

# EMBEDDING THEOREMS FOR SPACES OF $\mathbb{R}$ -PLACES OF RATIONAL FUNCTION FIELDS AND THEIR PRODUCTS

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ABSTRACT. We study spaces  $M(R(y))$  of  $\mathbb{R}$ -places of rational function fields  $R(y)$  in one variable. For extensions  $F|R$  of formally real fields, with  $R$  real closed and satisfying a natural condition, we find embeddings of  $M(R(y))$  in  $M(F(y))$  and prove uniqueness results. Further, we study embeddings of products of spaces of the form  $M(F(y))$  in spaces of  $\mathbb{R}$ -places of rational function fields in several variables. Our results uncover rather unexpected obstacles to a positive solution of the open question whether the torus can be realized as a space of  $\mathbb{R}$ -places.

## 1. INTRODUCTION

For any field  $K$ , the set of all orderings on  $K$ , given by their positive cones  $P$ , is denoted by  $\mathcal{X}(K)$ . This set is non-empty if and only if  $K$  is formally real. The **Harrison topology** on  $\mathcal{X}(K)$  is defined by taking as a subbasis the **Harrison sets**

$$H(a) := \{P \in \mathcal{X}(K) \mid a \in P\}, \quad a \in K \setminus \{0\}.$$

With this topology,  $\mathcal{X}(K)$  is a boolean space, i.e., it is compact, Hausdorff and totally disconnected (see [15, p. 32]).

Associated with every ordering  $P$  on  $K$  is an  $\mathbb{R}$ -place  $\lambda(P)$  of  $K$ , that is, a place of  $K$  with image contained in  $\mathbb{R} \cup \{\infty\}$ , which is compatible with the ordering in the sense that non-negative elements are sent to non-negative elements or  $\infty$ . The set of all  $\mathbb{R}$ -places of  $K$  will be denoted by  $M(K)$ . The Baer-Krull Theorem (see [14, Theorem 3.10]) shows that the mapping

$$\lambda : \mathcal{X}(K) \longrightarrow M(K)$$

(which we will also denote by  $\lambda_K$ ) is surjective. Through  $\lambda$ , we equip  $M(K)$  with the quotient topology inherited from  $\mathcal{X}(K)$ , making it a compact Hausdorff space (see [14, p. 74 and Cor. 9.9]), and  $\lambda$  a continuous closed mapping. According to [14, Theorem 9.11] the subbasis for the quotient topology on  $M(K)$  is given by the family of open sets of the form

$$U(a) = \{\zeta \in M(K) \mid \zeta(a) > 0\}$$

where  $a$  is in the **real holomorphy ring** of  $K$ , i.e.,  $\zeta(a) \neq \infty$  for all  $\zeta \in M(K)$ . Since for every  $b \in K$  the element  $\frac{b}{1+b^2}$  is in the real holomorphy ring of  $K$  (see

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[14, Lemma 9.5]), we have that

$$H'(b) := \{\zeta \in M(K) \mid \infty \neq \zeta(b) > 0\} = U\left(\frac{b}{1+b^2}\right)$$

is a subbasic set for every  $b \in K$ . So we can assume that the topology on  $M(K)$  is given by the subbasic sets  $H'(b)$ ,  $b \in K$ .

Throughout this paper,  $R(y)$  will always denote the rational function field in one variable over the field  $R$ . For the case of real closed  $R$ , we gave in [12] a handy criterion for two orderings on  $R(y)$  to be sent to the same  $\mathbb{R}$ -place by  $\lambda$ :

**Theorem 1.** *Take a real closed field  $R$  and two distinct orderings  $P_1, P_2$  of  $R(y)$ . Then  $\lambda(P_1) = \lambda(P_2)$  if and only if the cuts induced by  $y$  with respect to  $P_1$  and  $P_2$  in  $R$  are upper and lower edge of a ball in  $R$ .*

See Section 2 for the notions in this theorem and for more details.

If  $R$  is any real closed field, each ordering  $P$  on  $R(y)$  is uniquely determined by the cut  $(D, E)$  in  $R$  where  $D = \{d \in R \mid y - d \in P\}$  and  $E = R \setminus D$  (cf. [8]). Hence, if  $\mathcal{C}(R)$  is the set of all cuts in  $R$ , then we have a bijection

$$\chi : \mathcal{C}(R) \longrightarrow \mathcal{X}(R(y))$$

(which we will also denote by  $\chi_R$ ). With respect to the interval topology on  $\mathcal{C}(R)$  and the Harrison topology on  $\mathcal{X}(R(y))$ ,  $\chi_R$  is in fact a homeomorphism (see Proposition 15). Theorem 1 can be reformulated as: *Two distinct cuts in  $R$  are mapped by  $\lambda \circ \chi$  to the same place in  $M(R(y))$  if and only if they are upper and lower edge of a ball in  $R$ .*

In the present paper, we put this result to work in order to find, for given formally real extensions  $F$  of a real closed field  $R$ , continuous embeddings  $\iota$  of  $M(R(y))$  in  $M(F(y))$ , by finding suitable embeddings of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$ .

For any field extension  $L|K$ , the **restriction**

$$\text{res} = \text{res}_{L|K} : M(L) \ni \zeta \mapsto \zeta|_K \in M(K)$$

is continuous (see [5, 7.2.]). An embedding  $\iota : M(K) \rightarrow M(L)$  will be called **compatible with restriction** if  $\text{res} \circ \iota$  is the identity.

In order to determine when such embeddings of  $M(R(y))$  in  $M(F(y))$  exist, we have to look at the canonical valuations of the ordered fields  $R$  and  $F$ . The **canonical valuation**  $v$  of an ordered field is the valuation corresponding to its associated  $\mathbb{R}$ -place. If  $v$  is the canonical valuation of the ordered field  $F$ , then its restriction to  $R$  is the canonical valuation of the field  $R$  ordered by the restriction of the ordering of  $F$ , and we will denote it again by  $v$ . Recall that the ordering and canonical valuation of a real closed field are uniquely determined. By  $vF$  and  $vR$  we denote the respective value groups. Then  $vF|vR$  is an extension of ordered abelian groups. Note that  $vR = \{0\}$  if and only if  $R$  is archimedean ordered. In Section 5, we will prove:

**Theorem 2.** *Take a real closed field  $R$  and a formally real extension field  $F$  of  $R$ . A continuous embedding  $\iota$  of  $M(R(y))$  in  $M(F(y))$  compatible with restriction exists if and only if  $vR$  is a convex subgroup of  $vF$ , for some ordering of  $F$ . In particular, such an embedding always exists when  $R$  is archimedean ordered. If  $F$  is real closed, then there is at most one such embedding.*

For the case of  $F$  not being real closed, we prove a partial uniqueness result (Theorem 30).

Let us point out a somewhat surprising consequence of the above theorem. If  $R$  is a non-archimedean real closed field and  $F$  is an elementary extension (e.g., ultrapower) of  $R$  of high enough saturation, then  $vR$  will not be a convex subgroup of  $vF$  and there will be no such embedding  $\iota$ .

In Section 6 we consider the special case where  $R$  is archimedean ordered and give a more explicit construction of  $\iota$  and a more explicit proof of the uniqueness. The construction we give is of interest also when other spaces of places are considered (e.g., spaces of all places, together with the Zariski topology).

It is well known that for an archimedean real closed field  $R$ ,  $M(R(y))$  is homeomorphic to the circle (over  $\mathbb{R}$ , with the usual interval topology). In fact, this is an easy consequence of Theorem 1. Hence our embedding result shows that each  $M(F(y))$  contains the circle as a closed subspace.

While spaces of orderings are well understood, this is not the case for spaces of  $\mathbb{R}$ -places. Some important insight has been gained (see for instance [1], [2], [3], [6], [9], [12], [16], [17]), but several essential questions have remained unanswered. For example, it is still an open problem which compact Hausdorff spaces are realized as  $M(F)$  for some  $F$ . It is therefore important to determine operations on topological spaces (like passage to closed subspaces, taking finite disjoint unions, taking finite products) under which the class of realizable spaces is closed. It has been shown in [6] that closed subspaces and finite disjoint unions of realizable spaces are again realizable, as well as products of a realizable space with any boolean space.

It has remained an open question whether the product of two realizable spaces is realizable. A test case is the torus; it is not known whether the torus (or any other subspace of  $\mathbb{R}^n$  of dimension  $> 1$ ) is realizable.

As  $M(\mathbb{R}(y))$  is the circle,  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  is the torus. In Section 7 we generalize our construction given in Section 6 to obtain a natural embedding of  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  in  $M(\mathbb{R}(x, y))$ . In view of the above-mentioned negative result, this embedding cannot be continuous with an image that is closed in  $M(\mathbb{R}(x, y))$ , because otherwise it would follow from the realizability of closed subspaces that the torus is realizable. We show an even stronger negative assertion: the image of the embedding is dense in, while not being equal to,  $M(\mathbb{R}(x, y))$ . Hence, the image is not closed, and the embedding is not continuous.

In the final Section 9 we will show that for an arbitrary extension  $L|K$ , there is a continuous embedding of  $M(K)$  in  $M(L)$  compatible with restriction as soon as  $L$  admits a  **$K$ -rational place**, that is, a place trivial on  $K$  with image  $K \cup \{\infty\}$ . In particular, this applies when  $L$  is a rational function field over  $K$ .

## 2. CUTS, BALLS AND $\mathbb{R}$ -PLACES

Take any totally ordered set  $T$  and  $D, E \subseteq T$ . We will write  $D < E$  if  $d < e$  for all  $d \in D$  and  $e \in E$ . Note that  $\emptyset < T$  and  $T < \emptyset$ . For  $c \in T$ , we will write  $c > D$  if  $c > d$  for all  $d \in D$ , and  $c < E$  if  $c < e$  for all  $e \in E$ .

A pair  $C = (D, E)$  is called a **cut in  $T$**  if  $D < E$  and  $D \cup E = T$ . In this case,  $D$  is an **initial segment of  $T$** , that is, if  $d \in D$  and  $d > c \in T$ , then  $c \in D$ ; similarly,  $E$  is a **final segment of  $T$** , that is, if  $e \in E$  and  $e < c \in T$ , then  $c \in E$ .

We include the cuts  $C_{-\infty} = (\emptyset, T)$  and  $C_{\infty} = (T, \emptyset)$ ; the empty set is understood to be both initial and final segment of  $T$ . If  $C_1 = (D_1, E_1)$  and  $C_2 = (D_2, E_2)$  are two cuts, then we will write  $C_1 < C_2$  if  $D_1 \subsetneq D_2$ .

Take any non-empty subset  $A$  of  $T$ . By  $A^+$  we will denote the cut  $(D, T \setminus D)$  for which  $D$  is the smallest initial segment of  $T$  which contains  $A$ . Similarly, by  $A^-$  we will denote the cut  $(T \setminus E, E)$  for which  $E$  is the smallest final segment of  $T$  which contains  $A$ .

A cut  $(D, E)$  is called **principal** if  $D$  has a last element or  $E$  has a first element. In the first case, the cut is equal to  $\{d\}^+$ , where  $d$  is the last element of  $D$ ; in this case we will denote it by  $d^+$ . In the second case, the cut is equal to  $\{e\}^-$ , where  $e$  is the first element of  $E$ ; in this case we will denote it by  $e^-$ .

We will need the following fact:

**Lemma 3.** *If  $C_1, C_2$  are cuts in  $T$  such that  $C_1 < C_2$ , then  $C_1 \leq a^- < a^+ \leq C_2$  for some  $a \in T$ .*

*Proof:* Write  $C_1 = (D_1, E_1)$  and  $C_2 = (D_2, E_2)$ . If  $C_1 < C_2$ , then there is some  $a \in D_2 \setminus D_1$ . Then  $C_1 \leq a^- < a^+ \leq C_2$ .  $\square$

For any pair  $(D, E)$  such that  $D < E$ , we define the **between set**

$$\text{Betw}_T(D, E) := \{c \in T \mid D < c < E\}.$$

Now consider any ordered field  $F$  with its canonical valuation  $v$ . If  $D, E$  are any subsets of  $F$ , we set

$$v(E - D) := \{v(e - d) \mid e \in E, d \in D\} \subseteq vF \cup \{\infty\}.$$

The following observation is easy to prove.

**Lemma 4.** *Assume that  $D$  is an initial segment or  $E$  is a final segment of  $F$ . Then  $v(E - D)$  is an initial segment of  $vF \cup \{\infty\}$ .*

A subset  $B \subseteq F$  is called a **ball in  $F$**  (with respect to the valuation  $v$ ) if it is of the form

$$B = B_S(a, F) := \{b \in F \mid v(a - b) \in S \cup \{\infty\}\}$$

where  $a \in F$  and  $S$  is a final segment of  $vF$ . We consider  $S = \emptyset$  as a final segment of  $vF$ ; we have that  $B_{\emptyset}(a, F) = \{a\}$ .

The notion of “ball” does not refer to some space over  $F$ , but to the ultrametric underlying the natural valuation of  $F$ . Note that because of the ultrametric triangle law, every element of a ball is a center, that is, if  $b \in B_S(a, F)$  then  $B_S(a, F) = B_S(b, F)$ . Therefore,  $v(b - c) \in S$  for all  $b, c \in B_S(a, F)$ . A subset  $B$  of  $F$  is a ball if and only if for any choice of  $a, b \in B$  and  $c \in F$  such that  $v(a - c) \geq v(a - b)$  it follows that  $c \in B$ .

If  $0 \in B_S(a, F)$ , then  $B_S(a, F) = B_S(0, F)$  is a convex subgroup of the ordered additive group of  $F$ . Every ball in  $F$  is in fact a coset of a convex subgroup:  $B_S(a, F) = a + B_S(0, F)$ .

By a **ball complement** for the ball  $B = B_S(a, F)$  we will mean a pair  $(D, E)$  of subsets of  $F$  such that  $D < B < E$  and  $F = D \cup B \cup E$ . In this case again,  $D$  is an initial segment and  $E$  is a final segment of  $F$ .

**Lemma 5.** *If  $(D, E)$  is a ball complement for  $B = B_S(a, F)$ , then*

$$v(E - D) = v(E - B) = v(B - D) = vF \setminus S.$$

*Proof:* First, we show that  $v(E - D) = vF \setminus S$ . For  $d \in D$  and  $e \in E$ , we have that  $v(a - d) < S$  and  $v(e - a) < S$  because  $d, e \notin B$ . From  $d < a < e$  it then follows that  $v(e - d) = \min\{v(e - a), v(a - d)\} < S$ . This proves that  $v(E - D) < S$ .

Now take  $\alpha \in vF$ ,  $\alpha < S$ . Choose  $0 < c \in F$  such that  $vc = \alpha$ . Then  $v(a - (a - c)) = vc = \alpha$ , whence  $a - c \notin B$  and therefore,  $d := a - c \in D$ . Similarly,  $a + c \notin B$  and therefore,  $e := a + c \in E$ . Since  $d < a < e$ , we find  $\alpha = v(2c) = v(e - d) \in v(E - D)$ . Since  $v(E - D)$  is an initial segment of  $vF \cup \{\infty\}$  by Lemma 4, and  $S$  is a final segment, we can now conclude that  $v(E - D) = vF \setminus S$ .

Again by Lemma 4, also  $v(E - B)$  and  $v(B - D)$  are initial segments of  $vF \cup \{\infty\}$ . If  $d \in D$ ,  $e \in E$  and  $b \in B$ , then  $d < b < e$ , whence  $v(b - d) \geq v(e - d)$  and  $v(e - b) \geq v(e - d)$ . Consequently,  $v(E - D)$  is contained in  $v(E - B)$  and  $v(B - D)$ . On the other hand,  $d, e \notin B$  implies that  $v(b - d), v(e - b) < S$ . So by what we have proved earlier,  $v(b - d), v(e - b) \in v(E - D)$ . This shows that all three sets are equal.  $\square$

We will say that a cut is the lower edge of the ball  $B = B_S(a, F)$  if it is the cut  $B^-$ ; similarly, a cut is said to be the upper edge of the ball  $B$  if it is the cut  $B^+$ . Two cuts will be called **equivalent** if they are either equal or one is the lower edge  $B^-$  and the other is the upper edge  $B^+$  of a ball  $B$ .

A cut of the form  $B^+$  or  $B^-$  for  $B$  a ball will be called a **ball cut**. Principal cuts in  $F$  are ball cuts:  $a^+ = \{a\}^+ = B_\emptyset(a, F)^+$  and  $a^- = \{a\}^- = B_\emptyset(a, F)^-$ .

If a cut is neither the lower nor the upper edge of a ball, then we call it a **non-ball cut**. The equivalence class of a non-ball cut is a singleton. As the following lemma will show, the equivalence class of a ball cut consists of two distinct cuts.

**Lemma 6.** *If a cut is the upper or the lower edge of a ball in  $F$ , then the ball is uniquely determined. In particular,  $B_1^+ = B_2^-$  for two balls  $B_1$  and  $B_2$  is impossible. Therefore, equivalence classes of balls contain at most two cuts.*

*Proof:* We show the assertion for a cut  $B^+ = B_S(a, F)^+$ ; the case of  $B_S(a, F)^-$  is similar.

Take any  $d \in F$  and some final segment  $T$  of  $vF$ . Suppose that  $B^+ = B_T(d, F)^+$ . Since the balls  $B_S(a, F)$  and  $B_T(d, F)$  are final segments of the left cut set of  $B^+$ , their intersection is non-empty. So one of them is contained in the other. If they were not equal, the bigger one would contain an element which is bigger than all elements in the smaller ball, but that is impossible.

Now suppose that  $B^+ = B_T(d, F)^-$ . Then  $d > B_S(a, F)$ , so  $v(a - d) < S$ . Similarly,  $a < B_T(d, F)$ , so  $v(a - d) < T$ . Set  $d' := (d + a)/2$ ; then  $d < d' < a$  and  $v(a - d') = v(a - d) = v(d' - d)$ . Consequently,  $d' > B_S(a, F)$  and  $d' < B_T(d, F)$ , a contradiction.  $\square$

In combination with Theorem 1, this lemma shows that the mapping  $\lambda$  will glue not more than two orderings into one  $\mathbb{R}$ -place. The other, quite different way of proof is by an application of the Baer-Krull Theorem.

**Proposition 7.** *Take a real closed field  $F$ . Then for every  $\zeta \in M(F(y))$ , the preimage  $\lambda^{-1}(\zeta)$  consists of at most two orderings.*

Let us add the following observation:

**Proposition 8.** *For every formally real field  $F$ , the mapping  $\lambda : \mathcal{X}(F) \rightarrow M(F)$  induces **continuous glueings**, that is, if  $P_1, P_2 \in \mathcal{X}(F)$  such that for every pair of open neighborhoods  $U_1$  of  $P_1$  and  $U_2$  of  $P_2$  there are  $Q_1 \in U_1$  and  $Q_2 \in U_2$  with  $\lambda(Q_1) = \lambda(Q_2)$ , then  $\lambda(P_1) = \lambda(P_2)$ .*

Proof: Take two orderings  $P_1, P_2 \in \mathcal{X}(F)$  such that  $\lambda(P_1) \neq \lambda(P_2)$ . Since  $M(F)$  is Hausdorff, there are disjoint open neighborhoods  $U'_1$  of  $\lambda(P_1)$  and  $U'_2$  of  $\lambda(P_2)$ . Their preimages  $U_1 := \lambda^{-1}(U'_1)$  and  $U_2 := \lambda^{-1}(U'_2)$  are open neighborhoods of  $P_1$  and  $P_2$ , respectively. Since  $U_1 \cap U_2 = \emptyset$ , there cannot exist any orderings  $Q_1 \in U_1$  and  $Q_2 \in U_2$  such that  $\lambda(Q_1) = \lambda(Q_2)$ .  $\square$

### 3. TOPOLOGIES ON $\mathcal{C}(F)$ AND $\mathcal{X}(F)$

Take any ordered field  $F$ . We have already defined the ordering on  $\mathcal{C}(F)$ . Intervals are defined as in any other linearly ordered set. Note that the linear order of  $\mathcal{C}(F)$  has endpoints  $C_\infty$  and  $C_{-\infty}$ .

The **interval topology** on  $\mathcal{C}(F)$  (like on every other linearly ordered set with endpoints) is defined by taking as basic open sets all intervals of the form  $(C_1, C_2) = \{C \in \mathcal{C}(F) \mid C_1 < C < C_2\}$  for any two cuts  $C_1, C_2 \in \mathcal{C}(F)$ , together with  $(C_1, C_\infty]$  if  $C_1 \neq C_\infty$ , and  $[C_{-\infty}, C_2)$  if  $C_{-\infty} \neq C_2$ .

Note that in the interval topology on  $\mathcal{C}(F)$ , an open interval may have a first or a last element different from  $C_\infty, C_{-\infty}$ . Indeed, if  $C = a^+$  is a principal cut and  $C_1 < a^-$ , then  $(C_1, a^+)$  has last element  $C$ . Similarly, if  $C = a^-$  and  $a^+ < C_2$ , then  $(a^-, C_2)$  has first element  $C$ . However, this is the only way in which first and last elements will arise in open intervals:

**Lemma 9.** *Take an interval  $I$  that is open in the interval topology. If  $C$  is the first element of  $I$ , then  $C = a^+$  for some  $a \in F$ . If  $C$  is the last element of  $I$ , then  $C = a^-$  for some  $a \in F$ .*

Proof: A finite intersection or arbitrary union of intervals of the form  $(C_1, C_2)$  will only have a first or last element if that is already true for one of the intervals. Suppose that  $C$  is the first element of  $I$ ; the case of  $C$  being the last element is similar. Then  $C$  is the first element of an interval  $(C_1, C_2)$ , which means that there is no cut properly between  $C_1$  and  $C$ . Therefore, our assertion follows from Lemma 3.  $\square$

Let us also note that Lemma 3 implies:

**Lemma 10.** *The principal cuts lie dense in  $\mathcal{C}(F)$ .*

A subset of  $\mathcal{C}(F)$  will be called **full** if it is closed under equivalence. We define the **full topology** on  $\mathcal{C}(F)$  to consist of all full sets that are open in the interval topology. This topology is always strictly coarser than the interval topology because in the latter there are always open sets containing  $C_\infty$  without containing  $C_{-\infty}$ . Hence it is not Hausdorff, but it is quasi-compact.

**Proposition 11.** *Let  $B$  be a ball in  $F$ . Then the intervals  $[B^-, B^+]$ ,  $(B^-, B^+)$  and their complements are full in  $\mathcal{C}(F)$ .*

*Proof:* Take any ball  $B_1$  in  $F$ . If  $B_1 \cap B = \emptyset$ , then both  $B_1^+$  and  $B_1^-$  lie in the complement of  $(B^-, B^+)$ , and by Lemma 6, also in the complement of  $[B^-, B^+]$ .

If  $B_1 \cap B \neq \emptyset$ , then  $B_1 \subseteq B$  or  $B \subsetneq B_1$ . In the latter case again, both  $B_1^+$  and  $B_1^-$  lie in the complements of  $[B^-, B^+]$  and  $(B^-, B^+)$ . If  $B_1 \subsetneq B$ , then both  $B_1^+$  and  $B_1^-$  lie in  $[B^-, B^+]$  and in  $(B^-, B^+)$ . Finally, if  $B_1 = B$ , then both  $B_1^+$  and  $B_1^-$  lie in  $[B^-, B^+]$  and in the complement of  $(B^-, B^+)$ .  $\square$

Let us also observe:

**Lemma 12.** *If  $F|R$  is an extension of ordered fields, then the restriction mapping  $\text{res} : \mathcal{C}(F) \rightarrow \mathcal{C}(R)$  preserves  $\leq$  and equivalence and is continuous in both the interval and the full topology. The preimage of every full subset of  $\mathcal{C}(R)$  under  $\text{res}$  is again full.*

*Proof:* It is clear that  $\text{res}$  preserves  $\leq$ . Hence, the preimage of every convex set in  $\mathcal{C}(R)$  is convex in  $\mathcal{C}(F)$ . Therefore, if  $I$  is an open interval in  $\mathcal{C}(R)$ , then its preimage  $I'$  is convex, and if it has no smallest and no largest element, then it is open. If it has a smallest element  $C'$ , then  $\text{res}(C')$  is the smallest element of  $I$ , hence equal to  $C_{-\infty}$  in  $\mathcal{C}(R)$ . Therefore,  $I'$  contains the cut  $C_{-\infty}$  of  $\mathcal{C}(F)$ , whence  $C' = C_{-\infty}$ . Similarly, a largest element of  $I'$  can only be equal to  $C_\infty$  in  $\mathcal{C}(R)$ . It follows that  $I'$  is open. We have proved that  $\text{res}$  is continuous with respect to the interval topology.

Suppose that  $B$  is a ball in  $F$ . Then  $B_0 = B \cap R$  is either empty or a ball in  $R$ . In the first case,  $\text{res}B^- = \text{res}B^+$ , and in the second case,  $\text{res}B^- = B_0^-$  and  $\text{res}B^+ = B_0^+$ . This proves that  $\text{res}$  preserves equivalence. This implies that the preimage  $U'$  of a full set  $U$  is again full: if  $C_1 \in U'$  is equivalent to  $C_2$ , then  $\text{res}(C_1) \in U$  and  $\text{res}(C_2)$  are equivalent, whence  $\text{res}(C_2) \in U$  and  $C_2 \in U'$ . From this and the continuity shown above it follows that  $\text{res}$  is continuous with respect to the full topology.  $\square$

Take any ordered field  $L$ . The notion “full” was introduced in [10] for  $\mathcal{X}(L)$ , but only for the Harrison sets. We generalize the definition to arbitrary subsets  $Y$  of  $\mathcal{X}(L)$  by calling  $Y$  **full** if  $\lambda_L^{-1}(\lambda_L(Y)) = Y$ . We will call two orderings  $P_1, P_2 \in \mathcal{X}(L)$  **equivalent** if  $\lambda(P_1) = \lambda(P_2)$ . Hence,  $Y$  is full if and only if it is closed under equivalence.

Note that the intersection of finitely many full sets is again a full set and the union of any family of full sets is also a full set. We define the **full topology** on  $\mathcal{X}(L)$  by taking as open sets all full sets that are open in the Harrison topology. In general, this topology is strictly coarser than the Harrison topology and hence not Hausdorff, but it is always quasi-compact.

- Remark 13.** 1) If  $Y$  is a full open (or closed) subset of  $\mathcal{X}(L)$ , then  $\lambda(Y)$  is an open (or closed, respectively) subset of  $M(L)$ .  
 2) For any  $U \subset M(L)$ ,  $\lambda^{-1}(U)$  is a full subset of  $\mathcal{X}(L)$ .  
 3) Take any extension  $L|K$  of ordered fields. Then in the diagram

$$\begin{array}{ccc} \mathcal{X}(L) & \xrightarrow{\lambda_L} & M(L) \\ \text{res} \downarrow & & \text{res} \downarrow \\ \mathcal{X}(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

the restriction mappings are continuous, and the diagram commutes (see [5, 7.2.]). Being continuous mappings from compact spaces to Hausdorff spaces, the restriction mappings are also closed and proper.

The analogue of Lemma 12 is:

**Lemma 14.** *If  $L|K$  is an extension of ordered fields, then the restriction mapping  $\text{res} : \mathcal{X}(L) \rightarrow \mathcal{X}(K)$  preserves equivalence and is continuous w.r.t. both the Harrison and the full topology. The preimage of every full set in  $\mathcal{X}(R)$  under  $\text{res}$  is again full.*

*Proof:* The continuity in the Harrison topology has just been stated. The fact that  $\text{res}$  preserves equivalence follows from the commutativity of the above diagram. As in the proof of Lemma 12, this implies the last assertion, and it follows that  $\text{res}$  is also continuous with respect to the full topology.  $\square$

If  $R$  is any real closed field, each ordering  $P$  on  $R(y)$  is uniquely determined by the cut  $(D, E)$  in  $R$  where  $D = \{d \in R \mid y - d \in P\}$  and  $E = R \setminus D$  (cf. [8]). Hence, we have a bijection

$$\chi : \mathcal{C}(R) \longrightarrow \mathcal{X}(R(y)),$$

which we will also denote by  $\chi_R$ .

**Proposition 15.** *With respect to the interval topology on  $\mathcal{C}(R)$  and the Harrison topology on  $\mathcal{X}(R(y))$ ,  $\chi$  is a homeomorphism. The same holds with respect to the full topologies. For  $C_1, C_2 \in \mathcal{C}(R)$ ,  $C_1$  is equivalent to  $C_2$  if and only if  $\chi(C_1)$  is equivalent to  $\chi(C_2)$ .*

*Proof:* The first assertion is a consequence of [12, Prop.2.1]. For the proof of the second assertion, we first prove the third. By definition,  $\chi(C_1)$  is equivalent to  $\chi(C_2)$  if and only if  $\lambda(\chi(C_1)) = \lambda(\chi(C_2))$ . But by Theorem 1, this holds if and only if  $C_1$  and  $C_2$  are equivalent. It follows that the image of a full subset of  $\mathcal{C}(R)$  under  $\chi$  is again full, and the preimage of a full subset of  $\mathcal{X}(R(y))$  under  $\chi$  is again full. Now the second assertion follows from the first.  $\square$

This proposition, together with Theorem 1, gives us a description of  $M(R(y))$  as the quotient space of  $\mathcal{C}(R)$  with respect to the equivalence relation for cuts:



**Proposition 16.** *Via the mapping  $\lambda \circ \chi$ , the space  $M(R(y))$  with the topology induced by the Harrison topology is the quotient space of  $\mathcal{C}(R)$  with the full topology, where the quotient is taken modulo the equivalence of cuts. The full topology is the coarsest topology on  $\mathcal{C}(R)$  for which  $\lambda \circ \chi$  is continuous. The image of a full open set in  $\mathcal{C}(R)$  under  $\lambda \circ \chi$  is open.*

A place in  $M(R(y))$  is called **principal** if it is the image under  $\lambda \circ \chi$  of a principal cut in  $\mathcal{C}(R)$ . From Proposition 16 and Lemma 10 we obtain:

**Lemma 17.** *The principal cuts lie dense in  $M(R(y))$ .*

We will also need:

**Proposition 18.** *The restriction mappings in the following diagram are continuous (w.r.t. the interval and the Harrison topology as well as w.r.t. the full topologies), and the diagram commutes:*

$$\begin{array}{ccccc} \mathcal{C}(F) & \xrightarrow{\chi_F} & \mathcal{X}(F(y)) & \xrightarrow{\lambda_{F(y)}} & M(F(y)) \\ \text{res} \downarrow & & \text{res} \downarrow & & \text{res} \downarrow \\ \mathcal{C}(R) & \xrightarrow{\chi_R} & \mathcal{X}(R(y)) & \xrightarrow{\lambda_{R(y)}} & M(R(y)) \end{array}$$

**Proof:** In view of Lemmas 12 and 14 and part 3) of Remark 13, it just remains to prove that the square on the left hand side of the diagram commutes. This follows from the fact that the cut induced by  $y$  in  $R$  under the restriction of some ordering from  $F(y)$  is simply the restriction of the cut induced by  $y$  in  $F$  under this ordering.  $\square$

We note the following fact, which is straightforward to prove:

**Lemma 19.** *If  $\iota$  is an embedding of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$ , or of  $\mathcal{X}(K)$  in  $\mathcal{X}(L)$ , or of  $M(K)$  in  $M(L)$ , compatible with restriction, then the preimage of a set  $U$  under  $\iota$  is equal to its image under restriction.*

#### 4. EMBEDDINGS OF $\mathcal{C}(R)$ IN $\mathcal{C}(F)$

We consider an extension  $F|R$  of ordered fields. Our goal is to construct an embedding  $\iota$  of  $M(R(y))$  in  $M(F(y))$  under suitable assumptions on the extension; this will be done in Section 5. In view of Proposition 16, we first define an order preserving embedding of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$ .

From now on we will frequently have to compare cuts in  $R$  with cuts in  $F$ . If  $C = (D, E)$  is a cut in  $R$ , then we will say that the element  $a \in F$  **fills**  $C$  if  $D < a < E$  holds in  $F$ . Since a subset  $A$  of  $R$  is also a subset of  $F$ , we will have to distinguish whether  $A^+$  and  $A^-$  are taken in  $R$  or in  $F$ . If this is not clear from the context, we will write  $A^{+R}$  and  $A^{-R}$  for the former, and  $A^{+F}$  and  $A^{-F}$  for the latter.

To find a suitable embedding of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$ , we need to study the set of all elements in  $F$  that fill a cut in  $R$ . More generally, we have to consider the following situation.

**Lemma 20.** *Take two non-empty sets  $D < E$  in  $R$ . Assume that  $(D, E)$  is either a non-ball cut in  $R$  with  $\text{Betw}_F(D, E) \neq \emptyset$ , or a ball complement in  $R$ . Then*

$$\text{Betw}_F(D, E) = B_S(a, F)$$

for each  $a \in \text{Betw}_F(D, E)$ , where  $S$  is the largest final segment of  $vF$  disjoint from  $v(E - D)$  (or equivalently, the largest subset of  $vF$  such that  $S > v(E - D)$ ).

*Proof:* First, we show that  $\mathcal{B} := \text{Betw}_F(D, E)$  is contained in  $B_S(a, F)$ . Take any  $d \in D$ ,  $e \in E$  and  $b \in \mathcal{B}$ . As  $d < a < e$  and  $|a - b| < e - d$ , we have that  $v(a - b) \geq v(e - d)$ . We show that we must have  $v(a - b) > v(e - d)$ , which yields that  $b \in B_S(a, F)$ .

Suppose that  $v(a - b) = v(e - d)$ . We assume that  $b < a$ ; the case of  $b > a$  is symmetrical. Then it follows that  $v(a - d) = v(e - d)$  and  $v(b - d) \geq v(e - d)$ , so that  $v\left(\frac{a-b}{e-d}\right) = 0$ ,  $v\left(\frac{a-d}{e-d}\right) = 0$  and  $v\left(\frac{b-d}{e-d}\right) \geq 0$ . We consider the residues under  $v$ , which are real numbers. Firstly,  $v\left(\frac{a-b}{e-d}\right) = 0$  and  $\frac{a-b}{e-d} > 0$  imply that  $\left(\frac{a-b}{e-d}\right)v > 0$ , and  $v\left(\frac{b-d}{e-d}\right) \geq 0$  and  $\frac{b-d}{e-d} > 0$  imply that  $\left(\frac{b-d}{e-d}\right)v \geq 0$ . Secondly, we have that

$$0 \leq \left(\frac{b-d}{e-d}\right)v < \left(\frac{a-d}{e-d}\right)v,$$

where the last inequality holds because  $\left(\frac{a-d}{e-d}\right)v - \left(\frac{b-d}{e-d}\right)v = \left(\frac{a-d}{e-d} - \frac{b-d}{e-d}\right)v = \left(\frac{a-b}{e-d}\right)v > 0$ . So there are rational numbers  $q_1, q_2 > 0$  such that

$$\left(\frac{b-d}{e-d}\right)v < q_1 < q_2 < \left(\frac{a-d}{e-d}\right)v,$$

which yields

$$b - d < q_1(e - d) < q_2(e - d) < a - d,$$

whence

$$b < d + q_1(e - d) < d + q_2(e - d) < a.$$

Consequently,  $d + q_1(e - d)$ ,  $d + q_2(e - d) \in \text{Betw}_R(D, E)$ , which can only happen in the ball complement case. In this case,  $\text{Betw}_R(D, E)$  is a ball  $B_{S_0}(a_0, R)$  in  $R$ , with  $D < a_0 < E$ . By Lemma 5,  $S_0 = vR \setminus v(E - D)$ . But

$$v(d + q_2(e - d) - (d + q_1(e - d))) = v((q_2 - q_1)(e - d)) = v(e - d) < S_0,$$

in contradiction to  $d + q_1(e - d)$ ,  $d + q_2(e - d) \in B_{S_0}(a_0, R)$ . We have now proved that  $\mathcal{B}$  is contained in  $B_S(a, F)$ .

It remains to show that  $B_S(a, F)$  is contained in  $\mathcal{B}$ . If this were not the case, then for some  $b \in B_S(a, F)$  there would exist some  $d \in D$  with  $b \leq d$ , or some  $e \in E$  with  $b \geq e$ . We will assume the first case and deduce a contradiction; the second case is symmetrical. Since  $b \leq d < a$  and  $B_S(a, F)$  is convex, we have that  $d \in B_S(a, F)$ .

First, we consider the case of  $(D, E)$  being the complement of a ball  $B_{S_0}(a_0, R)$  in  $R$ . We have that  $a_0 \in \mathcal{B} \subseteq B_S(a, F)$ , so  $B_S(a, F) = B_S(a_0, F)$ . Further, we know from Lemma 5 that  $v(E - D) = vR \setminus S_0$ . By our choice of  $S$ , this implies that  $S \cap vR = S_0$ , and we obtain that  $d \in B_S(a_0, F) \cap R = B_{S_0}(a_0, R)$ , a contradiction.

In the non-ball case, we use that  $B_S(a, F) = B_S(d, F)$  to obtain that  $B_S(a, F) \cap R = B_T(d, R)$ , where  $T := S \cap vR$ . From  $S > v(E - D)$  it follows that  $B_T(d, R) < E$ . In the present case,  $\text{Betw}_R(D, E) = \emptyset$ , so we find that  $B_T(d, R)$  is contained in  $D$ . Since  $B_S(a, F)$  is convex and contains  $a > D$ , it follows that  $B_T(d, R)$  is a final segment of  $D$ . But this contradicts our assumption that  $(D, E)$  is a non-ball cut.  $\square$

**Remark 21.** In the case where  $(D, E)$  is the complement of a ball  $B_{S_0}(a_0, R)$  in  $R$ , we can choose  $a = a_0$ . Moreover,  $S$  is then equal to the largest final segment of  $vF$  disjoint from  $vR \setminus S_0$  (or equivalently, the largest subset of  $vF$  such that  $S > vR \setminus S_0$ ).

The next lemma tells us which cuts in  $F$  restrict to the same cut in  $R$ :

**Lemma 22.** *Take any cut  $C$  in  $R$ .*

a) *If  $C = (D, E)$ , then the set of all cuts in  $F$  that restrict to  $C$  is  $\{C' \in \mathcal{C}(F) \mid D^{+F} \leq C' \leq E^{-F}\}$ . (If  $D = \emptyset$ , then  $D^{+F}$  means the cut  $F^{-F}$ , and if  $E = \emptyset$ , then  $E^{-F}$  means the cut  $F^{+F}$ .)*

b) *Assume that  $C = B_0^{+R}$  or  $C = B_0^{-R}$  for a ball  $B_0 = B_{S_0}(a_0, R) \neq R$  in  $R$ , and take the ball  $B_S(a_0, F)$  as in Lemma 20. Then the set of all cuts in  $F$  that restrict to the cut  $C$  in  $R$  is  $\{C' \in \mathcal{C}(F) \mid B_0^{+F} \leq C' \leq B_S(a_0, F)^{+F}\}$  for  $C = B_0^{+R}$ , and  $\{C' \in \mathcal{C}(F) \mid B_S(a_0, F)^{-F} \leq C' \leq B_0^{-F}\}$  for  $C = B_0^{-R}$ . If  $vR$  is a convex subgroup of  $vF$  and  $C$  is not principal, then  $B_0^{+F} = B_S(a_0, F)^{+F}$ ,  $B_0^{-F} = B_S(a_0, F)^{-F}$ , and the above sets are singletons.*

*Proof:* The proof of part a) is straightforward. Now assume the hypotheses of part b). We prove the assertions for  $C = B_0^{+R}$ . For  $C = B_0^{-R}$ , the proof is symmetrical. If  $(D, E)$  is the ball complement of  $B_0$  in  $R$ , then  $C = (D \cup B_0, E)$ . By Lemma 20,  $\text{Betw}_F(D, E) = B_S(a, F)$ , which implies that  $\text{Betw}_F(D \cup B_0, E) = \{b \in B_S(a, F) \mid b > B_0\}$ . This implies the first assertion of part b).

For the proof of the second assertion, assume that  $vR$  is a convex subgroup of  $vF$  and that  $C$  is not principal. Then  $S_0$  is a non-empty final segment of  $vR$ , and  $S_0 \neq vR$  since  $B_{S_0}(a_0, R) \neq R$  by assumption. We wish to show that  $S_0$  is an initial segment of  $S$ . Since  $S_0$  is a final segment of  $vR$  and  $vR$  is convex in  $vF$ , also  $S_0$  is convex in  $vF$ . Hence if  $S_0$  were not an initial segment of  $S$ , then there would be an element  $\gamma \in S$  such that  $\gamma < S_0$ . On the other hand,  $S > vR \setminus S_0$ , whence  $S_0 > \gamma > vR \setminus S_0 \neq \emptyset$ . But this contradicts the convexity of  $vR$  in  $vF$ .

Since  $S_0$  is an initial segment of  $S$ , the ball  $B_{S_0}(a_0, R)$  is coinitial and cofinal in the ball  $B_S(a_0, F)$ . This yields that  $B_0^{+F} = B_S(a_0, F)^{+F}$  and  $B_0^{-F} = B_S(a_0, F)^{-F}$ .  $\square$

We define an order preserving embedding  $\tilde{i}$  of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$  as follows. Take a cut  $C$  in  $R$ . If  $C = (D, E)$  is a non-ball cut in  $R$ , then we set  $\tilde{i}(C) = D^{+F}$  or  $\tilde{i}(C) = E^{-F}$ . If  $C$  is the lower or upper edge of a ball  $B_0 \neq R$  in  $R$  and  $(D, E)$  is the ball complement of  $B_0$ , then we set  $\tilde{i}(C) = D^{+F}$  if  $C = B_0^{-R}$  is the lower edge, and  $\tilde{i}(C) = E^{-F}$  if  $C = B_0^{+R}$  is the upper edge. Finally, we set

$\tilde{\iota}(R^{-R}) = R^{-F}$  and  $\tilde{\iota}(R^{+R}) = R^{+F}$ . Note that  $\tilde{\iota}$  is uniquely determined by this definition if and only if no non-ball cut  $(D, E)$  in  $R$  is filled in  $F$  because then  $D^+ = E^-$  will still hold in  $F$ .

**Remark 23.** For a cut  $C$  in  $R$ , its image  $\tilde{\iota}(C)$  is a non-ball cut in  $F$  if and only if  $C$  is a non-ball cut in  $R$  that is not filled in  $F$ . Hence if  $\tilde{\iota}(C)$  is a non-ball cut in  $F$  then it is the only cut in  $F$  that restricts to  $C$ .

Indeed, if  $C$  is a ball cut in  $R$ , then by our definition of  $\tilde{\iota}$ , also  $\tilde{\iota}(C)$  is a ball cut. If  $C = (D, E)$  is a non-ball cut in  $R$  that is filled in  $F$ , then by Lemma 20,  $D^{+F} = B^{-F}$  and  $E^{-F} = B^{+F}$  for a ball  $B = B_S(a, F)$  in  $F$ , so  $\tilde{\iota}(C)$  is again a ball cut. But if the non-ball cut  $C = (D, E)$  is not filled in  $F$ , then it is also a non-ball cut in  $F$ , as the restriction to  $R$  of a ball cofinal in the left or cointial in the right cut set in  $F$  would be a ball in  $R$  cofinal in  $D$  or cointial in  $E$ .

The embedding  $\tilde{\iota}$  is order preserving since if  $(D_1, E_1) < (D_2, E_2)$  are two cuts in  $R$ , then  $E_1 \cap D_2 \neq \emptyset$  and therefore,  $D_1^{+F} \leq E_1^{-F} < D_2^{+F} \leq E_2^{-F}$ .

If  $B_{S_0}(a_0, R) \neq R$  is a ball in  $R$ , and if we take  $S$  as defined in Lemma 20, then by our definition,

$$\tilde{\iota}(B_{S_0}(a_0, R)^{-R}) = B_S(a_0, R)^{-F} \quad \text{and} \quad \tilde{\iota}(B_{S_0}(a_0, R)^{+R}) = B_S(a_0, R)^{+F}.$$

This together with  $\tilde{\iota}(R^{-R}) = R^{-F}$  and  $\tilde{\iota}(R^{+R}) = R^{+F}$  shows:

**Lemma 24.** *The embedding  $\tilde{\iota}$  sends equivalent cuts to equivalent cuts. Hence the preimage of a full set is full.*

Let us also note:

**Proposition 25.** *If  $vR$  is cofinal in  $vF$  (which implies that there is no  $f \in F$  such that  $f > R$ ), then  $\tilde{\iota}$  sends principal cuts to principal cuts. Otherwise, no principal cut is sent to a principal cut.*

Proof: A principal cut in  $R$  is the upper or lower edge of a ball  $B_\emptyset(a_0, R)$ . Take the ball  $B_S(a_0, F)$  as in Lemma 20. By definition,  $\tilde{\iota}(B_\emptyset(a_0, R)^{-R}) = B_S(a_0, F)^{-F}$  and  $\tilde{\iota}(B_\emptyset(a_0, R)^{+R}) = B_S(a_0, F)^{+F}$ . The latter cuts are principal if and only if  $S = \emptyset$ . By Remark 21,  $S = \emptyset$  if and only if there is no  $\gamma \in vF$  such that  $\gamma > vR$ , that is, if and only if  $vR$  is cofinal in  $vF$ .  $\square$

If there is at least one non-ball cut in  $R$  that is filled in  $F$ , then the embedding  $\tilde{\iota}$  will *not* be continuous with respect to the interval topology. Even worse:

**Proposition 26.** *Take any extension  $F|R$  of ordered fields. If there is at least one non-ball cut in  $R$  that is filled in  $F$ , then there exists no embedding of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$  that is continuous with respect to the interval topology and compatible with restriction.*

Proof: Take  $C = (D, E)$  to be a non-ball cut in  $R$  that is filled in  $F$ . Then Lemma 20 shows that  $\text{Betw}_F(D, E)$  is equal to a ball  $B$  in  $F$ . In order to be compatible with restriction, an embedding has to send  $C$  to a cut  $C'$  in  $F$  which is equal to  $B^{+F}$ ,  $B^{-F}$ , or a proper cut in  $B$ . Suppose that  $C' \neq B^{+F}$ . Take any cut  $C_1 < B^{-F}$  and consider the open interval  $I = (C_1, B^{+F})$  in  $\mathcal{C}(F)$ . Then the restriction of  $I$  to  $\mathcal{C}(R)$  is an interval in  $\mathcal{C}(R)$  with last element  $C$ . This shows

that the preimage of  $I$  under any embedding compatible with restriction is not open, as follows from Lemma 9 since  $C$  is not a principal cut.

In the case of  $C' = B^{+F}$ , choose  $C_2 \in \mathcal{C}(F)$  such that  $B^{+F} < C_2$  and consider the open interval  $I = (B^{-F}, C_2)$  in  $\mathcal{C}(F)$ . Its restriction to  $\mathcal{C}(R)$  is an interval with first element  $C$ , hence again not open.  $\square$

The problem is that an open interval in  $\mathcal{C}(F)$  can end in a set that fills a cut from  $R$ , in which case its preimage in  $\mathcal{C}(R)$  will include an endpoint. However, a full open set will have to enter the between set from both sides, and so we obtain the following positive result if we switch from the interval to the full topology:

**Proposition 27.** *Assume that  $vR$  is a convex subgroup of  $vF$ . Then the embeddings  $\tilde{i} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  constructed above are exactly the embeddings that are continuous with respect to the full topology and compatible with restriction.*

*Proof:* Take an embedding  $\tilde{i} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  as constructed above. In view of Lemma 22,  $\tilde{i}$  is compatible with restriction.

By virtue of Lemma 24, in order to show that  $\tilde{i}$  is continuous with respect to the full topology, it suffices to show that the preimage of any full open set  $U$  is open in the interval topology of  $\mathcal{C}(R)$ . Take  $C \in \mathcal{C}(R)$  with  $\tilde{i}(C) \in U$ . Since  $U$  is open in the interval topology of  $\mathcal{C}(F)$ , there is an open interval  $I \subseteq U$  which contains  $\tilde{i}(C)$ . The preimage of  $I$  under  $\tilde{i}$  is again an interval, and if  $C$  is not an endpoint of it, then  $C$  lies in some open subinterval of this preimage.

Now suppose that  $C$  is an endpoint of the preimage of  $I$ . Then either all cuts in  $I$  on the left side of  $\tilde{i}(C)$  restrict to  $C$ , or all cuts in  $I$  on the right side of  $\tilde{i}(C)$  restrict to  $C$ . In both cases, we have that more than one cut in  $F$  restricts to  $C$ . Since we have assumed  $vR$  to be a convex subgroup of  $vF$ , Lemma 22 shows that we are in one of the following cases:

- a)  $C$  is a non-ball cut,
- b)  $C$  is a principal cut,
- c)  $C = R^{-R}$  or  $C = R^{+R}$ .

In all three cases, by our construction of  $\tilde{i}$ , we have that  $\tilde{i}(C) = B^{-F}$  or  $\tilde{i}(C) = B^{+F}$  for some ball  $B$  in  $F$ . Denote the restriction of  $B^{-F}$  to  $R$  by  $C_1$ , and the restriction of  $B^{+F}$  to  $R$  by  $C_2$ . Then  $C = C_1$  or  $C = C_2$ .

Since  $U$  is assumed to be full,  $B^{-F}, B^{+F} \in U$  and since  $U$  is open,  $B^{-F} \in I_1$  and  $B^{+F} \in I_2$  for some open intervals  $I_1$  and  $I_2$  contained in  $U$ .

We first deal with cases a) and b). In both cases,  $B^{-F}$  is the smallest cut that reduces to  $C_1$  and  $B^{+F}$  is the largest cut that reduces to  $C_2$ . The open interval  $I_1$  contains a cut on the left of  $B^{-F}$ , which consequently restricts to a cut  $C'_1 < C_1$ . Similarly,  $I_2$  contains a cut on the right of  $B^{+F}$ , which consequently restricts to a cut  $C'_2 > C_2$ . For every  $C' \in (C_2, C'_2)$  we have that  $\tilde{i}(C_2) < \tilde{i}(C') < \tilde{i}(C'_2)$ , hence  $\tilde{i}(C') \in I_2$ . This shows that  $[C_2, C'_2)$  is contained in the preimage of  $I_2$ . Similarly, it is shown that  $(C'_1, C_1]$  is contained in the preimage of  $I_1$ .

In case a), both  $B^{-F}$  and  $B^{+F}$  restrict to  $C$ , so we have  $C = C_1 = C_2$ . In case b), where  $C = a^{-R}$  or  $C = a^{+R}$  for some  $a \in R$ ,  $B^{-F}$  restricts to  $a^{-R}$  and  $B^{+F}$  restricts to  $a^{+R}$ . In both cases,  $(C'_1, C_1] \cup [C_2, C'_2) = (C'_1, C'_2)$ . It follows that  $C$  has the open neighborhood  $(C'_1, C'_2)$  which is contained in the preimage of  $U$ .

Now we consider case c). In this case,  $\tilde{i}(C) = R^{-F}$ , the largest cut that restricts to  $C_1 = R^{-R}$ , or  $\tilde{i}(C) = R^{+F}$ , the smallest cut that restricts to  $C_2 = R^{+R}$ . The open interval  $I_1$  contains a cut on the right of  $R^{-F}$ , which consequently restricts to a cut  $C'_1 > R^{-R}$ . Similarly,  $I_2$  contains a cut on the left of  $R^{+F}$ , which consequently restricts to a cut  $C'_2 < R^{+R}$ . For every  $C' \in (C'_2, R^{+R})$  we have that  $\tilde{i}(C'_2) < \tilde{i}(C') < \tilde{i}(R^{+R})$ , hence  $\tilde{i}(C') \in I_2$ . This shows that  $(C'_2, R^{+R}]$  is contained in the preimage of  $I_2$ . Similarly, it is shown that  $[R^{-R}, C'_1)$  is contained in the preimage of  $I_1$ . Now one of these two intervals is an open neighborhood of  $C$ .

It follows in all three cases that  $C$  has an open neighborhood which is contained in the preimage of  $U$ . This proves that the restriction of  $U$  is open.

Now assume that  $\tilde{i}'$  is an embedding of  $\mathcal{C}(R)$  in  $\mathcal{C}(F)$ , compatible with restriction. Suppose that there is a cut  $C$  in  $\mathcal{C}(R)$  such that its image  $\tilde{i}'(C)$  is not in accordance with our above construction.

First, we consider the case of  $C = (D, E)$  being a non-ball cut. Then our assumption and the compatibility with restriction yield that  $D^{+F} < \tilde{i}'(C) < E^{-F}$  in  $\mathcal{C}(F)$ . If the ball  $B_S(a, F)$  is chosen as in Lemma 20, then  $D^{+F} = B_S(a, F)^{-F}$  and  $E^{-F} = B_S(a, F)^{+F}$ . Therefore, the open interval  $(D^{+F}, E^{-F})$  in  $\mathcal{C}(F)$  is full by Lemma 11. But the preimage of this interval is the singleton  $\{C\}$ , hence not open.

Now we consider the case of  $C = B_0^{+R}$  for some ball  $B_0$  in  $R$ ; the case of  $C = B_0^{-R}$  is symmetrical. If  $(D, E)$  is the ball complement of  $B_0$  in  $R$ , then our assumption and the compatibility with restriction yield that  $D^{+F} < B_0^{+F} \leq \tilde{i}'(C) < E^{-F}$  in  $\mathcal{C}(F)$ . The same argument as before shows that  $(D^{+F}, E^{-F})$  is a full open interval in  $\mathcal{C}(F)$ . Its preimage in  $\mathcal{C}(R)$  has  $C$  as its last element. Since  $C$  is an upper edge of a ball not equal to  $R$ , it follows that this interval is not open.

Finally, we consider the case of  $C = R^{+R}$ ; the case of  $C = R^{-R}$  is symmetrical. Then our assumption and the compatibility with restriction yield that  $R^{+F} < \tilde{i}'(C)$  in  $\mathcal{C}(F)$ . The open set  $[C_{-\infty}, R^{-F}) \cup (R^{+F}, C_{\infty}]$  in  $\mathcal{C}(F)$  is full by Lemma 11. But the preimage of it is either  $\{R^{+R}\}$  or  $\{R^{-R}, R^{+R}\}$ , hence not open.  $\square$

Our positive result is contrasted by the following negative result:

**Proposition 28.** *Assume that  $vR$  is not a convex subgroup of  $vF$ . Then there are no embeddings  $\tilde{i} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  that are continuous with respect to the full topology and compatible with restriction.*

*Proof:* If  $vR$  is not a convex subgroup of  $vF$ , then there are  $\alpha, \beta \in vR$  and  $\gamma \in vF \setminus vR$  such that  $\alpha < \gamma < \beta$ . Take  $S_0 := \{\delta \in vR \mid \gamma < \delta\}$  and  $B_0 := B_{S_0}(0, R)$ . Note that  $B_0 \neq R$  because  $\alpha \notin S_0$ , and that  $B_0$  is not a singleton because  $\beta \in S_0$ .

Now if  $B_S(0, F)$  is as in Lemma 20, then it follows from Remark 21 that  $\gamma \in S \setminus S_0$ . This implies that  $B_{S_0}(0, R)$  is not cofinal in  $B_S(0, F)$ , whence  $B_0^{+F} < B_S(0, F)^{+F}$ . Now assume that  $\tilde{i} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  is an embedding compatible with restriction. Then by Lemma 22,  $B_0^{+F} \leq \tilde{i}(B_0^{+R}) \leq B_S(0, F)^{+F}$ . Suppose first that  $B_0^{+F} < \tilde{i}(B_0^{+R})$ . By Lemma 11, the open neighborhood  $U := [C_{-\infty}, B_0^{+F}) \cup$

$(B_0^{+F}, C_\infty]$  of  $\tilde{\iota}(B_0^{+R})$  in  $\mathcal{C}(F)$  is full. But  $\tilde{\iota}^{-1}(U) = [C_{-\infty}, B_0^{-R}) \cup [B_0^{+R}, C_\infty]$  or  $\tilde{\iota}^{-1}(U) = [C_{-\infty}, B_0^{-R}] \cup [B_0^{+R}, C_\infty]$  in  $\mathcal{C}(R)$ , both of which are not open since  $B_0$  is not a singleton and therefore  $B_0^{+R}$  is not the immediate successor of  $B_0^{-R}$ .

Suppose now that  $B_0^{+F} = \tilde{\iota}(B_0^{+R})$ . Again by Lemma 11, the open neighborhood  $U := (B_S(0, F)^{-F}, B_S(0, F)^{+F})$  of  $\tilde{\iota}(B_0^{+R})$  in  $\mathcal{C}(F)$  is full. But  $\tilde{\iota}^{-1}(U) = (B_0^{-R}, B_0^{+R}]$  or  $\tilde{\iota}^{-1}(U) = [B_0^{-R}, B_0^{+R}]$  in  $\mathcal{C}(R)$ , both of which are not open since  $B_0 \neq R$ .  $\square$

## 5. EMBEDDINGS OF $M(R(y))$ IN $M(F(y))$

We will now consider an extension of formally real fields  $F|R$ , with  $R$  real closed, but not necessarily archimedean. We will first consider the case where also  $F$  is real closed.

We assume that  $vR$  is convex in  $vF$  and start from one of the embeddings  $\tilde{\iota} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  constructed in the previous section (cf. Proposition 27). We define an embedding

$$\iota : M(R(y)) \longrightarrow M(F(y))$$

in the following way. If  $M(R(y)) \ni \zeta = \lambda_{R(y)} \circ \chi_R(C)$  for a cut  $C$  in  $R$ , then we set

$$\iota(\zeta) := \lambda_{F(y)} \circ \chi_F(\tilde{\iota}(C)) .$$

Since  $\tilde{\iota}$  is compatible with the equivalence of cuts, the embedding  $\iota$  is well-defined and the diagram

$$\begin{array}{ccc} \mathcal{C}(F) & \xrightarrow{\lambda_{F(y)} \circ \chi_F} & M(F(y)) \\ \tilde{\iota} \uparrow & & \iota \uparrow \\ \mathcal{C}(R) & \xrightarrow{\lambda_{R(y)} \circ \chi_R} & M(R(y)) \end{array}$$

commutes.

**Theorem 29.** *Take an extension  $F|R$  of real closed fields. If  $vR$  is convex in  $vF$ , then the embedding  $\iota$  as defined above does not depend on the particular choice of  $\tilde{\iota}$  and is continuous and compatible with restriction.*

*Conversely, if  $\iota : M(R(y)) \rightarrow M(F(y))$  is continuous and compatible with restriction, then it induces an embedding  $\tilde{\iota} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  continuous w.r.t. the full topology and compatible with restriction, such that the above diagram commutes, and  $vR$  is convex in  $vF$ .*

**Proof:** Take  $\tilde{\iota}$  as constructed in the previous section. We show that  $\iota$  is continuous. Take any open set  $U$  in  $M(F(y))$ . By Proposition 16, its preimage  $U_1$  in  $\mathcal{C}(F)$  is a full open set. Then by Proposition 27, the preimage  $U_2$  of  $U$  in  $\mathcal{C}(R)$  is a full open set. Again by Proposition 16, the image  $U_3$  of  $U_2$  in  $M(R(y))$  is open. From Lemma 19 we know that  $\text{res}(U)$  is the preimage of  $U$  under  $\iota$ . But from the commutativity of the diagram in Proposition 18 we know that

$$\text{res}(U) = \text{res} \circ \lambda_{F(y)} \circ \chi_F(U_1) = \lambda_{R(y)} \circ \chi_R \circ \text{res}(U_1) = U_3 .$$

So the preimage of  $U$  under  $\iota$  is open. This proves the continuity of  $\iota$ .

In the construction of  $\tilde{\iota}$  in the previous section the only freedom we had was to choose either the upper or the lower edge of the ball which fills a non-ball cut in  $R$ ; but these cuts correspond to the same  $\mathbb{R}$ -place in  $M(F(y))$ . This shows that all embeddings  $\tilde{\iota}$  constructed in the previous section determine the same embedding  $\iota$ .

We will now prove the second assertion. Take  $\iota$  as in the assumption. For each  $C \in \mathcal{C}(R)$ , we wish to define  $\tilde{\iota}(C)$  such that

$$\lambda_{F(y)} \circ \chi_F \circ \tilde{\iota}(C) = \iota \circ \lambda_{R(y)} \circ \chi_R(C).$$

Set  $\xi := \lambda_{R(y)} \circ \chi_R(C) \in M(R(y))$  and  $\xi' := \iota(\xi)$ . Since  $\iota$  is compatible with restriction,  $\xi$  is the restriction of  $\xi'$  to  $R(y)$ . By the commutativity of the diagram in Proposition 18, we find that if  $C' \in \mathcal{C}(F)$  is sent to  $\xi'$  by  $\lambda_{F(y)} \circ \chi_F$ , then  $\text{res}(C')$  must be sent to  $\xi$  by  $\lambda_{R(y)} \circ \chi_R$ .

If  $C$  is a non-ball cut, then choose any  $C' \in \mathcal{C}(F)$  such that  $\lambda_{F(y)} \circ \chi_F(C'_1) = \xi'$  and define  $\tilde{\iota}(C) := C'$ . Since  $C$  is the only cut in  $R$  that is sent to  $\xi$  by  $\lambda_{R(y)} \circ \chi_R$ , it follows that  $\text{res}(C') = C$ .

If  $C$  is a ball cut, that is,  $C = B_0^{-R}$  or  $C = B_0^{+R}$  for some ball  $B_0$  in  $R$ , then we have to find images for both  $B_0^{-R}$  and  $B_0^{+R}$ . We claim that the continuity of  $\iota$  implies that the preimage of  $\xi'$  under  $\lambda_{F(y)} \circ \chi_F$  is  $\{B^{-F}, B^{+F}\}$  for some ball  $B$  in  $F$  with  $\text{res}(B^{-F}) = B_0^{-R}$  and  $\text{res}(B^{+F}) = B_0^{+R}$ . We treat the case of  $B_0 \neq R$  and leave the case of  $B_0 = R$  to the reader.

We write  $B_0 = B_{S_0}(a_0, R)$ , take  $S$  as in Lemma 20, and set  $B := B_S(a_0, F)$ . Suppose the preimage of  $\xi'$  is not  $\{B^{-F}, B^{+F}\}$ . Take  $C'$  in the preimage. Then by what we have shown above,  $C'$  restricts to  $B_0^{-R}$  or  $B_0^{+R}$ . We assume the latter case; the former is symmetrical. Then  $B_0^{+F} \leq C' < B^{+F}$ . By Proposition 11, the open interval  $(B^{-F}, B^{+F})$  is full, so  $U := \lambda_{F(y)} \circ \chi_F((B^{-F}, B^{+F}))$  is open in  $M(F(y))$  and contains  $\xi'$ . The restriction  $I$  of  $(B^{-F}, B^{+F})$  to  $\mathcal{C}(R)$  has  $B_0^{+R} = \text{res}(C')$  as its largest element, hence it is not open. The same argument as in the first part of this proof shows that the preimage  $U'$  of  $U$  under  $\iota$  is equal to  $\lambda_{F(y)} \circ \chi_F \circ \text{res}((B^{-F}, B^{+F})) = \lambda_{F(y)} \circ \chi_F(I)$ , which is not open. But this contradicts the continuity of  $\iota$ . We see that the preimage of  $\xi'$  must be  $\{B^{-F}, B^{+F}\}$ . So we set  $\tilde{\iota}(B_0^{-R}) = B^{-F}$  and  $\tilde{\iota}(B_0^{+R}) = B^{+F}$  and note that  $\text{res}(B^{-F}) = B_0^{-R}$  and  $\text{res}(B^{+F}) = B_0^{+R}$ .

We have now defined a mapping  $\tilde{\iota} : \mathcal{C}(R) \rightarrow \mathcal{C}(F)$  which is compatible with the restriction. Therefore,  $\tilde{\iota}$  must be injective, and since the restriction preserves  $\leq$  by Lemma 12,  $\tilde{\iota}$  must preserve  $<$ . By definition,  $\tilde{\iota}$  also preserves equivalence.

It remains to show that  $\tilde{\iota}$  is continuous w.r.t. the full topology. Take a full open set  $U$  in  $\mathcal{C}(F)$ . By Proposition 16,  $U_1 := \lambda_{F(y)} \circ \chi_F(U)$  is open. By Lemma 19,  $U_2 := \text{res}(U_1)$  is the preimage of  $U_1$  under  $\iota$ , hence open since  $\iota$  is continuous. By the commutativity of the diagram in Proposition 18,

$$U_2 = \text{res} \circ \lambda_{F(y)} \circ \chi_F(U) = \lambda_{R(y)} \circ \chi_R \circ \text{res}(U).$$

Thus, the full set  $\text{res}(U)$  in  $\mathcal{C}(R)$  is the preimage of  $U_2$ , hence open by Proposition 16. Again by Lemma 19, the full open set  $\text{res}(U)$  is the preimage of  $U$  under  $\tilde{\iota}$ . This proves the continuity of  $\tilde{\iota}$ .  $\square$



Now we will consider the case of  $F$  not being real closed. We choose a real closure  $R'$  of  $F$  and take  $\iota' : M(R(y)) \rightarrow M(R'(y))$  to be the embedding constructed above. Since  $\text{res}_{R'(y)|F(y)}$  is continuous (cf. Remark 13, part 3))

$$\iota := \text{res}_{R'(y)|F(y)} \circ \iota'$$

is a continuous mapping from  $M(R(y))$  to  $M(F(y))$ . Since  $\iota'$  is compatible with the restriction

$$\text{res}_{R'(y)|R(y)} = \text{res}_{F(y)|R(y)} \circ \text{res}_{R'(y)|F(y)},$$

we see that  $\iota$  is compatible with the restriction. For this reason, it is also injective.

As the real closure  $R'$  can be taken with respect to any ordering on  $F$ , we may lose the uniqueness of  $\iota$ . However, we are able to show the following partial uniqueness result:

**Theorem 30.** *Take two orderings  $P_1$  and  $P_2$  of  $F$  which induce the same  $\mathbb{R}$ -place,  $R'_1$  and  $R'_2$  the respective real closures of  $F$ , and  $\iota'_i : M(R(y)) \rightarrow M(R'_i(y))$ ,  $i = 1, 2$ , the unique continuous embeddings compatible with restriction. Consider the following commuting diagram:*

$$\begin{array}{ccc} & M(R'_1(y)) & \\ \iota'_1 \nearrow & & \searrow \text{res}_1 \\ M(R(y)) & \xleftarrow{\text{res}} & M(F(y)) \\ \iota'_2 \searrow & & \nearrow \text{res}_2 \\ & M(R'_2(y)) & \end{array}$$

Then

$$\text{res}_1 \circ \iota'_1 = \text{res}_2 \circ \iota'_2.$$

*Proof:* We will first show that the mappings coincide on all  $\mathbb{R}$ -places of  $R(y)$  determined by the principal cuts.

Suppose that  $\zeta = \lambda \circ \chi(a^+) = \lambda \circ \chi(a^-)$ , where  $a \in R$ . Note that for the corresponding valuation  $v_\zeta$  on  $R(y)$ , we have that  $vR < v_\zeta(a - y)$ . Let  $\zeta_i := \iota'_i(\zeta)$ , for  $i = 1, 2$ . By the definition of the embedding  $\iota'_i$ , we have that  $\zeta_i$  is determined by the upper and lower edge of the ball  $B_{S_i}(a, R'_i)$  where  $S_i = \{\alpha \in vR'_i \mid \alpha > vR\}$ . Then for the corresponding valuation  $v_{\zeta_i}$  on  $R'_i(y)$  we have that  $vR < v_{\zeta_i}(a - y) < S_i$  in  $v_{\zeta_i}R'_i$ . Since these value groups are divisible (by [7, Theorem 4.3.7],  $R'_i$  being real closed fields), the values  $v_{\zeta_i}(a - y)$  are rationally independent over these value groups. Therefore, the valuations  $v_{\zeta_i}$  are uniquely determined by the natural valuations on  $R'_i$  and the values  $v_{\zeta_i}(a - y)$ . The same remains true when we restrict to  $F(y)$ . There, by our assumption, the restrictions of the natural valuations on  $R'_i$  coincide, so the restrictions of the valuations  $v_{\zeta_i}$  to  $F(y)$  must coincide, too. Further, the residue fields of  $v_{\zeta_i}$  on  $F(y)$  are equal to the residue field of  $F$  because  $v_{\zeta_i}(a - y)$  is rationally independent over  $vF$ . Since the restrictions to  $F$  of  $\zeta_1$  and  $\zeta_2$  coincide, the restrictions to  $F(y)$  of these  $\mathbb{R}$ -places coincide, as well. Therefore,  $\text{res}_1 \circ \iota'_1(\zeta) = \text{res}_2 \circ \iota'_2(\zeta)$ .

Now take  $\zeta_1 = \text{res}_1 \circ \iota'_1(\zeta)$  and  $\zeta_2 = \text{res}_2 \circ \iota'_2(\zeta)$  for some  $\zeta \in M(R(y))$  and suppose they are distinct. Since  $M(F(y))$  is Hausdorff, there are disjoint open neighborhoods  $U_1 \ni \zeta_1$  and  $U_2 \ni \zeta_2$ . The preimages of  $U_1$  and  $U_2$  in  $M(R(y))$  are open, and  $\zeta$  lies in their intersection. So this intersection is not empty, and by the density of the principal places in  $M(R(y))$  (cf. Lemma 17), there is a principal place  $\zeta_0$  in this intersection. But the images of  $\zeta_0$  under the two embeddings are equal and hence must lie in  $U_1 \cap U_2$ , a contradiction.  $\square$

## 6. EMBEDDINGS OF $M(R(y))$ IN $M(F(y))$ FOR ARCHIMEDEAN $R$

In this section we will consider an extension of formally real fields  $F|R$  in the special case where  $R$  is archimedean real closed. The general case has been treated in the previous section. Here, we wish to give a different, more explicit construction of a continuous embedding  $\iota$  of  $M(R(y))$  in  $M(F(y))$  which is compatible with restriction.

We choose any real place  $\xi$  of  $F$ . Then  $\overline{F} := \xi(F) \subseteq \mathbb{R}$ . Since  $R$  is archimedean, we can assume that  $\xi|_R = \text{id}_R$  and that  $\overline{F}|R$  is an extension of archimedean ordered fields. By  $\xi_y$  we denote the **constant extension** of  $\xi$  to  $F(y)$ , i.e., the unique extension of  $\xi$  which is trivial on  $R(y)$ . Its valuation ring is the smallest subring of  $F(y)$  containing both the valuation ring of  $\xi$  and  $R(y)$ . The valuation associated with  $\xi_y$  is the **Gauß** or **functorial valuation** on  $F(y)$  extending the valuation associated with  $\xi$  on  $F$ . On polynomials in  $F[y]$  with coefficients in the valuation ring of  $\xi$ ,  $\xi_y$  acts by applying  $\xi$  to the coefficients. Therefore, the residue field of  $\xi_y$  is  $\xi(F)(y)$ .

For every  $\zeta \in M(R(y))$  we define the constant extension  $\zeta_{\overline{F}}$  of  $\zeta$  to  $\overline{F}(y)$  as follows. As  $\zeta$  is trivial on the archimedean field  $R$ , it is determined by an irreducible polynomial  $p(y) \in R[y]$  (or by  $1/y$ ). Since  $R$  is real closed and  $\overline{F}$  is formally real, such a polynomial  $p$  remains irreducible over  $\overline{F}$  and thus,  $p$  (or  $1/y$ , respectively) determines a unique extension of  $\zeta$  to  $\overline{F}(y)$  which is trivial on  $\overline{F}$ . We set  $\iota_{\overline{F}|R}(\zeta) := \zeta_{\overline{F}}$ .

**Lemma 31.** *The mapping  $\iota_{\overline{F}|R} : M(R(y)) \rightarrow M(\overline{F}(y))$  is a continuous embedding compatible with the restriction. If  $\overline{F}$  is real closed, then it is a homeomorphism.*

*Proof:* Since  $\overline{F}|R$  is an extension of archimedean ordered fields,  $R$  lies dense in  $\overline{F}$ . It follows from [12, Theorem 3.2] that the restriction mapping from  $M(\overline{F}(y))$  to  $M(R(y))$  is a homeomorphism if  $\overline{F}$  is real closed. Hence in this case,  $\iota_{\overline{F}|R}$  is a homeomorphism.

If  $\overline{F}$  is not real closed, then we consider a real closure  $R'$  of  $\overline{F}$ . By what we have shown already,  $\iota_{R'|R}$  is a homeomorphism. Since  $\text{res}_{R'(y)|R(y)}$  is continuous, the same holds for  $\iota_{\overline{F}|R} = \text{res}_{R'(y)|\overline{F}(y)} \circ \iota_{R'|R}$ .  $\square$

Now we define

$$(1) \quad \iota(\zeta) := \zeta_{\overline{F}} \circ \xi_y.$$

**Theorem 32.** *The mapping  $\iota : M(R(y)) \rightarrow M(F(y))$  is a continuous embedding.*

*Proof:* Take  $a \in F(y)$ . We have to show that the preimage of a subbasis set  $H'(a)$  under  $\iota$  is open in  $M(R(y))$ . If  $\xi_y(a)$  is 0 or  $\infty$ , then the same holds for  $\zeta_{\overline{F}} \circ \xi_y$  for every  $\zeta \in M(R(y))$ . In this case,  $H'(a)$  is empty and we are done.

Assume now that  $\xi_y(a) \neq 0, \infty$ . Then  $\xi_y(a)$  is a nonzero rational function  $g(y) \in \overline{F}(y)$ . The preimage of  $H'(a)$  is then the set of all real places  $\zeta \in M(R(y))$  such that  $\zeta_{\overline{F}}(g) > 0$ . In the case of  $\overline{F} = R$  (which for instance holds when  $R = \mathbb{R}$ ), this is precisely  $H'(g)$  in  $M(R(y))$ . For the general case, we apply Lemma 31 to conclude that the preimage of  $H'(g)$  under the constant extension mapping  $\zeta \mapsto \zeta_{\overline{F}}$ , and hence the preimage of  $H'(a)$  under  $\iota$ , is open.  $\square$

From Theorem 30, where we take  $F = R(x)$ , we now obtain:

**Theorem 33.** *The mapping  $\iota$  defined in (1) is the unique continuous embedding of  $M(R(y))$  in  $M(R(x, y))$  that is compatible with restriction and such that all places in the image of  $\iota$  have restriction  $\xi$  to  $R(x)$ .*

We have chosen to give a direct proof of Theorem 33 although it can be derived from the theorems of the last section. In order to do this, we have to show that the embedding defined in (1) coincides with the embedding we have constructed before. To this end, we consider an ordering  $P$  of  $R(y)$  and the cut  $C$  it induces in the archimedean real closed field  $R$ . If  $R = \mathbb{R}$ , then the only possibilities are  $C = C_\infty$ ,  $C = C_{-\infty}$ , or  $C = r^+, r^-$  for  $r \in \mathbb{R}$ . If  $R \neq \mathbb{R}$ ,  $C$  can also be a cut induced in  $R$  by some real number  $r \in \mathbb{R} \setminus R$ .

If  $C = C_\infty$  or  $C = C_{-\infty}$ , we have that  $y > F$  or  $y < F$  under the corresponding orderings. In this case,  $0 < vy^{-1} < vF^{>0}$ , where  $vF^{>0}$  denotes the set of positive elements of  $vF$ .

In the case of  $C = r^+, r^-$ , we have that  $\iota(C)$  is the upper or lower edge of  $B_{vF^{>0}}(r, F)$ . This ball is  $r + \mathcal{M}$  where  $\mathcal{M}$  is the valuation ideal of infinitesimals in  $F$ . Since  $C$  is induced by  $y$ , we find that  $0 < v(y - r) < vF^{>0}$ .

In the final case, we have two subcases. If  $C$  is not filled in  $F$ , then  $v(y - f) \leq 0$  for every  $f \in F$ . If  $C$  is filled by some element in  $F$ , then we can identify this element with the real number  $r$  that fills the cut  $C$ . In this case, we obtain the same result as in the previous case.

In all three cases, we find the constant extension  $\xi_y$  of  $\xi$  must be trivial on  $R(y)$ , which implies that  $\iota(\zeta)$  must be of the form  $\zeta_{\overline{F}} \circ \xi_y$ .

In the case of  $R = \mathbb{R}$ , we can show the above more directly:

**Proposition 34.** *Take  $\iota$  to be an embedding of  $M(\mathbb{R}(y))$  in  $M(\mathbb{R}(x, y))$ , compatible with restriction and such that all places in the image of  $\iota$  have the same restriction to  $\mathbb{R}(x)$ . If for some  $\xi \in \text{im}(\iota)$  such that  $\xi(x) = a$  and  $\xi(y) = b$  we have that for some  $n \in \mathbb{N}$ ,*

$$0 < v_\xi(x - a) < nv_\xi(y - b),$$

*then the embedding is not continuous. The same holds if  $\xi(x) = \infty$  and  $x - a$  is replaced by  $1/x$  and/or  $\xi(y) = \infty$  and  $y - b$  is replaced by  $1/y$ .*

*Proof:* Take

$$f(x, y) = \frac{x - a + (y - b)^n}{x - a}.$$

Then  $H'(f) \cap \text{im}(\iota)$  is the singleton  $\{\xi\}$ . Indeed,  $\xi \in H'(f)$  since  $\xi(f) = 1$ . But if  $\xi' = \iota(\zeta) \neq \xi$ , then  $\zeta(y) \neq b$ , whence  $\xi'(f) = \infty$ . The cases of  $\xi(x) = \infty$  and/or  $\xi(y) = \infty$  are similar.  $\square$

It is possible to generalize the approach of this section to the general setting of the previous section by replacing the  $\mathbb{R}$ -place  $\xi$  of  $F$  by the finest coarsening  $\xi'$  whose residue field contains  $R$ . (The valuation ring of  $\xi'$  is the compositum of the valuation ring of  $\xi$  and the subfield  $R$  of  $F$ .) But we would need an analogue of Lemma 31 for the case of non-archimedean fields  $R$  and  $\bar{F} = \xi'(F)$ . We found that the tools developed to deal with this analogue can be directly applied to construct the embedding of  $M(R(y))$  in  $M(F(y))$  in the setting of the previous section.

## 7. EMBEDDINGS OF $\prod_{i=1}^n M(\mathbb{R}(x_i))$ IN $M(\mathbb{R}(x_1, \dots, x_n))$

In order to study possible embeddings of the torus in spaces of real places, we wish to consider embeddings of  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  in  $M(\mathbb{R}(x, y))$ . Initially, we will treat the more general case of  $n$  variables. We consider the projection mapping

$$\rho : M(\mathbb{R}(x_1, \dots, x_n)) \ni \xi \mapsto (\xi|_{\mathbb{R}(x_1)}, \dots, \xi|_{\mathbb{R}(x_n)}) \in \prod_{i=1}^n M(\mathbb{R}(x_i)).$$

**Lemma 35.** *The mapping  $\rho$  is surjective.*

We describe a general construction that will prove the lemma. Take  $\mathbb{R}$ -places  $\xi_i \in M(\mathbb{R}(x_i))$ . We wish to associate to them an  $\mathbb{R}$ -place  $\xi$  of  $\mathbb{R}(x_1, \dots, x_n)$  whose restriction to  $\mathbb{R}(x_i)$  is  $\xi_i$ . We may assume that  $\xi_i(x_i) \neq \infty$ ; otherwise, we can replace  $x_i$  by  $1/x_i$ . For  $1 \leq i < n$ , let  $\xi'_i$  be the place of  $\mathbb{R}(x_i, \dots, x_n)$  which is trivial on  $\mathbb{R}(x_{i+1}, \dots, x_n)$  and such that  $\xi'_i(x_i) = \xi_i(x_i)$ . Its residue field is  $\mathbb{R}(x_{i+1}, \dots, x_n)$ . Then the place

$$(2) \quad \xi = \xi_n \circ \xi'_{n-1} \circ \dots \circ \xi'_1.$$

satisfies the above conditions. This construction can be replaced by the symmetric ones where the  $x_i$  are permuted.

**Remark 36.** There are many more possibilities for choosing a common extension  $\xi$  of the  $\xi_i$ . Set  $\xi_i(x_i) = a_i$ . Choose any rationally independent elements  $r_1, \dots, r_n \in \mathbb{R}$ . Then there is a (uniquely determined)  $\mathbb{R}$ -place  $\xi$  of  $\mathbb{R}(x_1, \dots, x_n)$  such that for the valuation  $v$  associated with  $\xi$  we have that  $v(x_i - a_i) = r_i$ . The value group of  $\xi$  is generated by the values  $r_1, \dots, r_n$  and is thus archimedean. In contrast to this, the value group of the place in (2) has rank  $n$  and is thus not archimedean if  $n > 1$ .

The surjectivity shows that there exist embeddings

$$(3) \quad \iota : \prod_{i=1}^n M(\mathbb{R}(x_i)) \hookrightarrow M(\mathbb{R}(x_1, \dots, x_n)).$$

Such an embedding will be called **compatible** if  $\rho \circ \iota$  is the identity.

**Theorem 37.** *The image of every compatible embedding  $\iota$  as in (3) lies dense in  $M(\mathbb{R}(x_1, \dots, x_n))$ . But for  $n > 1$ , every non-empty basic open subset of  $M(\mathbb{R}(x_1, \dots, x_n))$  contains infinitely many places that are not in the image of  $\iota$ .*

Proof: Take non-zero elements  $f_1, \dots, f_m \in \mathbb{R}(x_1, \dots, x_n)$  such that

$$U := H'(f_1) \cap \dots \cap H'(f_m) \neq \emptyset.$$

Take  $\zeta \in U$  and write  $f_i(x_1, \dots, x_n) = \frac{g_i(x_1, \dots, x_n)}{h_i(x_1, \dots, x_n)}$ . Choose an ordering on  $\mathbb{R}(x_1, \dots, x_n)$  compatible with  $\zeta$ . Then the existential sentence

$$\exists X_1 \dots \exists X_n : \bigwedge_{1 \leq i \leq m} h_i(X_1, \dots, X_n) \neq 0 \wedge \frac{g_i(X_1, \dots, X_n)}{h_i(X_1, \dots, X_n)} > 0$$

holds in  $\mathbb{R}(x_1, \dots, x_n)$  with this ordering. By Tarski's Transfer Principle, it also holds in  $\mathbb{R}$  with the usual ordering. That is, there exist  $a_1, \dots, a_n \in \mathbb{R}$  such that  $h_i(a_1, \dots, a_n) \neq 0$  and  $\frac{g_i(a_1, \dots, a_n)}{h_i(a_1, \dots, a_n)} > 0$  for  $1 \leq i \leq m$ . Hence for every  $\mathbb{R}$ -place  $\zeta \in M(\mathbb{R}(x_1, \dots, x_n))$  such that  $\zeta(x_i) = a_i$  we will have that  $\zeta(f_i) = \frac{g_i(a_1, \dots, a_n)}{h_i(a_1, \dots, a_n)} > 0$ . Among all such  $\zeta$  there is precisely one in  $\text{im}(\iota)$ . For this  $\zeta$ , we have that  $\zeta \in U \cap \text{im}(\iota)$ . This proves that  $\text{im}(\iota)$  lies dense in  $M(\mathbb{R}(x_1, \dots, x_n))$ .

For  $n > 1$ , Remark 36 shows that there are infinitely many  $\mathbb{R}$ -places  $\zeta \in M(\mathbb{R}(x_1, \dots, x_n))$  such that  $\zeta(x_i) = a_i$ . As only one of them is in  $\text{im}(\iota)$ ,  $U \setminus \text{im}(\iota)$  is infinite.  $\square$

**Corollary 38.** *A compatible embedding  $\iota$  as in (3) cannot be continuous with respect to the product topology on  $\prod_{i=1}^n M(\mathbb{R}(x_i))$ .*

Proof: Suppose we have a continuous compatible embedding. Under the product topology, the space  $\prod_{i=1}^n M(\mathbb{R}(x_i))$  is compact. As the continuous image of a compact space in a Hausdorff space is again compact (cf. [11], Chapter 5, Theorem 8), we find that the image is closed in  $M(\mathbb{R}(x_1, \dots, x_n))$ . As it is also dense in  $M(\mathbb{R}(x_1, \dots, x_n))$  by Theorem 37, it must be equal to  $M(\mathbb{R}(x_1, \dots, x_n))$ . But this contradicts the second assertion of Theorem 37. Hence the embedding cannot be continuous.  $\square$

**Remark 39.** All of the above can be generalized to the case of infinitely many elements  $x_i$ ,  $i \in I$  that are algebraically independent over  $\mathbb{R}$ . After choosing some well-ordering on  $I$ , the construction of the embedding

$$\iota : \prod_{i \in I} M(\mathbb{R}(x_i)) \hookrightarrow M(\mathbb{R}(x_i \mid i \in I))$$

proceeds by (possibly transfinite) induction. The above theorem and corollary remain valid. The proof of the theorem still works, as in the finitely many polynomials  $f_1, \dots, f_m$  only finitely many variables  $x_i$  can appear. For infinite  $I$ , it is no longer true that the choice of the elements  $a_1, \dots, a_n$  determines a unique place in  $\text{im}(\iota)$ . Still, an application of Remark 36 shows that  $U \setminus \text{im}(\iota)$  is infinite.

We will now reprove the result of the corollary in the case of  $n = 2$  by looking more closely at the topologies that are involved here. Every embedding of  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  in  $M(\mathbb{R}(x, y))$  will induce a topology on  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  whose open sets are the preimages of the intersections of the open sets of  $M(\mathbb{R}(x, y))$  with the image of the embedding.

**Theorem 40.** *For every compatible embedding  $\iota$ , the topology induced on  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  is finer than the product topology.*

**Proof:** Take a basic open set in the product topology of  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  which is the interior or exterior of a circle given by  $(x - a)^2 + (y - b)^2 = r^2$ , where  $a, b, r \in \mathbb{R}$ . We set

$$f(x, y) = r^2 - (x - a)^2 - (y - b)^2.$$

Then the set  $\text{im}(\iota) \cap H'(f)$  is precisely the image of the interior of the circle, and set  $\text{im}(\iota) \cap H'(-f)$  is precisely the image of the exterior of the circle. This proves that the induced topology is equal to or finer than the product topology.

It remains to present an induced open set in  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  which is not open in the product topology. Take the unique  $\xi$  in  $\text{im}(\iota)$  such that  $\xi(x) = 0$  and  $\xi(y) = 0$ .

$$f(x, y) = \begin{cases} 1 + \frac{x}{y} & \text{if } \xi\left(\frac{x}{y}\right) = 0 \\ 1 + \frac{y}{x} & \text{if } \xi\left(\frac{y}{x}\right) = 0 \\ \frac{y^2}{x^2} & \text{otherwise.} \end{cases}$$

It follows in all three cases that  $\xi \in H'(f)$ . The preimage of  $\xi$  under  $\iota$  is  $(\xi_1, \xi_2)$  where  $\xi_1(x) = 0$  and  $\xi_2(y) = 0$ . If the subset  $U$  induced by  $H'(f)$  in  $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$  would be open, then it would contain the interior of a circle  $x^2 + y^2 = r^2$  for some  $r > 0$ . But this is impossible since whenever  $(\xi_1, \xi_2) \in U$ , then for the first choice of  $f$ ,  $\xi_2(y) = 0$  must imply  $\xi_1(x) = 0$ , and for the two other choices of  $f$ ,  $\xi_1(x) = 0$  must imply  $\xi_2(y) = 0$ .  $\square$

**Open Problem:** What is the induced topology? Is it one-dimensional or two-dimensional?

## 8. EMBEDDINGS OF MORE GENERAL PRODUCTS

For simplicity, we will only consider the product of two spaces  $M(F_1)$  and  $M(F_2)$ ; a generalization to any finite products can be achieved along the lines of the last section. We will also assume that  $F_1$  and  $F_2$  both contain  $\mathbb{R}$ . Then we can assume them embedded in some extension field of  $\mathbb{R}$  such that  $F_1$  and  $F_2$  are linearly disjoint over  $\mathbb{R}$ . We denote by  $F$  the field compositum of  $F_1$  and  $F_2$ , that is, the smallest subextension of the given extension of  $\mathbb{R}$  that contains both  $F_1$  and  $F_2$ .

As before, we consider the corresponding projection mapping

$$\rho : M(F) \ni \xi \mapsto (\xi|_{F_1}, \xi|_{F_2}) \in M(F_1) \times M(F_2).$$

We show that  $\rho$  is surjective. Take  $(\xi_1, \xi_2) \in M(F_1) \times M(F_2)$ . Then there is an extension  $\xi'_1$  of  $\xi_1$  from  $F_1$  to  $F$  such that the residue field of  $\xi'_1$  is  $F_2$ . Then take  $\iota(\xi_1, \xi_2) = \xi_2 \circ \xi'_1$ . Here again, one obtains a different place of  $F$  by interchanging  $F_1$  and  $F_2$ , showing that  $\rho$  is not injective.

The surjectivity shows that there exist embeddings

$$\iota : M(F_1) \times M(F_2) \longrightarrow M(F)$$

As before,  $\iota$  will be called **compatible** if  $\rho \circ \iota$  is the identity.

If  $F_1|\mathbb{R}$  and  $F_2|\mathbb{R}$  are function fields, we can again prove that the image of every compatible embedding  $\iota$  lies dense in  $M(F)$ . We will need the following fact. For a proof, see the second half of the proof of the lemma on p. 190 of [13].

**Lemma 41.** *Take a field  $k$  and a function field  $K = k(x_1, \dots, x_d, z)$  where  $x_1, \dots, x_d$  are algebraically independent over  $k$  and  $z$  is separable-algebraic over  $k(x_1, \dots, x_d)$ . If  $f \in k[x_1, \dots, x_d, Z]$  is the irreducible polynomial of  $z$  over  $k(x_1, \dots, x_d)$  and if  $a_1, \dots, a_d, b \in k$  are such that*

$$f(a_1, \dots, a_d, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial Z}(a_1, \dots, a_d, b) \neq 0,$$

*then  $K$  admits a  $k$ -rational place  $\xi$  such that  $\xi(x_i) = a_i$  for  $1 \leq i \leq d$ , and  $\xi(z) = b$ .*

**Theorem 42.** *If  $F_1|\mathbb{R}$  and  $F_2|\mathbb{R}$  are function fields of transcendence degree  $\geq 1$ , then the image of every compatible embedding  $\iota$  lies dense in  $M(F)$ . But every non-empty basic open subset of  $M(F)$  contains infinitely many places that are not in the image of  $\iota$ .*

*Proof:* We write  $F_1 = \mathbb{R}(x_1, \dots, x_d, z_1)$  and  $F_2 = \mathbb{R}(x_{d+1}, \dots, x_{d+e}, z_2)$  with  $x_1, \dots, x_{d+e}$  algebraically independent over  $\mathbb{R}$ ,  $z_1$  separable-algebraic over  $\mathbb{R}(x_1, \dots, x_d)$ , and  $z_2$  separable-algebraic over  $\mathbb{R}(x_{d+1}, \dots, x_{d+e})$ . Then  $F = \mathbb{R}(x_1, \dots, x_{d+e}, z_1, z_2)$ . Let  $G_1 \in k[x_1, \dots, x_d, Z_1]$  be the irreducible polynomial of  $z_1$  over  $k(x_1, \dots, x_d)$  and  $G_2 \in k[x_{d+1}, \dots, x_{d+e}, Z]$  be the irreducible polynomial of  $z_2$  over  $k(x_{d+1}, \dots, x_{d+e})$ .

Take non-zero elements  $f_1, \dots, f_m \in F$  such that  $U := H'(f_1) \cap \dots \cap H'(f_m) \neq \emptyset$ . Take  $\zeta \in U$  and write

$$f_i(x_1, \dots, x_{d+e}, z_1, z_2) = \frac{g_i(x_1, \dots, x_{d+e}, z_1, z_2)}{h_i(x_1, \dots, x_{d+e})}$$

with polynomials  $g_i \in \mathbb{R}[X_1, \dots, X_{d+e}, Z_1, Z_2]$  and  $h_i \in \mathbb{R}[X_1, \dots, X_{d+e}]$ . Choose an ordering on  $F$  compatible with  $\zeta$ . Then the existential sentence

$$\begin{aligned} & \exists X_1 \dots \exists X_{d+e} \exists Z_1 \exists Z_2 : \\ & G_1(X_1, \dots, X_d, Z_1) = 0 \wedge \frac{\partial G_1}{\partial Z_1}(X_1, \dots, X_d, Z_1) \neq 0 \wedge \\ & G_2(X_{d+1}, \dots, X_{d+e}, Z_2) = 0 \wedge \frac{\partial G_2}{\partial Z_2}(X_{d+1}, \dots, X_{d+e}, Z_2) \neq 0 \wedge \\ & \bigwedge_{1 \leq i \leq m} h_i(X_1, \dots, X_{d+e}) \neq 0 \wedge \frac{g_i(X_1, \dots, X_{d+e}, Z_1, Z_2)}{h_i(X_1, \dots, X_{d+e})} > 0 \end{aligned}$$

holds in  $F$  with this ordering. By Tarski's Transfer Principle, it also holds in  $\mathbb{R}$  with the usual ordering. That is, there exist  $a_1, \dots, a_{d+r}, b_1, b_2 \in \mathbb{R}$  such that

$$(4) \quad G_1(a_1, \dots, a_d, b_1) = 0 \wedge \frac{\partial G_1}{\partial Z_1}(a_1, \dots, a_d, b_1) \neq 0$$

$$(5) \quad G_2(a_{d+1}, \dots, a_{d+e}, b_2) = 0 \wedge \frac{\partial G_2}{\partial Z_2}(a_{d+1}, \dots, a_{d+e}, b_2) \neq 0$$

$$(6) \quad \bigwedge_{1 \leq i \leq m} h_i(a_1, \dots, a_{d+e}) \neq 0 \wedge \frac{g_i(a_1, \dots, a_{d+e}, b_1, b_2)}{h_i(a_1, \dots, a_{d+e})} > 0$$

Hence for every  $\mathbb{R}$ -place  $\zeta \in M(F)$  such that  $\zeta(x_i) = a_i$  and  $\zeta(z_j) = b_j$  we will have that  $\zeta(f_i) > 0$ ,  $1 \leq i \leq m$ . By Lemma 41, (4) guarantees that there is  $\zeta_1 \in M(F_1)$  such that  $\zeta_1(x_i) = a_i$ ,  $1 \leq i \leq d$ , and  $\zeta_1(z_1) = b_1$ , and (5) guarantees that there is  $\zeta_2 \in M(F_2)$  such that  $\zeta_2(x_i) = a_i$ ,  $d+1 \leq i \leq d+e$ , and  $\zeta_2(z_2) = b_2$ . Consequently, there is  $\zeta \in \text{im}(\iota)$  with  $\zeta(x_i) = a_i$  and  $\zeta(z_j) = b_j$ . It follows that  $\zeta \in U \cap \text{im}(\iota)$ . This proves that the image of our construction lies dense in  $M(F)$ .

From Remark 36 it again follows that there are infinitely many  $\mathbb{R}$ -places  $\zeta$  of  $\mathbb{R}(x_1, \dots, x_{d+e})$  such that  $\zeta(x_i) = a_i$ . These places can be extended to  $F$  by setting  $\zeta(z_j) = b_j$ . All of them have archimedean value group. In contrast, all places in  $\text{im}(\iota)$  are compositions of two non-trivial places and therefore have non-archimedean value group. This shows that  $U \setminus \text{im}(\iota)$  is infinite.  $\square$

As before, one proves:

**Corollary 43.** *If  $F_1|\mathbb{R}$  and  $F_2|\mathbb{R}$  are function fields, then a compatible embedding cannot be continuous with respect to the product topology on  $M(F_1) \times M(F_2)$ .*

## 9. RAISING THE TRANSCENDENCE DEGREE

In this final section, we show how to use previous constructions to embed  $M(K)$  in  $M(L)$ , for an arbitrary field  $K$  and suitable transcendental extensions  $L$  of  $K$ .

**Theorem 44.** *Assume that  $L$  admits a  $K$ -rational place  $\xi$ . Then*

$$\iota : M(K) \ni \zeta \mapsto \zeta \circ \xi \in M(L)$$

*is a continuous embedding compatible with restriction.*

**Proof:** It is clear that the embedding is compatible with restriction. For the continuity, take  $f \in L$  and assume that  $H'(f) \cap \text{im}(\iota) \neq \emptyset$ . Pick  $\zeta \in M(K)$  such that  $\zeta \circ \xi = \iota(\zeta) \in H'(f)$ . It follows that  $(\zeta \circ \xi)(f) \neq \infty$  and therefore,  $\infty \neq \xi(f) \in K$ . For arbitrary  $\zeta \in M(K)$ , we have that  $(\zeta \circ \xi)(f) = \zeta(\xi(f))$ , so  $\zeta \circ \xi \in H'(f) \Leftrightarrow \zeta \in H'(\xi(f))$ . Hence,  $\iota^{-1}(H'(f)) = H'(\xi(f))$ , which proves that  $\iota$  is continuous.  $\square$

There are fields  $L$  of arbitrary transcendence degree over  $K$  which allow a unique  $K$ -rational place  $\xi$ . This fact has been used in [6] to show that a given space of  $\mathbb{R}$ -places can be realized over arbitrarily large fields. The other extreme is:



**Corollary 45.** *Take a collection  $x_i, i \in I$ , of elements algebraically independent over  $K$ . Then there are at least  $|K|^{|I|}$  many distinct continuous embeddings of  $M(K)$  in  $M(K(x_i \mid i \in I))$ , all of them compatible with restriction and having mutually disjoint images.*

This follows from the fact that for every choice of elements  $a_i \in K$  there is a  $K$ -rational place  $\xi$  of  $L$  such that  $\xi(x_i) = a_i$ .

**Corollary 46.** *There are at least  $2^{\aleph_0}$  many continuous embeddings of  $M(\mathbb{R}(x))$  in  $M(\mathbb{R}(x, y))$ , all of them compatible with restriction and having mutually disjoint images.*

It should be noted that Theorem 2 does not follow from Theorem 44. The condition that  $vR$  is a convex subgroup of  $vF$  does by no means imply that  $F(y)$  admits a  $R(y)$ -rational place.

## REFERENCES

- [1] E. Becker, D. Gondard: *Notes on the space of real places of a formally real field*. Real Analytic and Algebraic Geometry, W. de Gruyter (1995), 21–46
- [2] R. Brown: *Real places and ordered fields*. Rocky Mount. J. Math. **1** (1971), 633–636
- [3] R. Brown: *Real-valued places on the function field of an algebraic curve*. Houston J. Math. **6** (1980), 227–243
- [4] T.C. Craven, *The topological space of orderings of rational function field*. Duke Math. J. **41** (1974), 339–347
- [5] D.W. Dubois, *Infinite primes and ordered fields*, Dissertationes Math. **69** (1970), 1–43
- [6] I. Efrat, K. Osiak, *Topological spaces as spaces of  $\mathbb{R}$ -places*, J. Pure Applied Alg. **215** (2011), 839–846
- [7] A.J. Engler, A. Prestel: *Valued Fields*, Springer 2005
- [8] R. Gilmer: *Extension of an order to a simple transcendental extension*. Contemp. Math. **8** (1981), 113–118
- [9] D. Gondard, M. Marshall: *Towards an abstract description of the space of real places*. Contemp. Math. **253** (2000), 77–113
- [10] J. Harman, *Chains of higher level orderings*, Contemp. Math. **8**, (1982), 141–174.
- [11] J. L. Kelley, *General Topology*. Springer Graduate Texts in Mathematics (1955)
- [12] F.-V. Kuhlmann, M. Machura, K. Osiak, *Spaces of  $\mathbb{R}$ -places of function fields over real closed fields*, Comm. in Alg. **39** (2011), 3166–3177
- [13] F.-V. Kuhlmann, A. Prestel, *On places of algebraic function fields*, J. reine angew. Math. **353** (1984), 181–195
- [14] T.Y. Lam, *Orderings, valuations and quadratic forms*. CBMS Regional Conf. Ser. Math. **52**. Published for the Conf. Board of the Math. Sciences, Washington (1983)
- [15] T.Y. Lam, *Ten lectures on quadratic forms over fields*. Conference on Quadratic Forms-1976 (Proc. Conf., Queen’s Univ., Kingston, Ont., 1976), pp. 1-102. Queen’s Papers in Pure and Appl. Math., No. 46, Queen’s Univ., Kingston, Ont. (1977)
- [16] M. Machura, M. Marshall, K. Osiak: *Metrizability of the space of  $\mathbb{R}$ -places of a real function field*, Math. Z. **266** (2010), 237–242
- [17] H.W. Schülting: *On real places of a field and their holomorphy ring*. Comm. Alg. **10** (1982), 1239–1284

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