TOPICS IN HIGHER RAMIFICATION THEORY

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Abstract. We introduce and study several notions in the setting of higher ramification theory, in particular ramification ideals and differents. After general results on the computation of ramification ideals, we discuss their connection with defect and compute them for Artin-Schreier extensions and Kummer extensions of prime degree equal to the residue characteristic, with or without defect. We present an example that shows that nontrivial defect in an extension of degree not a pime may not imply the existence of a nonprincipal ramification ideal. We compute differents for the mentioned extensions of prime degree, after computing the necessary traces, and discuss the question when they are equal to the annihilator of the Kähler differentials of the extension. Further, we introduce and study the ideal generated by the differents of the elements of the upper valuation rings in such extensions.

1. Introduction

Higher ramification theory is the theory of valued field extensions $\mathcal{E} = (L|K, v)$ where (K, v) has positive residue characteristic p and is its own **absolute ramification field** (see Section 2.2). The latter means that (K, v) is henselian, its **value group** vK is divisible by all primes different from p, and its **residue field** Kv is separable-algebraically closed. The **absolute Galois group** $\mathrm{Gal}\, K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K, is then a p-group. This implies that every finite Galois extension of K is a tower of Galois extensions of degree p. In **equal characteristic**, i.e., if $\mathrm{char}\, K = \mathrm{char}\, Kv = p$, the latter are **Artin-Schreier extensions**, and in **mixed characteristic**, i.e., if $\mathrm{char}\, K = 0$ and $\mathrm{char}\, Kv = p$, they are **Kummer extensions** because K contains all p-th roots of unity (see Section 3.3).

Since (K, v) is henselian, the extension is **unibranched**, that is, the extension of v from K to L is unique. We will assume this for all extensions that we discuss in the sequel.

Our interest in higher ramification theory owes its existence to the following well known deep open valuation theoretical problems in positive characteristic:

1) local uniformization, the local form of resolution of singularities in arbitrary dimension,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, and of its perfect hull, where q is a power of a prime p.

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Both problems are connected with the structure theory of valued function fields of positive characteristic p. The main obstruction here is the phenomenon of the **defect**, which we define in Section 2.1. For background on the defect and its impact on the above problems, see [10, 11, 12, 13, 15, 17, 21, 18].

Via ramification theory, the study of defect in extensions of arbitrary finite degree can be reduced to the investigation of purely inseparable extensions and of Galois extensions of degree $p = \mathrm{char}\, Kv > 0$. This is explained e.g. in [4, Section 2.1]. Defects of Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree have been classified ("dependent" vs. "independent" defect) first in [14] for the equal characteristic case and then in [23] in general. Theorem 1.4 of [23] presents various criteria for independent defect. These use:

i) The ramification ideal $I_{\mathcal{E}}$, which we define in Section 2.4. Section 3 is then devoted to the computation of ramification ideals. Starting with a first approach described by Ribenboim in [30] we develop more elaborate computations. Of particular interest is the case of extensions that have valuation bases; for this notion, see Section 2.5. Based on this, we treat towers of two Galois extensions where the upper one has a valuation basis, which we need for the example in Section 3.4 below.

In Section 3.2 we discuss the correlation between defect and the existence of nonprincipal ramification ideals. While it is true that a finite Galois extension without defect has only principal ramification ideals, the converse does not hold. We give an example for this phenomenon in Section 3.4 for the equal characteristic case. An example for the mixed characteristic case will be added in a later version of this manuscript.

In Section 3.3 we first compute the unique ramification ideals $I_{\mathcal{E}}$ for Galois extensions $\mathcal{E} = (L|K, v)$ of degree $p = \mathrm{char}\, Kv$ without defect; the results are applied in [5]. We then take a closer look at the unique ramification ideals $I_{\mathcal{E}}$ for Galois extensions $\mathcal{E} = (L|K, v)$ of degree $p = \mathrm{char}\, Kv$ with defect which are computed in [4].

ii) The trace $\mathrm{Tr}_{L|K}$ of the maximal ideal $\mathcal{M}_L$ of the valuation ring $\mathcal{O}_L$ of (L, v). In Section 4.1 we compute the trace $\mathrm{Tr}_{L|K} I$ for arbitrary (possibly fractional) $\mathcal{O}_L$-ideals I. This is then used to compute differents in Section 4.2. The **different** of $(L|K, v)$ is $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) := \mathcal{O}_L :_L \mathcal{C}(\mathcal{O}_L|\mathcal{O}_K)$, where $\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) := (z \in L \mid \mathrm{Tr}\,(z\mathcal{O}_L) \subseteq \mathcal{O}_K)$ is the fractional $\mathcal{O}_L$-ideal called the **complementary ideal** (cf. [35, Ch. V, §11]). The different for Galois extensions $(L|K, v)$ of prime degree with defect is computed in [4], making use of the fact that $vL = vK$ in this case; see Theorem 4.14. For the case without defect it is not computed in [5], so we present the computations here.

If $b \in \mathcal{O}_L$ and h_b is its minimal polynomial over K, then $h_b'(b)$ is called the **different of** b. In Section 4.3 we study the $\mathcal{O}_L$-ideal generated by the differents of the elements in $\mathcal{O}_L \setminus \mathcal{O}_K$, which we call the **naive different ideal**, and compare it to differents and ramification ideals.

iii) The Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$, i.e., the module of relative differentials of the ring extension $\mathcal{O}_L|\mathcal{O}_K$. For Galois extensions of prime degree with defect they are computed in [4], and for the corresponding case without defect in [5]. In both papers, their annihilators are determined using tools from [20]. In Section 4.2 we summarize our results on the annihilators $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ and compare them to the differents $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$. In the classical cases they are equal, but in general this is not true, and we classify the cases where equality holds in Proposition 4.13 and Theorem 4.14.

Finally, we compute the norms $\mathrm{N}_{L|K}$ of the ramification ideals $I_{\mathcal{E}}$ in Section 4.4.

2. Preliminaries

2.1. **The defect.**

For basic facts from valuation theory, see [6], [7], [28], [34], [36].

Take a valued field (K, v). We denote its value group by vK, its residue field by Kv, its valuation ring by $\mathcal{O}_K$, and its maximal ideal by $\mathcal{M}_K$. For $a \in K$, we write va for its value and av for its residue.

A valued field extension $(L|K, v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation. A valued field (K, v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, if all of its algebraic extensions are unibranched.

If $(L|K, v)$ is a finite unibranched extension, then by the Lemma of Ostrowski [36, Corollary to Theorem 25, Section G, p. 78]),

$$[L : K] \;=\; \tilde{p}^{\nu} \cdot (vL : vK)[Lv : Kv]\,, \tag{1}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K, v) := \tilde{p}^{\nu}$ is the **defect** of the extension $(L|K, v)$. We call $(L|K, v)$ a **defect extension** if $d(L|K, v) > 1$, and a **defectless extension** if $d(L|K, v) = 1$. Nontrivial defect only appears when $\operatorname{char} Kv = p > 0$, in which case $\tilde{p} = p$. A henselian field (K, v) is called a **defectless field** if all of its finite extensions are defectless.

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 2.1. *Take a valued field (K, v). If $L|K$ and $M|L$ are finite extensions and the extension of v from K to M is unique, then*

$$\mathrm{d}(M|K, v) = \mathrm{d}(M|L, v) \cdot \mathrm{d}(L|K, v) \tag{2}$$

In particular, $(M|K, v)$ is defectless if and only if $(M|L, v)$ and $(L|K, v)$ are defectless.

Lemma 2.2. *Take a unibranched algebraic extension $(K(a)|K, v)$ and an extension of v from $K(a)$ to the algebraic closure $\tilde{K}$. Denote by (K^h, v) the henselization of (K, v) in $(\tilde{K}, v)$. Then:*

a) $K(a)|K$ is linearly disjoint from $K^h|K$,

b) $(K^h(a)|K^h, v)$ is a defect extension if and only if $(K(a)|K, v)$ is, and

c) $v(a - K^h) = v(a - K)$.

Proof. Our first assertion follows from [2, Lemma 2.1]. For the proof of the second assertion, recall that henselizations are immediate extensions, so we have $vK^h =$

vK and $K^hv = Kv$. Further, we have $K^h(a) = K(a)^h$ since on the one hand, $K^h(a)$ is henselain, being an algebraic extension of K^h, and on the other hand, it contains $K(a)$. Hence, $vK^h(a) = vK(a)$ and $K^h(a)v = K(a)v$. Since $K(a)|K$ is linearly disjoint from $K^h|K$, we also have $[K^h(a) : K^h] = [K(a) : K]$. As an algebraic extension of a henselian field, $(K^h(a)|K^h, v)$ is unibranched. It follows that

$$\begin{aligned} d(K^h(a)|K^h, v) &= [K^h(a) : K^h]/(vK^h(a) : vK^h)[K^h(a)v : K^hv] \\ &= [K(a) : K]/(vK(a) : vK)[K(a)v : Kv] \\ &= d(K(a)|K, v)\,. \end{aligned}$$

This proves our second assertion.

Suppose that $v(a - K^h) \neq v(a - K)$. Since $v(a - K)$ is an initial segment of $vK = vK^h$, this means that there must be some $z \in K^h$ such that $v(a - z) > v(a - K)$. However, as $K(a)|K$ is linearly disjoint from $K^h|K$, we know from [16, Theorem 2] that this cannot be true. This proves our third assertion. □

2.2. **The ramification field.**

In order to reduce the study of arbitrary finite defect extensions to purely inseparable extensions and Galois extensions of degree $p = \operatorname{char} Kv > 0$, we fix an extension of v from K to its algebraic closure $\tilde{K}$. The **absolute ramification field of** (K, v) (with respect to the chosen extension of v), denoted by (K^r, v), is the ramification field of the Galois extension $(K^{\mathrm{sep}}|K, v)$. The **ramification field** of a Galois extension $(L|K, v)$ with Galois group $G = \operatorname{Gal}(L|K)$ is the fixed field in L of the **ramification group**

$$G^r := \left\{ \sigma \in G \;\middle|\; \frac{\sigma b - b}{b} \in \mathcal{M}_L \text{ for all } b \in L^\times \right\}. \tag{3}$$

If $L|K$ is finite and $(L|K, v)$ is a defect extension, then $(L.K^r|K^r, v)$ is a defect extension with the same defect (see [23, Proposition 2.12]). On the other hand, $K^{\mathrm{sep}}|K^r$ is a p-extension (see [14, Lemma 2.7]), so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p. Note that $(K, v) = (K^r, v)$ if and only if (K, v) is henselian, vK is divisible by all primes different from $\operatorname{char} Kv$, and Kv is separable-algebraically closed.

2.3. **Immediate extensions.**

An arbitrary extension $(L|K, v)$ is called **immediate** if $(vL : vK) = 1 = [Lv : Kv]$, i.e., the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto. Throughout this paper, when we talk of a **defect extension** $(L|K, v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. Then it follows from (1) that $[L : K] = p = \operatorname{char} Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$, that is, $(L|K, v)$ is an immediate extension. Let us give more details about immediate extensions.

Lemma 2.3. *Take an arbitrary extension $(L|K, v)$ and $b \in L$. Then there is $c \in K$ such that $v(b - c) > vb$ if and only if $vb \in vK$ and $c'bv \in Kv$ for every $c' \in K$ such that $vc'b = 0$.*

Proof. Assume first that $v(b - c) > vb$. Then $vb = vc \in vK$ and for any $c' \in K$ such that $vc'b = 0$ we have $v(c'b - c'c) > 0$ so that $c'bv = c'cv \in Kv$. Now assume that $vb \in vK$ and $c'bv \in Kv$ for every $c' \in K$ such that $vc'b = 0$. Take $c_1 \in K$ such that $vc_1 = vb$ and set $c' = c_1^{-1}$. Then $vc'b = 0$, hence by assumption, $c'bv \in Kv$. Take $c_2 \in K$ such that $c'bv = c_2 v$, so that $v(c'b - c_2) > 0$. Multiplying with c_1 we obtain $v(b - c_1c_2) > vc_1 = vb$. □

It follows that an extension $(L|K, v)$ is immediate if and only if for all $b \in L$ there is $c \in K$ such that $v(b-c) > vb$. This lays the basis for the proof of the next theorem. For every extension $(L|K, v)$ of valued fields and $a \in L$ we define

$$v(a - K) \;:=\; \{v(a-c) \,|\, c \in K\} \,.$$

The set $v(a - K) \cap vK$ is an initial segment of vK. For more information on its properties, see [19]. For the following theorem, see [8, Theorem 1] and [19, Lemma 2.29].

Theorem 2.4. *If $(L|K, v)$ is an immediate extension of valued fields, then for every element $a \in L \setminus K$ the set $v(a - K)$ is an initial segment of vK without maximal element.*

The following partial converse of this theorem also holds (see [1, Lemma 4.1], cf. also [14, Lemma 2.21]):

Lemma 2.5. *Assume that $(K(a)|K, v)$ is a unibranched extension of prime degree such that $v(a - K)$ has no maximal element. Then the extension $(K(a)|K, v)$ is immediate and hence a defect extension.*

2.4. **Higher ramification groups and ramification ideals.**

Take a valued field extension $\mathcal{E} = (L|K, v)$. Assume that $L|K$ is a Galois extension, and let $G = \mathrm{Gal}\, L|K$ denote its Galois group. We define the **upper series of ramification groups**

$$G_I \;:=\; \left\{ \sigma \in G \;\middle|\; \frac{\sigma b - b}{b} \in I \;\text{ for all } b \in L^\times \right\} , \tag{4}$$

where I runs through all $\mathcal{O}_L$-ideals (cf. [36], §12). Note that $G_{\mathcal{M}_L}$ is the ramification group of $(L|K, v)$. Every G_I is a normal subgroup of G ([36] (d) on p.79). We call G_I a **higher ramification group** if it is a subgroup of $G_{\mathcal{M}_L}$. We call $\mathcal{E}$ a purely wild extension if $\mathrm{Gal}\, L|K = G_{\mathcal{M}_L}$; this matches the (more general) definition of "purely wild extension" in [22].

The function

$$\varphi :\; I \;\mapsto\; G_I \tag{5}$$

preserves $\subseteq$, that is, if $I \subseteq J$, then $G_I \subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the upper series of ramification groups is linearly ordered by inclusion. Note that in general, φ will neither be injective nor surjective as a function to the set of normal subgroups of G. This gives rise to the task to dertermine the smallest ideal that is sent by φ to a group G_I in its image. To this end, we define the $\mathcal{O}_L$-ideals

$$I_H \;:=\; \left(\frac{\sigma b - b}{b} \;\middle|\; \sigma \in H \,,\, b \in L^\times \right) \;=\; \left(\frac{\sigma b}{b} - 1 \;\middle|\; \sigma \in H \,,\, b \in L^\times \right) \tag{6}$$

and consider the function

$$\psi:\ H\ \mapsto\ I_H \tag{7}$$

from the set of all subgroups H of G to the set of all $\mathcal{O}_L$-ideals. Also ψ preserves $\subseteq$ and is in general neither injective nor surjective. However, it is easy to see that $G_{(0)}=\{\mathrm{id}\}$ and $I_{\{\mathrm{id}\}}=(0)$. If I_H is nontrivial and contained in $\mathcal{M}_L$, then we call it a **ramification ideal**. We note:

Proposition 2.6. *1) For every $\mathcal{O}_L$-ideal I, the ramification group G_I is the largest of all subgroups H' of G such that $I_{H'}\subseteq I$.*

2) For every subgroup H of G, the ideal I_H is the smallest of all $\mathcal{O}_L$-ideals I' such that $H\subseteq G_{I'}$.

3) If $I=I_H$ for some subgroup H of G, then $I_{G_I}=I$. If $H=G_I$ for some $\mathcal{O}_L$-ideal I, then $G_{I_H}=H$. Hence φ is an inclusion preserving bijection from the set of all $\mathcal{O}_L$-ideals onto the set of all ramification groups, with ψ its inverse.

4) The function φ induces an inclusion preserving bijection from the set of all ramification ideals onto the set of all nontrivial higher ramification groups, with its inverse induced by ψ.

5) A subgroup H of G is a higher ramification group if and only if it is a subgroup of $G_{\mathcal{M}_L}$ and for every subgroup H' of G we have $H\subsetneq H'\Rightarrow I_H\subsetneq I_{H'}$.

6) An $\mathcal{O}_L$-ideal I is a ramification ideal if and only if it is nontrivial and contained in $\mathcal{M}_L$ and for every $\mathcal{O}_L$-ideal I' we have $I'\subsetneq I\Rightarrow G_{I'}\subsetneq G_I$.

7) If $\mathcal{E}=(L|K,v)$ is a nontrivial purely wild Galois extension, then I_G is its largest ramification ideal. If in addition $\mathcal{E}$ is of prime degree, then I_G is its unique ramification ideal.

Proof. 1) and 2) follow directly from the definitions of G_I and I_H.

3): If $I=I_H$, then it follows from part 1) that $H\subseteq G_I$. Thus $I=I_H\subseteq I_{G_I}\subseteq I$, so $I_{G_I}=I$. If $H=G_I$, then it follows from part 2) that $I_H\subseteq I$. Thus $H\subseteq G_{I_H}\subseteq G_I=H$, so $G_{I_H}=H$.

4): If I_H is a ramification ideal, then I_H is nonzero and contained in $\mathcal{M}_L$, hence by part 3), $H=G_{I_H}\subseteq G_{\mathcal{M}_L}$ which is a nontrivial higher ramification group. This shows that φ sends ramification ideals to nontrivial higher ramification groups.

If G_I is a higher ramification group, then $G_I\subseteq G_{\mathcal{M}_L}$, hence again by part 3), $I=I_{G_I}\subseteq I_{G_{\mathcal{M}_L}}=\mathcal{M}_L$, and if G_I is nontrivial, then $I=I_{G_I}$ is nonzero. This shows that ψ sends nontrivial higher ramification groups to ramification ideals. Now the assertion of part 4) follows from part 3).

5): It suffices to show that H is a ramification group if and only if it is a subgroup of $G_{\mathcal{M}_L}$ and for every subgroup H' of G we have $H\subsetneq H'\Rightarrow I_H\subsetneq I_{H'}$.

Assume first that H is a ramification group, and take an $\mathcal{O}_L$-ideal I such that $H=G_I$. Take a subgroup H' of G which properly contains G_I. Then by part 1), $I_H=I_{G_I}\subsetneq I_{H'}$.

For the converse, assume that H is a subgroup of G such that for every subgroup H' of G we have $H\subsetneq H'\Rightarrow I_H\subsetneq I_{H'}$. By part 1), G_{I_H} is the largest of all subgroups H' of G such that $I_{H'}\subseteq I_H$. Therefore $G_{I_H}=H$, which shows that H is a ramification group.

6): It suffices to show that there is a subgroup H of G such that $I = I_H$ if and only if for every $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I \Rightarrow G_{I'} \subsetneq G_I$.

Assume first that $I = I_H$. Take an $\mathcal{O}_L$-ideal I' properly contained in I_H. Then by parts 2) and 3), $G_{I'} \subsetneq H = G_{I_H} = G_I$.

For the converse, assume that I is an $\mathcal{O}_L$-ideal such that for every $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I \Rightarrow G_{I'} \subsetneq G_I$. By part 2), I_{G_I} is the smallest of all $\mathcal{O}_L$-ideals I' such that $G_I \subseteq G_{I'}$. Therefore $I = I_{G_I}$.

7): Since $\mathcal{E}$ is nontrivial, also G is nontrivial, which by definition of I_G implies that $I_G \neq (0)$. Since $G = G_{\mathcal{M}_L}$, we have $I_G \subseteq \mathcal{M}_L$. Thus I_G is a ramification ideal. As ψ preserves inclusion, I_G is the largest ramification ideal of $\mathcal{E}$.

If in addition $\mathcal{E}$ is of prime degree, then the only subgroups of G are G and $\{\mathrm{id}\}$. Since $I_{\{\mathrm{id}\}} = (0)$ is not a ramification ideal, I_G is then the unique ramification ideal of $\mathcal{E}$. □

The function

$$v:\ I \mapsto \Sigma_I := \{vb \mid 0 \neq b \in I\} \tag{8}$$

is an order preserving bijection from the set of all nontrivial, possibly fractional, ideals of $\mathcal{O}_L$ onto the set of all nonempty final segments of vL. This set is again linearly ordered by inclusion, and the function (8) is order preserving: $J \subseteq I$ holds if and only if $\Sigma_J \subseteq \Sigma_I$ holds. The inverse of the above function is the order preserving function

$$\Sigma \mapsto I_\Sigma := \{a \in L \mid va \in \Sigma\} \cup \{0\}\,. \tag{9}$$

Now the ramification groups can be represented as

$$G_\Sigma := G_{I_\Sigma} = \left\{ \sigma \in G \;\middle|\; v\,\frac{\sigma b - b}{b} \in \Sigma \cup \{\infty\} \text{ for all } b \in L^\times \right\},$$

where Σ runs through all (possibly empty) final segments of $(vL)^{>0}$.

Like the function (5), also the function $\Sigma \mapsto G_\Sigma$ is in general neither injective nor surjective. We call a nonempty final segment Σ of $(vL)^{>0}$ a **ramification jump** if and only if

$$\Sigma' \subsetneq \Sigma \Rightarrow G_{\Sigma'} \subsetneq G_\Sigma$$

for every final segment Σ' of $(vL)^{>0}$.

Proposition 2.7. *Take a nontrivial purely wild Galois extension $\mathcal{E} = (L|K, v)$. Then a nonempty final segment Σ of $(vL)^{>0}$ is a ramification jump if and only if I_Σ is a ramification ideal.*

Proof. First note that for every nonempty final segment Σ of $(vL)^{>0}$ the ideal I_Σ is nontrivial, and contained in $\mathcal{M}_L$ by our assumption on $\mathcal{E}$. Now a nonempty final segment Σ of $(vL)^{>0}$ is a ramification jump if and only if for every nonempty final segment Σ' of $(vL)^{>0}$ we have $\Sigma' \subsetneq \Sigma \Rightarrow G_{I_{\Sigma'}} \subsetneq G_{I_\Sigma}$. This holds if and only if for every nontrivial $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I_\Sigma \Rightarrow G_{I'} \subsetneq G_{I_\Sigma}$. By Proposition 2.6, this in turn holds if and only if I_Σ is a ramification ideal. □

By Propositions 2.6 and 2.7, the number of ramification ideals and ramification jumps in a purely wild Galois extension is bounded by the number of nontrivial subgroups of its Galois group. It may not always be equal to this number, as an example given in Section 3.4 below will show.

In this paper we are particularly interested in the case where $\mathcal{E} = (L|K, v)$ is a purely wild Galois extension of prime degree p. Then by Lemma 2.6, $\mathcal{E}$ has the unique ramification ideal I_G, and we denote it by $I_{\mathcal{E}}$. Hence $\Sigma_{\mathcal{E}} := \Sigma_{I_{\mathcal{E}}}$ is the unique ramification jump of $\mathcal{E}$. As we will show in the next section, ramification jump and ramification ideal carry important information about the extension $(L|K, v)$.

Remark 2.8. In [23] we also included empty final segments in the definitions of the function (8). However, classically ramification jumps have always been defined as integers in the case of discrete valuations, and as real numbers in the case of valuations of rank one, and the intended meaning of "jump" does not fit well with the value $v0 = \infty$. #

Further, we want to quickly discuss the **lower series of ramification groups**

$$G_I^l := \{\sigma \in G \mid \sigma b - b \in I \text{ for all } b \in \mathcal{O}_L\} \tag{10}$$

(see [36], \$12). Again, for every ideal I of $\mathcal{O}_L$, G_I^l is a normal subgroup of G ([36] (d) on p.79), and $G_I \subseteq G_I^l$. But in the case of an immediate extension $(L|K, v)$, the two groups coincide, as follows from the next, more general, result:

Lemma 2.9. *If $vL = vK$, then $G_I = G_I^l$ for all nontrivial ideals I of $\mathcal{O}_L$ contained in $\mathcal{M}_L$.*

Proof. It suffices to show that $G_I^l \subseteq G_I$. Take $\sigma \in G_I^l$ and $f \in \mathcal{O}_L \setminus \{0\}$. Since $vL = vK$, we can pick some $c \in K$ such that $vcb = 0$. As $\sigma \in G_I^l$, we have that $\sigma(cb) - cb \in I$. Since $vcb = 0$, it follows that

$$\frac{\sigma b - b}{b} = \frac{\sigma(cb) - cb}{cb} \in I .$$

This shows that $\sigma \in G_I$. □

2.5. Valuation bases.

Take a an extension $(L|K, v)$. The elements $b_1, \ldots, b_n \in L$ are called **valuation independent** (over K) if for all choices of $c_1, \ldots, c_n \in K$,

$$v \sum_{i=1}^{n} c_i b_i = \min_i vc_i b_i .$$

If in addition these elements form a basis of $L|K$, then they are called a **valuation basis** of $(L|K, v)$. If the valuation basis contains 1, we will speak of a **valuation basis with 1**.

Recall that $(L|K, v)$ is defectless if it satisfies the fundamental equality $[L : K] = \mathrm{e} \cdot \mathrm{f}$, where $\mathrm{e} = (vL : vK)$ is the ramification index and $\mathrm{f} = [Lv : Kv]$ is the inertia degree. In this case, $(L|K, v)$ admits a **standard valuation basis**, which we construct as follows: we take $y_1, \ldots, y_{\mathrm{e}} \in L$ such that $vy_1 + vK, \ldots, vy_{\mathrm{e}} + vK$ are the cosets of vK in vL, and $z_1, \ldots, z_{\mathrm{f}} \in L$ such that $z_1 v, \ldots, z_{\mathrm{f}} v$ are a basis of $Lv|Kv$. Then the products $y_i z_j$, $1 \leq i \leq \mathrm{e}$, $1 \leq j \leq \mathrm{f}$, form a valuation basis of $(L|K, v)$ (see [7, Lemma 3.2.2]). Note that we can always choose $y_1 = z_1 = 1$ so that $y_1 z_1 = 1$. We will then speak of a **standard valuation basis with 1**.

The next result has been shown in the proof of [14, Lemma 2.1].

Lemma 2.10. *Take an extension $(L|K,v)$ of prime degree p. If for $b \in L$, either $vb \notin vK$ or there is some $c \in K$ such that $vcb = 0$ and $cbv \notin Kv$, then $1, b, \ldots, b^{p-1}$ forms a standard valuation basis with 1 of $(L|K,v)$.*

For the following, cf. [3, Proposition 3.4].

Lemma 2.11. *Take a finite unibranched extension $(L|K,v)$. Then the following are equivalent:*

a) is defectless,
b) $(L|K,v)$ admits a valuation basis,
c) $(L|K,v)$ admits a standard valuation basis,
d) $(L|K,v)$ admits a standard valuation basis with 1.

Proof. Implication a)⇒d) has just been shown above. Implications d)⇒c) and c)⇒b) are trivial. For the implication b)⇒a), see the proof of [3, Proposition 3.4]. □

In particular, for a finite unibranched defectless extension there is always a valuation basis with 1.

Lemma 2.12. *Take a finite unibranched defectless extension $(L|K,v)$ and $a \in L$. Then the set $\{v(a-c) \mid c \in K\}$ has a maximum. More precisely, if we choose a valuation basis $b_1 = 1, b_2, \ldots, b_n$ for $(L|K,v)$ and write*

$$a \;=\; \sum_{i=1}^{n} c_i b_i \;,$$

then $v(a-c_1)$ is the maximum of $\{v(a-c) \mid c \in K\}$.

Proof. For every $c \in K$,

$$\begin{aligned} v(a-c_1) &= v\sum_{i=2}^{n} c_i b_i \;=\; \min_{2\le i\le n} vc_ib_i \;\ge\; \min\{v(c_1-c)\,,\, vc_ib_i \mid 2\le i\le n\} \\ &= v\left(c_1 - c + \sum_{i=2}^{n} c_i b_i\right) \;=\; v(a-c)\;. \end{aligned}$$

□

Corollary 2.13. *Take a unibranched defectless extension $(L|K,v)$ of prime degree and $a_0 \in L$. Then there is some $c \in K$ such that for $a = a_0 - c$, the elements $1, a, \ldots, a^{p-1}$ form a valuation basis.*

Proof. By Lemma 2.12 there is some $c \in K$ such that $v(a_0-c) = \max\{v(a_0-c) \mid c \in K\}$. By Lemma 2.3 this can only happen if either $v(a_0-c) \notin vK$ or there is some $d \in K$ such that $vd(a_0-c) = 0$ and $d(a_0-c)v \notin Kv$. We set $a = a_0 - c$; then in both cases, the elements $1, a, \ldots, a^{p-1}$ form a valuation basis by Lemma 2.10. □

For a more general setting, see Lemma 2.10 and Corollary 2.11 of [3].

3. Computation of ramification ideals

3.1. Basic computations.

Proposition 3.1. *Take a finite unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ with Galois group G. Then every ramification ideal is principal.*

Take a nontrivial subgroup H of G. We will prove the proposition by giving an algorithm for the computation of an element $b_{\min}$ such that for some $\sigma \in H$,

$$v\left(\frac{\sigma b_{\min}}{b_{\min}} - 1\right) \;=\; \min\left\{ v\left(\frac{\sigma b}{b} - 1\right) \,\middle|\, b \in L^\times \,,\, \sigma \in H \right\}\,, \tag{11}$$

which means that $\frac{\sigma}{b_{\min}} - 1$ generates the ramification ideal (6).

Remark 3.2. This proposition was proven in 1970 by P. Ribenboim in [30]. Ribenboim assumes that (L,v) has rank 1, that is, vL is archimedean ordered. Our computations presented below are inspired by his. As they will show, the assumption of rank 1 is not necessary.

A different version of the computation was presented by M. Marshall in [26]. He does not assume that (L,v) has rank 1, but that (K,v) is maximally complete and that the extension $Lv|Kv$ is separable. Because of the latter assumption, $v\left(\frac{\sigma b_j}{b_j} - 1\right) = 0$ for all j and $\sigma \neq \mathrm{id}$ and therefore, the elements b_j are not needed in the computation. The assumption that (K,v) is maximally complete means that it has no nontrivial immediate extensions, and this implies that (K,v) is defectless and henselian.

In [29] Ribenboim attempts to prove Proposition 3.1 for all non-discrete valuations and all finite unibranched Galois extensions, but this is false. (We will present counterexamples below.) Ribenboim's mistake was noticed by J. L. Chabert. In [30] Ribenboim then gives a correct proof of Proposition 3.1 for all finite defectless unibranched Galois extensions in the case of rank one valuations. #

We shall now present computations that will not only prove the above proposition, but will also be used later for more advanced results. Let us start with some useful basic principles.

Lemma 3.3. *Let K be any field and take and $\sigma \in \mathrm{Gal}\, K^{\mathrm{sep}}|K$.*
1) For all $a,b \in K^{\mathrm{sep}}$ and $c \in K$,

$$\frac{\sigma cab}{cab} - 1 \;=\; \frac{\sigma ab}{ab} - 1 \;=\; \left(\frac{\sigma a}{a} - 1\right)\left(\frac{\sigma b}{b} - 1\right) \;+\; \left(\frac{\sigma a}{a} - 1\right) \;+\; \left(\frac{\sigma b}{b} - 1\right)\,. \tag{12}$$

2) Assume that v is a valuation on K^{sep} and that $a \in K^{\mathrm{sep}}$ is such that $v\left(\frac{\sigma a}{a} - 1\right) > 0$. Take $i \in \mathbb{N}$ and assume that $i < \mathrm{char}\, K$ if $\mathrm{char}\, K > 0$. Then

$$v\left(\frac{\sigma a^i}{a^i} - 1\right) \;=\; v\left(\frac{\sigma a}{a} - 1\right)\,. \tag{13}$$

Proof. 1): We leave the straightforward proof to the reader.
2): By our assumption on i, we have $vi = 0$. Using this together with equation (12), one proves equation (13) by induction on i. □

Further, we will need the following fact.

Lemma 3.4. *Take a valued field extension $(L|L_0, v)$ and pick elements $a_1, \ldots, a_n \in L_0$. Assume that the elements $b_1, \ldots, b_n \in L$ are valuation independent over L_0 and set*

$$b = \sum_{i=1}^{n} a_i b_i \,. \tag{14}$$

Then for each embedding $\sigma : L \to \tilde{L}$,

$$v\left(\frac{\sigma b}{b} - 1\right) \geq \min_i v\left(\frac{\sigma a_i b_i}{a_i b_i} - 1\right) . \tag{15}$$

If in addition σ is trivial on all b_i, then $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_i}{a_i} - 1$.

Proof. We have

$$\frac{\sigma b}{b} - 1 = \sum_i \left(\frac{\sigma a_i b_i}{a_i b_i} - 1\right) \cdot \frac{a_i b_i}{b} \,. \tag{16}$$

Since $vb \leq va_i b_i$ for $1 \leq i \leq n$, this implies (15) and that $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_i b_i}{a_i b_i} - 1$. If in addition $\sigma b_i = b_i$, then $\frac{\sigma a_i b_i}{a_i b_i} - 1 = \frac{\sigma a_i}{a_i} - 1$. □

We note that if $(L|K, v)$ is a unibranched Galois extension, then for every $\sigma \in \operatorname{Gal} L|K$ and $b \in L^\times$,

$$\frac{\sigma b}{b} - 1 \in \mathcal{O}_L \,. \tag{17}$$

Lemma 3.5. *Assume that $(L|K, v)$ is is a finite purely wild Galois extension. Then for every $\sigma \in \operatorname{Gal} L|K$ and all $a, b \in L^\times$,*

$$\frac{\sigma b}{b} - 1 \in \mathcal{M}_L \tag{18}$$

and

$$v\left(\frac{\sigma cab}{cab} - 1\right) \geq \min\left\{ v\left(\frac{\sigma a}{a} - 1\right), v\left(\frac{\sigma b}{b} - 1\right) \right\} , \tag{19}$$

with equality holding if $v\left(\frac{\sigma a}{a} - 1\right) \neq v\left(\frac{\sigma b}{b} - 1\right)$.

Proof. Equation (18) holds since by the definition of "purely wild extension", $\operatorname{Gal} L|K = G_{\mathcal{M}_L}$. Equation (19) follows from equation (18). □

Proposition 3.6. *Assume that $\mathcal{E} = (L|K, v)$ is a finite unibranched Galois extension with Galois group G.*

1) Assume that $\mathcal{E}$ is defectless and choose a valuation basis b_i, $1 \leq i \leq n$. Set

$$\gamma := \min\left\{ v\left(\frac{\sigma b_i}{b_i} - 1\right) \,\middle|\, 1 \leq i \leq n\,,\, \sigma \in G \right\} . \tag{20}$$

Then $\gamma \geq 0$ *and*

$$\gamma \;=\; \min\left\{ v\left(\frac{\sigma b}{b}-1\right) \,\middle|\, b\in L^\times\,,\ \sigma\in G\right\}\;. \tag{21}$$

Hence $b_{\min}$ *can be chosen to be* b_i *for suitable* i.

2) Assume in addition that $\mathcal{E}$ *is purely wild and choose a standard valuation basis* y_iz_j , $1\leq i\leq e$, $1\leq j\leq f$ *of* $(L|K,v)$ *as described in Section 2.5. Then (21) holds for*

$$\gamma \;:=\; \min\left\{ v\left(\frac{\sigma y_i}{y_i}-1\right),\, v\left(\frac{\sigma z_j}{z_j}-1\right) \,\middle|\, 1\leq i\leq \mathrm{e}\,,\ 1\leq j\leq \mathrm{f}\,,\ \sigma\in G\right\}\;. \tag{22}$$

3) Assume in addition that $\mathcal{E}$ *is purely wild and that* L_0 *is an intermediate field of* $\mathcal{E}$ *such that* $\mathcal{E}_1=(L|L_0,v)$ *is defectless and that* b_i , $1\leq i\leq n$ *is a valuation basis of* $(L|L_0,v)$. *Define* γ *as in (20). Assume further that there is* $\gamma_0\in vL$ *such that*

$$v\left(\frac{\sigma a}{a}-1\right) \;\geq\; \gamma_0 \quad \textit{for all } a\in L_0^\times \textit{ and } \sigma\in G\,. \tag{23}$$

Then

$$v\left(\frac{\sigma b}{b}-1\right) \;\geq\; \min\{\gamma_0\,,\,\gamma\} \quad \textit{for all } b\in L^\times \textit{ and } \sigma\in G\,. \tag{24}$$

If "$>$" holds in (23) and $\gamma\leq\gamma_0$, *then*

$$\gamma \;=\; \min\left\{ v\left(\frac{\sigma b}{b}-1\right) \,\middle|\, b\in L^\times,\ \sigma\in G\right\}\;. \tag{25}$$

Proof. 1): It follows from (17) that $\gamma_{\mathcal{E}}\geq 0$. We apply Lemma 3.4 with $L_0=K$, so $\sigma a=a$ for $a\in L_0$. Take $b\in L$ and write it in the form (14). Then (15) reads as

$$v\left(\frac{\sigma b}{b}-1\right) \;\geq\; \min_i v\left(\frac{\sigma b_i}{b_i}-1\right)\;.$$

This proves (21).

2): Take $b\in L$ and write it in the form $b=\sum_{i,j} c_{i\,j}y_iz_j$. Part 1) together with (19) shows that for all $b\in K$,

$$\begin{aligned} v\left(\frac{\sigma b}{b}-1\right) \;&\geq\; \min\left\{ v\left(\frac{\sigma y_iz_j}{y_iz_j}-1\right) \,\middle|\, 1\leq i\leq \mathrm{e}\,,\ 1\leq j\leq \mathrm{f},\ \sigma\in G\right\}\\ &=\; \min\left\{ v\left(\frac{\sigma y_i}{y_i}-1\right),\, v\left(\frac{\sigma z_j}{z_j}-1\right) \,\middle|\, 1\leq i\leq \mathrm{e}\,,\ 1\leq j\leq \mathrm{f},\ \sigma\in G\right\}, \end{aligned}$$

which proves our assertion.

3): Take $b\in L$ and write it in the form (14). Using (15) together with (19), we obtain:

$$\begin{aligned} v\left(\frac{\sigma b}{b}-1\right) \;&\geq\; \min\left\{ v\left(\frac{\sigma a_ib_i}{a_ib_i}-1\right) \,\middle|\, a_i\in L_0^\times\,,\ 1\leq i\leq n,\ \sigma\in G\right\}\\ &=\; \min\left\{ v\left(\frac{\sigma a_i}{a_i}-1\right),\, v\left(\frac{\sigma b_i}{b_i}-1\right) \,\middle|\, a_i\in L_0^\times\,,\ 1\leq i\leq n,\ \sigma\in G\right\}\\ &=\; \min\{\gamma_0\,,\,\gamma\}\,, \end{aligned}$$

which proves (24).

Now assume in addition that " $>$ " holds in (23) and that $\gamma \leq \gamma_0$. Then

$$v\left(\frac{\sigma a}{a} - 1\right) > \gamma$$

for all $\sigma \in G$ and $a \in L_0^\times$. Together with (24) and the definition of γ, this implies (25). □

Proof of Proposition 3.1:
With γ as in part 1) of Proposition 3.6, equation (21) yields $I_\mathcal{E} = (a \in L \mid va \geq \gamma)$, which is principal.

Now take any nontrivial subgroup H of G and denote its fixed field in L by K'. Then also $L|K'$ is a finite unibranched Galois extension, by Lemma 2.1 it is again defectless, and its Galois group is H. Hence by what we have just shown, also I_H is principal. This proves our proposition. □

Finally, we prove a generalization of a fact that has been used in [32, Section 7.1]. For information on tame and purely wild extensions, see [17, 22].

Proposition 3.7. *Take a henselian field (K, v), a finite purely wild Galois extension $(L|K, v)$ and a tame extension $(K'|K, v)$. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K', v)$ is a purely wild Galois extension of degree $[L : K]$, and*

$$I \mapsto I\mathcal{O}_{L'} \tag{26}$$

is a bijection between the ramification ideals of $(L|K, v)$ and those of $(L'|K', v)$.

Proof. The extensions $L|K$ and $K'|K$ are linearly disjoint and therefore, $L'|K'$ is a Galois extension with its Galois group G isomorphic to the Galois group of $L|K$ via the restriction of its elements to L. Every finite subextension $(K'_0|K, v)$ is again tame, and so is $(L'_0|L, v)$ for the field compositum $L'_0 = L.K'_0$. Hence the extension $(L'_0|L, v)$ admits a valuation basis $b_1, \ldots, b_n$.

Each element $b \in L'$ already lies in the compositum $L'_0 = L.K'_0$ for a finite subextension $K'_0|K$ of $K'|K$, so it can be written as $b = \sum_{1 \leq i \leq n} a_i b_i$ with $b_1, \ldots, b_n$ a valuation basis of $(L'_0|L, v)$ and suitable elements $a_i \in L$. Hence by Lemma 3.4 with L'_0 in place of L and L in place of L_0 , $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_{L'_0}$-ideal generated by the elements $\frac{\sigma a_i}{a_i} - 1$.

Now take a ramification ideal $I = I_H$ of $(L|K, v)$ where H is a nontrivial subgroup of G. If $b \in L'$ is written as above and $\sigma \in H$, then since $\frac{\sigma a_i}{a_i} - 1 \in I_H$, we obtain that

$$\frac{\sigma b}{b} - 1 \in I_H\mathcal{O}_{L'_0} \subseteq I_H\mathcal{O}_{L'} . \tag{27}$$

This shows that the ramification ideal I'_H of $(L'|K', v)$ is a subset of $I_H\mathcal{O}_{L'}$. On the other hand, since $L \subseteq L'$ it is immediate from the definition that $I_H \subseteq I'_H$. Thus,

$$I_H\mathcal{O}_{L'} = I'_H .$$

This proves that the function (26) sends ramification ideals of $(L|K, v)$ to ramification ideals of $(L'|K', v)$. It also shows that I'_H is the collection of all elements in L' whose value is not less than the value of some element in I_H . This implies that $I'_H \cap \mathcal{O}_L$ is the collection of all elements in L whose value is not less than the value of some element in I_H. In other words, $I'_H \cap \mathcal{O}_L = I_H\mathcal{O}_L = I_H$. Hence, $I_H\mathcal{O}_{L'} \cap \mathcal{O}_L = I_H$, which proves that the function (26) is a bijection. □

Remark 3.8. In [32, Section 7.1] only the special case is considered where (K,v) is a henselian field of mixed characteristic, $L|K$ has prime degree p and $K' = K(\zeta_p)$ where ζ_p is a p-th root of unity. The latter implies that $(K'|K,v)$ is a tame extension. This case is of interest when $L|K$, though being Galois, is not a Kummer extension, since $L'|K'$ will be a Kummer extension. #

With a proof adapted from the one of the previous proposition, the following can be shown:

Proposition 3.9. *Take a henselian field (K,v), a finite immediate Galois extension $(L|K,v)$ and an extension $(K'|K,v)$ for which every finite subextension is defectless. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K',v)$ is an immediate Galois extension of degree $[L:K]$, and (26) is again a bijection between the ramification ideals of $(L|K,v)$ and those of $(L'|K',v)$.* □

3.2. **Ramification ideals and defect.**

Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p with Galois group G. For every $\sigma \in G \setminus \{\mathrm{id}\}$ we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma b - b}{b}\right) \,\middle|\, b \in L^\times \right\} . \tag{28}$$

The next theorem follows from [23, Theorems 3.4 and 3.5] together with Theorem 2.4.

Theorem 3.10. *For every generator $a \in L$ of $\mathcal{E}$ and every $\sigma \in G \setminus \{\mathrm{id}\}$,*

$$\Sigma_\sigma = -v(a-K) + v(a-\sigma a) , \tag{29}$$

and this set is a final segment of $vK^{>0} = \{\alpha \in vK \mid \alpha > 0\}$ without a smallest element. Moreover, Σ_σ does not depend on the choice of $\sigma \in G \setminus \{\mathrm{id}\}$, and G is the unique ramification group of $\mathcal{E}$.

Our theorem shows that for every Galois defect extension of prime degree, the set (29) is independent of the choice of a and σ, so we denote it by $\Sigma_{\mathcal{E}}$.

Corollary 3.11. *In the situation of Theorem 3.10, the unique ramification ideal of $\mathcal{E} = (L|K,v)$ is the nonprincipal ideal*

$$I_{\mathcal{E}} := I_{\Sigma_{\mathcal{E}}} = \left(\frac{\sigma_0\, a - a}{a-c} \mid c \in K\right) = \left(\frac{\sigma_0(a-c)}{a-c} - 1 \mid c \in K\right) , \tag{30}$$

where σ_0 is any generator of G and a is any generator of $L|K$.

Proof. This follows from Theorem 3.10. Since $\Sigma_{\mathcal{E}}$ has no smallest element, showing that $I_{\Sigma_{\mathcal{E}}}$ does not contain an element of smallest value and is thus nonprincipal. □

In what follows, let $(L|K,v)$ be a finite unibranched Galois extension. Denote its ramification field ("Verzweigungskörper" in German) by V. Assuming that $V \neq L$, we wish to investigate the ramification ideals of the Galois extension $(L|V,v)$. Since $\mathrm{Gal}\, L|V$ is a p-group, $L|V$ is a tower

$$V = K_0 \subset \ldots \subset K_n = L \tag{31}$$

of Galois extensions of degree p such that each extension $K_i|V$ is again a p-extension, $1 \le i \le n$. By the multiplicativity of the defect, $(L|K,v)$ is a defect extension if and only if at least one extension of degree p in the tower is a defect extension.

Proposition 3.12. *If the extension $(L|K,v)$ is such that for some $i \le n$ the extension $(K_i|K_{i-1},v)$ in the tower (31) is not defectless, then the smallest ramification ideal of $(K_i|K,v)$ is nonprincipal. In particular, if $(K_n|K_{n-1},v)$ is not defectless, then the smallest ramification ideal of $(L|K,v)$ is nonprincipal.*

Proof. After replacing K_i by L if necessary, it suffices to prove the second assertion. Set $H = \mathrm{Gal}\, K_n|K_{n-1} \subseteq \mathrm{Gal}\, L|K$. We know that I_H is a ramification ideal of $(L|K,v)$. It is the smallest since H has no nontrivial subgroup. As it is at the same time the unique ramification ideal of the extension $(K_n|K_{n-1},v)$ by part 7) of Proposition 2.6, we know from Corollary 3.11 that it is nonprincipal. □

Theorem 3.13. *Take a finite unibranched Galois extension $(L|K,v)$. The extension is defectless if and only if for every Galois subextension $(L'|K,v)$ every ramification ideal is principal.*

Proof. First assume that $(L|K,v)$ is defectless. Then by the multiplicativity of the defect, also every Galois subextension is defectless, and it is again unibranched. Hence by Proposition 3.1, each of its ramification ideals is principal.

Now assume that $(L|K,v)$ is not defectless. Then at least one of the extensions $(K_i|K_{i-1},v)$ in the tower (31) is not defectless. Setting $L' = K_i$, we obtain that $L'|K$ is a Galois extension, and we can infer from Proposition 3.12 that not every ramification ideal of $(L'|K,v)$ is principal. □

Proposition 3.12 and Theorem 3.13 are best possible, as shown by Proposition 3.20 below.

3.3. **Unibranched Galois extensions of prime degree.**

A Galois extension of degree p of a field K of characteristic $p > 0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** $X^p - X - c$ with $c \in K$. A Galois extension of degree p of a field K of characteristic 0 which contains all p-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^p \in K$. For these facts, see [25, Chapter VIII, §8].

If $(L|K,v)$ is a Galois defect extension of degree p of fields of characteristic 0, then a Kummer generator of $L|K$ can be chosen to be a 1-unit. Indeed, choose any Kummer generator η. Since $(L|K,v)$ is immediate, we have that $v\eta \in vK(\eta) = vK$, so there is $c \in K$ such that $vc = -v\eta$. Then $v\eta c = 0$, and since $\eta cv \in K(\eta)v = Kv$, there is $d \in K$ such that $dv = (\eta cv)^{-1}$. Then $v(\eta cd) = 0$ and $(\eta cd)v = 1$. Hence ηcd is a 1-unit. Furthermore, $K(\eta cd) = K(\eta)$ and $(\eta cd)^p = \eta^p c^p d^p \in K$. Thus we can replace η by ηcd and assume from the start that η is a 1-unit. It follows that also $\eta^p \in K$ is a 1-unit.

Throughout this article, whenever we speak of "Artin-Schreier extension" we refer to fields of positive characteristic, and with "Kummer extension" we refer to fields of characteristic 0.

3.3.1. *The defectless case.*

The following proposition is taken from [4]. For the convenience of the reader, and as an illustration of the usefulness of Lemma 2.12, we include its proof here.

Proposition 3.14. *1) Take a valued field (K,v) of equal positive characteristic p and a unibranched defectless Artin-Schreier extension $(L|K,v)$.*

If $\mathrm{f}\,(L|K,v)=p$, then the extension has an Artin-Schreier generator ϑ of value $v\vartheta\le 0$ such that $Lv=Kv(\tilde{c}\vartheta v)$ for every $\tilde{c}\in K$ with $v\tilde{c}\vartheta=0$; the extension $Lv|Kv$ is separable if and only if $v\vartheta=0$.

If $\mathrm{e}\,(L|K,v)=p$, then the extension has an Artin-Schreier generator ϑ such that $vL=vK+\mathbb{Z}v\vartheta$. Every such ϑ satisfies $v\vartheta<0$.

2) Take a valued field (K,v) of mixed characteristic and a unibranched defectless Kummer extension $(L|K,v)$ of degree $p=\mathrm{char}\,Kv$. Then the extension has a Kummer generator η such that:

a) if $\mathrm{f}\,(L|K,v)=p$, then either ηv generates the residue field extension, in which case it is inseparable, or η is a 1-unit and for some $\tilde{c}\in K$, $\tilde{c}(\eta-1)v$ generates the residue field extension;

b) if $\mathrm{e}\,(L|K,v)=p$, then either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta-1)$ generates the value group extension.

Proof. 1): Take any Artin-Schreier generator y of $(L|K,v)$. Then by Lemma 2.12 there is $c\in K$ such that either $v(y-c)\notin vK$, or for every $\tilde{c}\in K$ such that $v\tilde{c}(x-c)=0$ we have $\tilde{c}(y-c)v\notin Kv$. Since p is prime, in the first case it follows that $\mathrm{e}\,(L|K,v)=p$ and that $v(y-c)$ generates the value group extension. In the second case it follows that $\mathrm{f}\,(L|K,v)=p$ and that $\tilde{c}(y-c)v$ generates the residue field extension. In both cases, $\vartheta=y-c$ is an Artin-Schreier generator. Let $\vartheta^p-\vartheta=b\in K$.

Assume that $\mathrm{f}\,(L|K,v)=p$. If $v\vartheta<0$, then $v(\vartheta^p-b)=v\vartheta>pv\vartheta=v\vartheta^p$, whence $v((\tilde{c}\vartheta)^p-\tilde{c}^pb)=v\tilde{c}^p\vartheta>v(\tilde{c}\vartheta)^p$ for $\tilde{c}\in K$ with $v\tilde{c}\vartheta=0$ and therefore, $(\tilde{c}\vartheta)^pv=\tilde{c}^pbv\in Kv$. In this case, the residue field extension is inseparable. Now assume that $v\vartheta\ge 0$ and hence also $vb\ge 0$. The reduction of X^p-X-b to $Kv[X]$ is a separable polynomial, so $Lv|Kv$ is separable. The polynomial X^p-X-bv cannot have a zero in Kv, since otherwise the p distinct roots of this polynomial give rise to p distinct extensions of v from K to L, contradicting our assumption that $(L|K,v)$ is unibranched. Consequently, $bv\ne 0$, whence $vb=0$ and $v\vartheta=0$.

Assume that $\mathrm{e}\,(L|K,v)=p$. If $v\vartheta\ge 0$, then $vb\ge 0$ and ϑv is a root of X^p-X-bv. If this polynomial does not have a zero in Kv, then ϑv generates a nontrivial residue field extension, contradicting our assumption that $\mathrm{e}\,(L|K,v)=p$. If the polynomial has a zero in Kv, then similarly as before one deduces that $(L|K,v)$ is not unibranched, contradiction. Hence $v\vartheta<0$.

2): Take any Kummer generator y of $(L|K,v)$. If there is a Kummer generator η such that $v\eta\notin vK$, then it follows as before that $\mathrm{e}\,(L|K,v)=p$ and that $v\eta$ generates the value group extension. Now assume that there is no such η.

If there is a Kummer generator y and some $\tilde{c} \in K$ such that $v\tilde{c}y = 0$ and $\tilde{c}yv \notin Kv$, then it follows as before that $\mathrm{f}(L|K,v) = p$ and that $\tilde{c}yv$ generates the residue field extension. We set $\eta = \tilde{c}y$ and observe that also η is a Kummer generator. Since $(\eta v)^p \in Kv$, $Lv|Kv$ is purely inseparable in this case.

Now assume that the above cases do not appear, and choose an arbitrary Kummer generator y of $(L|K,v)$. Consequently, we have that $vy \in vK$ and $\tilde{c}yv \in Kv$ for all $\tilde{c} \in K$ with $v\tilde{c}y = 0$. Then as described at the start of this section, there are $c_1, c_2 \in K$ such that c_2c_1y is a Kummer generator of $(L|K,v)$ which is a 1-unit. We replace y by c_2c_1y.

By Lemma 2.12 there is $c \in K$ such that $v(y-c)$ is maximal in $v(y-K)$ and either $v(y-c) \notin vK$ or there is some $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since y is a 1-unit, we know that $v(y-1) > 0$, hence also $v(y-c) > 0 = vy$, showing that also c is a 1-unit. Then $\eta := c^{-1}y$ is again a Kummer generator of $(L|K,v)$ which is a 1-unit. Since $vc = 0$, we know that $v(\eta - 1) = vc(\eta - 1) = v(y-c)$. Hence if $v(y-c) \notin vK$, then $v(\eta - 1)$ generates the value group extension.

Now assume that there is $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since c is a 1-unit, it follows that $v\tilde{c}(\eta - 1) = v\tilde{c}c(\eta - 1) = v\tilde{c}(y-c) = 0$ and $\tilde{c}(\eta - 1)v = \tilde{c}c(\eta - 1)v = \tilde{c}(y-c)v$. We find that $\tilde{c}(\eta - 1)v$ generates the residue field extension. □

From this proposition we deduce:

Theorem 3.15. *Take a unibranched defectless Galois extension $(L|K,v)$ of prime degree p.*

1) If $\mathcal{E} = (L|K,v)$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis for $(L|K,v)$. The element $b_{\min}$ as in (11) can be chosen to be ϑ, so that

$$I_{\mathcal{E}} = \left(\frac{1}{\vartheta}\right) . \tag{32}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v\vartheta = 0$, and this holds if and only if $Lv|Kv$ is separable of degree p.

2) Let $\mathcal{E} = (L|K,v)$ be a Kummer extension. Then there are two cases:

a) $(L|K,v)$ admits a Kummer generator η such that $v\eta \geq 0$ and $1, \eta, \ldots, \eta^{p-1}$ form a valuation basis for $(L|K,v)$. In this case, $b_{\min}$ can be chosen to be η and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$ and

$$I_{\mathcal{E}} = (\zeta_p - 1) . \tag{33}$$

b) $(L|K,v)$ admits a Kummer generator η such that η is a 1-unit with $v(\eta - 1) \leq v(\zeta_p - 1)$ and $1, \eta - 1, \ldots, (\eta - 1)^{p-1}$ is a valuation basis for $(L|K,v)$. In this case, $b_{\min}$ can be chosen to be $\eta - 1$ and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1) - v(\eta - 1)$ and

$$I_{\mathcal{E}} = \left(\frac{\zeta_p - 1}{\eta - 1}\right) . \tag{34}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v(\eta - 1) = v(\zeta_p - 1)$, and this holds if and only if $Lv|Kv$ is separable of degree p.

Proof. Throughout the proof we use part 1) of Proposition 3.6,

1): By part 1) of Proposition 3.14 there exists an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $v\vartheta$ generates the value group extension, or $v\tilde{c}\vartheta = 0$ and $Lv = Kv(\tilde{c}\vartheta v)$ for some $\tilde{c} \in K$. By Lemma 2.10, it follows that $1, \vartheta, \ldots, \vartheta^{p-1}$ is a valuation basis for $(L|K, v)$.

If $v\vartheta < 0$, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{\sigma\vartheta - \vartheta}{\vartheta}\right) = -v\vartheta = v\left(\frac{1}{\vartheta}\right) > 0 \tag{35}$$

for every $\sigma \in \operatorname{Gal} L|K \setminus \{\mathrm{id}\}$ since then $\sigma\vartheta - \vartheta \in \mathbb{F}_p \setminus \{0\}$. Hence by Lemma 3.3, for $1 \leq j \leq p-1$ we have

$$v\left(\frac{\sigma\vartheta^j}{\vartheta^j} - 1\right) = v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{1}{\vartheta}\right) .$$

This proves that $b_{\min}$ can be chosen to be ϑ in this case.

If $v\vartheta = 0$, which by part 1) of Proposition 3.14 holds if and only if $Lv|Kv$ is separable of degree p, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{1}{\vartheta}\right) = 0 ,$$

and as the value γ defined in (20) is non-negative, this is equivalent to $I_{\mathcal{E}} = \mathcal{O}_L$.

2): By part 2) of Proposition 3.14 there exists a Kummer generator η such that either

a) $v\eta$ generates the value group extension, or ηv generates the residue field extension, or

b) η is a 1-unit and $v(\eta - 1)$ generates the value group extension or for some $\tilde{c} \in K$, $\tilde{c}(\eta - 1)v$ generates the residue field extension.

We first consider case a). By Lemma 2.10, it follows that $1, \eta, \ldots, \eta^{p-1}$ is a valuation basis for $(L|K, v)$. If $v\eta$ generates the value group extension, we can assume that $v\eta \geq 0$ because if $v\eta$ generates the value group extension, then so does $v\eta^{-1}$. For $1 \leq j \leq p-1$,

$$v\left(\frac{\sigma\eta^j}{\eta^j} - 1\right) = v\left(\frac{\sigma\eta^j - \eta^j}{\eta^j}\right) = v\left(\frac{\zeta_p^k\eta^j - \eta^j}{\eta^j}\right) = v(\zeta_p^k - 1) = v(\zeta_p - 1)$$

for some $k \in \mathbb{N}$; the last equation holds since $v(\zeta - 1) = vp/(p-1)$ for every primitive p-th root of unity (cf. [4, Lemma 2.5]). This proves that in case a), $b_{\min}$ can be chosen to be η and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$.

Now we consider case b). Again by Lemma 2.10, $1, \eta - 1, \ldots, (\eta - 1)^{p-1}$ is a valuation basis for $(L|K, v)$. Since $v\eta = 0$, we have

$$v\left(\frac{\sigma\eta - 1}{\eta - 1} - 1\right) = v\left(\frac{\sigma\eta - \eta}{\eta - 1}\right) = v(\zeta_p - 1) - v(\eta - 1) .$$

This value must be non-negative since it is not less than $\gamma_{\mathcal{E}}$. If it is equal to 0, then it must be equal to $\gamma_{\mathcal{E}}$. If it is positive, then we can apply Lemma 3.3, obtaining that for $1 \le j \le p-1$,

$$v\left(\frac{\sigma(\eta-1)^j}{(\eta-1)^j}-1\right) \;=\; v\left(\frac{\sigma\eta-1}{\eta-1}-1\right)$$

and consequently, this value is again equal to $\gamma_{\mathcal{E}}$. Hence in case b), $b_{\min}$ can be chosen to be $\eta-1$ and we have $\gamma_{\mathcal{E}} = v(\zeta_p-1)-v(\eta-1)$. We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if the ramification field of $(L|K,v)$ is equal to L, which means that p does not divide $\mathrm{e}\,(L|K,v)$ and $Lv|Kv$ must be separable. Since $(L|K,v)$ is assumed to be unibranched and defectless of degree p, this can only hold if and only if $Lv|Kv$ is separable of degree p. □

Remark 3.16. Equation (34) also holds in case 2 a) of the previous theorem since in this case, $v(\eta-1)=0$. Indeed, in that case we have $v\eta \ge 0$, and $1,\eta,\dots,\eta^{p-1}$ form a valuation basis for $(L|K,v)$. If $v\eta>0$, then $v(\eta-1)=0$. If $v\eta=0$, then $1,\eta v,\dots,(\eta v)^{p-1}$ form a basis of $Lv|Kv$, so $\eta v \ne 1$, whence $v(\eta-1)=0$ again. #

3.3.2. *The defect case.*

The next results follow from Corollary 3.11 and are part of [23, Theorems 3.4 and 3.5].

Theorem 3.17. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree with Galois group G. If $(L|K,v)$ is an Artin-Schreier defect extension with any Artin-Schreier generator ϑ, then*

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta-K)\,. \tag{36}$$

If K contains a primitive root of unity ζ_p and $(L|K,v)$ is a Kummer extension with Kummer generator η of value 0, then

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p-1) \;-\; v(\eta-K) \;=\; \frac{vp}{p-1} \;-\; v(\eta-K)\,. \tag{37}$$

Theorem 3.18. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree p.*

1) If $(L|K,v)$ is an Artin-Schreier extension with Artin-Schreier generator ϑ, then

$$\begin{aligned} I_{\mathcal{E}} \;&=\; \left(\frac{1}{\vartheta-c} \mid c\in K\right)\\ &=\; \left(\frac{1}{b} \mid b \text{ an Artin-Schreier generator of } L|K\right)\,. \end{aligned}$$

2) Let $(L|K,v)$ be a Kummer extension with a Kummer generator η which is a 1-unit, and ζ_p a primitive p-th root of unity. Then

$$\begin{aligned} I_{\mathcal{E}} \;&=\; \left(\frac{\zeta_p-1}{\eta-c} \mid c\in K \text{ a 1-unit}\right)\\ &=\; \left(\frac{\zeta_p-1}{b-1} \mid b \text{ a Kummer generator of } L|K \text{ which is a 1-unit}\right)\,. \end{aligned}$$

Proof. 1): The first equation follows from equation (30) of Corollary 3.11, where we take σ_0 such that $\sigma_0\vartheta=\vartheta+1$. The ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation because $\vartheta-c$

is again an Artin-Schreier generator for every $c \in K$. Further, by Corollary 3.11 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the second equation follows from the first.

2): The first equation follows from equation (30) of Corollary 3.11, where we take σ_0 such that $\sigma_0\eta = \zeta_p\eta$, because then $\sigma_0(\eta - c) - (\eta - c) = (\zeta_p - 1)\eta$ and we can drop η since it is a unit. Further, we can restrict c to 1-units since if c is not a 1-unit, then $v(\eta - c) \leq 0 < v(\eta - 1)$ and $\frac{\zeta_p-1}{\eta-c} \in \left(\frac{\zeta_p-1}{\eta-1}\right)$.

When c is a 1-unit, then $\eta - c = c(\frac{\eta}{c} - 1)$, the quotient $b = \frac{\eta}{c}$ is again a Kummer generator which is a 1-unit, and we can drop the unit factor c. This shows that the ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation. Further, by Corollary 3.11 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the second equation again follows from the first. □

3.4. **An example.**

We are going to give an example of a Galois defect extension $(L|K, v)$ of degree p^2, $p = \operatorname{char} K > 0$, which is a tower of two Galois extensions of degree p, the upper one defectless and the lower a defect extension, but has only one ramification ideal, this being principal.

We will construct a tower of two Galois extensions $L|L_0$ and $L_0|K$ of degree $p = \operatorname{char} K$. We need a criterion for $L|K$ to be Galois. We set $\wp(X) := X^p - X$. The following is Lemma 2.9 in [27]:

Lemma 3.19. *Take Artin-Schreier extensions $L|L_0$ and $L_0|K$, and an Artin-Schreier generator ϑ of $L|L_0$ with $\vartheta^p - \vartheta = b \in L_0$. Then $L|K$ is a Galois extension if and only if $\sigma_0 b - b \in \wp(L_0)$ for some generator σ_0 of* $\operatorname{Gal} L_0|K$.

Consider the rational function field $\widetilde{F_p}(t)$ with the t-adic valuation $v = v_t$. Extend v to its algebraic closure and let $K_0 = \widetilde{F_p}(t)^r$ be the respective ramification field. Then vK_0 is a subgroup of $\mathbb{Q}$ divisible by each prime other than p, but vt is not divisible by p in vK_0. Choose a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ in vK_0 with upper bound $-1/p$ and starting with $q_1 = -1$. Define

$$s := \sum_{i\in\mathbb{N}} t^{pq_i} \in \widetilde{F_p}((t^{\mathbb{Q}})) .$$

Take (K, v) to be the henselization of $(K_0(s), v)$.

Let ϑ_0 be a root of the Artin-Schreier polynomial $X^p - X - s$. Define

$$c_k := \sum_{i=1}^{k} t^{q_i} \in K .$$

We compute:

$$v(\vartheta_0 - c_k)^p = v(\vartheta_0^p - c_k^p) = v(\vartheta_0 + s - c_k^p) = \min\{v\vartheta_0, v(s - c_k^p)\} .$$

Since $vs = -pvt < 0$, we have $v\vartheta_0 = -vt$. Further, $v(s - c_k^p) = pq_{k+1}vt < -vt$ since $q_{k+1} < -1/p$. It follows that $v(\vartheta_0 - c_k)^p = pq_{k+1}vt$, so that $v(\vartheta_0 - c_k) = q_{k+1}vt$. This increasing sequence of values is contained in $v(\vartheta_0 - K)$. It must be cofinal, showing that $v(\vartheta_0 - K)$ has no maximal element, because the pseudo Cauchy sequence $(c_k)_{k\in\mathbb{N}}$ has no limit in (K,v). It thus follows from Lemma 2.5 that for $L_0 := K(\vartheta_0)$, the extension $\mathcal{E}_0 := (L_0|K,v)$ is immediate and thus a defect extension. From Theorem 3.17 we know that

$$vI_{\mathcal{E}_0} \;=\; -v(\vartheta_0 - K)\;,$$

which has no minimal element and lower bound $\gamma := vt/p \notin vI_{\mathcal{E}_0}$. Hence $I_{\mathcal{E}_0}$ is nonprincipal. However, we will construct the extension $(L|K,v)$ such that $I_{\mathcal{E}_0}$ is not a ramification ideal of it.

Let ϑ be a root of the Artin-Schreier polynomial $X^p - X - \vartheta_0$, and set $L := L_0(\vartheta) = K(\vartheta_0,\vartheta)$. Since $v\vartheta_0 = -vt < 0$, We have $v\vartheta = -vt/p \notin vK = vL_0$. Hence by Corollary 2.13, the elements $1,\vartheta,\ldots,\vartheta^{p-1}$ form a valuation basis of $\mathcal{E}_1 := (L|K(\vartheta_0),v)$, showing that this extension is defectless. By part 1) of Theorem 3.15,

$$I_{\mathcal{E}_1} \;=\; \left(\frac{1}{\vartheta}\right)\;,$$

so the minimum of $vI_{\mathcal{E}_1}$ is $-v\vartheta = vt/p = \gamma$, which is smaller than the values of all elements of $vI_{\mathcal{E}_0}$.

Since $\vartheta^p - \vartheta = \vartheta_0$, we have $L = K(\vartheta)$. To show that $L|K$ is a Galois extension, take some generator σ_0 of $\mathrm{Gal}\,L_0|K$. Since $\sigma_0\vartheta_0$ is also a root of $X^p - X - s$, we have $\sigma_0\vartheta_0 - \vartheta_0 = i$ for some $i \in \mathbb{F}_p$. As K contains $\widetilde{\mathbb{F}_p}$, it contains the Artin-Schreier roots of i, i.e., $i \in \wp(K) \subseteq \wp(L_0)$. Now Lemma 3.19 shows that $L|K$ is a Galois extension. However, by Corollary 2.10 of [27] it is not cyclic, and the discussion leading up to this corollary shows the following. Take $\sigma \in G = \mathrm{Gal}\,L|K$ such that $\sigma\vartheta_0 - \vartheta_0 = 1$. Then $\zeta := \sigma\vartheta - \vartheta$ satisfies $\zeta^p - \zeta = 1$ and is therefore an elment of $\widetilde{\mathbb{F}_p} \subset K_0$. Further, take $\tau \in G$ such that $\tau\vartheta - \vartheta = 1$. Then τ is trivial on L_0 and σ and τ commute. Thus the subgroups of G of order p are generated by the automorphisms τ and $\sigma\tau^i$, $0 \le i \le p-1$.

Let us first consider the subgroup $\langle\tau\rangle$ of G. Since $\langle\tau\rangle = \mathrm{Gal}\,L|L_0$, the ramification ideal $I_{\langle\tau\rangle}$ is the ramification ideal $I_{\mathcal{E}_1}$ of the extension $\mathcal{E}_1$.

Let us now consider the subgroups $\langle\sigma\tau^i\rangle$ of G, for $0 \le i \le p-1$. Since τ is trivial on L_0, the restrictions of all elements of each subgroup $\langle\sigma\tau^i\rangle$ form the Galois group of $\mathcal{E}_0$. Therefore,

$$v\left(\frac{\rho a}{a} - 1\right) \;>\; \gamma \quad \text{for all } a \in L_0^\times \text{ and } \rho \in \mathrm{Gal}\,\mathcal{E}_0\,. \tag{38}$$

For $1 \le k \le p-1$ we have $(\sigma\tau^i)^k = \sigma^k\tau^{ik}$ and

$$\sigma^k\tau^{ik}\vartheta - \vartheta \;=\; k\zeta + ik \;\in\; \widetilde{\mathbb{F}_p}\;,$$

hence $v(\sigma^k\tau^{ik}\vartheta - \vartheta) = 0$ and

$$v\left(\frac{\sigma^k\tau^{ik}\vartheta}{\vartheta} - 1\right) \;=\; -v\vartheta \;=\; \gamma\,.$$

Applying part 2) of Lemma 3.3, we find that for $1 \le \ell \le p-1$,

$$v\left(\frac{\sigma^k\tau^{ik}\vartheta^\ell}{\vartheta^\ell} - 1\right) \;=\; \gamma\,. \tag{39}$$

Now we can apply part 3) of Proposition 3.6 to deduce that (25) holds with $\langle\sigma\tau^i\rangle$ in place of G. This shows that also the ramification ideals $I_{\langle\sigma\tau^i\rangle}$ are equal to $I_{\mathcal{E}_1}$.

Finally, since $\mathrm{Gal}\, L|K$ is the union of all subgroups listed above, it follows that (25) also holds for $G = \mathrm{Gal}\, L|K$. Hence, $I_G = I_{\mathcal{E}_1}$. We have now proved:

Proposition 3.20. *There are Galois extensions of degree p^2 of valued fields in equal characteristic p that have only one ramification group, and this ramification group is principal although the extension is not defectless.*

4. Traces, differents, and norms

4.1. **Traces.**

Traces are used in the definition of differents. For their computation in the next section, we will do the necessary computations of traces in this section.

For a finite Galois extension $L|K$ we will denote the trace by $\mathrm{Tr}_{L|K}$. In what follows we abbreviate it by Tr when the extension $L|K$ is fixed. When L carries a valuation v, we denote by $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$ the $\mathcal{O}_L$-ideal generated by $\mathrm{Tr}\,(\mathcal{O}_L)$. In [4, Lemma 5.5] the following is proven:

Lemma 4.1. *Take an extension $(L|K,v)$ of valued fields with $vL = vK$, an $\mathcal{O}_L$-ideal I, and $n \in \mathbb{N}_{>0}$. Then the $\mathcal{O}_L$-ideal J generated by $(I \cap K)^n$ equals I^n.*

The proof of the following fact can be found in [9, Section 6.3].

Lemma 4.2. *Take a separable field extension $K(a)|K$ of degree n and let $f(X) \in K[X]$ be the minimal polynomial of a over K. Then*

$$\mathrm{Tr}_{K(a)|K}\left(\frac{a^m}{f'(a)}\right) = \begin{cases} 0 & \text{if } 1 \leq m \leq n-2 \\ 1 & \text{if } m = n-1\,. \end{cases} \tag{40}$$

□

4.1.1. *The defectless case.*

The following facts will be used for the computation of differents in the next section. For certain assertions we will need the assumption that $vK^{<0}$ is cofinal in $vL^{<0}$. This holds if and only if $vK^{>0}$ is coinitial in $vL^{>0}$, and this in turn holds if and only if either vK is densely ordered or vL and vK have the same smallest positive elements.

Proposition 4.3. *Take a unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ of prime degree p.*

1) Assume that $\mathcal{E}$ is an Artin-Schreier extension with Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis for $(L|K,v)$. Then for every $z \in \mathcal{O}_L$,

$$\mathrm{Tr}\,(z\mathcal{O}_L) = \{b \in K \mid vb \geq vz - (p-1)v\vartheta\}\,. \tag{41}$$

We have $\mathrm{Tr}\,(z\mathcal{O}_L) \subseteq \mathcal{O}_K$ if $vz \geq (p-1)v\vartheta$, and if $vK^{<0}$ is cofinal in $vL^{<0}$, then

$$\mathrm{Tr}\,(z\mathcal{O}_L) \subseteq \mathcal{O}_K \Leftrightarrow vz \geq (p-1)v\vartheta\,. \tag{42}$$

2) Assume that $\mathcal{E}$ is a Kummer extension. Then there are two cases:

a) $(L|K,v)$ admits a Kummer generator η such that $v\eta \geq 0$ and $1, \eta, \ldots, \eta^{p-1}$ form a valuation basis for $(L|K,v)$. In this case, for every $z \in \mathcal{O}_L$,

$$\mathrm{Tr}\,(z\mathcal{O}_L) = \{b \in K \mid vb \geq vz + vp\}\,. \tag{43}$$

We have $\mathrm{Tr}\,(z\mathcal{O}_L)\subseteq\mathcal{O}_K$ *if* $vz\geq -vp$, *and if* $vK^{<0}$ *is cofinal in* $vL^{<0}$, *then*

(44) $$\mathrm{Tr}\,(z\mathcal{O}_L)\;\subseteq\;\mathcal{O}_K\;\Leftrightarrow\;vz\geq -vp\,.$$

b) $(L|K,v)$ *admits a Kummer generator* η *such that* η *is a* 1*-unit with* $v(\eta-1)\leq v(\zeta_p-1)$ *and* $1,\eta-1,\ldots,(\eta-1)^{p-1}$ *is a valuation basis for* $(L|K,v)$*. In this case, for every* $z\in\mathcal{O}_L$ *,*

(45) $$\mathrm{Tr}\,(z\mathcal{O}_L)\;=\;\{b\in K\mid vb\;\geq\;vz+vp-(p-1)v(\eta-1)\}\,.$$

We have $\mathrm{Tr}\,(z\mathcal{O}_L)\subseteq\mathcal{O}_K$ *if* $vz\geq -vp+(p-1)v(\eta-1)$, *and if* $vK^{<0}$ *is cofinal in* $vL^{<0}$, *then*

(46) $$\mathrm{Tr}\,(z\mathcal{O}_L)\;\subseteq\;\mathcal{O}_K\;\Leftrightarrow\;vz\geq -vp+(p-1)v(\eta-1)\,.$$

4) In all cases,

(47) $$\mathrm{Tr}\,(z\mathcal{O}_L)\;=\;zI_{\mathcal{E}}^{p-1}\cap K\,.$$

Proof. 1): Take $a\in L$ and write $a=\sum_{i=0}^{p-1}c_i\vartheta^i$ with $c_i\in K$. Then $\mathrm{Tr}\,a=-c_{p-1}$ by Lemma 4.2. We have

$$a\in z\mathcal{O}_L\;\Leftrightarrow\;va\geq vz\;\Leftrightarrow\;\forall i:\;vc_i\vartheta^i\geq vz\;\Leftrightarrow\;\forall i:\;vc_i\geq vz-iv\vartheta\,.$$

Hence if $a\in z\mathcal{O}_L$, then $v\mathrm{Tr}\,a=vc_{p-1}\geq vz-(p-1)v\vartheta$, which proves the inclusion "$\subseteq$" in (41). To prove the converse inclusion, take some $b\in K$ such that $vb\geq vz-(p-1)v\vartheta$. Set $a=-b\vartheta^{p-1}$ so that $b=\mathrm{Tr}\,a$. As $va=vb+(p-1)\vartheta\geq vz$, we have $a\in z\mathcal{O}_L$. This proves the inclusion "$\supseteq$" in (41).

Assume that $vz\geq(p-1)v\vartheta$, i.e., $vz-(p-1)v\vartheta\geq 0$. Then it follows from (41) that $\mathrm{Tr}\,(z\mathcal{O}_L)\subseteq\mathcal{O}_K$. To prove the converse, assume that $vK^{<0}$ is cofinal in $vL^{<0}$ and that $vz-(p-1)v\vartheta<0$. Then there is $b\in K$ such that $vz-(p-1)v\vartheta\leq b<0$. Then by (41), $b\in\mathrm{Tr}\,(z\mathcal{O}_L)$, but $b\notin\mathcal{O}_K$.

2)a): Take $a\in L$ and write $a=\sum_{i=0}^{p-1}c_i\eta^i$ with $c_i\in K$. Since $(\eta^i)^p\in K$, we have

(48) $$\mathrm{Tr}\,_{K(\eta)|K}(\eta^i)\;=\;0$$

for $1\leq i\leq p-1$. This implies that $\mathrm{Tr}\,a=pc_0$. Hence if $a\in z\mathcal{O}_L$, then $v\mathrm{Tr}\,a=vp+vc_0\geq vp+vz$, which proves the inclusion "$\subseteq$" in (43). To prove the converse inclusion, take some $b\in K$ such that $vb\geq vp+vz$. Set $a=p^{-1}b$ so that $b=\mathrm{Tr}\,a$. As $va=vb-vp\geq vz$, we have $a\in z\mathcal{O}_L$. This proves the inclusion "$\supseteq$" in (43).

Assume that $vz\geq -vp$, i.e., $vz+vp\geq 0$. Then it follows from (43) that $\mathrm{Tr}\,(z\mathcal{O}_L)\subseteq\mathcal{O}_K$. To prove the converse, assume that $vK^{<0}$ is cofinal in $vL^{<0}$ and that $vz+vp<0$. Then there is $b\in K$ such that $vz+vp\leq b<0$. Then by (43), $b\in\mathrm{Tr}\,(z\mathcal{O}_L)$, but $b\notin\mathcal{O}_K$. This proves (44).

2)b): Take $a\in L$ and write $a=\sum_{i=0}^{p-1}c_i(\eta-1)^i$ with $c_i\in K$. We compute:

$$(\eta-1)^i\;=\;\sum_{j=1}^{i}\binom{i}{j}\eta^j(-1)^{i-j}\;+\;(-1)^i\,.$$

Thus by (48), for every $c\in K$,

(49) $$\mathrm{Tr}\,_{K(\eta)|K}(c(\eta-1)^i)\;=\;pc(-1)^i\,.$$

Hence,

$$\mathrm{Tr}\, a \;=\; \sum_{i=0}^{p-1} pc_i(-1)^i \,. \tag{50}$$

We have that

$$c_i(\eta-1)^i \;\in\; z\mathcal{O}_L \;\Leftrightarrow\; vc_i \geq vz - iv(\eta-1)\,. \tag{51}$$

If $a\in z\mathcal{O}_L$, then $vc_i\geq vz-iv(\eta-1)$ for $0\leq i\leq p-1$, hence by (50) and (51),

$$v\mathrm{Tr}\, a \;=\; vp\sum_{i=0}^{p-1} c_i(-1)^i \;\geq\; vp+vz-iv(\eta-1) \;\geq\; vp+vz-(p-1)v(\eta-1)\,,$$

where the last inequality holds because $v(\eta-1)>0$. This proves the inclusion "$\subseteq$" in (45). To show the converse inclusion, take some $b\in K$ such that $vb\geq vz+vp-(p-1)v(\eta-1))$. Set $a=-\frac{b}{p}(\eta-1)^{p-1}$ so that $b=\mathrm{Tr}\, a$. As $va=vb-vp+(p-1)(\eta-1)\geq vz$, we have $a\in z\mathcal{O}_L$. This proves the inclusion "$\supseteq$" in (45).

Assume that $vz\geq -vp+(p-1)v(\eta-1)$, i.e., $vz+vp-(p-1)v\eta\geq 0$. Then it follows from (45) that $\mathrm{Tr}\,(z\mathcal{O}_L)\subseteq\mathcal{O}_K$. To prove the converse, assume that $vK^{<0}$ is cofinal in $vL^{<0}$ and that $vz+vp-(p-1)v(\eta-1)<0$. Then there is $b\in K$ such that $vz+vp-(p-1)v(\eta-1)\leq b<0$. Then by (41), $b\in\mathrm{Tr}\,(z\mathcal{O}_L)$, but $b\notin\mathcal{O}_K$. This proves (46).

4): In case 1), $-(p-1)v\vartheta$ is the minimal value of $I_{\mathcal{E}}^{p-1}$ by part 1) of Theorem 3.15. In cae 2)a), $vp=(p-1)v(\zeta_p-1)$ is the minimal value of $I_{\mathcal{E}}^{p-1}$ by part 2)a) of Theorem 3.15. In cae 2)b), $vp-(p-1)v(\eta-1)=(p-1)(v(\zeta_p-1)-v(\eta-1))$ is the minimal value of $I_{\mathcal{E}}^{p-1}$ by part 2)b) of Theorem 3.15. □

Remark 4.4. Assume that $vK^{<0}$ is not cofinal in $vL^{<0}$. Then since vL is contained in the divisible hull of vK as $L|K$ is algebraic, it follows that vK has a smallest positive element π_K which is not equal to the smallest positive element π_L of vL. Since

$$\{b\in K\mid vb\;\geq\;0\}\;=\;\{b\in K\mid vb\;>\;-\pi_K\}$$

we obtain from the previous proposition:

$$\mathrm{Tr}\,(z\mathcal{O}_L)\;\subseteq\;\mathcal{O}_K\;\Leftrightarrow\;vz>(p-1)v\vartheta-\pi_K\;\Leftrightarrow\;vz\geq(p-1)v\vartheta-\pi_K+\pi_L\,.$$

Similarly, in case 2)a),

$$\mathrm{Tr}\,(z\mathcal{O}_L)\;\subseteq\;\mathcal{O}_K\;\Leftrightarrow\;vz>-vp-\pi_K\;\Leftrightarrow\;vz\geq -vp-\pi_K+\pi_L\,,$$

and in case 2)b),

$$\begin{aligned}\mathrm{Tr}\,(z\mathcal{O}_L)\;\subseteq\;\mathcal{O}_K\;&\Leftrightarrow\;vz>-vp+(p-1)v(\vartheta-1)-\pi_K\\ &\Leftrightarrow\;vz\geq -vp+(p-1)v(\vartheta-1)-\pi_K+\pi_L\,.\end{aligned}$$

#

As an immediate application of Proposition 4.3, we obtain:

Proposition 4.5. *Take any (possibly fractional) $\mathcal{O}_L$-ideal I. Under the assumptions of Proposition 4.3, we have*

$$\operatorname{Tr} I \;=\; I_{\mathcal{E}}^{p-1} I \cap K\,.$$

In particular,

$$\operatorname{Tr} \mathcal{M}_L \;=\; I_{\mathcal{E}}^{p-1} \mathcal{M}_L \cap K\,,$$

In case 1) of Proposition 4.3, this is equal to $\mathcal{M}_K$ if $v\vartheta = 0$. In case 2)a) of Proposition 4.3, this is equal to $p\mathcal{M}_L = \mathcal{M}_K$.

From this, taking $I = \mathcal{O}_L$, together with Lemma 4.1, we obtain:

Corollary 4.6. *If $\mathcal{E} = (L|K,v)$ is a unibranched defectless Galois extension of prime degree p with $vL = vK$, then*

$$\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) \;=\; I_{\mathcal{E}}^{p-1}\,. \tag{52}$$

4.1.2. *The defect case.*

In [4, Theorem 1.5 and Lemma 5.5] the following is proven:

Theorem 4.7. *Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree $p =$ $\operatorname{char} Kv$. If $\operatorname{char} K = 0$, then assume that K contains all p-th roots of unity. Then*

$$\operatorname{Tr}\,(\mathcal{O}_L) \;=\; \operatorname{Tr}\,(\mathcal{M}_L) \;=\; (b \in K \mid vb \in (p-1)\Sigma_{\mathcal{E}}) \;=\; (I_{\mathcal{E}} \cap K)^{p-1} \tag{53}$$

and $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$,

4.2. **Differents.**

Throughout this section, we assume that $\mathcal{E} = (L|K,v)$ a unibranched Galois extension of prime degree $p = \operatorname{char} Kv$ and if $\operatorname{char} K = 0$, then K contains all p-th roots of unity.

4.2.1. *The defectless case.*

In this subsection, we assume in addition that $\mathcal{E}$ is defectless.

Proposition 4.8. *Assume first that $vK^{<0}$ is cofinal in $vL^{<0}$. Then*

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) \;=\; (\vartheta)^{p-1} \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \left(\frac{1}{\vartheta}\right)^{p-1} \;=\; I_{\mathcal{E}}^{p-1}$$

in case 1) of Proposition 4.3,

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \left(\frac{1}{p}\right) \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; (p) \;=\; I_{\mathcal{E}}^{p-1}$$

in case 2)a) of Proposition 4.3, and

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \left(\frac{\eta-1}{\zeta_p-1}\right)^{p-1} \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \left(\frac{\zeta_p-1}{\eta-1}\right)^{p-1} \;=\; I_{\mathcal{E}}^{p-1}$$

in case 2)b) of Proposition 4.3.

Now assume that $vK^{<0}$ is not cofinal in $vL^{<0}$ and that $\pi_K \in K$ and $\pi_L \in L$ such that $v\pi_K$ is the smallest positive element in vK and $v\pi_L$ is the smallest positive element in vL. Then

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \frac{\pi_L}{\pi_K}(\vartheta)^{p-1} \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \frac{\pi_K}{\pi_L}\left(\frac{1}{\vartheta}\right)^{p-1} \;=\; \frac{\pi_K}{\pi_L} I_{\mathcal{E}}^{p-1}$$

in case 1) of Proposition 4.3,

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \frac{\pi_L}{\pi_K}\left(\frac{1}{p}\right) \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; \frac{\pi_K}{\pi_L}(p) \;=\; \frac{\pi_K}{\pi_L} I_{\mathcal{E}}^{p-1}$$

in case 2)a) of Proposition 4.3, and

$$\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) = \frac{\pi_L}{\pi_K}\left(\frac{\eta-1}{\zeta_p-1}\right)^{p-1} \quad \textit{and} \quad \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \frac{\pi_K}{\pi_L}\left(\frac{\zeta_p-1}{\eta-1}\right)^{p-1} = \frac{\pi_K}{\pi_L} I_{\mathcal{E}}^{p-1}$$

in case 2)b) of Proposition 4.3.

Proof. All results for $\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K)$ follow from Proposition 4.3 and Remark 4.4. The results for $\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K)$ follow since if $I = (a)$ is principal, then $\mathcal{O}_L :_L I = (a^{-1})$. □

Corollary 4.9. *Under the above assumptions, if in addition $vL = vK$, then*

$$\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \;=\; I_{\mathcal{E}}^{p-1} \;=\; \operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}\,.$$

Proof. This follows from Proposition 4.8 together with [5, Theorems 4.4 and 4.6]. □

To treat the case of unibranched defectless Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree p with $vL \neq vK$, we need some more preparation. In this case, Theorem 3.2 of [5] tells us that there is $x \in L$ such that

$$\mathcal{O}_L \;=\; \bigcup_{c\in K \text{ with } vcx>0} \mathcal{O}_K[cx]\,. \tag{54}$$

With this element x, define the $\mathcal{O}_L$-ideal

$$I_x \;:=\; (cx \mid c \in K \text{ with } vcx > 0)\,. \tag{55}$$

The following result is part of Theorem 3.3 of [5]:

Proposition 4.10. *Under the above assumptions, I_x is the maximal ideal of a valuation ring that contains $\mathcal{O}_L$.*

We will denote I_x by $\mathcal{M}_{\mathcal{E}}$ and the associated valuation ring by $\mathcal{O}_{\mathcal{E}}$. Let us determine $\mathcal{M}_{\mathcal{E}}$ in an important special case; the proof is straightforward.

Lemma 4.11. *Assume that $vK^{<0}$ is not cofinal in $vL^{<0}$ and that $\pi_K \in K$ and $\pi_L \in L$ such that $v\pi_K$ is the smallest positive element in vK and $v\pi_L$ is the smallest positive element in vL. Then $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$ and $\mathcal{M}_{\mathcal{E}} = (\pi_L) = \mathcal{M}_L$.*

Lemma 4.12. *1) We have $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}$ if and only if $vK^{<0}$ is not cofinal in $vL^{<0}$. In this case, $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$ and $\mathcal{M}_L$ is principal.*
2) We have $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}})^{p-1}$ if and only if $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_L$ and $vK^{<0}$ is cofinal in $vL^{<0}$.

Proof. From Proposition 4.8 we know that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$ if $vK^{<0}$ is cofinal in $vL^{<0}$, and $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \frac{\pi_K}{\pi_L} I_{\mathcal{E}}^{p-1}$ if $vK^{<0}$ is not cofinal in $vL^{<0}$. In both cases, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is a principal $\mathcal{O}_L$-ideal.

1): Assume first that $vK^{<0}$ is cofinal in $vL^{<0}$. Then $I_{\mathcal{E}}^{p-1}$ is a principal $\mathcal{O}_L$-ideal, but $(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}$ is principal only if $I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}$ is. If $\mathcal{M}_{\mathcal{E}} \neq \mathcal{M}_L$, then $\mathcal{M}_{\mathcal{E}}$ and $I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}$ are not principal. If $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$ and $\mathcal{M}_L$ is principal, then $I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}$ is principal, but properly contains $I_{\mathcal{E}}$. Hence in all cases, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \neq (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}$.

Now assume that $vK^{<0}$ is not cofinal in $vL^{<0}$. In this case, by Lemma 4.11, $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L = \pi_L\mathcal{O}_L$. Without loss of generality we can assume that $\pi_L^p = \pi_K$, whence $\frac{\pi_K}{\pi_L} = \pi_L^{p-1}$. It follows that $(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1} = (I_{\mathcal{E}}\pi_L\mathcal{O}_L)^{p-1} = \pi_L^{p-1}I_{\mathcal{E}}^{p-1} = \frac{\pi_K}{\pi_L} I_{\mathcal{E}}^{p-1} = \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$.

2): Since $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is a principal $\mathcal{O}_L$-ideal, hence for $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}})^{p-1}$ to hold, the same must be true for $I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}}$. But if $\mathcal{O}_{\mathcal{E}} \neq \mathcal{O}_L$, then $\mathcal{O}_{\mathcal{E}}$ and hence also $I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}}$ are not principal, so we must have $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_L$. Consequently, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}})^{p-1}$ holds if and only if $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_L$ (or equivalently, $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$) and $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$, that is, $vK^{<0}$ is cofinal in $vL^{<0}$. □

Proposition 4.13. *Assume that $vL \neq vK$. Then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}$ if $\mathcal{M}_{\mathcal{E}}$ is a principal $\mathcal{O}_{\mathcal{E}}$-ideal, and $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}})^{p-1} = I_{\mathcal{E}}^{p-1}\mathcal{O}_{\mathcal{E}}$ if $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal.*

The equality $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ holds if and only if $vK^{<0}$ is not cofinal in $vL^{<0}$, or $vK^{<0}$ is cofinal in $vL^{<0}$, $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_L$ and $\mathcal{M}_L$ is a nonprincipal $\mathcal{O}_L$-ideal (i.e., vL has no snallest positive element).

Proof. The first two assertions follow from [5, Theorems 4.5 and 4.7] together with [5, Corollary 3.5].

To prove the last assertion, we use Lemma 4.12. Assume first that $vK^{<0}$ is not cofinal in $vL^{<0}$. Then $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$ and $\mathcal{M}_L$ is principal, hence $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1} = \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$. Now assume that $vK^{<0}$ is cofinal in $vL^{<0}$. Then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$ which is equal to $(I_{\mathcal{E}}\mathcal{O}_{\mathcal{E}})^{p-1}$ if and only if $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_L$, or equivalently, $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$. If this holds, then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$ if and only if $\mathcal{M}_L$ is a nonprincipal $\mathcal{O}_L$-ideal. □

4.2.2. *The defect case.*

The following is part of Theorem 1.6 of [4]; it gives more details on $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ which we will not state here.

Theorem 4.14. *In addition to our general assumptions, let $\mathcal{E}$ be a defect extension.*

1) We have that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$ if and only if $vI_{\mathcal{E}}^{p-1}$ has no infimum in vL. If $vI_{\mathcal{E}}^{p-1}$ has infimum va in vL for some $a \in L$, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = a\mathcal{O}_L \neq I_{\mathcal{E}}^{p-1}$ and $I_{\mathcal{E}}^{p-1} = \mathcal{M}_L\,\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$.

2) If (K, v) has rank 1, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

4.3. **The "naive different ideal" $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)$.**
Take an algebraic extension $(L|K,v)$ of valued fields. If $b\in\mathcal{O}_L$ and h_b is its minimal polynomial over K, then $h_b'(b)$ is called the **different of** b. The $\mathcal{O}_L$-ideal

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;:=\; (h_b'(b)\mid b\in\mathcal{O}_L\setminus\mathcal{O}_K) \tag{56}$$

generated by the differents of all elements in $\mathcal{O}_L\setminus\mathcal{O}_K$ appears to be occasionally called the naive different ideal, and we will adopt this name. We will use the abbreviation $\delta(b):=h_b'(b)$.

Proposition 4.15. *Take a unibranched separable-algebraic extension $(L|K,v)$ of valued fields and assume that*

$$\mathcal{O}_L \;=\; \bigcup_\alpha \mathcal{O}_K[b_\alpha] \tag{57}$$

where α runs through some index set S. Then

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (\delta(b_\alpha)\mid\alpha\in S)\,. \tag{58}$$

Proof. Take any $b\in\mathcal{O}_L\setminus\mathcal{O}_K$ and let h be its minimal polynomial over K. For each $i\geq 1$ and $\sigma\in\operatorname{Gal}K$ we have

$$b^i-\sigma b^i \;=\; b^i-(\sigma b)^i \;=\; b^i-(b+(\sigma b-b))^i \;=\; \sum_{j=0}^{i-1}\binom{i}{j}b^j(\sigma b-b)^{i-j}\,.$$

Since the extension is unibranched, we have $v\sigma b=vb$, whence $v(b-\sigma b)\geq vb\geq 0$. Consequently,

$$v(b^i-\sigma b^i)\;\geq\; v(\sigma b-b)\;=\;v(b-\sigma b)\,.$$

Every $b\in\mathcal{O}_K[b_\alpha]\setminus\mathcal{O}_K$ is of the form

$$b\;=\;\sum_{i=0}^{n-1}c_ib_\alpha^i$$

with $c_i\in\mathcal{O}_K$. We write

$$\delta(b)\;=\;\prod_{\sigma\in G_b}(b-\sigma b)$$

where G_b is a subset of $\operatorname{Gal}K$ with $\deg h_b-1$ many elements such that σb, $\sigma\in G_b$, are all conjugates of b that are different from b. Then

$$\begin{aligned} v\delta(b) \;&=\; v\prod_{\sigma\in G_b}(b-\sigma b)\;=\;\sum_{\sigma\in G_b}v\left(\sum_{i=0}^{n-1}c_ib_\alpha^i-\sigma\sum_{i=0}^{n-1}c_i(b_\alpha)^i\right)\\ &=\;\sum_{\sigma\in G_b}v\sum_{i=1}^{n-1}c_i(b_\alpha^i-\sigma b_\alpha^i)\,.\end{aligned}$$

For $1\leq i\leq n-1$, we have

$$vc_i(b_\alpha^i-\sigma b_\alpha^i)\;\geq\; v(b_\alpha^i-\sigma b_\alpha^i)\;\geq\; v(b_\alpha-\sigma b_\alpha)\,,$$

showing that

$$v\sum_{i=1}^{n-1}c_i(b_\alpha^i-\sigma b_\alpha^i)\;\geq\; v(b_\alpha-\sigma b_\alpha)\,.$$

Hence,

$$v\delta(b) \;\geq\; \sum_{\sigma\in G_b} v(b_\alpha - \sigma b_\alpha) \;=\; vh'_\alpha(b_\alpha)\,.$$

Using (57), we now obtain:

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(\delta(b)\mid b\in\mathcal{O}_K[b_\alpha]\setminus\mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(\delta(b_\alpha)) \;=\; (\delta(b_\alpha)\mid\alpha\in S)\,.$$

□

We will now determine $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)$ for unibranched Galois extensions $\mathcal{E} = (L|K,v)$.

4.3.1. *The defectless case.*

Proposition 4.16. *Assume that $\mathcal{E} = (L|K,v)$ is a unibranched defectless Galois extension of prime degree $p = \operatorname{char} Kv$. If K has characteristic* 0*, then we assume in addition that it contains all p-th roots of unity.*

1) If $p = \mathrm{f}\,(L|K,v)$, then

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; I_{\mathcal{E}}^{p-1} = \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)\,. \tag{59}$$

2) If $p = \mathrm{e}\,(L|K,v)$, then

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}\,. \tag{60}$$

This is equal to $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ if and only if $vK^{<0}$ is not cofinal in $vL^{<0}$.

Proof. 1): This follows from [5, Lemmas 3.7 and 3.9] together with Proposition 4.8.

2): Equation (60) follows from [5, Lemmas 3.8 and 3.10]. The second assertion follows from Lemma 4.12. □

4.3.2. *The defect case.*

Let us consider an immediate not necessarily algebraic extension $(K(x)|K,v)$. Then by [14, Theorem 2.19] the set $v(x-K)\subseteq vK$ is a final segment of vK; in particular, it has no maximal element. If $g\in K[X]$ and there is $\alpha\in v(x-K)$ such that for all $c\in K$ with $v(x-c)\geq\alpha$ the value $vg(c)$ is constant, then we will say that **the value of g is ultimately fixed over** K. We call $(K(x)|K,v)$ **pure in** x if the value of every $g(X)\in K[X]$ of degree smaller than $[K(x):K]$ is ultimately fixed over K. Note that we set $[K(x):K]=\infty$ if x is transcendental over K.

The following is Lemma 2.3 of [4]:

Lemma 4.17. *Every unibranched immediate extension $(K(x)|K,v)$ of prime degree is pure in x.*

For every $c\in\mathcal{O}_K$ we know that $v(x-c)\in vK$ since the extension is immediate, so we may choose $t_c\in K$ such that $vt_c = -v(x-c)$ and set

$$x_c \;:=\; t_c(x-c) \;\in\; \mathcal{O}_{K(x)}^{\times}\,.$$

Lemma 4.18. *Assume that the immediate extension $(K(x)|K,v)$ is pure. Then for every $g(x)\in\mathcal{O}_{K(x)}\cap K[x]$ there is $c\in K$ such that $g(x)\in\mathcal{O}_K[x_c]$. If in addition $K(x)|K$ is algebraic of degree n, then*

$$\mathcal{O}_{K(x)} \;=\; \bigcup_{c\in K}\mathcal{O}_K[x_c]$$

and

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (\,(x-c)^{1-n}\delta(x)\mid c\in K)\,. \tag{61}$$

Proof. The first assertions are proven in [4, Lemma 3.1]. It remains to prove the last assertion. From Proposition 4.15 we know that $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)=(\delta(x_c)\mid c\in K)$. We take G_x to be a subset of $\mathrm{Gal}\,K$ with $n-1$ many elements such that σx, $\sigma\in G_x$, are all conjugates of b that are different from x. Now we compute:

$$\delta(x_c) \;=\; \prod_{\sigma\in G_x}(x_c-\sigma x_c) \;=\; \prod_{\sigma\in G_x}(t_c(x-c)-\sigma t_c(x-c)) \tag{62}$$

$$\;=\; t_c^{n-1}\prod_{\sigma\in G_x}(x-\sigma x) \;=\; t_c^{n-1}\delta(x)\,. \tag{63}$$

Since $vt_c=-v(x-c)$, we thus have

$$(\delta(x_c)\mid c\in K) \;=\; (t_c^{n-1}\delta(x)\mid c\in K) \;=\; (\,(x-c)^{1-n}\delta(x)\mid c\in K)\,.$$

□

Proposition 4.19. *Take a unibranched Galois defect extension $\mathcal{E}=(K(x)|K,v)$ of prime degree $p=\mathrm{char}\,Kv$. Then*

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; I_{\mathcal{E}}^{p-1}\,. \tag{64}$$

Proof. We set $G:=\mathrm{Gal}\,L|K$. Under the assumptions of our proposition, we have

$$\delta(x) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}}(x-\sigma x)\,.$$

Choose $\sigma_0\in G\setminus\{\mathrm{id}\}$ such that $v(x-\sigma_0 x)=\max\{v(x-\sigma x)\mid \mathrm{id}\neq\sigma\in G\}$. Since $(L|K,v)$ is unibranched, we have $v(x-\sigma_0x)=v\sigma_0(x-\sigma_0x)=v(\sigma_0x-\sigma_0^2x)$, whence $v(x-\sigma_0^2x)\geq\min\{v(x-\sigma_0x)\,,v(\sigma_0x-\sigma_0^2x)\}=v(x-\sigma_0x)$. By our choice of σ_0, this shows that $v(x-\sigma_0^2x)=v(x-\sigma_0x)$. By induction, one shows that $v(x-\sigma_0^ix)=v(x-\sigma_0x)$ for $1\leq i\leq p-1$. Since σ_0 generates G, we conclude that

$$v\delta(x) \;=\; \sum_{\sigma\in G\setminus\{\mathrm{id}\}}v(x-\sigma x) \;=\; (p-1)v(x-\sigma_0x)\,.$$

Consequently,

$$v(x-c)^{1-p}\delta(x) \;=\; (p-1)(-v(x-c)+v(x-\sigma_0x))\,.$$

Using (61) we find that

$$v\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (p-1)(-v(x-K)+v(x-\sigma_0x)) \;=\; (p-1)\Sigma_{\mathcal{E}}\,,$$

where the second equality holds by Theorem 3.10. This proves (64). □

4.4. **Norms.**
Throughout, we assume that $\mathcal{E} = (L|K, v)$ is a unibranched Galois extension of prime degree $p = \operatorname{char} Kv$. We will compute the norm of the ramification ideal $I_{\mathcal{E}}$. With different methods, this has also been done in [31, 32, 33].

The proof of the following fact is straightforward:

Lemma 4.20. *Let $(L|K, v)$ is a unibranched Galois extension of degree p.*
1) For every $b \in L$ we have $v\mathrm{N}_{L|K}b = pvb$, hence

$$v\mathrm{N}_{L|K}b \leq v\mathrm{N}_{L|K}b' \Leftrightarrow vb \leq vb' . \tag{65}$$

2) If $I = (b_i \mid i \in \mathcal{I})$ as $\mathcal{O}_L$-ideal for some index set $\mathcal{I}$, then $\mathrm{N}_{L|K}I = (\mathrm{N}_{L|K}b_i \mid i \in \mathcal{I})$ as $\mathcal{O}_K$-ideal.

Proof. 1): The proof is straightforward.
2): Take any $b \in I$. Then there is $i \in \mathcal{I}$ such that $vb_i \leq vb$. By part 1), this is equivalent to $v\mathrm{N}_{L|K}b_i \leq v\mathrm{N}_{L|K}b$, and this in turn is equivalent to $\mathrm{N}_{L|K}b \in (\mathrm{N}_{L|K}b_i) \subseteq (\mathrm{N}_{L|K}b_i \mid i \in \mathcal{I})$. □

4.4.1. *The defectless case.*

Proposition 4.21. *Take*
1) Assume that $L|K$ is an Artin-Schreier extension. Then it admits an Artin-Schreier generator ϑ such that the $\mathcal{O}_K$-ideal generated by $\mathrm{N}_{L|K}I_{\mathcal{E}}$ is

$$(\mathrm{N}_{L|K}I_{\mathcal{E}}) \;=\; \left(\frac{1}{\vartheta^p - \vartheta}\right) .$$

2) Assume that $L|K$ is a Kummer extension. Then it admits a Kummer generator η such that the $\mathcal{O}_K$-ideal generated by $\mathrm{N}_{L|K}I_{\mathcal{E}}$ is equal to

$$((\zeta_p - 1)^p) \quad \text{or} \quad \left(\frac{(\zeta_p - 1)^p}{\eta^p - 1}\right) ,$$

depending on whether case a) or case b) holds in part 2) of Theorem 3.15.

Proof. Our assertions follow from Theorem 3.15 together with part 2) of Lemma 4.20. Here we also use that $\mathrm{N}_{L|K}\vartheta = \pm(\vartheta^p - \vartheta)$, and $\mathrm{N}_{L|K}(\eta - 1) = 1 - \eta^p$ as the minimal polynomial of $\eta - 1$ is $(X + 1)^p - \eta^p$ whose constant term is $1 - \eta^p$. □

We turn our attention to the defect case.

4.4.2. *The defect case.*

Proposition 4.22. *Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree $p = \operatorname{char} Kv$.*
1) Assume that $L|K$ is an Artin-Schreier extension with Artin-Schreier generator ϑ. Then the $\mathcal{O}_K$-ideal generated by $\mathrm{N}_{L|K}I_{\mathcal{E}}$ is

$$\begin{aligned}(\mathrm{N}_{L|K}I_{\mathcal{E}}) \;&=\; \left(\frac{1}{b^p - b} \;\middle|\; b = \vartheta - c \text{ with } c \in K\right) \\ &=\; \left(\frac{1}{b^p - b} \;\middle|\; b \text{ an Artin-Schreier generator of } L|K\right) .\end{aligned}$$

2) Assume that $L|K$ is a Kummer extension with Kummer generator η. Then the $\mathcal{O}_K$-ideal generated by $\mathrm{N}_{L|K} I_{\mathcal{E}}$ is

$$\begin{aligned}(\mathrm{N}_{L|K} I_{\mathcal{E}}) &= \left(\frac{(\zeta_p - 1)^p}{\eta^p - c^p} \;\middle|\; c \in K \right) \\ &= \left(\frac{(\zeta_p - 1)^p}{b^p - 1} \;\middle|\; b \text{ a Kummer generator of } L|K \text{ which is a 1-unit} \right) .\end{aligned}$$

Proof. 1): If $b \in L$ is an Artin-Schreier generator of $L|K$, then for every $\sigma \in \mathrm{Gal}\, L|K \setminus \{\mathrm{id}\}$,

$$\mathrm{N}_{L|K} \frac{\sigma b - b}{b} = \frac{i}{b^p - b} , \tag{66}$$

where i is in $\mathbb{F}_p$ and hence a unit in (K, v). For every $c \in K$, also $\vartheta - c$ is an Artin-Schreier generator of $L|K$. This proves the inclusion "$\supseteq$" in the first equation. The inclusion "$\subseteq$" follows via equation (66) from the first equation in part 1) of Theorem 3.18 together with part 2) of Lemma 4.20.

This proves the first equation. The argument for the validity of the second equation is as in the proof of part 1) of Theorem 3.18.

2): If b is any Kummer generator of $L|K$ and $c \in K$, then $\mathrm{N}_{L|K}(b - c) = c^p - b^p$ as the minimal polynomial of $b - c$ is $(X + c)^p - b^p$ whose constant term is $c^p - b^p$. Hence, $\mathrm{N}_{L|K} \left(\frac{\zeta_p - 1}{b - c} \right) = \left(\frac{(\zeta_p - 1)^p}{b^p - c^p} \right)$. This proves the inclusion "$\supseteq$" in the first equation. The inclusion "$\subseteq$" follows from the first equation in part 2) of Theorem 3.18 together with part 2) of Lemma 4.20.

This proves the first equation. The argument for the validity of the second equation is as in the proof of part 2) of Theorem 3.18. □

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