and b) is straightforward. Trivially, c) implies b), and part 2) of our lemma shows that a) implies d). To show that d) implies c), take $\hat{a} \in \mathcal{O}_{\hat{K}}$. Then by (DRvr), using the equivalence of a) and b) with $\hat{K}$ in place of K, there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $\hat{a} \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. We take $c \in \mathcal{O}_K$ such that $c \equiv \hat{c} \mod p\mathcal{O}_{\hat{K}}$. Then $\hat{a} \equiv \hat{c}^p \equiv c^p \mod p\mathcal{O}_{\hat{K}}$, whence $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$.

4): Take $(L|K,v)$ to be a subextension of $(\hat{K}|K,v)$. Then $\hat{L} = \hat{K}$, and in the case of $\operatorname{char} K = p > 0$ our assertion follows from part 1).

Now assume that (K,v) is of mixed characteristic and satisfies (DRvr). Then by the implication d)⇒c) of part 3), for every $\hat{a} \in \mathcal{O}_{\hat{K}} = \mathcal{O}_{\hat{L}}$ there is $c \in \mathcal{O}_K \subseteq \mathcal{O}_L$ such that $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$. Hence (45) is surjective in (L,v), and the implication a)⇒d) of part 3) shows that (L,v) satisfies (DRvr). □

Lemma 4.2. *If (K,v) satisfies (DRvr), then the following assertions hold:*

1) The residue fields Kv and $\operatorname{crf}(K,v)$ are perfect.

2) If $\operatorname{char} K = p > 0$, then vK is p-divisible and (K,v) is a semitame field.

Proof. To prove part 1), take any $a \in \mathcal{O}$. By assumption, there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $a \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. From this we obtain that $av = \hat{c}^p v = (\hat{c}v)^p \in \hat{K}v = Kv$. Hence Kv is perfect. If (K,v) is of mixed characteristic, then the same holds with $v_0 \circ v_p$ in place of v, which shows that $\operatorname{crf}(K,v)$ is perfect.

To prove part 2), assume that $\operatorname{char} K = p > 0$. Then by part 1) of Lemma 4.1, (DRvr) implies that $\hat{K}$ is perfect, so $vK = v\hat{K}$ is p-divisible and (DRst) holds, showing that (K,v) is a semitame field. □

Take any $d \in \mathcal{M}_K$. If for every $a \in \mathcal{O}_K$ there is $c \in \mathcal{O}_K$ such that $a \equiv c^p \mod d\mathcal{O}_K$, we will say that the function

$$\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K \tag{46}$$

is **surjective modulo** $d\mathcal{O}_K$. This implies that the function

$$\mathcal{O}_K^\times \ni x \mapsto x^p \in \mathcal{O}_K^\times \tag{47}$$

is **surjective modulo** $d\mathcal{O}_K$ (with the obvious modification of the above definition).

Lemma 4.3. *For a valued field (K,v) of mixed characteristic, the following assertions hold:*

1) If (K,v) is an rdr field, then $(vK)_{vp}$ is p-divisible; in particular, vK contains $\frac{1}{p^\infty}\mathbb{Z}vp$. If in addition $(vK)_{vp} = vK$, then (K,v) is a semitame field.

2) Assume that for $d \in \mathcal{M}_K$ the function (47) is surjective modulo $d\mathcal{O}_K$. Then for every $a \in K$ with p-divisible value va there is $c \in K$ such that

$$v(a - c^p) \;\geq\; va + vd\,. \tag{48}$$

If in addition $vd \in (vK)_{vp}$ and $(vK)_{vp}$ is p-divisible, then the function (46) is surjective modulo $d\mathcal{O}_K$.

Proof. 1): First, let us show that every $\alpha \in vK$ with $0 \leq \alpha < vp$ is divisible by p. Take $a \in \mathcal{O}$ such that $va = \alpha$. From (DRvr) we obtain that there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $a \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. Since $va < vp$, this yields that $va = v\hat{c}^p = pv\hat{c}$, showing that $\alpha = va$ is divisible by p in $v\hat{K} = vK$.

By assumption, vp is not the smallest positive element in vK, hence there is $\alpha \in vK$ such that $0 < \alpha < vp$, and we know that α is divisible by p. We may assume that $2\alpha \geq vp$ since otherwise we replace α by $vp - \alpha$. In this way we make sure that $(vK)_{vp}$ is equal to the smallest convex subgroup containing α. This implies that for every $\beta \in (vK)_{vp}$ there is some $n \in \mathbb{Z}$ such that $0 \leq \beta - n\alpha < vp$. Then by what we have already shown, $\beta - n\alpha$ is divisible by p. Since also α is divisible by p, the same is consequently true for β.

If in addition $(vK)_{vp} = vK$, then vK is p-divisible, and since (DRvr) holds by assumption, (K, v) is a semitame field.

2): Take $a \in K$ with p-divisible value. Then there is $b \in K$ such that $pvb = va$. Hence $vb^{-p}a = 0$ and by assumption, there is $c_0 \in K$ such that $v(b^{-p}a - c_0^p) \geq vd$, whence

$$v(a - (bc_0)^p) \;=\; pvb + v(b^{-p}a - c_0^p) \;\geq\; va + vd\,.$$

With $c := bc_0$, this yields (48).

Now assume in addition that $vd \in (vK)_{vp}$ and $(vK)_{vp}$ is p-divisible, and take $a \in \mathcal{O}_K$. If $va > (vK)_{vp}$, then $a \equiv 0^p \mod d\mathcal{O}_K$. If $va \in (vK)_{vp}$, then va is p-divisible and by what we have already shown there is $c \in K$ such that $a \equiv c^p \mod d\mathcal{O}_K$. This proves that (46) is surjective modulo $d\mathcal{O}_K$. □

Proposition 4.4. *Take a valued field (K, v) of mixed characteristic such that $(vK)_{vp}$ is p-divisible. Further, take $d \in \mathcal{M}_K$ such that $vd < vp$ and $nvd \geq vp$ for some $n \in \mathbb{N}$. Then the following assertions are equivalent:*

a) the function (45) is surjective, so (K, v) is an rdr field,

b) the function (46) is surjective modulo $d\mathcal{O}_K$,

c) the function (47) is surjective modulo $d\mathcal{O}_K$.

Proof. Since $vd < vp$, we have that $p\mathcal{O}_K \subset d\mathcal{O}_K$. Hence the proof of implication a)⇒c) is straightforward. Implication c)⇒b) is the content of part 2) of Lemma 4.3.

b)⇒a): Assume that assertion b) holds, and take any $a \in \mathcal{O}_K$. By part 2) of Lemma 4.3, there is $c_1 \in K$ such that $v(a - c_1^p) \geq vd$. Now we proceed by induction: having chosen c_k such that

$$v(a - c_1^p - \ldots - c_k^p) \;\geq\; kvd\,,$$

we can employ part 2) of Lemma 4.3 again to find $c_{k+1} \in K$ such that

$$v(a - c_1^p - \ldots - c_k^p - c_{k+1}^p) \geq kvd + vd\,.$$

After n many steps we have:

$$v(a - c_1^p - \ldots - c_n^p) \;\geq\; nvd \geq vp\,.$$

Using part 1) of Lemma 2.16, we obtain:

$$a \;\equiv\; c_1^p + \ldots + c_n^p \;\equiv\; (c_1^p + \ldots + c_n)^p \quad \mod p\mathcal{O}_K\,.$$

This proves that the function (45) is surjective. By part 2) of Lemma 4.1, (DRvr) holds. By assumption, $(vK)_{vp}$ is p-divisible, hence also (DRvp) holds. This proves that (K,v) is an rdr field. □

Lemma 4.5. *Assume that (K,v) is of mixed characteristic with $(vK)_{vp}$ p-divisible and Kv perfect, and take $\eta \in \tilde{K}$ such that $\eta^p \in \mathcal{O}_K$. Then either $v(\eta - K)$ does not admit a maximal element, or its maximal element is not smaller than $\frac{vp}{p}$.*

Proof. Take $c \in K$ such that $0 \leq v(\eta - c) < \frac{vp}{p}$. Then by use of (14) it follows that $v(\eta^p - c^p) = v(\eta - c)^p < vp$. Since $(vK)_{vp}$ is p-divisible, there is some $d_1 \in K$ such that $vd_1^p(\eta^p - c^p) = 0$, and since Kv is perfect, there is some $d_2 \in K$ such that $(d_2^p d_1^p(\eta^p - c^p))v = 1$. With $d = (d_1 d_2)^{-1}$ it follows that $v(d^{-p}(\eta^p - c^p) - 1) > 0$, whence $v(\eta^p - c^p - d^p) > v(\eta^p - c^p)$. Again by (14), we obtain that $(\eta - c - d)^p \equiv \eta^p - c^p - d^p \mod p\mathcal{O}$, and it follows that $v(\eta - c - d) > v(\eta - c)$. □

4.2. **Proof of Theorem 1.2.**

1): Assume that (K,v) is nontrivially valued. The implication tame field ⇒ separably tame field is obvious, and so is the implication semitame field ⇒ deeply ramified field. To prove the implication deeply ramified field ⇒ rdr field, we first observe that if $\operatorname{char} K = p > 0$, then $vp = \infty$ which is not the smallest positive element of vK. If $\operatorname{char} K = 0$, then vp is not the smallest positive element of vK since otherwise, if Γ is the largest convex subgroup of vK not containing vp, then $(vK)_{vp}/\Gamma \simeq \mathbb{Z}$ in contradiction to (DRvg).

Now assume that (K,v) is a separably tame field. If $\operatorname{char} K > 0$, then by [22, Corollary 3.12], (K,v) is dense in its perfect hull. Then the completion of the perfect hull is also the completion of (K,v). Since the completion of a perfect valued field is again perfect, we obtain that the completion of (K,v) is perfect. Now part 1) of Lemma 4.1 shows that (K,v) is a semitame field.

Assume that $\operatorname{char} K = 0$. Then the separably tame field (K,v) is a tame field. By [22, Lemma 3.1 and Theorem 3.2], (K,v) is defectless, vK is p-divisible and Kv is perfect. Take any $b \in K$ that is not a p-th power, and take $\eta \in \tilde{K}$ with $\eta^p = b$. The unibranched extension $(K(\eta)|K,v)$ is defectless, hence by Lemma 2.7, $v(\eta - K)$ has a maximal element. By Lemma 4.5, this maximal element is not smaller than $\frac{vp}{p}$. Now part 3) of Lemma 2.16 shows the existence of $c \in K$ such that $b \equiv c^p \mod p\mathcal{O}_K$. This proves that (K,v) is a semitame field.

2): Assume that (K,v) is an rdr field of rank 1 and mixed characteristic. Since the rank is 1, we have that $(vK)_{vp} = vK$. Hence by part 1) of Lemma 4.3, (K,v) is a semitame field. This together with part 1) of our theorem shows the required equivalence in the case of mixed characteristic. For the case of equal characteristic, it will be shown in the proof of part 3).

3): Assume that (K,v) is a nontrivially valued field of characteristic $p > 0$.

The implications a)⇒b)⇒c) have already been shown in part 1).

c)⇒d): This holds by definition.

d)⇒e): This holds by part 1) of Lemma 4.1.

e)⇒f): If the completion of (K,v) is perfect, then it contains the perfect hull of K; since (K,v) is dense in its completion, it is then also dense in its perfect hull.

f)⇒g): If (K,v) is dense in its perfect hull, then in particular it is dense in $K^{1/p} = \{a^{1/p} \mid a \in K\}$. Since $x \mapsto x^p$ is an isomorphism which preserves valuation divisibility, the latter holds if and only if (K^p, v) is dense in (K,v).

g)⇒f): Assume that (K^p, v) is dense in (K,v). Since for each $i \in \mathbb{N}$, $x \mapsto x^{p^i}$ is an isomorphism which preserves valuation divisibility, it follows that $(K^{1/p^{i-1}}, v)$ is dense in $(K^{1/p^i}, v)$. By transitivity of density we obtain that (K,v) is dense in $(K^{1/p^i}, v)$ for each $i \in \mathbb{N}$, and hence also in its perfect hull.

f)⇒e): This implication was already shown in the proof of part 1) of our theorem.

e)⇒a): Assume that $\hat{K}$ is perfect. The extension $(\hat{K}|K,v)$ is immediate, so $vK = v\hat{K}$, which is p-divisible. Hence (DRst) holds. By part 1) of Lemma 4.1, also (DRvr) holds.

4): The assertion follows from the implication f)⇒a) of part 3) as a perfect field is equal to its perfect hull. □

4.3. **Proof of Propositions 1.3 and 1.4.**

For the proof of Propositions 1.3 and 1.4, we will need some preparation.

Lemma 4.6. *Assume that (K,v) is of mixed characteristic, and set $w := v_p \circ \overline{v}$. Then (K,v) is an rdr field if and only if (Kv_0, w) is an rdr field.*

Proof. First assume that (K,v) is an rdr field. Then vp is not the smallest positive element in vK, which implies that wp is not the smallest element in $w(Kv_0)$. Take any $b \in \mathcal{O}_{Kv_0}$. Then choose $a \in \mathcal{O}_K$ such that $av_0 = b$. Since (K,v) is an rdr field, there is some $c \in \mathcal{O}_K$ such that $a - c^p \in p\mathcal{O}_K$. It follows that $cv_0 \in \mathcal{O}_{Kv_0}$ with $b - (cv_0)^p = (a - c^p)v_0 \in p\mathcal{O}_{Kv_0}$, showing that (Kv_0, w) satisfies (DRvr) by part 3) of Lemma 4.1. Hence (Kv_0, w) is an rdr field.

Now assume that (Kv_0, w) is an rdr field. Then wp is not the smallest element in $w(Kv_0)$, which implies that vp is not the smallest positive element in vK. Take any $a \in \mathcal{O}_K$. Then $av_0 \in \mathcal{O}_{Kv_0}$ and there is some $d \in \mathcal{O}_{Kv_0}$ such that $av_0 - d^p \in p\mathcal{O}_{Kv_0}$. Choose $c \in \mathcal{O}_K$ such that $cv_0 = d$. It follows that $a - c^p \in p\mathcal{O}_K$. Again using part 3) of Lemma 4.1, we conclude that (K,v) is an rdr field. □

Proof of Proposition 1.3.
The equivalence of assertions a) and b) is proved in Lemma 4.6. To prove the equivalence of assertions a) and c), we may assume that v_0 is trivial, that is, $Kv_0 = K$, $v = v_p \circ \overline{v}$ and $vK = (vK)_{vp}$. The assertion is trivial if $\overline{v}$ is trivial, so we assume that it is not. This implies that vp is not the smallest positive element in vK.

Let us first assume that (K,v) is an rdr field. Then $\frac{vp}{p} \in vK$ by part 1) of Lemma 4.3, so $\frac{v_p p}{p} \in v_pK$, showing that $v_p p$ is not the smallest positive element in v_pK. It remains to show that (K,v_p) satisfies (DRvr); by part 3) of Lemma 4.1 it suffices to prove that (45) is surjective in (K,v_p). Take any $a \in \mathcal{O}_{v_p}$. Since (K,v) is an rdr field, by part 2) of Lemma 4.3 there is $c \in K$ such that $v(a-c^p) \geq va + vp$, whence $v_p(a-c^p) \geq v_p a + v_p p \geq v_p p$.

Now assume that (K,v_p) is a deeply ramified field, hence an rdr field. As $\overline{v}$ is not trivial, we know already that (DRvp) holds in (K,v), so it remains to show that (45) holds. Since (K,v_p) is an rdr field, for every $a \in \mathcal{O}_v \subseteq \mathcal{O}_{v_p}$ there is some

$c\in K$ such that $v_p(a-c^p)\geq v_pp$, whence $v(a-c^p)>\frac{vp}{p}$. Choosing $d\in K$ with $vd=\frac{vp}{p}$ and applying Proposition 4.4, we conclude that (K,v) is an rdr field.

We turn to the equivalence of assertions a) and d). The implication a)⇒d) follows from part 1) of Lemma 4.3. Conversely, if vK is roughly p-divisible, then vp itself is divisible by p, so (DRvp) holds. □

Proof of Proposition 1.4.
Take an arbitrary valued field (K,v) and assume that $v=w\circ\overline{w}$ with w and $\overline{w}$ nontrivial. Assume first that $\operatorname{char}K>0$. Then by part 3) of Theorem 1.2, the properties "semitame", "deeply ramified" and "rdr" are equivalent, so we have to prove the assertions of the lemma only for "rdr".

As w is nontrivial and a coarsening of v, the topologies generated by v and w are equal, and it follows that (K,v) is dense in its perfect hull if and only if the same holds for (K,w). By the equivalence of assertions c) and f) in part 3) of Theorem 1.2, it follows that (K,v) is an rdr field if and only if (K,w) is an rdr field. If the latter is the case, then from Lemma 4.2 we see that Kw is perfect, and as it is of positive characteristic like K, we obtain from part 3) of Theorem 1.2 that $(Kw,\overline{w})$ is also an rdr field.

Now we assume that $\operatorname{char}K=0$ and prove the assertions for the property "rdr". First we discuss the case where $\operatorname{char}Kw>0$ and write w in the same way as we do for v: $w=w_0\circ w_p\circ\overline{w}$. Then $v_0=w_0$, $v_p=w_p$, and $\overline{w}$ is a (possibly trivial) coarsening of $\overline{v}$. Hence it follows from Proposition 1.3 that (K,v) is an rdr field if and only if (K,w) is an rdr field. If the latter is the case, then because of $\operatorname{char}Kw>0$ it follows as before that $(Kw,\overline{w})$ is also an rdr field.

Now we discuss the case where $\operatorname{char}Kw=0$. Then (K,w) is trivially an rdr field, and w_0 is a coarsening of v_0. We write $\overline{w}=\overline{w}_0\circ\overline{w}_p\circ\overline{\overline{w}}$ as for v. We obtain that $\overline{w}_p=v_p$, $\overline{\overline{w}}=\overline{v}$, and $\overline{w}_0$ is possibly trivial, with $w\circ\overline{w}_0=v_0$. It follows that $(Kv_0,v_p)=((Kw)\overline{w}_0,\overline{w}_p)$. Using Proposition 1.3, we conclude that (K,v) is an rdr field if and only if $(Kw,\overline{w})$ is an rdr field.

It remains to consider the properties "semitame" and "deeply ramified". We observe that if $\operatorname{char}Kv=p>0$, then vK is p-divisible if and only if the same is true for wK and $\overline{w}(Kw)$. Likewise, all archimedean components of vK are densely ordered if and only if the same is true for wK and $\overline{w}(Kw)$. From what we have proved before, it thus follows that (K,v) is a semitame (or deeply ramified) field if and only if both (K,w) and $(Kw,\overline{w})$ are semitame (or deeply ramified, respectively).

Further, we recall that in the case of $\operatorname{char}Kw>0$, (K,w) being an rdr field implies that Kw is perfect, and so $\overline{w}(Kw)$ is p-divisible and thus all of its archimedean components are densely ordered. This proves that (K,v) is a semitame (or deeply ramified) field already if (K,w) is. □

4.4. **Proof of Theorems 1.5 and 1.8 for the equal characteristic case.**

Proposition 4.7. *Take an algebraic extension $(L|K,v)$ of valued fields of positive characteristic. If (K,v) is an rdr field, then so is (L,v). If $L|K$ is finite and (L,v) is an rdr field, then so is (K,v). Both statements also hold for "deeply ramified" and "semitame" in place of "rdr".*

Proof. In view of part 3) of Theorem 1.2, our assertions only need to be proved for rdr fields. By part 3) of Theorem 1.2, a valued field (K,v) of positive characteristic is an rdr field if and only if its completion $(\hat{K},v)$ is perfect.

Assume that (K,v) is an rdr field. Then the completion $(\hat{L},v)$ of (L,v) contains $(\hat{K},v)$. Since $\hat{K}$ is perfect, so is $L.\hat{K}$. Since $(\hat{L},v)$ is also the completion of $(L.\hat{K},v)$, it is perfect too. Hence (L,v) is an rdr field.

Now assume that $L|K$ is finite and (L,v) is an rdr field. Then $\hat{L} = L.\hat{K}$ is perfect. As $L.\hat{K}|\hat{K}$ is finite, it follows that $\hat{K}$ is perfect. Thus (K,v) is an rdr field. □

4.5. **Proof of Theorem 1.6 and Corollary 1.7.**

Our next goal is the proof of Theorem 1.6. First, we prove the upward direction. By Proposition 4.7, we only need to prove it in the mixed characteristic case.

Lemma 4.8. *Assume that (K,v) is a henselian rdr field of mixed characteristic with residue characteristic $p>0$, and that $(L|K,v)$ is a finite extension. Then the following assertions hold.*

1) If $[L:K]=[Lv:Kv]$, then also (L,v) is an rdr field.

2) Take a prime q different from p. Assume that $L=K(a)$ with $a^q\in K$ and $va\notin vK$. Then also (L,v) is an rdr field.

Proof. Like (K,v), also (L,v) satisfies (DRvp). Hence by part 3) of Lemma 4.1, (L,v) will be an rdr field once (45) is surjective.

In order to prove part 1), we take a finite extension $(L|K,v)$ such that $[L:K]= [Lv:Kv]$. Since Kv is perfect by Lemma 4.2, $Lv|Kv$ is separable and we write $Lv=Kv(\xi)$ with $\xi\in Lv$. Since also Lv is perfect, there are $\xi_0,\ldots,\xi_n\in Kv$ with $n=[Lv:Kv]-1$ such that $\xi=(\xi_n\xi^n+\ldots+\xi_1\xi+\xi_0)^p$. Let F be the extension of $\mathbb{F}_p$ generated by the coefficients of the minimal polynomial of ξ over Kv and the elements $\xi_0,\ldots,\xi_n$. As a finitely generated extension of the perfect field $\mathbb{F}_p$, F is separably generated, that is, it admits a transcendence basis $t_1,\ldots,t_k$ such that $F|\mathbb{F}_p(t_1,\ldots,t_k)$ is separable-algebraic. We have that $F\subseteq Kv$, so we may choose $x_1,\ldots,x_k\in K$ such that $x_iv=t_i$. Then $v\mathbb{Q}(x_1,\ldots,x_k)=v\mathbb{Q}=\mathbb{Z}vp$ and $\mathbb{Q}(x_1,\ldots,x_k)v=\mathbb{F}_p(t_1,\ldots,t_k)$ (cf. [4, chapter VI, §10.3, Theorem 1]). Using Hensel's Lemma, we find an extension K_0 of $\mathbb{Q}(x_1,\ldots,x_k)$ within the henselian field K such that $K_0v=F$ and $vK_0=v\mathbb{Q}(x_1,\ldots,x_k)=\mathbb{Z}vp$.

Using Hensel's Lemma again, we find $a\in L$ such that $av=\xi$, $[K_0(a):K_0]= [F(\xi):F]$ and $vK_0(a)=vK_0=\mathbb{Z}vp$. By construction, $\xi^{1/p}\in F(\xi)$, so we can choose $b\in K_0(a)$ such that $bv=\xi^{1/p}$. Then $av=(bv)^p=b^pv$, so $v(a-b^p)>0$ and thus $v(a-b^p)\geq vp$.

We observe that since F contains all coefficients of the minimal polynomial of ξ over Kv,

$$[Kv(\xi):Kv] \;=\; [F(\xi):F] \;=\; [K_0(a):K_0] \;\geq\; [K(a):K] \;\geq\; [Kv(\xi):Kv]\,.$$

Hence equality holds everywhere; in particular, $K(a)=L$. Also, we obtain that $1,a,\ldots,a^n$ is a basis of $K(a)|K$ with the residues $1,av,\ldots,a^nv$ linearly independent over Kv. Hence if we write an arbitrary element of $K(a)$ as $\sum_{i=0}^n c_ia^i$ with $c_i\in K$, then

$$v\sum_{i=0}^n c_ia^i \;=\; \min_{0\leq i\leq n} vc_i\,.$$

Thus, for the sum to have non-negative value, all c_i must have non-negative value. Since (K,v) is an rdr field, there exists $d_i\in K$ such that $c_i\equiv d_i^p \mod p\mathcal{O}_K\,$.

Consequently,

$$\sum_{i=0}^{n} c_i a^i \;\equiv\; \sum_{i=0}^{n} d_i^p (b^p)^i \;\equiv\; \left(\sum_{i=0}^{n} d_i b^i\right)^p \mod p\mathcal{O}_L\,,$$

where the last equivalence holds by part 1) of Lemma 2.16. This shows that (45) is surjective, which proves that (L,v) is an rdr field.

In order to prove part 2), we take a prime q different from p and a finite extension $(L|K,v)$ such that $L=K(a)$ with $a^q\in K$ and $va\notin vK$. We obtain that $[K(a):K]=q=(vK(a):vK)$. As p and q are coprime, also $pva=va^p$ generates $vK(a)$ over vK, and $K(a)=K(a^p)$. Therefore, $1,a^p,\ldots,a^{p(q-1)}$ is a basis of $K(a)|K$ with the values $v1,va^p,\ldots,va^{p(q-1)}$ belonging to distinct cosets of vK. Hence if we write an arbitrary element b of $K(a)$ as $b=\sum_{i=0}^{q-1}c_ia^{pi}$ with $c_i\in K$, then

$$vb \;=\; v\sum_{i=0}^{q-1} c_i a^{pi} \;=\; \min_{0\le i<q} vc_i + iva^p\,.$$

Assume that $vb\geq 0$. Then all c_ia^{pi} must have non-negative value. However, for $i>0$ this does not imply that $vc_i\geq 0$; we only know that $vc_ia^{pi}>0$ since $iva^p\notin vK$, whence $va^{pi}>-vc_i$.

Suppose that va is not equivalent to an element in vK modulo $(vL)_{vp}$. Then the same holds for vc_i+piva in place of va, for $1\leq i<q$, so that $vc_ia^{pi}\notin (vL)_{vp}$. In this case, b is equivalent to c_0 modulo $p\mathcal{O}_L$. Since (K,v) is an rdr field, there is $d_0\in K$ such that $b\equiv c_0\equiv d_0^p \mod p\mathcal{O}_L$. Hence we may now assume that va is equivalent to an element $\delta\in vK$ modulo $(vL)_{vp}$. We choose $d\in K$ with $vd=\delta$ and replace a by a/d, so from now on we can assume that $va\in (vL)_{vp}$.

As (K,v) is an rdr field, $(vK)_{vp}$ is p-divisible by part 1) of Lemma 4.3. It follows that $p(vK)_{vp}$ lies dense in $(vL)_{vp}$ and thus there is $b_i\in K$ such that $-vc_i\leq pvb_i\leq va^{pi}$, whence $vc_ib_i^p\geq 0$ and $vb_i^{-p}a^{pi}\geq 0$. Again since (K,v) is an rdr field, there are $d_i\in K$ such that $c_ib_i^p\equiv d_i^p \mod p\mathcal{O}_K$. Hence we obtain that

$$\sum_{i=0}^{q-1} c_i a^{pi} \;=\; \sum_{i=0}^{q-1}(c_ib_i^p)(b_i^{-p}a^{pi}) \;\equiv\; \sum_{i=0}^{q-1} d_i^p b_i^{-p} a^{pi} \;\equiv\; \left(\sum_{i=0}^{q-1} d_i b_i a^i\right)^p \mod p\mathcal{O}_L\,,$$

where the last equivalence holds by part 1) of Lemma 2.16. Again, this shows that (45) is surjective, which proves that (L,v) is an rdr field. □

Proposition 4.9. *Take a valued field (K,v) of mixed characteristic, fix any extension of v to $\tilde{K}$, and let (K^r,v) be the corresponding absolute ramification field of (K,v). If (K,v) is an rdr field, then so is (K^r,v).*

Proof. In this proof we will freely make use of facts from ramification theory; for details, see [9, 10, 22].

We let L be a maximal extension of K inside of K^r that is again an rdr field; since the union over an ascending chain of rdr fields is again an rdr field, L exists by Zorn's Lemma.

First we will show that (L,v) is henselian. The decomposition $v=v_0\circ v_p\circ\overline{v}$ for v on K carries over to L with extensions of the respective valuations v_0, v_p and $\overline{v}$. We note that v is henselian on L if and only if v_0, v_p and $\overline{v}$ are.

Suppose that v_0 is not henselian on L. As (K^r,v) is henselian, so is (K^r,v_0) which therefore contains a henselization $L^{h(v_0)}$ of L with respect to v_0. As henselizations are immediate extensions, we know that $L^{h(v_0)}v_0 = Lv_0$; by Proposition 1.3, (Lv_0,v_p) is an rdr field. Using the same proposition again, we find that also $(L^{h(v_0)},v_0)$ is an rdr field. By the maximality of L we conclude that $L^{h(v_0)} = L$, so v_0 is henselian on L.

Next, suppose that v_p is not henselian on Lv_0. As (K^r,v) is henselian, so is $(K^r v_0,v_p)$ which therefore contains a henselization $Lv_0^{h(v_p)}$ of Lv_0 with respect to v_p. We know already that (Lv_0,v_p) is an rdr field. As its rank is 1, its henselization lies in its completion. Hence by part 4) of Lemma 4.1, $(Lv_0^{h(v_p)},v_p)$ satisfies (DRvr). Since (DRvp) holds in (Lv_0,v_p), it also holds in $(Lv_0^{h(v_p)},v_p)$, so the latter is an rdr field. The extension $Lv_0^{h(v_p)}|Lv_0$ is separable-algebraic, so we can use Hensel's Lemma to find an extension L' of L within K^r such that $L'v_0 = Lv_0^{h(v_p)}$. Using Proposition 1.3 again, we find that (L',v) is an rdr field. Hence $L' = L$ by the maximality of L, that is, $Lv_0 = Lv_0^{h(v_p)}$, showing that (Lv_0,v_p) is henselian.

Finally, suppose that $\overline{v}$ is not henselian on Lv_0v_p. As (K^r,v) is henselian, so is $(K^r v_0v_p,\overline{v})$ which therefore contains a henselization $Lv_0v_p^{h(\overline{v})}$ of Lv_0v_p with respect to $\overline{v}$. Suppose that $Lv_0v_p^{h(\overline{v})}|Lv_0v_p$ is nontrivial, so it contains a finite separable subextension. Using Hensel's Lemma, we lift it to a subextension $F|L$ of $K^r|L$ such that $[F:L] = [Fv_0v_p : Lv_0v_p]$. By what we have shown already, (L,v_0v_p) is henselian, and by definition it is of mixed characteristic. Therefore, we can employ part 1) of Lemma 4.8 to deduce that (F,v_0v_p) is an rdr field. By Proposition 1.3, also (F,v) is an rdr field. This contradiction to the maximality of L shows that $Lv_0v_p^{h(\overline{v})} = Lv_0v_p$, that is, $(Lv_0v_p,\overline{v})$ is henselian. Altogether, we have now shown that (L,v) is henselian.

The residue field of K^r is the separable-algebraic closure of Kv. Suppose that Lv is not separable-algebraically closed, so it admits a finite separable-algebraic extension. Using Hensel's Lemma, we lift it to a subextension $F|L$ of $K^r|L$ such that $[F:L] = [Fv:Lv]$. Again by part 1) of Lemma 4.8, (F,v) is an rdr field, contradicting the maximality of L. Hence Lv is separable-algebraically closed.

The value group of K^r is the closure of vK under division by all primes other than p. Suppose that $vL \neq vK^r$. Then there is some prime $q \neq p$ and $\alpha \in vK^r \setminus vL$ with $q\alpha \in vL$. Take $a \in \tilde{K}$ such that $a^q \in L$ with $va^q = q\alpha$. It follows that $(L(a)|L,v)$ is a tame extension, hence a lies in the maximal tame extension L^r of L. Since $K \subseteq L \subset K^r$, we know that $K^r = L^r$, so $a \in K^r$. By part 2) of Lemma 4.8, also $(L(a),v)$ is an rdr field, which again contradicts the maximality of (L,v). We conclude that $vL = vK^r$.

By what we have shown, $Lv = K^r v$ and $vL = vK^r$. As $K^r = L^r$, we see that $(K^r|L,v)$ is a tame extension. Together with the equality of the value groups and residue fields, this implies that $L = K^r$. Thus (K^r,v) is an rdr field. □

Proposition 4.10. *Assume that (K,v) is an rdr field of mixed characteristic, and take $a \in \mathcal{O}_K$.*

1) Assume that $va = 0$. Then for every $c \in \mathcal{O}_K$ with $0 < v(a-c^p) \in (vK)_{vp}$ there is $c_1 \in \mathcal{O}_K$ such that

$$v(a-c_1^p) \;=\; vp + \frac{1}{p}v(a-c^p)\,.$$

2) Assume that $va \in (vK)_{vp}$ *and that* $\mathrm{dist}\,(a, K^p) < va + \frac{p}{p-1}vp$. *Then*

$$va + vp \;<\; \mathrm{dist}\,(a, K^p) \;=\; va + \frac{p}{p-1}vp \,+\, H^- \,,$$

where H *is a convex subgroup of* vK *not containing* vp.

Proof. 1) Set $\alpha := v(a - c^p) > 0$. Since (K, v) is an rdr field, part 2) of Lemma 4.3 shows that there is $\tilde{c} \in K$ such that:

$$v(a - c^p - \tilde{c}^p) \;\geq\; vp + \alpha \,. \tag{49}$$

It follows that $v\tilde{c}^p = \alpha > 0$. Since $vc = va = 0$,

$$v((c + \tilde{c})^p - c^p - \tilde{c}^p) \;=\; v\sum_{i=1}^{p-1}\binom{p}{i}c^{p-i}\tilde{c}^i \;=\; vp + v\tilde{c} \;=\; vp + \frac{\alpha}{p}\,. \tag{50}$$

From (49) and (50), we obtain for $c_1 := c + \tilde{c}$:

$$v(a - c_1^p) \;=\; \min\left\{vp + \alpha, vp + \frac{\alpha}{p}\right\} \;=\; vp + \frac{\alpha}{p}\,.$$

2) First we prove the assertion in the case of $va = 0$. Since (K, v) is an rdr field, there is some $c \in K$ such that $v(a - c^p) \geq vp$, so $\mathrm{dist}\,(a, K^p) \geq vp$.

We will use the following observation. If $(vK)_{vp} \ni v(a - c^p) \geq \frac{p}{p-1}vp - \varepsilon > 0$ for some $c \in K$ and positive $\varepsilon \in vK$, then by part 1) there is $d \in \mathcal{O}_K$ such that

$$v(a - d^p) \;=\; vp + \frac{v(a - c^p)}{p} \;\geq\; vp + \frac{vp}{p-1} - \frac{1}{p}\,\varepsilon \;=\; \frac{p}{p-1}vp - \frac{1}{p}\,\varepsilon\,.$$

By assumption, $\mathrm{dist}\,(a, K^p) < \frac{p}{p-1}vp$. Hence the set of all convex subgroups H' of vK such that $v(a - K^p) < \frac{p}{p-1}vp + H'$ is nonempty as it contains $\{0\}$. The set is closed under arbitrary unions, so it contains a maximal subgroup H. Since $0 \in v(a - K^p)$, we see that H cannot contain vp.

Take any positive $\delta \notin H$. Then by the definition of H, there is some $n \in \mathbb{N}$ such that $v(a - K^p)$ contains a value $\geq \frac{p}{p-1}vp - n\delta$. We set $\varepsilon := \min\left\{\frac{p}{p-1}vp - vp, n\delta\right\}$ and observe that there is $c \in K$ such that

$$v(a - c^p) \;\geq\; \frac{p}{p-1}vp - \varepsilon \;\geq\; vp \;>\; 0\,.$$

Note that $v(a - c^p) \in (vK)_{vp}$ since $\mathrm{dist}\,(a, K^p) < \frac{p}{p-1}vp$. Using our above observation, by induction starting from $c_0 = c$ we find $c_i \in K$ such that

$$v(a - c_i^p) \;\geq\; \frac{p}{p-1}vp - \frac{1}{p^i}\varepsilon\,.$$

We choose some $j \in \mathbb{N}$ such that $\frac{n}{p^j} < 1$. Then

$$\frac{1}{p^j}\varepsilon \leq \frac{n}{p^j}\delta < \delta$$

and consequently,

$$v(a - c_j^p) \;>\; \frac{p}{p-1}vp - \delta\,.$$

This together with the definition of H shows that

$$vp \;<\; \mathrm{dist}\,(a, K^p) \;=\; \frac{p}{p-1}vp \,+\, H^-\,. \tag{51}$$

If $0 \neq va \in (vK)_{vp}$, then since (K,v) is an rdr field, part 1) of Lemma 4.3 shows that there is $b \in K$ such that $vb^p = va$. By what we have already shown, (51) holds for $b^{-p}a$ in place of a. We have that

$$v(a-(bc)^p) \;=\; vb^p + v(b^{-p}a - c^p) \;=\; va + v(b^{-p}a - c^p)\,,$$

whence

$$\mathrm{dist}\,(a,K^p) \;=\; va + \mathrm{dist}\,(b^{-p}a, K^p)\,,$$

which together with (51) for $b^{-p}a$ in place of a proves assertion 2) of our lemma. □

We pause to note the following consequence of Proposition 4.10 which was mentioned in the Introduction, but will not be needed any further.

Proposition 4.11. *Take a valued field (K,v) of mixed characteristic such that $(vK)_{vp}$ is p-divisible. Further, take $d' \in \mathcal{M}_K$ such that $vp \leq vd' < \frac{p}{p-1}vp + H_0^-$ for the largest convex subgroup H_0 of vK not containing vp. Then the following assertions are equivalent:*

a) the function (45) is surjective, so (K,v) is an rdr field,

b) the function (46) is surjective modulo $d'\mathcal{O}_K$.

Proof. Since $d'\mathcal{O}_K \subseteq p\mathcal{O}_K$, and in view of the equivalence of a) and b) in part 3) of Lemma 4.1, the implication b)⇒a) is trivial.

a)⇒b): Assume that assertion a) holds, and take any $a \in \mathcal{O}_K$. We may assume that $va \in (vK)_{vp}$ since otherwise, $va > vd'$ and there is nothing to show. By our choice of H_0 and part 2) of Proposition 4.10, we now obtain:

$$\mathrm{dist}\,(a,K^p) \;\geq\; va + \frac{p}{p-1}vp \,+\, H_0^- \;\geq\; \frac{p}{p-1}vp \,+\, H_0^-\,.$$

Therefore, there is $c \in K$ such that $v(a-c^p) \geq vd'$. This proves assertion b). □

The next two propositions will describe the relation between rdr and independent defect fields.

Proposition 4.12. *Every rdr field containg all p-th roots of unity is an independent defect field.*

Proof. Assume first that $\mathrm{char}\,K > 0$. Then by part 3) of Theorem 1.2, the perfect hull of (K,v) lies in its completion. Now part 2) of Proposition 1.14 (which has already been proved at the end of Section 3.3) shows that (K,v) is an independent defect field.

Now assume that $\mathrm{char}\,K = 0$, and take a Galois defect extension $(L|K,v)$ of prime degree. As shown in the beginning of Section 3.2, we can assume that $L = K(\eta)$ with a Kummer generator η which is a 1-unit.

Suppose that there is some $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p-1}$. Since the defect extension $(K(\eta)|K,v)$ is immediate, $v(\eta - c)$ has no maximal element, and so there will also be some element $c \in K$ such that $v(\eta - c) > \frac{vp}{p-1}$. Then by Lemma 2.19, η lies in some henselization K^h. But this is impossible since by Lemma 2.4, the unibranched extension $(K(\eta)|K,v)$ is linearly disjoint from $K^h|K$. We conclude that $v(\eta - K) < \frac{vp}{p-1}$. By Lemma 2.17, this is equivalent to $v(\eta^p - K^p) < \frac{p}{p-1}vp$. Therefore, we can apply part 2) of Proposition 4.10 to $a = \eta^p$. We find that

$$\mathrm{dist}\,(\eta^p,K^p) \;=\; \frac{p}{p-1}vp \,+\, H^-\,, \tag{52}$$

where H is a convex subgroup of vK not containing vp. By part 1) of Lemma 4.3, $(vK)_{vp}$ is p-divisible. Since $H \subset (vK)_{vp}$, we can again apply Lemma 2.17 to obtain that (52) is equivalent to (42). By part 3) of Proposition 3.7 it follows that $(K(\eta)|K,v)$ has independent defect. This proves that (K,v) is an independent defect field. □

Proposition 4.13. *Assume that $(vK)_{vp}$ is p-divisible and* $\mathrm{crf}\,(K,v)$ *is perfect. If (K,v) is an independent defect field, then it is an rdr field.*

Proof. From our assumption that $(vK)_{vp}$ is p-divisible it follows that (DRvp) holds. It remains to show that (K,v) satisfies (DRvr).

Assume first that $\mathrm{char}\,K > 0$. Then by assumption, vK is p-divisible and Kv is perfect, hence the perfect hull of K is an immediate extension of (K,v). Thus by part 2) of Proposition 1.14, our assumption that (K,v) is an independent defect field implies that the perfect hull of K lies in its completion. This means that (K,v) lies dense in its perfect hull. Now part 3) of Theorem 1.2 shows that (K,v) is an rdr field.

Now assume that $\mathrm{char}\,K = 0$, and set $w := v_0 \circ v_p$. By Proposition 1.4 it suffices to prove that (K,w) is an rdr field. Assume that $b \in K$ is not a p-th power, and take $\eta \in \tilde{K}$ with $\eta^p = b$. Then from Lemma 4.5 with w in place of v we infer that either $w(\eta - K)$ has a maximal element $\geq \frac{wp}{p}$, or it has no maximal element at all. In the first case, part 3) of Lemma 2.16 shows the existence of $c \in K$ such that $b \equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Assume that $w(\eta - K)$ has no maximal element. If it is not bounded from above in $(wK)_{wp}$, then there is some $c \in K$ such that $w(\eta - c) \geq \frac{wp}{p}$, which by part 3) of Lemma 2.16 gives us that $b \equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Now assume that $w(\eta - K)$ is bounded from above in $(wK)_{wp}$. Then in particular, $w\eta \in (wK)_{wp}$. It follows that $(\eta v_0)^p = bv_0 \in Kv_0$ and that $v_p(\eta v_0 - Kv_0)$ has no maximal element but is bounded from above in $v_p(Kv_0) = (wK)_{wp}$. Hence by Lemma 2.9, $(Kv_0(\eta v_0)|Kv_0, v_p)$ is a defect extension of degree p. From this it follows that also $(K(\eta)|K,v)$ is a defect extension of degree p. We set $K' := K(\zeta_p)$ where ζ_p is a primitive p-th root of unity. Then by (13) of Lemma 2.12, also $(K'(\eta)|K',v)$ is a defect extension of degree p, with $\mathrm{dist}\,(\eta, K') = \mathrm{dist}\,(\eta, K)$. By assumption, this defect extension is independent, so

$$\mathrm{dist}\,(\eta, K) \;=\; \mathrm{dist}\,(\eta, K') \;=\; \frac{vp}{p-1} \,+\, H^-$$

for some proper convex subgroup H of vK with $vp \notin H$. Hence there is some $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$; thus as before, $b \equiv c^p \mod p\mathcal{O}_K$. This implies that $b \equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Altogether, we have shown that (45) is surjective. Hence by part 2) of Lemma 4.1, (DRvr) holds. □

Lemma 4.14. *Fix any extension of v from K to $\tilde{K}$, and let (K^r,v) be the corresponding absolute ramification field of (K,v). If (K^r,v) is an rdr field, then so is (K,v), and if (K^r,v) is a semitame field, then so is (K,v).*

Proof. Assume that (K^r,v) is an rdr field and hence an independent defect field by Proposition 4.12. By Lemmas 4.2 and 4.3, $(vK^r)_{vp}$ is p-divisible and $\mathrm{crf}\,(K^r,v)$ is perfect. Since vK^r/vK has no p-torsion, it follows that $(vK)_{vp}$ is p-divisible. From Lemma 2.15 we infer that the extension $\mathrm{crf}\,(K^r,v)\,|\,\mathrm{crf}\,(K,v)$ is separable, so

$\operatorname{crf}(K,v)$ is perfect. We set $K' := K(\zeta_p) \subseteq K^r$ as before. From Proposition 3.8 we conclude that (K',v) is an independent defect field. Hence by definition, the same holds for (K,v). Proposition 4.13 now shows that (K,v) is an rdr field.

Now assume that (K^r,v) is a semitame field. Then by Theorem 1.2, (K^r,v) is an rdr field, hence so is (K,v). Since vK^r is p-divisible and vK^r/vK has no p-torsion, also vK is p-divisible. Hence by definition, (K^r,v) is a semitame field. □

Proof of Theorem 1.6:
It has been proven already in Lemma 4.14 that if (K^r,v) is an rdr field, then so is (K,v), and if (K^r,v) is a semitame field, then so is (K,v). Let us now assume that (K,v) is an rdr field. If $\operatorname{char} K > 0$, then (K,v) is an rdr field by Proposition 4.7. The case of rdr fields of mixed characteristic has been settled in Proposition 4.9. Being an rdr field, (K^r,v) is also deeply ramified, as its value group is divisible by every prime $q \neq \operatorname{char} Kv$ and thus satisfies (DRvg).

Assume now that (K,v) is a semitame field. Then by part 1) of Theorem 1.2, (K,v) is an rdr field. As shown above, it follows that the same is true for (K^r,v). Since vK is p-divisible, vK^r is p-divisible too. Hence (K^r,v) is a semitame field. □

Proof of Corollary 1.7:
Part 1) is an immediate consequence of both the upward and the downward direction of Theorem 1.6. As (K^h,v) is a subextension of (K^r,v), the assertions of part 2) for rdr and semitame fields follow immediately from part 1). Also the assertion for the case of deeply ramified fields follows since the extension $(K^h|K,v)$ is immediate, so (K^h,v) satisfies (DRvg) if and only if (K,v) does. □

4.6. **Proof of Theorem 1.5.**

We will need some preparations.

Proposition 4.15. *Assume that (K,v) is an rdr field of mixed characteristic containing all p-th roots of unity, and that $(L|K,v)$ is a Galois defect extension of prime degree. Then also (L,v) is an rdr field.*

Proof. Let p be the residue characteristic of (K,v). By part 1) of Lemma 4.3, vK contains $\frac{1}{p^\infty}\mathbb{Z}vp$. We choose $d \in K$ such that

$$vd \;=\; \frac{vp}{p}\,.$$

By Proposition 4.4 with L in place of K, in order to show that (L,v) is an rdr field, it suffices to show that the function $\mathcal{O}_L \ni x \mapsto x^p \in \mathcal{O}_L$ is surjective modulo $d\mathcal{O}_L$.

From Section 3.2 we know that the extension admits a Kummer generator which is a 1-unit $1+a$ with $a \in \mathcal{M}_L$. Proposition 4.12 shows that (K,v) is an independent defect field. By Proposition 3.7, $\operatorname{dist}(1+a,K) = \frac{vp}{p-1} + H^-$ for some convex subgroup H of vK that does not contain vp. Hence for every positive $\alpha < \frac{vp}{p-1}$ in $\frac{1}{p^\infty}\mathbb{Z}vp$ there is some $b \in K$ such that $v(1+a-b) \geq \alpha$. Then b must itself be a 1-unit, say $b = 1+c$. Now $v(1+a-(1+c)) = v(a-c)$ and the 1-unit

$$1+a_c \;:=\; \frac{1+a}{1+c} \;=\; 1+(a-c) - \frac{c(a-c)}{1+c}$$

satisfies $va_c = v(a-c) \geq \alpha$. Since $b \in K$, $1+a_c$ is also a Kummer generator of the extension.

We note that $\frac{1}{p^\infty}\mathbb{Z}vp$ is dense in $\mathbb{Q}vp$. Thus we can choose α so close to $\frac{vp}{p-1}$ that

$$vd \;=\; \frac{vp}{p} \;\leq\; \alpha - 2p\left(\frac{vp}{p-1} - \alpha\right) \;<\; \alpha \;<\; \frac{vp}{p-1} \;\leq\; vp\,. \tag{53}$$

By what we have shown above, we may from now on assume that $L|K$ admits a Kummer generator which is a 1-unit $\eta = 1 + a$ with $va \geq \alpha$.

Take an element in $\mathcal{O}_L$ and write it as $f(\eta)$ where $f(X) = \sum_{i=0}^{p-1} c_i X^i$ with $c_i \in K$. The problem is that even though $f(\eta)$ lies in $\mathcal{O}_L$, the coefficients c_i do not necessarily lie in $\mathcal{O}_K$. (This is in contrast to the case of defectless extensions, such as extensions within the absolute ramification field, where for a suitably chosen η, the value of $f(\eta)$ is equal to the minimum of the values of the summands $c_i\eta^i$.) Since $v(\eta - K)$ has no maximal element, Lemma 2.10 shows that there must be some $\gamma \in v(\eta - K)$ such that for all $b \in K$ with $v(\eta - b) \geq \gamma$, the monomials $\partial_i f(b)(\eta - b)^i$ in

$$f(\eta) \;=\; \sum_{i=0}^{p-1} \partial_i f(b)(\eta - b)^i$$

have distinct and thus non-negative values, and that for each i, the values $v\partial_i f(b)$ are constant, say equal to β_i. Consequently,

$$\beta_i + i\gamma \;\geq\; 0 \quad \text{for } 0 \leq i \leq p-1\,.$$

As all of this will remain true if we replace γ by any larger value in $v(\eta - K)$, we can assume that $\gamma > va \geq \alpha > 0$. We fix one b with $v(\eta - b) \geq \gamma$. Then also b must be a 1-unit, and we write $b = 1 + c$ with $c \in \mathcal{M}_K$. Thus, $v(a - c) = v(\eta - b) \geq \gamma$, and it follows that $vc = va \geq \alpha$. We set

$$\eta_c \;:=\; \frac{1 + a}{1 + c}$$

and observe that

$$a - c \;=\; \eta_c - 1 + \frac{c(a - c)}{1 + c} \;\equiv\; \eta_c - 1 \mod c(a - c)\mathcal{O}_L\,. \tag{54}$$

We choose some $z \in K$ with value $vz = v(a - c)$. Then

$$f(\eta) \;=\; \sum_{i=0}^{p-1} \partial_i f(b)(a - c)^i \;=\; \sum_{i=0}^{p-1} \partial_i f(b) z^i \left(\frac{\eta_c - 1}{z} + \frac{c(a - c)}{z(1 + c)}\right)^i . \tag{55}$$

Now $\partial_i f(b)z^i \in \mathcal{O}_K$ for all i. Further, $v\frac{\eta_c - 1}{z} = 0$ and $\frac{c(a-c)}{z(1+c)} \in c\mathcal{O}_L$. Consequently,

$$f(\eta) \;\equiv\; \sum_{i=0}^{p-1} \partial_i f(b) z^i \left(\frac{\eta_c - 1}{z}\right)^i \mod c\mathcal{O}_L\,. \tag{56}$$

Since $vc = va \geq \alpha > vd$, this congruence also holds modulo $d\mathcal{O}_L$. Hence in order to show that $f(\eta)$ is a p-th power in $\mathcal{O}_L$ modulo $d\mathcal{O}_L$, it suffices to show that this is true for the polynomial on the right hand side of (56). With the element C as in Lemma 2.20, this polynomial is equal to

$$\sum_{i=0}^{p-1} \partial_i f(b) C^i \left(\frac{\eta_c - 1}{C}\right)^i \;=\; \sum_{i=0}^{p-1} \partial_i f(b) C^i \vartheta^i\,, \tag{57}$$

where

$$\vartheta \;:=\; \vartheta_{\eta_c} \;=\; \frac{\eta_c - 1}{C}\,.$$

Since

$$vC \;=\; \frac{vp}{p-1} \;>\; v(\eta - b) \;=\; v(a-c) \;=\; vz\,,$$

the coefficients $\partial_i f(b)C^i$ still lie in $\mathcal{O}_K$. We note that $v\vartheta = v(\eta_c - 1) - vC = v(a-c) - vC < 0$. Further,

$$0 \;>\; v\vartheta \;\geq\; \gamma - vC \;>\; \alpha - vC\,. \tag{58}$$

From (21) and (22) we know that ϑ satisfies the equation

$$\vartheta \;=\; \vartheta^p - \frac{\eta_c^p - 1}{C^p} + g(\vartheta)\,,$$

where

$$g(\vartheta) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p}\vartheta^i\,.$$

We compute for $2 \leq i \leq p-1$:

$$\begin{aligned} v\binom{p}{i}C^{i-p}\vartheta^i \;&=\; vp + (i-p)vC + iv\vartheta \;=\; (i-1)vC + iv\vartheta \\ &\geq\; vC + pv\vartheta \;>\; \alpha + pv\vartheta \;=\; \alpha + 2pv\vartheta - pv\vartheta \\ &>\; \alpha - 2p(vC-\alpha) - pv\vartheta \;\geq\; vd - pv\vartheta\,, \end{aligned}$$

where the last inequality holds by (53). Hence $g(\vartheta) \in d\vartheta^{-p}\mathcal{O}_L$ and

$$\vartheta \;\equiv\; \vartheta^p - \frac{\eta_c^p - 1}{C^p} \quad \bmod d\vartheta^{-p}\mathcal{O}_L\,. \tag{59}$$

As $v\vartheta < 0$, we have that $v\vartheta > v\vartheta^p$, so that

$$v\frac{\eta_c^p - 1}{C^p} \;=\; v\vartheta^p\,.$$

Since (K,v) is an rdr field, using part 2) of Lemma 4.3 we can find elements $t, t_i \in \mathcal{O}_K$ such that

$$t^p \;\equiv\; \frac{\eta_c^p - 1}{C^p} \quad \bmod pt^p\mathcal{O}_K \;=\; p\vartheta^p\mathcal{O}_L \tag{60}$$

and for $0 \leq i \leq p-1$,

$$t_i^p \;\equiv\; \partial_i f(b)C^i \quad \bmod p\mathcal{O}_K \;\subseteq\; d\mathcal{O}_L\,. \tag{61}$$

We have that

$$\vartheta^p - t^p \;\equiv\; (\vartheta - t)^p \quad \bmod p\vartheta^p\mathcal{O}_L$$

and consequently,

$$\vartheta^p - \frac{\eta_c^p - 1}{C^p} \;\equiv\; (\vartheta - t)^p \quad \bmod p\vartheta^p\mathcal{O}_L\,. \tag{62}$$

From (53) and (58) we derive that

$$vd\vartheta^{-2p} \;=\; vd - 2pv\vartheta \;<\; vd + 2p(vC - \alpha) \;\leq\; \alpha \;<\; vp\,,$$

so that $p\vartheta^p\mathcal{O}_L \subseteq d\vartheta^{-p}\mathcal{O}_L$. Hence by (59) and (62),

$$\vartheta \;\equiv\; (\vartheta - t)^p \quad \bmod d\vartheta^{-p}\mathcal{O}_L\,.$$

We write $(\vartheta - t)^p = \vartheta + d\vartheta^{-p}s$ with $s \in \mathcal{O}_L$. Then for $0 \le i \le p-1$,

$$(\vartheta - t)^{ip} \;=\; \vartheta^i \,+\, \sum_{j=1}^{i} \binom{i}{j} \vartheta^{i-j}(d\vartheta^{-p}s)^j \,.$$

Since $v\vartheta < 0 < vd\vartheta^{-p}s$, the summand of least value in the sum on the right hand side is the one for $j = 1$. This shows that

$$\vartheta^i \;\equiv\; (\vartheta - t)^{ip} \mod d\vartheta^{-p+i-1}\mathcal{O}_L \,.$$

Here, each $d\vartheta^{-p+i-1}\mathcal{O}_L$ can be replaced by the larger ideal $d\mathcal{O}_L$. Combining this with (61), we obtain:

$$\sum_{i=0}^{p-1} \partial_i f(b) C^i \vartheta^i \;\equiv\; \sum_{i=0}^{p-1} t_i^p (\vartheta - t)^{ip} \mod d\mathcal{O}_L \,. \tag{63}$$

We observe that the corresponding summands in the sums on the right hand sides of (55), (56), (57) and (63) all have the same non-negative value. Consequently,

$$\sum_{i=0}^{p-1} t_i^p (\vartheta - t)^p \;\equiv\; \left(\sum_{i=0}^{p-1} t_i (\vartheta - t) \right)^p \mod p\mathcal{O}_L \,.$$

Together with (63), this leads to

$$f(\eta) \;\equiv\; \sum_{i=0}^{p-1} \partial_i f(b) C^i \left(\frac{\eta_c - 1}{C} \right)^i \;\equiv\; \left(\sum_{i=0}^{p-1} t_i (\vartheta - t) \right)^p \mod d\mathcal{O}_L \,,$$

which completes our proof. □

Proposition 4.16. *Assume that (K, v) is an rdr field of mixed characteristic with algebraically closed residue field. Take a defectless unibranched Galois extension $(L|K, v)$ of degree $p = \mathrm{char}\, Kv$. Then also (L, v) is an rdr field.*

Proof. Since $(L|K, v)$ is unibranched and defectless, equation (1) shows that $p = [L : K] = (vL : vK)[Lv : Kv]$. However, as Kv is algebraically closed, $[Lv : Kv] = 1$. Hence $(vL : vK) = p$. By part 1) of Lemma 4.3, $v_p \circ \overline{v}(Kv_0) = (vK)_{vp}$ is p-divisible. It follows that $(v_0 L : v_0 K) = p$ and therefore, $Lv_0 = Kv_0$. Applying Proposition 1.3 to (K, v), we find that $(Kv_0, v_p) = (Lv_0, v_p)$ is an rdr field, and applying the proposition again, we conclude that (L, v) is an rdr field. □

Proof of Theorem 1.5. For the case of deeply ramified fields of positive characteristic we have given the proof already in Proposition 4.7, so let us assume that (K, v) is an rdr field of mixed characteristic and $(L|K, v)$ an algebraic extension. By Theorem 1.6, (K^r, v) is a deeply ramified field. Hence $K^r v$ is perfect by Lemma 4.2, but as it is also separable-algebraically closed, it must be algebraically closed.

We let L' be a maximal extension of K^r inside of $L.K^r$ that is again an rdr field; since the union over an ascending chain of rdr fields is again an rdr field, L' exists by Zorn's Lemma. Since K^r contains all p-th roots of unity, so does L', and since $K^r v$ is algebraically closed, so is $L'v$.

Suppose that $L' \neq L.K^r$. Since $\tilde{K}|K^r$ is a p-extension, the same holds for $\tilde{K}|L'$. Consequently, $L.K^r|L'$ contains a Galois subextension $(L''|L', v)$ of degree p. If this is a defect extension, then it follows from Proposition 4.15 that (L'', v) is an rdr field. If the extension is defectless, then it follows from Proposition 4.16 that (L'', v)

is an rdr field. In both cases we have obtained a contradiction to the maximality of L'. This proves that $(L.K^r, v)$ is an rdr field. Since $L.K^r = L^r$ by [9, (20.15) b)], we now obtain from Theorem 1.6 that (L, v) is an rdr field.

It remains to deal with deeply ramified fields and with semitame fields. For them the proof follows immediately from what we have already shown, since deeply ramified fields are just the rdr fields that satisfy (DRvg), and semitame fields are just the rdr fields with p-divisible value groups. All of these properties are preserved under algebraic extensions. □

4.7. **Proof of Theorem 1.8.**

The equal characteristic case has already been settled in Proposition 4.7. Thus we assume now that $(L|K, v)$ is a finite extension of valued fields of mixed characteristic and that (L, v) is an rdr field. We wish to show that (K, v) is an rdr field. In order to derive a contradiction, we suppose that this is not the case.

We take an extension of v to $\tilde{K} = \tilde{L}$. This determines the absolute ramification field (K^r, v) of (K, v). By [9, (20.15) b)], $(L.K^r, v)$ is the absolute ramification field (L^r, v) of (L, v). By Theorem 1.6, (L^r, v) is an rdr field. From Lemma 2.13 we know that $L.K^r|K^r$ is a finite tower of Galois extensions of degree p. By our assumption and Theorem 1.6, (K^r, v) is not an rdr field. Then there is a maximal field (N, v) in the tower that is not an rdr field, and a Galois extension (N', v) of (N, v) of degree p that is an rdr field.

By part 1) of Lemma 4.3, vN' contains $\frac{1}{p^\infty}\mathbb{Z}vp$. Since $(N'|N, v)$ is a finite extension, also vN contains $\frac{1}{p^\infty}\mathbb{Z}vp$. By part 1) of Lemma 4.2, $\mathrm{crf}\,(N', v)$ is perfect. As $\mathrm{crf}\,(N', v)|\mathrm{crf}\,(N, v)$ is a finite extension, also $\mathrm{crf}\,(N, v) = Nv_0v_p$ is perfect. Hence the same holds for Nv.

Since (N, v) is not an rdr field, Proposition 4.4 shows that for every $d \in N$ with $vd \in \frac{1}{p^\infty}\mathbb{Z}vp$ and $0 < vd \leq vp$ there must be some $b_d \in \mathcal{O}_N^\times$ such that there is no $c \in N$ with $b_d - c^p \in d\mathcal{O}_N$. We choose $\eta_d \in \tilde{N}$ such that $\eta_d^p = b_d$. Then there is no $c \in N$ such that $v(\eta_d - c) \geq \frac{vd}{p}$ since this would imply $v(b_d - c^p) = v(\eta_d^p - c^p) \geq vd$ as $\eta_d^p - c^p \equiv (\eta_d - c)^p \mod p\mathcal{O}_N$. Lemma 4.5 shows that $v(\eta_d - N)$ has no maximal element. Hence by Lemma 2.7, $(N(\eta_d)|N, v)$ is a Galois defect extension, and by Proposition 3.7, it has dependent defect.

We distinguish two cases. First, let us assume that $(N'|N, v)$ is not a defect extension. Then by Lemma 2.3, $(N'(\eta_d)|N', v)$ is a Galois defect extension with $\mathrm{dist}\,(\eta_d, N') = \mathrm{dist}\,(\eta_d, N)$, which shows that also this extension has dependent defect. Therefore, (N', v) is not an independent defect field and thus by Proposition 4.12, it is not an rdr field. This contradicts our assumption.

Now let us assume that $(N'|N, v)$ is a defect extension. Since K^r contains all p-th roots of unity, the same holds for N. Therefore, the extension $N'|N$ admits a Kummer generator η, and we can assume that it is a 1-unit. Since Nv_0v_p is perfect, it follows that there is some $c \in N$ such that $v_0 \circ v_p(\eta - c) > 0$, and thus we can choose some $d \in N$ as above such that $\frac{vd}{p} \in v(\eta - N)$. It follows that

$$v(\eta_d - N) \subsetneq v(\eta - N)\,. \tag{64}$$

This means that $\mathrm{dist}\,(\eta_d, N) < \mathrm{dist}\,(\eta, N)$. Note that $v(\sigma\eta_d - \eta_d) = v(\sigma\eta - \eta)$ as both η_d and η are Kummer generators of value 0 of the extensions $N(\eta_d)|N$ and $N'|N$, respectively.

If $v(\eta_d - N') = v(\eta_d - N)$, then again by Lemma 2.3, $(N'(\eta_d)|N', v)$ is a Galois defect extension with $\mathrm{dist}\,(\eta_d, N') = \mathrm{dist}\,(\eta_d, N)$, yielding a contradiction as before.

Suppose that $\eta_d \in N'$. Then inequality (64) leads to

$$-v(\eta_d - N) + v(\sigma\eta_d - \eta_d) \;\neq\; -v(\eta - N) + v(\sigma\eta - \eta)\,, \tag{65}$$

which in view of equation (32) together with Theorem 3.5 is a contradiction. Hence we can assume that $\eta_d \notin N'$.

Now our proof will be complete once we show that $v(\eta_d - N') \neq v(\eta_d - N)$ is impossible. In order to derive a contradiction, suppose that the two sets are not equal. Then there is some $\tilde{\eta} \in N'$ such that $v(\eta_d - \tilde{\eta}) \notin v(\eta_d - N)$. Since $v(\eta_d - N)$ is an initial segment of $vN' = vN$, it follows that $v(\eta_d - \tilde{\eta}) > v(\eta_d - N)$. By part 1) of Lemma 2.1,

$$v(\eta_d - N) \;=\; v(\tilde{\eta} - N)$$

holds for all $\tilde{\eta} \in N'$ with $v(\eta_d - \tilde{\eta}) > v(\eta_d - N)$. As $\tilde{\eta} \in N' \setminus N$ and $[N' : N] = p$, also $\tilde{\eta}$ is a generator of $N'|N$.

For $\sigma \in \mathrm{Gal}\,(\tilde{K}|K)$ with $\sigma\tilde{\eta} \neq \tilde{\eta}$, we compute:

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;\geq\; \min\{v(\sigma\tilde{\eta} - \sigma\eta_d), v(\sigma\eta_d - \eta_d), v(\eta_d - \tilde{\eta})\}\,. \tag{66}$$

As an algebraic extension of (K^r, v), also (N, v) is henselian. Hence we have that $v(\sigma\tilde{\eta} - \sigma\eta_d) = v\sigma(\eta_d - \tilde{\eta}) = v(\eta_d - \tilde{\eta})$. Suppose that

$$v(\eta_d - \tilde{\eta}) \;\geq\; v(\sigma\eta_d - \eta_d)\,.$$

As $(vN')_{vp}$ is p-divisible and $N'v$ is perfect, $v(\eta_d - N')$ does not have a maximum inside of $(vN')_{vp}$, so we may assume that $v(\eta_d - \tilde{\eta}) > v(\sigma\eta_d - \eta_d)$. Thus in all cases, we may assume that $v(\eta_d - \tilde{\eta}) \neq v(\sigma\eta_d - \eta_d)$. Hence by (66),

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;=\; \min\{v(\sigma\eta_d - \eta_d), v(\eta_d - \tilde{\eta})\}\,. \tag{67}$$

If $v(\sigma\tilde{\eta} - \tilde{\eta}) = v(\sigma\eta_d - \eta_d)$, then $v(\sigma\tilde{\eta} - \tilde{\eta}) = v(\sigma\eta - \eta)$ and we obtain a contradiction exactly as in (65) with η_d replaced by $\tilde{\eta}$. Hence we now assume that

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;=\; v(\eta_d - \tilde{\eta}) \;<\; v(\sigma\eta_d - \eta_d)\,.$$

Again because $v(\eta_d - N')$ does not have a maximum inside of $(vN')_{vp}$, we can choose $\tilde{\eta}_1 \in N'$ such that

$$v(\eta_d - \tilde{\eta}_1) \;>\; v(\eta_d - \tilde{\eta}) \;>\; v(\eta_d - N)\,.$$

Like $\tilde{\eta}$, also $\tilde{\eta}_1$ is a generator of $N'|N$. With the same computations as before, we arrive at (67) with $\tilde{\eta}$ replaced by $\tilde{\eta}_1$. We must have that $v(\eta_d - \tilde{\eta}_1) < v(\sigma\eta_d - \eta_d)$ since otherwise, we would obtain a contradiction as before. Therefore,

$$v(\sigma\tilde{\eta}_1 - \tilde{\eta}_1) \;=\; v(\eta_d - \tilde{\eta}_1)$$

and

$$v(\tilde{\eta}_1 - N) \;=\; v(\eta_d - N) \;=\; v(\tilde{\eta} - N)\,.$$

Combining everything, we find:

$$\begin{aligned} -v(\tilde{\eta}_1 - N) + v(\sigma\tilde{\eta}_1 - \tilde{\eta}_1) &= -v(\eta_d - N) + v(\eta_d - \tilde{\eta}_1) \\ &\neq -v(\eta_d - N) + v(\eta_d - \tilde{\eta}) \\ &= -v(\tilde{\eta} - N) + v(\sigma\tilde{\eta} - \tilde{\eta})\,, \end{aligned}$$

which again by equation (32) together with Theorem 3.5 is a contradiction. □

4.8. **Proof of Theorems 1.10 and of Proposition 1.14.**

Proof of Theorem 1.10: As before we set $K' = K(\zeta_p)$. Then for any extension of v to $\tilde{K}$, $(K(\zeta_p), v)$ is contained in the corresponding absolute ramification field.

1): Assume that (K, v) is an rdr field. The assertions on $(vK)_{vp}$ and $\mathrm{crf}\,(K, v)$ have been proven in Lemmas 4.2 and 4.3. By part 1) of Corollary 1.7, also (K', v) is an rdr field. It follows from Proposition 4.12 that (K', v) is an independent defect field. Thus by definition, (K, v) is an independent defect field. The converse is the content of part 1) of Proposition 4.13.

2): We note that every unibranched Galois extension of prime degree different from the residue characteristic is automatically tame.

First, we assume that (K, v) is a semitame field. Then by part 1) of Corollary 1.7, also (K', v) is a semitame field, so vK' is p-divisible. By Lemma 4.2, $K'v$ is perfect. Therefore, equation (1), with K' in place of K, shows that every unibranched Galois extension $(L|K', v)$ of degree p either has defect p, or satisfies $[Lv : Kv] = p$ with $Lv|K'v$ a separable extension. In the latter case, the extension has no defect and is tame. Otherwise, it is a defect extension of degree p. Then, as (K', v) is an rdr field by Theorem 1.2, part 1) of our theorem shows that it must be an independent defect extension.

For the converse, we first show that our assumptions yield that vK', and hence also $(vK')_{vp}$, is p-divisible, and that $\mathrm{crf}\,(K', v)$ is perfect. Indeed, if $\alpha \in vK'$ is not divisible by p and we take $a \in K'$ with $va = \alpha$, then taking a p-th root of a induces a Galois extension that is neither tame nor immediate, contradicting the hypothesis. The same holds if $a \in K'$ is such that $va = 0$ and av does not have a p-th root in $K'v$, hence $K'v$ is perfect.

Suppose that $\mathrm{crf}\,(K', v)$ is not perfect. Pick $a \in K'$ such that $av_0 \circ v_p$ has no p-th root and choose some $b \in \tilde{K}$ such that $b^p = a$. Since vK' is p-divisible, the same holds for $\overline{v}(K'v_0 \circ v_p)$. In addition, $(K'v_0 \circ v_p)\overline{v} = K'v$ is perfect. It follows that the extension $(K'v_0 \circ v_p(bv_0 \circ v_p)|K'v_0 \circ v_p, \overline{v})$ is immediate of degree $p = [K'(b) : K']$, which implies that also $(K'(b)|K', v)$ is immediate. Further, the former extension is unibranched as it is purely inseparable. Since also the extension $(K'(b)|K, v_0 \circ v_p)$ is unibranched as its inertia degree is p, also $(K'(b)|K', v)$ is unibranched. By assumption, its defect must be independent since defect extensions of degree p are not tame. But then there must be $c \in K'$ such that $v(b - c) > \frac{vp}{p}$, whence $bv_0 \circ v_p \in K'v_0 \circ v_p$, contradiction.

Our assumption yields that every Galois defect extension of (K', v) of degree p is independent. Hence we obtain from part 1) that (DRvr) holds, so (K', v) is a semitame field. By part 1) of Corollary 1.7, also (K, v) is a semitame field. □

Proof of Proposition 1.14. Part 1) follows from Proposition 3.8. Part 2) has already been proved at the end of Section 3.3. □

4.9. **Proof of Proposition 1.1.**

It is well known that first order properties of the value group vK of a valued field (K, v) can be encoded in (K, v) in the language of valued fields. The axiomatization for (DRvp) and (DRst) is straightforward. Further, (DRvg) holds in an ordered abelian group $(G, <)$ if and only if for each positive $\alpha \in G$ there is $\beta \in G$ such that $2\beta \leq \alpha \leq 3\beta$.

If (K,v) is of mixed characteristic, then (DRvr) is equivalent to the surjectivity of (45), and this in turn holds if and only if for each $a\in K$ with $va\geq 0$ there is $b\in K$ such that $v(a-b^p)\geq vp$. Hence the classes of semitame, deeply ramified and rdr fields of mixed characteristic are first order axiomatizable.

If (K,v) is of equal positive characteristic, then part 3) of Theorem 1.2 shows that semitame, deeply ramified and rdr fields form the same class. This class can be axiomatized by saying that (K^p,v) is dense in (K,v), or in other words, for every $\alpha\in vK$ and every $a\in K$ there is $b\in K$ such that $v(a-b^p)\geq\alpha$.

In the case of equal characteristic 0, (DRvp), (DRvr) and (DRst) are trivial and all valued fields are semitame and rdr fields, while the class of deeply ramified fields consists of those which satisfy (DRvg). □

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Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland
Email address: fvk@usz.edu.pl

Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, 40-007 Katowice, Poland
Email address: anna.rzepka@us.edu.pl