METRIZABILITY OF SPACES OF $\mathbb{R}$-PLACES OF FUNCTION FIELDS OF TRANSCENDENCE DEGREE 1 OVER REAL CLOSED FIELDS

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Abstract. In this paper we discuss the question "When do different orderings of the rational function field R(X) (where R is a real closed field) induce the same $\mathbb{R}$-place?". We use this to show that if R contains a dense real closed subfield R', then the spaces of $\mathbb{R}$-places of $R(X)$ and $R'(X)$ are homeomorphic. For the function field $K = R(X)$ we prove that its space $M(K)$ of $\mathbb{R}$-places is metrizible if and only if R contains a countable dense subfield. Moreover, we show that this condition is neccessary for the metrizability of $M(F)$ for any function field F of transcendence degree 1 over R.

1. Introduction

The metrizability of spaces of $\mathbb{R}$-places of function fields of transcendence degree at least 2 over real closed fields has been studied in the paper [15]. The case of transcendence degree 1 has remained open — surprisingly, it cannot be settled by the methods used in the mentioned paper. By introducing new methods, we partially settle this case in the present paper. We prove that for any real closed field R, the space of $\mathbb{R}$-places of a function field F of transcendence degree 1 over R is metrizable only if R contains a countable dense subfield. If $F = R(X)$, then this condition is also sufficient. The same holds for any F if R is archimedean because then the space is homeomorphic to a union of circles (cf. [4], Theorem 2.1). The general non-archimedean case remains open as we do not know yet how the structure of the spaces changes under finite field extensions.

For the background on $\mathbb{R}$-real places, we refer the reader to [2], [3], [4], [5], [7], [11], [12], [14], and [17]. We shall briefly outline some basic notions.

Take an ordered field K. The set $\mathcal{X}(K)$ of all orderings of K carries a Boolean (i.e., compact, Hausdorff and totally disconnected) topology with a subbasis consisting of the Harrison sets

$$H_K(a) := \{P \in \mathcal{X}(K) \mid a \in P\}, \;\; a \in \dot{K} = K \setminus \{0\}.$$

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Every ordering P of K determines a valuation ring

$$A(P) := \{a \in K \mid \exists q \in \mathbb{Q}^+(-q <_P a <_P q)\}$$

with maximal ideal

$$I(P) := \{a \in K \mid \forall q \in \mathbb{Q}^+(-q <_P a <_P q)\}.$$

We denote the set of units of $A(P)$ by $U(P)$. The ordering determined by P on the residue field $k_P := A(P)/I(P)$ is Archimedean. Thus k_P is naturally embedded in the field $\mathbb{R}$; this embedding composed with the place $K \to k_P \cup \{\infty\}$ gives a real-valued place ξ_P, or an $\mathbb{R}$-place for short. The valuation associated with the place ξ_P will be denoted by v_P. Note that we identify equivalent valuations.

Conversely, every place ξ of K with values in $\mathbb{R}$ is induced by some ordering of K in the way described above (see [14, Prop. 9.1]). We will use v_ξ to denote the valuation associated with the place ξ.

The set of all $\mathbb{R}$-places of the field K will be denoted by $M(K)$. Through the canonical surjection

$$\lambda_K : \mathcal{X}(K) \longrightarrow M(K),$$

we equip $M(K)$ with the quotient topology inherited from $\mathcal{X}(K)$, making it a compact, Hausdorff space (see [14, Cor. 9.9]).

Throughout, we shall denote by $\dot{S}$ the set $S \setminus \{0\}$, for any subset S of a field. If A, B are subsets of an ordered set S, then by $A < B$ we mean that $a < b$ for every $a \in A$ and every $b \in B$. We shall also use the familiar notation of intervals, with endpoints written in lowercase, whereas the notation (A, B) will denote a cut.

2. The $\mathbb{R}$-places of $R(X)$

Let R be a real closed field with its unique ordering $\dot{R}^2$. Denote by v the natural valuation of R, i.e., associated to $A(\dot{R}^2)$, by Γ the value group of v and by k the residue field of v. Since R is real closed, Γ is a divisible group and k is a real closed field (see [9, Th. 4.3.7]). Moreover, using Hensel's Lemma one can show that k can be considered as a subfield of R.

There is a one-to-one correspondence between orderings P of $R(X)$ and cuts of R (see [10], [18]). The cut (A_P, B_P) corresponding to P is given by $A_P = \{a \in R \mid a <_P X\}$ and $B_P = \{b \in R \mid b >_P X\}$. Conversely, if (A, B) is a cut in R, then the set

$$P = \{f \in R(X) \mid \exists a \in A \; \exists b \in B \; \forall c \in (a, b) \;\; (f(c) \in \dot{R}^2)\}$$

is an ordering of $R(X)$ with $(A_P, B_P) = (A, B)$.

The cuts $(\emptyset, R)$ and $(R, \emptyset)$ are called the *improper cuts.* The corresponding orderings will be denoted by P_∞^- and P_∞^+, respectively.

If A has a maximal element or B has a minimal element, then (A, B) is called *a principal cut.* Every $a \in R$ defines two principal cuts: $((-\infty, a), [a, \infty))$ and $((-\infty, a], (a, \infty))$, with the corresponding orderings denoted by P_a^- and P_a^+, respectively.

The correspondence between cuts in R and orderings of $R(X)$ makes the set $\mathcal{X}(R(X))$ linearly ordered: if Q is another ordering of $R(X)$, then let

$$P \prec Q \iff A_P \subset A_Q.$$

Proposition 2.1. *The Harrison topology on the space $\mathcal{X}(R(X))$ coincides with the topology induced by the ordering defined above.*

Proof. Take the Harisson set $H_{R(X)}(\frac{f}{g}) = H_{R(X)}(fg) \subset \mathcal{X}(R(X))$. Note that $H_{R(X)}(fg)$ is a finite union of intervals (P_a^-, P_b^+) such that:

1. $a, b \in R \cup \{\infty\}$ and if $a, b \in R$, then they are roots of fg;
2. fg has positive values on (a, b).

So, $H_{R(X)}(\frac{f}{g})$ is open in the order topology of $\mathcal{X}(R(X))$.

On the other hand, an interval $(P, Q) \subset \mathcal{X}(R(X))$ can be replaced by a union of Harrison sets $H_{R(X)}(f)$, where f runs through all quadratic polynomials with roots $a < b \in B_P \cap A_Q$ such that f is positive on (a, b). □

The mapping

$$\lambda_{R(X)} : \mathcal{X}(R(X)) \longrightarrow M(R(X))$$

is described in [18], where cuts in R are represented by cut symbols ([18, Prop.3.2.2]). By using [18, Prop.4.3.2] for $n = 1$ one can find which cut symbols determine the same ℝ-place. We shall obtain a more intuitive criterion in a direct way; it will be used in the next section.

A non-empty subset D of a valued field (K, v) is called a *ball* if for all $a, b \in D$ and $c \in K$, $v(c - a) \geq v(b - a)$ implies that $c \in D$.

From now on let (A_1, B_1) and (A_2, B_2) be cuts in R corresponding to distinct orderings $P_1 \prec P_2$ of the rational function field $R(X)$, respectively. We have $A_1 \subset A_2$, $B_2 \subset B_1$ and $B_1 \cap A_2 \neq \emptyset$. The remainder of this section is devoted to the proof of the following theorem:

Theorem 2.2. *Let R be a real closed field and v its natural valuation. Then $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$ if and only if $B_1 \cap A_2$ is a ball in (R, v).*

We set $D = B_1 \cap A_2$, $\xi_i = \lambda_{R(X)}(P_i)$ and $v_i = v_{\xi_i}$, for $i = 1, 2$. The next four lemmas hold even in the case where R is not real closed, since also in this general case, every ordering of $R(X)$ gives rise to a cut in R.

Lemma 2.3. *If D is not a ball, then $\xi_1 \neq \xi_2$.*

Proof. Take $a, b \in D$ and $c \in R \setminus D$ such that $v(c - a) \geq v(b - a)$. Then we also have that $v(b-c) \geq \min\{v(c-a), v(b-a)\} = v(b-a)$. We may assume that $a < b$ (note that $a = b$ would imply $c = a \in D$). As D is convex, we have $c < a$ or $c > b$. We assume that $c > b$; the other case is symmetrical. If $v_1 \neq v_2$, then we are done. So let us assume that $v_1 = v_2$.

With respect to P_1, we have that $X < a < b$, hence $X - a < 0 < b - a$. With respect to P_2, we have that $a < b < X < c$, hence $0 < b - a < X - a < c - a$ and thus $v_1(X - a) = v_2(X - a) \geq v_2(c - a) \geq v(b - a) \geq v_2(X - a)$, so all of these values are equal. It follows that for $i = 1, 2$, $\xi_i\left(\frac{X-a}{b-a}\right)$ are finite, and

$$\xi_1\left(\frac{X-a}{b-a}\right) \;<\; 0 \;<\; \xi_2\left(\frac{X-a}{b-a}\right) \; .$$

Hence, $\xi_1 \neq \xi_2$. □

We leave the easy proof of the following lemma to the reader:

Lemma 2.4. *Suppose that a, b, X are elements of an ordered field with natural valuation v. If $v(X-a) \geq v(b-a)$, then there are integers $n_1, n_2 \neq 0$ such that for $c_i = a+n_i(b-a)$, $i = 1, 2$,*

$$c_1 \;<\; X \;<\; c_2 \quad \textit{and} \quad v(c_1 - a) \;=\; v(c_2 - a) \;=\; v(b - a) \; .$$

We set $S = \{v(b-a) \mid a, b \in D\}$ and $T = \{v(c-a) \mid a \in D, c \in R \setminus D\}$. Every element in an ultrametric ball is a center; therefore,

Lemma 2.5. *If D is a ball and $a \in D$, then $S = \{v(b - a) \mid b \in D\}$ and $T = \{v(c - a) \mid c \in R \setminus D\}$.*

Lemma 2.6. *If D is a ball, then (T, S) is a cut in Γ and for each $a \in D$,*

$$T \;<\; v_i(X - a) \;<\; S \qquad \textit{for } i = 1, 2 \; .$$

Proof. Since D is a ball, we have that $T < S$. Take some $\delta \in \Gamma \setminus S$; we have to show that $\delta \in T$. Take $a \in D$ and choose $d \in R$ such that $v(d) = \delta$. Set $c = d + a$. Then $v(c - a) = v(d) = \delta$, hence $c \notin D$ and $\delta \in T$.

Now we show that $T < v_1(X-a) < S$; the proof for $v_2(X-a)$ is symmetrical. Suppose that $v_1(X-a) < S$ does not hold. Then $v_1(X-a) \geq \beta$ for some $\beta \in S$. By Lemma 2.5, there is $b \in D$ such that $\beta = v(b-a)$. As we are working with the ordering P_1, we have $X < b$ (as well as $X < a$, which we will use later). By Lemma 2.4, there is some $c \in R$ such that $c < X < b$ and $v(c-a) = v(b-a) \in S$. But $c \notin D$, a contradiction to $T < S$.

Now suppose that $T < v_1(X-a)$ does not hold. Then $v_1(X-a) \leq \beta$ for some $\beta \in T$. By Lemma 2.5, there is $c \in R \setminus D$ such that $\beta = v(c-a)$. Now $v_1(X-a) \leq v(c-a)$ implies that there is some positive integer n such that $|X-a| > \frac{1}{n}|c-a|$, whence $X - a < -\frac{1}{n}|c-a| < 0$. Setting $d = a - \frac{1}{n}|c-a|$, we obtain that $X < d < a$. Hence $d \in D$. But $S \ni v(d-a) = v\left(\frac{1}{n}|c-a|\right) = v(c-a) \in T$, a contradiction to $T < S$. □

Proof of Theorem 2.2: In view of Lemma 2.3, we only have to show that for R real closed, $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$ holds if $B_1 \cap A_2$ is a ball. Suppose the latter is the case. Then it follows from Lemma 2.6 that $v_1(X-a), v_2(X-a) \notin \Gamma$. As Γ is divisible, both values are rationally independent over Γ. Therefore, they determine the valuations v_1 and v_2 uniquely because for every polynomial $f(X) = c_nX^n + \ldots + c_0 \in R[X]$,

$$v_i f(X) \;=\; \min_{0\leq j\leq n} v_i(c_jX^j) \quad \text{for} \quad i = 1,2\,. \tag{2.1}$$

Moreover, since $v_1(X-a)$ and $v_2(X-a)$ satisfy the same cut over Γ, sending $v_1(X-a)$ to $v_2(X-a)$ induces an isomorphism of the value group $\Gamma \oplus \mathbb{Z}v_1(X-a)$ of $(R(X), v_1)$ to the value group $\Gamma \oplus \mathbb{Z}v_2(X-a)$ of $(R(X), v_2)$. Therefore, v_1 and v_2 are equivalent valuations. Further, (2.1) implies that the residue fields of $R(X)$ under v_1 and v_2 are both equal to the residue field of R. Since the latter is real closed, we must have that $\xi_1 = \xi_2$. This completes the proof. □

Remark 2.7. Theorem 2.2 shows that two distinct orderings of $R(X)$ induce the same $\mathbb{R}$-place if and only if their corresponding cuts in R are given by the upper and the lower edge of a ball in (R, v).

At most two orderings determine the same $\mathbb{R}$-place. Let $P_1 \prec P_2 \prec P_3$ be orderings of $R(X)$ with corresponding cuts $(A_1, B_1), (A_2, B_2)$ and (A_3, B_3) in R. Then $B_1 \cap A_3$ is the disjoint union of $B_1 \cap A_2$ and $B_2 \cap A_3$. But the disjoint union of two balls is never a ball. This also shows that there is no cut (A, B) in R such that there is a ball which is a final segment of A and another ball which is an initial segment of B.

3. $M(R(X))$ and $M(R'(X))$ are homeomorphic if R' is dense in R

Let L/K be an extension of ordered fields. Then we have restriction mappings

$$\rho : \mathcal{X}(L) \to \mathcal{X}(K), \qquad \rho(P) = P \cap K,$$

and

$$\rho : M(L) \to M(K), \qquad \rho(\xi) = \xi\mid_K .$$

The restriction mappings are continuous and the diagram

$$\begin{array}{ccc} \mathcal{X}(L) & \xrightarrow{\lambda_L} & M(L) \\ \rho\Big\downarrow & & \rho\Big\downarrow \\ \mathcal{X}(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

commutes (see [7, 7.2.]). Being continuous mappings from compact spaces to Hausdorff spaces, the restriction mappings are also closed and proper.

Note that surjectivity of the mapping $\rho : \mathcal{X}(L) \to \mathcal{X}(K)$ implies surjectivity of the mapping $\rho : M(L) \to M(K)$.

Lemma 3.1. *Let $R' \subset R$ be an extension of real closed fields and let P be an ordering of $R(X)$ with corresponding cut (A, B) in R. Then $(A \cap R', B \cap R')$ is a cut in R' whose corresponding ordering $P' \in \mathcal{X}(R'(X))$ is the restriction of P. The mappings $\rho : \mathcal{X}(R(X)) \to \mathcal{X}(R'(X))$ and $\rho : M(R(X)) \to M(R'(X))$ are surjective.*

Proof. It is easy to see that $(A \cap R', B \cap R')$ is a cut in R'. If (A, B) is an improper cut in R, then $(A \cap R', B \cap R')$ is an improper cut in R', as well.

Recall that if (A, B) is a proper cut in R, then $f \in P$ iff there exist $a \in A$ and $b \in B$ such that $f(c) > 0$ for every $c \in (a, b)$. Since R' is a real closed field, all real roots of a polynomial $f \in R'[X]$ are in R'. This implies that $P \cap R'(X) = P'$.

To show the last assertion, take $P' \in \mathcal{X}(R'(X))$ with corresponding cut (A', B') in R'. Set $A = \{a \in R \mid a < B'\}$ and $B = R \setminus A$. Then (A, B) is a cut in R and $(A \cap R', B \cap R') = (A', B')$. Let $P \in \mathcal{X}(R(X))$ be the ordering corresponding to this cut. By what we have already proved, $P \cap R'(X) = P'$. □

Theorem 3.2. *Let $R' \subset R$ be an extension of real closed fields. Then R' is dense in R if and only if $\rho : M(R(X)) \to M(R'(X))$ is a homeomorphism.*

Proof. The restriction mapping $\rho : M(R(X)) \longrightarrow M(R'(X))$ is surjective and continuous. Since both spaces are compact and Hausdorff, it is a homeomorphism if and only if it is injective.

Assume that R' is dense in R and take two distinct places $\xi_1, \xi_2 \in M(R(X))$ with corresponding orderings $P_1 \prec P_2$ of $R(X)$, and (A_1, B_1) and (A_2, B_2) the cuts in R associated with them. Denote by v the natural valuation corresponding to the unique ordering of R. Since $\xi_1 \neq \xi_2$, we know from Theorem 2.2 that $B_1 \cap A_2 \neq \emptyset$ is not a ball. Hence, there are $a, b \in B_1 \cap A_2$ and $c \in R \setminus B_1 \cap A_2$ such that $v(c - a) \geq v(b - a)$. In particular, $a \neq b$. We may assume that $c \in A_1$; the remaining case is symmetrical.

Set $A'_i = A_i \cap R'$ and $B'_i = B_i \cap R'$ for $i = 1, 2$. By the density of R' in R, there are $a', b', c' \in R'$ such that a', b' lie inbetween a and b, $v(a' - a) > v(c - a)$, $v(b' - b) > v(c - a)$, $c' < c$ and $v(c' - c) > v(c - a)$. It follows that $a', b' \in B_1 \cap A_2$ so that $a', b' \in B'_1 \cap A'_2$, $v(c' - a') = v(c - a) \geq v(b - a) = v(b' - a')$. On the other hand, $c' < c < B'_1 \cap A'_2$. This proves that $B'_1 \cap A'_2$ is not a ball. Hence by Theorem 2.2, the restrictions of ξ_1 and ξ_2 to $R'(X)$ remain distinct.

For the converse, assume that R' is not dense in R. Then there are two elements $c < d$ in R such that no element of R' lies between them. So the two distinct cuts c^+ and d^- induce the same cut in R'. Hence if $\xi_1, \xi_2 \in M(R(X))$ are the places corresponding to these two cuts, then their restrictions to R' coincide. On the other hand, as the cuts c^- and c^+ determine the same place and at most two cuts can determine the same place, we see that $\xi_1 \neq \xi_2$. Hence, ρ is not injective. □

4. Metrizibility of the space $M(R(X))$

First we shall recall some basic topological facts. By Urysohn's metrization theorem (see [13, p. 125]), a compact Hausdorff space is metrizable if and only if it is second-countable. Every second-countable space is separable, that is, there is a countable dense subset. Recall that the cellularity of a topological space M is

$$\sup\{|\mathcal{F}| \mid \mathcal{F} \text{ is a family of pairwise disjoint open subsets of } M\}.$$

The cellularity is not bigger than the density of M (i.e., the infimum of the cardinalities of dense subsets of M). Hence if the cellularity of a compact Hausdorff space is uncountable, then the space is not metrizable.

Recall that *the real holomorphy ring* $\mathcal{H}_K$ of a formally real field K is the intersection of all real valuation rings of K, i.e.,

$$\mathcal{H}_K = \bigcap\{A(P), P \in \mathcal{X}(K)\}.$$

By [14, Th. 9.11], a subbasis for the space $M(K)$ is given by the family of the sets

$$U(a) \;=\; \{\xi \in M(K) \mid \xi(a) > 0\} \;,$$

where $a \in \mathcal{H}_K$. If K is countable, then this subbasis (and consequently, also a basis) of $M(K)$ is countable, so $M(K)$ is second-countable. So we have:

Corollary 4.1. *If K is a countable field, then $M(K)$ is metrizable.*

As before we consider a real closed field R with natural valuation v, value group Γ, and residue field $k \subset R$.

Lemma 4.2. *Let N be a dense subset in $M(R(X))$. Then $\lambda^{-1}_{R(X)}(N)$ is a dense subset of $\mathcal{X}(R(X))$.*

Proof. Take a basic open set in $\mathcal{X}(R(X))$, i.e., the set of all cuts in an interval $(a,b) \subset R$. Consider a polynomial $f(X) \in R[X]$, $f(X) = \frac{-4(X-a)(X-b)}{(b-a)^2}$ and let $g = \frac{f}{1+f^2}$. Note that g is positive only on the interval (a,b), and that $g(\frac{a+b}{2}) = \frac{1}{2}$. Therefore the subbasic set $U(g)$ is non-empty (the $\mathbb{R}$-place determined by the principal cuts in $\frac{a+b}{2}$ belongs to $U(g)$), and by density of N in $M(R(X))$, there exists $\xi \in N \cap U(g)$. Let $P \in \lambda^{-1}_{R(X)}(\xi)$ and let (A,B) be a cut corresponding to P. Since $\xi(g) > 0$, $g \in P$. So there exists $a' \in A, b' \in B$ such that for every $c \in (a', b')$, $g(c) > 0$. So, $(a',b') \subseteq (a,b)$ and P corresponds to a cut in (a,b). □

For every proper cut (A,B) in R, we set $v(B-A) = \{v(b-a) \mid a \in A, b \in B\}$. We leave it to the reader to prove that this is an initial segment of Γ.

Proposition 4.3. *Assume that Γ and k are countable and $M(R(X))$ is metrizable. Then R contains a countable dense subfield.*

Proof. Since $M(R(X))$ is metrizable, it is separable. Let N be a countable, dense subset of $M(R(X))$. Then, by the previous lemma, the set $\lambda^{-1}_{R(X)}(N)$ is dense in $\mathcal{X}(R(X))$. Using this set we shall describe a construction of a countable, dense subset of R.

For every $\gamma \in \Gamma$ choose an element $c_\gamma \in \dot{R}^2$ such that $v(c_\gamma) = \gamma$.

Let (A,B) be a cut in R with corresponding ordering $P \in \lambda^{-1}_{R(X)}(N)$. For every γ which is not the maximal element in $v(B-A)$ choose a pair of elements $a^P_\gamma \in A$ and $b^P_\gamma \in B$ such that $v(b^P_\gamma - a^P_\gamma) = \gamma$. If γ is the maximal element in $v(B-A)$ then choose $a \in A$, $b \in B$ such that $v(b-a) = \gamma$. As pointed out earlier in the paper, we may assume that the residue field k is a subfield of R. Then

$$(\{\bar{d} \in k \mid a + \bar{d}c_\gamma \in A\}, \{\bar{e} \in k \mid a + \bar{e}c_\gamma \in B\})$$

is a cut in k. We note that 0 is in the left cut set, and that $\bar{e}$ is in the right cut set whenever $k \ni \bar{e} > \frac{b-a}{c_\gamma}$. Hence, this cut is proper. For every $\bar{c} \in \dot{k}^2$ we can thus choose $\bar{d}$

in the left and $\bar{e}$ in the right cut set such that $\bar{e} - \bar{d} = \bar{c}$. Setting $a^P_{\bar{c}} = a + \bar{d}c_\gamma \in A$ and $b^P_{\bar{c}} = a + \bar{e}c_\gamma \in B$ we obtain that $v(b^P_{\bar{c}} - a^P_{\bar{c}}) = \gamma$ and $\xi_{\dot{R}^2}(\frac{b^P_{\bar{c}} - a^P_{\bar{c}}}{c_\gamma}) = \bar{c}$.

Let $\mathcal{A}_P$ be the set of all $a^P_\gamma, b^P_\gamma, a^P_{\bar{c}}, b^P_{\bar{c}}$ with $\gamma \in v(B - A)$, $\bar{c} \in \dot{k}^2$. Note that $\mathcal{A}_P$ is a countable set because $v(B-A)$ and $\dot{k}^2$ are countable. Let $\mathcal{A} = \bigcup\{\mathcal{A}_P \mid P \in \lambda^{-1}_{R(X)}(N)\}$. Then $\mathcal{A}$ is countable. We will show that it is dense in R.

Suppose that $a < b \in R$. By density of $\lambda^{-1}_{R(X)}(N)$ in $\mathcal{X}(R(X))$, there exists $P \in \lambda^{-1}_{R(X)}(N)$ such that $P^+_a \prec P \prec P^-_b$. Let (A, B) be a cut in R corresponding to P. Then $a \in A$ and $b \in B$. If $v(a-b)$ is not the maximal element in $v(B-A)$, then $v(a-b) < \gamma$ for some $\gamma \in v(B-A)$. In this case, consider $a^P_\gamma, b^P_\gamma \in \mathcal{A}_P$. Since $v(b-a) < v(b^P_\gamma - a^P_\gamma)$, we have $a^P_\gamma \in (a,b)$ or $b^P_\gamma \in (a,b)$. If $\gamma = v(b-a)$ is the maximal element in $v(B-A)$, then $\bar{d} = \xi_{\dot{R}^2}(\frac{b-a}{c_\gamma}) \in \dot{k}^2$. Take $\bar{c} \in \dot{k}^2$, $\bar{c} < \bar{d}$. Then for $a^P_{\bar{c}}, b^P_{\bar{c}} \in \mathcal{A}_P$,

$$\xi_{\dot{R}^2}(\frac{b-a}{c_\gamma} - \frac{b^P_{\bar{c}} - a^P_{\bar{c}}}{c_\gamma}) \; = \; \bar{d} - \bar{c} \; > \; 0\,.$$

Thus, $(b-a) - (b^P_{\bar{c}} - a^P_{\bar{c}}) > 0$.

If $a^P_{\bar{c}} < a$, then $0 < (b-a) - (b^P_{\bar{c}} - a^P_{\bar{c}}) < b - b^P_{\bar{c}}$, and thus $b^P_{\bar{c}} < b$. Similarly, if $b < b^P_{\bar{c}}$, then $a < a^P_{\bar{c}}$. So the interval (a,b) contains an element from $\mathcal{A}$. Since $\mathcal{A}$ is dense in R the field $k(\mathcal{A})$ is dense in R and countable, because $\mathcal{A}$ is countable. □

Lemma 4.4. *Take $a \in R$ and a non-empty final segment S in Γ without smallest element. Then the set*

$$U_{a,S} = \{\xi \in M(R(X)) \mid v_\xi(X-a) \geq \gamma \textit{ for some } \gamma \in S\}$$

is open in $M(R(X))$.

Proof. If Γ is the trivial group, then there is no such set S. So we assume that Γ is not trivial.

We shall show that

$$\lambda^{-1}_{R(X)}(U_{a,S}) = \bigcup_{c \in \dot{R}^2, v(c) \in S} (P^-_{a-c}, P^+_{a+c}).$$

This implies our assertion as each (P^-_{a-c}, P^+_{a+c}) is an open interval in $\mathcal{X}(R(X))$.

Suppose that $P \in \lambda^{-1}_{R(X)}(U_{a,S})$. Then there exists $\gamma \in S$ such that $v_P(X-a) \geq \gamma$. Since S has no smallest element, there exists $c \in \dot{R}^2$ such that $\gamma > v(c) \in S$. Then $-c <_P X - a <_P c$, and thus $a - c <_P X <_P a + c$, so $P \in (P^-_{a-c}, P^+_{a+c})$.

Now suppose that $P \in (P^-_{a-c}, P^+_{a+c})$ for some $c \in \dot{R}^2, v(c) \in S$, i.e., $a - c \leq_P X \leq_P a + c$. Then $-c \leq_P X - a \leq_P c$ and thus $v_P(X-a) \geq v(c) \in S$. □

Take a non-Archimedean real closed field R, with residue field $k \subset R$ and value group Γ. Pick an element $a \in R$ and a value $\gamma \in \Gamma$. Set $S_\gamma := \{\delta \in \Gamma \mid \delta > \gamma\}$ and define

$$U_{a,\gamma} \;:=\; U_{a,S_\gamma} \;=\; \{\xi \in M(R(X)) \mid v_\xi(X-a) > \gamma\}\;.$$

Using the previous lemma, we will now describe two constructions of pairwise disjoint open subsets of $U_{a,\gamma}$ that will not only give us information about the cellularity of $M(R(X))$ but also enable us to derive results about the finite extensions of $R(X)$. For this reason, we will also give in the following an alternate proof of Proposition 4.3.

1) Take values $\alpha, \beta \in \Gamma$ such that $\gamma < \beta \leq \alpha$, and suppose that $\{d_j \mid j \in J\}$ is a set of elements of value β such that $v(d_j - d_\ell) \leq \alpha$ whenever $j, \ell \in J$, $j \neq \ell$. Consider the collection

$$\{U_{a+d_j,\alpha} \mid j \in J\}$$

of sets which are all open by the previous lemma. Note that each $U_{a+d_j,\alpha}$ is non-empty, because the place determined by the principal cuts in $a + d_j$ belongs to $U_{a+d_j,\alpha}$. If $\xi \in U_{a+d_j,\alpha}$, i.e., $v_\xi(X-a-d_j) > \alpha \geq \beta$, then $v(d_j) = \beta$ implies that $v_\xi(X-a) = \beta > \gamma$, whence $\xi \in U_{a,\gamma}$. Therefore, each $U_{a+d_j,\alpha}$ is a subset of $U_{a,\gamma}$.

Suppose that $\xi \in U_{a+d_j,\alpha} \cap U_{a+d_\ell,\alpha}$ for $j, \ell \in J$. Then $v_\xi(X-a-d_j) > \alpha$ and $v_\xi(X-a-d_\ell) > \alpha$, thus $v(d_j - d_\ell) = v_\xi((X-a-d_j) - (X-a-d_\ell)) > \alpha$. This implies that $j = \ell$ by our choice of the d_j. This shows that $U_{a+d_j,\alpha}$ and $U_{a+d_\ell,\alpha}$ are disjoint for $j \neq \ell$.

2) For every $\delta \in \Gamma$, $\delta > \gamma$, choose an element $d_\delta \in R$ with $v(d_\delta) = \delta$. Consider the collection

$$\{U_{a+d_\delta,\delta} \mid \gamma < \delta \in \Gamma\}$$

of sets which are again all open and non-empty, as before. If $\xi \in U_{a+d_\delta,\delta}$, i.e., $v_\xi(X-a-d_\delta) > \delta$ then since $v(d_\delta) = \delta$ we have that $v_\xi(X-a) = \delta > \gamma$, hence $\xi \in U_{a,\gamma}$. Therefore, each $U_{a+d_\delta,\delta}$ is a subset of $U_{a,\gamma}$.

Suppose that $\xi \in U_{a+d_\alpha,\alpha} \cap U_{a+d_\beta,\beta}$ for $\alpha, \beta \in \Gamma$ both greater than γ. Then $v_\xi(X-a-d_\alpha) > \alpha$ and $v_\xi(X-a-d_\beta) > \beta$, thus $v(d_\alpha - d_\beta) = v_\xi((X-a-d_\alpha) - (X-a-d_\beta)) > \min\{\alpha, \beta\}$. This implies that $\alpha = \beta$ since otherwise, $v(d_\alpha - d_\beta) = \min\{\alpha, \beta\}$. This shows that $U_{a+d_\alpha,\alpha}$ and $U_{a+d_\beta,\beta}$ are disjoint for $\alpha \neq \beta$.

Theorem 4.5. *Let R be a real closed field that does not admit a countable dense subfield. Pick some $a \in R$ and $\gamma \in \Gamma$. Then $U_{a,\gamma}$ contains uncountably many pairwise disjoint open sets. In particular, $M(R(X))$ has uncountable cellularity and is not metrizable.*

Proof. If the residue field $k \subset R$ is an uncountable field, then we choose some $b \in R$ with $\beta = v(b) > \gamma$ and apply the first construction with $\alpha = \beta$, $J = \dot{k}$ and $d_j = jb$ for $j \in \dot{k}$. This yields the desired subsets of $U_{a,\gamma}$.

If the value group Γ of R is an uncountable group, then also $\{\alpha \in \Gamma \mid \alpha > \gamma\}$ is uncountable. In this case, the second construction yields the desired subsets of $U_{a,\gamma}$.

Now assume that value group and residue field are countable, but R does not admit a countable dense subfield. Let R_0 be any countable subfield of R having the same value group and residue field as R. One can construct such a field by choosing arbitrary representatives in R for all values in the value group and residues in the residue field; then adjoin these elements to $\mathbb{Q}$ and take R_0 to be a real closure of the resulting field.

To find the elements d_j used in the first construction, we take d_1 to be an element of R that does not lie in the completion of R_0. Having constructed d_μ for all $\mu < \nu$ where ν is a countable ordinal, we know that the countable field $R_0(d_\mu \mid \mu < \nu)$ is not dense in R, and so we can take some d_ν in R that does not lie in the completion of $R_0(d_\mu \mid \mu < \nu)$. By induction, we find elements d_ν for all $\nu < \aleph_1$. Pick any $\beta > \gamma$. Since R_0 and R have the same value group, we can multiply each d_ν with a suitable element from R_0 to obtain that all of them have value β.

It remains to find a suitable $\alpha \in \Gamma$. As d_ν is not in the completion of $R_0(d_\mu \mid \mu < \nu)$, there is $\alpha_\nu \in \Gamma$ such that $v(d_\nu - c) < \alpha_\nu$ for all $c \in R_0(d_\mu \mid \mu < \nu)$. As Γ is countable, there is some $\alpha \in \Gamma$, $\alpha > \beta$, such that $\alpha > \alpha_\nu$ for uncountably many $\nu < \aleph_1$. Delete all members d_ν from the sequence for which $\alpha \leq \alpha_\nu$. The resulting uncountable sequence has the properties needed in the first construction, from which we now obtain the desired subsets of $U_{a,\gamma}$. This completes the proof of our assertion. □

Remark 4.6. The proof shows that, without the assumption that R be real closed, the cellularity of $M(R(X))$ is bigger or equal to $\max\{|k|, |\Gamma|\}$.

Theorem 4.7. *Let R be a real closed field. Then $M(R(X))$ is metrizable if and only if R contains a countable dense subfield.*

Proof. If R does not contain a countable dense subfield, then Theorem 4.5 shows that $M(R(X))$ is not metrizable.

Now suppose that K is a countable, dense subfield of R. Let R' be the real closure of K inside of R. Then $R' \subset R$ and R' is countable and dense in R. By Theorem 3.2, $M(R'(X)) \cong M(R(X))$, and by Corollary 4.1, $M(R'(X))$ is metrizable. □

The following example will present a modification of our construction 1) above.

Example 4.8. Let k be a countable, Archimedean field and Γ a countable, nontrivial, ordered, divisible group. The field $k((\Gamma))$ is real closed, with its natural valuation v being its t-adic valuation with value group Γ and residue field k. Take R to be the real closure of $k(\Gamma)$ in $k((\Gamma))$.

Consider the function field $R(X)$. Since R is countable, $M(R(X))$ is metrizable. We shall show that $M(k((\Gamma))(X))$ is not metrizable.

Since Γ is divisible, $\mathbb{Q} \subseteq \Gamma$. Fix an increasing sequence of rational numbers (γ_n) converging to 0. Consider a Cantor set given as a family of functions

$$\sigma \in \{0,1\}^{\{\gamma_n | n \in \mathbb{N}\}}.$$

Now define a family of sets U_σ of cardinality $2^{\aleph_0}$ as follows: U_σ contains all $\mathbb{R}$-places determined by cuts of the interval (a^σ, b^σ), where

$$a^\sigma_\delta = \begin{cases} \sigma(\delta) & \delta = \gamma_n \\ -1 & \delta = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad b^\sigma_\delta = \begin{cases} \sigma(\delta) & \delta = \gamma_n \\ +1 & \delta = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Take distinct $\sigma, \tau \in \{0,1\}^{\{\gamma_n | n \in \mathbb{N}\}}$, a cut (A_1, B_1) in (a^σ, b^σ) and a cut (A_2, B_2) in (a^τ, b^τ). We have that $a^\sigma \neq a^\tau$ and we may assume that $a^\sigma < a^\tau$. Then $b^\sigma, a^\tau \in B_1 \cap A_2$. Taking m to be the smallest element in $\mathbb{N}$ such that $\sigma(\gamma_m) \neq \tau(\gamma_m)$, we have that $v(b^\sigma - a^\tau) = \gamma_m < 0 = v(b^\sigma - a^\sigma)$. This shows that $B_1 \cap A_2$ is not a ball, and hence by Theorem 2.2, the $\mathbb{R}$-places determined by orderings of $k((\Gamma))(X)$ associated to the cuts (A_1, B_1) and (A_2, B_2) are distinct. Therefore $U_\sigma \cap U_\tau = \emptyset$ for $\sigma \neq \tau$, and thus the cellularity of $M(k((\Gamma))(X))$ is uncountable.

More generally, take any real closed subfield R' of $k((\Gamma))$. If it is included in a subfield of $k((\Gamma))$ that is of countable transcendence degree over the completion of R, then by Theorem 4.7 $M(R'(X))$ is metrizable. It can be shown that also the converse is true: if the compositum of R' with the completion is of uncountable transcendence degree over the completion, then there are again uncountably many $\sigma \in R'$ that one can use for the above definition of the intervals U_σ.

Finally, let us consider finite extensions of $R(X)$, that is, function fields F of transcendence degree 1 over R. The important fact we will be using is that by the Open Mapping Theorem ([8], Theorem 4.9), the restriction mapping from $\mathcal{X}(F)$ to $\mathcal{X}(R(X))$ is open. (It is easy to see that the analogue for spaces of $\mathbb{R}$-places does not hold: Take a finite extension $F|R(X)$ such that the restriction mapping from $M(F)$ to $M(R(X))$ is not onto. As the restriction is a closed mapping, the image is closed. But as $M(R(X))$ is connected, the image cannot be open.)

Theorem 4.9. *Take a real closed field R that does not admit a countable dense subfield. Further, take a formally real function field F of transcendence degree 1 over R. Then $M(F)$ is not metrizable.*

Proof. Since F is formally real, there is at least one ordering of $R(X)$ which extends to F. So there is a non-empty open set in $\mathcal{X}(F)$. By the Open Mapping Theorem, its image under restriction to $\mathcal{X}(R(X))$ is again open. So it will contain some open interval. For every non-empty open interval of cuts in R one can find elements $b_1 < b_2$ such that $(P_{b_1}^+, P_{b_2}^-)$ is contained in the interval. Pick $a \in R$ such that $b_1 < a < b_2$, and set $\gamma := \max\{v(a - b_1), v(a - b_2)\}$. Now any $\xi \in M(R(X))$ with $v_\xi(X - a) > \gamma$ will correspond to cuts in $(P_{b_1}^+, P_{b_2}^-)$, and all of the corresponding orderings extend to F. Hence if $\xi \in U_{a,\gamma}$, then ξ extends to F.

Now assume that R does not admit a countable dense subfield. Then by Theorem 4.5, $U_{a,\gamma}$ contains uncountably many pairwise disjoint open sets. As the restriction mapping from $M(F)$ to $M(R(X))$ is continuous, the preimages of these open sets are again pairwise disjoint open subsets of $M(F)$. By what we have just shown, they are all non-empty and hence all distinct. This shows that the cellularity of $M(F)$ is uncountable. Hence $M(F)$ is not metrizable. □

It is an open problem whether the converse of this theorem holds.

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