ABELIAN GROUPS WITH CONTRACTIONS II: WEAK O-MINIMALITY

FRANZ–VIKTOR KUHLMANN
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
D-69120 Heidelberg, Germany

Abstract. A special map, called "contraction", on ordered abelian groups is studied. A contraction contracts every archimedean class $\neq \{0\}$ to a set $\{a, -a\}$ of two points. A weakly o-minimal theory of divisible ordered abelian groups with surjective contraction is given. It is shown not to have the algebraic exchange property. Contractions appear in a natural way in the theory of nonarchimedean exponential fields.

Key words: Ordered abelian group, contraction, weak o-minimality, algebraic exchange property, exponential field

1. Introduction

In [K], the author studied the model theory of abelian groups with special maps, called contractions. Natural models of abelian groups with such maps are the value groups of nonarchimedean exponential fields with respect to their natural (also called archimedean) valuations. In that case, the contraction map is induced by the exponential, as discussed in Chapter 3 of [K–K1]. Our interest in the structure of nonarchimedean exponential fields was promoted by the remarkable progress that has recently been achieved in the model theory of the reals with exponentiation (cf. [W], [D–M–M]). In [KS], [K–K1], [K–K2] and [K–K–S], this structure has been studied, using a valuation theoretical approach.

All abelian groups will be written additively. Let $(G,<)$ be a totally ordered abelian group and χ a map from G into G. Then χ is called a **precontraction** and (G,χ) is called a **precontraction group** if χ satisfies the axioms

(C0) $\chi x = 0 \Leftrightarrow x = 0\,,$

(C$\leq$) χ preserves $\leq$,

(C$-$) $\chi(-x) = -\chi x\,,$

(CA) if x is archimedean equivalent to y and $\mathrm{sign}(x) = \mathrm{sign}(y)$, then $\chi x = \chi y\,.$

Note that axiom (CA) that the preimage $\chi^{-1}(x)$ for every element $x \neq 0$ is infinite. If a precontraction χ in addition satisfies

(CS) χ is surjective,

then χ will be called a **contraction** and the group is a **contraction group**. A contraction has an amazing property: although "contracting" archimedean classes, it is surjective. Nontrivial contraction groups must be quite "big". In fact, axioms (CA) and (CS) together show that every archimedean ordered contraction group must be trivial.

Further, $(G,+,-,0,<,\chi)$ will be called **centripetal**, if it satisfies

(CP) $\forall x\in G\setminus\{0\}:\ |x|>|\chi x|$.

An abelian group G is **divisible** if it satisfies

(D) $\forall x\,\exists y:\ ny=x \qquad (0\neq n\in\mathbb{N})$.

All axioms introduced above are elementary in the language $\mathcal{L}_{\rm cg}=\{+,-,0,<,\chi\}$ where $<$ is a binary relation symbol, $+$ is a binary function symbol, and $-$ and χ are unary function symbols. See [K] for details. In that article, we proved:

Theorem 1.1 *The elementary theory of divisible centripetal contraction groups admits elimination of quantifiers in the language* $\mathcal{L}_{\rm cg}$.

In the present paper, we shall study the terms built up with contractions. This leads to a description of the definable sets definable (first, explicitly, of those definable without constants in Theorem 3.3, and later, implicitly, of those definable with constants). This description together with Theorem 1.1 will show:

Theorem 1.2 *The elementary theory of divisible centripetal contraction groups is weakly o-minimal.*

A theory (in which an order $<$ is given or definable) is called **weakly o-minimal** if in every model of this theory, every (elementarily) definable subset is a finite union of convex subsets. In contrast to o-minimality, it is not required that these convex subsets be intervals.

Let $\mathcal{M}$ be a model of a given elementary theory $\mathcal{T}$ and A a subset of $\mathcal{M}$. Then an element $a\in\mathcal{M}$ is said to be **algebraic** over A if it satisfies an elementary formula (with constants from A) which is satisfied only by finitely many elements of $\mathcal{M}$. The set of all elements of $\mathcal{M}$ which are algebraic over A is called the **algebraic closure** of A in $\mathcal{M}$. A theory $\mathcal{T}$ is said to have the **algebraic exchange property** if the following holds in every model $\mathcal{M}$ of $\mathcal{T}$: given $A\subseteq\mathcal{M}$ and $a,b\in\mathcal{M}$ such that a lies in the algebraic closure of $A\cup\{b\}$ but not in the algebraic closure of A, then b lies in the algebraic closure of $A\cup\{a\}$. For example, the theory of abelian groups and the theory of fields have the algebraic exchange property. Divisible ordered abelian groups and real closed fields are examples for o-minimal theories having the algebraic exchange property. But the situation may change drastically if we add new symbols to the language from which the formulas are built up. The theory of divisible ordered abelian groups with order-compatible valuation is an example of a weakly o-minimal theory also having the algebraic exchange property. In contrast to this, we have:

Observation. *The elementary theory of divisible centripetal contraction groups does not have the algebraic exchange property.*

This is particularly remarkable since it is the first known naturally appearing weakly o-minimal theory not having the algebraic exchange property. See [MP–M–S] for a different, more “artificial” example. Note that by a theorem of Pillay and Steinhorn [P–S], every o-minimal theory has the algebraic exchange property. For the above observation, let us give a quick proof which is due to D. Macpherson: Choose an $\aleph_0$-saturated model (G,χ) of the theory. Take $A=\emptyset$ and G_0 to be the algebraic closure of $\emptyset$. Since our language of precontraction groups is finite, $\aleph_0$-saturation yields the existence of some $a \in G \setminus G_0$. Take G_1 to be the algebraic closure of $\{a\}$. Since G_1 is finitely generated and the preimage $\chi^{-1}(a)$ is infinite, $\aleph_0$-saturation yields the existence of some $b \in \chi^{-1}(a) \setminus G_1$. On the other hand, $a = \chi(b)$ shows that a lies in the algebraic closure of $\{b\}$. Since it does not lie in G_0, this witnesses the failure of the algebraic exchange property.

Here, we have not even used the fact that $G_0 = \{0\}$. While this is easy to see, the following will be a nontrivial consequence of our analysis of the terms built up with contractions:

Theorem 1.3 *If A is a set of elements in the divisible centripetal contraction group (G, χ), then the algebraic closure of A in G is precisely the divisible hull of the precontraction group generated by A in G.*

Note that in every ordered group, the algebraic closure of a subset A is equal to the definable closure of A. Indeed, in a finite set of algebraic elements, every element can be singled out by means of the order.

From this theorem, one can deduce that the element b of our above example does not even lie in the least convex subgroup of G which contains the algebraic closure of $\{a\}$.

At this point, I would like to thank my wife, Salma Kuhlmann, for her support and many invaluable discussions, and Dugald Macpherson for his very useful comments and suggestions.

2. Preliminaries

Before we continue, let us put together some technical preliminaries. Let G be an ordered abelian group. We write $G^{<0} = \{g \in G \mid g < 0\}$ and $G^{>0} = \{g \in G \mid g > 0\}$. If $S_1, S_2 \subseteq G$ and $a \in G$, we will write $a < S_2$ if $a < b$ for all $b \in S_2$, and further, we will write $S_1 < S_2$ if $a < S_2$ for all $a \in S_1$. Similarly, we use the relations $>$, $\leq$ and $\geq$. We set $\operatorname{sign}(0) = 0$ and for $a \in G$, we set $\operatorname{sign}(a) = 1$ if $a > 0$, and $\operatorname{sign}(a) = -1$ if $a < 0$. Further, we set $|a| := \max\{a, -a\} = \operatorname{sign}(a) \cdot a$. Two elements $a, b \in G$ are called **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. Let va denote the equivalence class of a. The set of equivalence classes is ordered as follows: $va < vb$ if and only if $|a| > |b|$ and $va \neq vb$. We write $\infty := v0$; this is the maximal element in the ordered set of equivalence classes. The map $a \mapsto va$ is a group valuation, that is, it satisfies

(V0) $\forall x:\ vx = \infty \Leftrightarrow x = 0$,

(V1) $\forall x, y:\ v(x-y) \geq \min\{vx, vy\}$.

From these rules, we may deduce

(V2) $\forall x:\ vx = v(-x)$,

(V3) $v(\sum_{1\leq i\leq n} x_i) = \min_{1\leq i\leq n} vx_i$ if all nonzero x_i have different values,

(V4) $\forall x, y:\ v(x-y) > \min\{vx, vy\} \Rightarrow vx = vy$.

The ordered set $vG := \{vg \mid 0 \neq g \in G\}$ will be called the **value set** of G.

The valuation that we have defined above, is canonically associated to every ordered group and is called the **natural valuation**. G is archimedean if and only if vG consists of just one element. Note that for every $a \in G$ and every $n \in \mathbb{Z} \setminus \{0\}$, the element $na \in G$ is archimedean equivalent to a and so, the natural valuation satisfies the axiom scheme

(NV1) $\forall x : \; v(nx) = vx \qquad (0 \neq n \in \mathbb{Z})\,.$

Further, the natural valuation is compatible with the ordering, in the following sense:

(NV2) $\forall x, y \in G^{<0} : \; vx < vy \Rightarrow x < y.$

Let us also note:

(NV3) $\mathrm{sign}(\sum_{1\leq i\leq n} x_i) = \mathrm{sign}(x_m) \quad \text{if } vx_m < vx_i \text{ for all } i \neq m,$

(NV4) $\forall x, y : \; v(x-y) > vx \Rightarrow \mathrm{sign}(x) = \mathrm{sign}(y).$

Further, (NV3) may be generalized to

(NV5) $x_m < x'_m \Rightarrow \sum_{1\leq i\leq n} x_i < \sum_{1\leq i\leq n} x'_i$ if $vx_m < vx_i$ and $vx'_m < vx'_i$ for all $i \neq m$.

In the sequel, v will always denote the natural valuation.

Using v, we may express axiom (CA) in the following way:

(CV1) $\forall x, y : \; vx = vy \wedge \mathrm{sign}(x) = \mathrm{sign}(y) \Longrightarrow \chi x = \chi y\,.$

Sometimes, we will only be interested in equality up to the sign; instead of writing $|a| = |b|$ we will then write $a = \pm b$. Then, (CV1) reads as follows:

(CV2) $\forall x, y : \; vx = vy \Longrightarrow \chi x = \pm \chi y\,.$

From (V3), (NV3) and (V4), (NV4) we infer:

(CV3) $\chi(\sum_{1\leq i\leq n} x_i) = \chi x_m \quad \text{if } vx_m < vx_i \text{ for all } i \neq m,$

(CV4) $\forall x, y : \; v(x-y) > vx \Rightarrow \chi x = \chi y\,.$

From (CV2) and (NV2) together with (C$\leq$), one deduces

(CV5) $\forall x, y : \; vx \leq vy \Longrightarrow |\chi x| \geq |\chi y|\,.$

Observe that axioms (C0), (C$-$) and (CA) together imply

(CZ) $\forall x \in G : \; \chi(zx) = \mathrm{sign}(z) \cdot \chi x \qquad (z \in \mathbb{Z})\,.$

Observe further that by axiom (C$\leq$), $x \leq y \leq z$ and $\chi x = \chi z$ implies $\chi y = \chi z$.

Lemma 2.1 *Let (G, χ) be a precontraction group. Then χ is centripetal if and only if $v\chi a > va$ for all $a \in G \setminus \{0\}$.*

Proof: Assume that χ is centripetal. Then for every $a \in G \setminus \{0\}$, we have $|\chi a| < |a|$ and thus, $v\chi a \geq va$. But for every $b \in G$ with $va = vb$ it follows from (CV2) and (CP) that $|\chi a| = |\chi b| < |b|$. Hence $b \neq \chi a$, that is, $v\chi a = va$ is impossible. The converse follows from (NV2) and (C-). □

We define the following convex subgroups of G for $0 \neq a \in G$:

$$\mathcal{O}_G^{va} := \{g \in G \mid vg \geq va\} \quad \text{and} \quad \mathcal{M}_G^{va} := \{g \in G \mid vg > va\}\,.$$

Note that for every $c \in G$, if M is a convex subset of G, then this also holds for $c + M = \{c + m \mid m \in M\}$; in particular, $c + \mathcal{O}_G^{va}$ and $c + \mathcal{M}_G^{va}$ are convex subsets of G. We also see that for every $va \in vG$, the set $\mathcal{O}_G^{va} \setminus \mathcal{M}_G^{va} = \{g \in G \mid vg = va\}$ is the union of two convex sets, namely $\{g \in G \mid vg = va \wedge \operatorname{sign}(g) = \operatorname{sign}(a)\}$ and $\{g \in G \mid vg = va \wedge \operatorname{sign}(g) = -\operatorname{sign}(a)\}$.

Lemma 2.2 *Let $a, b, d \in G$.*
a) If $a + \mathcal{M}_G^{vd} \cap b + \mathcal{M}_G^{vd} \neq \emptyset$, then $a + \mathcal{M}_G^{vd} = b + \mathcal{M}_G^{vd}$.
b) If $v(a-b) = \min\{va, vb\}$, then $a + \mathcal{M}_G^{va} \cap b + \mathcal{M}_G^{vb} = \emptyset$.
If $v(a-b) > \min\{va, vb\}$, then $a + \mathcal{M}_G^{va} = b + \mathcal{M}_G^{vb}$.

Proof: a): Suppose $c \in a + \mathcal{M}_G^{vd} \cap b + \mathcal{M}_G^{vd}$. Then $v(a-c) > vd$ and $v(b-c) > vd$, hence $v(a-b) > vd$. Let $b' \in b + \mathcal{M}_G^{vd}$. Then $v(a-b') \geq \min\{v(a-b), v(b-b')\} > vd$, hence $b' \in a + \mathcal{M}_G^{vd}$. Symmetrically, one proves $a' \in b + \mathcal{M}_G^{vd}$ for every $a' \in a + \mathcal{M}_G^{vd}$. This proves the equality of the two sets.

b): Note that by (V1), $v(a-b) \geq \min\{va, vb\}$. If $v(a-b) = \min\{va, vb\}$, say $= va$, then for every $b' \in b + \mathcal{M}_G^{vb}$ we have $v(b-b') > vb \geq va$ which yields that $v(a-b') = \min\{v(a-b), v(b-b')\} = va$ by (V3), showing that $b' \notin a + \mathcal{M}_G^{va}$. If $v(a-b) > \min\{va, vb\}$, then by (V4), $va = vb$, and we find $b \in a + \mathcal{M}_G^{va}$. By part a), this yields equality of the two sets. □

We will need the notion of an **open** convex set, which is just a convex set that does not contain a least and does not contain a greatest element. There are two basic examples: the open intervals (a, b) in G and the nontrivial convex subgroups of G (or their cosets). We will also view $\emptyset$ as an open convex subset.

The following lemma states an important property of maps which preserve $\leq$. Its proof is straightforward.

Lemma 2.3 *Assume that $(S, <)$ is a totally ordered set and $\chi : S \to S$ is monotone (i.e., χ preserves $\leq$ on all of S or reverses $\leq$ on all of S). Then the preimage of every convex set is convex. If in addition χ is surjective, then the image of every convex set is convex, and the preimage of every open convex set is open convex.*

3. Weak o-minimality

We are now going to show that the theory of divisible centripetal contraction groups is weakly o-minimal. Let us consider a divisible centripetal contraction group (G, χ). By Theorem 1.1, its theory admits quantifier elimination in the language $\mathcal{L}_{\text{cg}}$. It follows that every formula in one free variable x and with constants from G is equivalent to a finite boolean combination of formulas of the type $t_1(x) < t_2(x)$ or $t_1(x) = t_2(x)$, where t_1, t_2 are terms in the language $\mathcal{L}_{\text{cg}}(G, x)$. Here, $\mathcal{L}_{\text{cg}}(G, x)$ denotes the language $\mathcal{L}_{\text{cg}}$ augmented by the free variable x and constant names for the elements of G. But since "$-$" is included in $\mathcal{L}_{\text{cg}}$, the above atomic formulas are equivalent to $t_2(x) - t_1(x) > 0$ resp. $t_2(x) - t_1(x) = 0$. So every formula in one free variable x is equivalent to a finite boolean combination of formulas of the type $t(x) > 0$ or $t(x) = 0$. Every term $t(x)$ may be viewed as a map from G into G.

It follows that every definable set in a divisible centripetal contraction group is a finite boolean combination of sets of the form $t^{-1}(G^{>0})$ and $t^{-1}(\{0\})$. If such sets are already finite unions of convex sets, then the same is true for an arbitrary finite boolean combination. Hence:

Lemma 3.1 *If for every divisible centripetal contraction group (G,χ) and every term $t(x)\in\mathcal{L}_{\rm cg}(G,x)$, the sets $t^{-1}(G^{>0})$ and $t^{-1}(\{0\})$ are convex subsets of G, then the theory of divisible centripetal contraction groups is weakly o-minimal.*

On the other hand, in a contraction group (G,χ) the preimage $\chi^{-1}(a)$ of a nonzero element a can never be an interval. Indeed, by (CV1) and (C$\leq$), such a preimage is of the form $\{g\in G^{>0}\mid vg\in\Gamma\}$ if $a>0$, or $\{g\in G^{<0}\mid vg\in\Gamma\}$ if $a<0$, for a convex subset $\Gamma\subseteq vG$. These convex sets are convex but not intervals: they have no minimal or maximal elements, and the same is true for their complements on the left and on the right. Hence, contraction groups cannot be o-minimal.

In this section, let (G,χ) always be a centripetal precontraction group.

Let us first analyze the subsets of G which are definable *without constants*, that is, in the language $\mathcal{L}_{\rm cg}(x)$. A special role will be played by terms of the form

$$z_n\chi^n x+\ldots+z_1\chi x+z_0x \tag{1}$$

with $z_0,\ldots,z_n\in\mathbb{Z}$. Such a term will be called a χ-**polynomial**. For the following, we set $\chi^0x=x$.

Lemma 3.2 *Let $t(x)$ be a term in the language $\mathcal{L}_{\rm cg}(x)$.*
a) As a map from G into itself, t equals some χ-polynomial. If t equals a nonzero χ-polynomial f of the form (1) and m is the minimal index such that $z_m\neq 0$, then χt equals $\mathrm{sign}(z_m)\chi^{m+1}x$.
b) As a map from G into itself, t is monotone. More precisely, if f and m are as in a), then t is monotonically increasing if $z_m>0$ and monotonically decreasing if $z_m<0$. If $m=0$, then t is strictly monotone.

Proof: a): The proof proceeds by induction on the complexity of the terms built up in the language $\mathcal{L}_{\rm cg}(x)$. Since the only constants, 0 and x, are themselves χ-polynomials, we just have to show that adding χ-polynomials as well as applying the functions "$-$" and "χ" to a χ-polynomial produces again a term which equals a χ-polynomial as a map. For adding and applying "$-$", this is in fact trivial. Now if $t(x)$ equals a χ-polynomial $f(x)$ of the form (1), then we have to consider the term $\chi f(x)$. If f is the zero polynomial, that is, $t(g)=0$ for all $g\in G$, then also $\chi t(g)=0$ for all $g\in G$ and $\chi t(x)$ equals again the zero polynomial. Now assume that f is a nonzero χ-polynomial. Then there is a minimal index m such that $z_m\neq 0$. Since χ is assumed to be centripetal, we know that $v\chi^ig>v\chi^mg$ for all $i>m$, $g\in G\setminus\{0\}$ (unless $\chi^ig=0=\chi^mg$). Since $vz_i\chi^ig=v\chi^ig$ whenever $z_i\neq 0$, we find that $z_m\chi^mg$ is the summand of least value in $f(g)$. By (CV3), this implies $\chi t(g)=\chi f(g)=\chi(z_m\chi^mg)=\mathrm{sign}(z_m)\chi^{m+1}g$ for all $g\in G$. We have shown that as a map, $\chi t(x)$ again equals a χ-polynomial.

b): If t equals the zero polynomial, there is nothing to show. Now assume f and m to be as in a). Let $a, b \in G$ such that $a \leq b$. If $\chi^m a = \chi^m b$ then $\chi^i a = \chi^i b$ for all $i \geq m$, which yields $z_i \chi^i a = z_i \chi^i b$ for all i, $1 \leq i \leq n$, and thus, $t(a) = t(b)$. Now assume that $\chi^m a \neq \chi^m b$. Since χ preserves $\leq$, the same is true for χ^m. We thus find that $\chi^m a < \chi^m b$. Hence $z_m \chi^m a < z_m \chi^m b$ if $z_m > 0$, and $z_m \chi^m a > z_m \chi^m b$ if $z_m < 0$. Since χ is assumed to be centripetal, $v\chi^i g > v\chi^m g$ for all $i > m$, $g \in G$ (unless $\chi^i g = 0 = \chi^m g$). Consequently, $z_m \chi^m a < z_m \chi^m b$ implies $t(a) < t(b)$ and $z_m \chi^m a > z_m \chi^m b$ implies $t(a) > t(b)$ by virtue of (NV5). This proves the monotonicity.

If $m = 0$, that is, if $z_0 \neq 0$, then $a \neq b$ always implies $z_m \chi^m a \neq z_m \chi^m b$. Then it follows that the monotonicity is strict. This completes our proof. $\square$

From part b) of this lemma together with Lemma 2.3 and Theorem 1.1, we now deduce:

Theorem 3.3 *If in a centripetal precontraction group (G, χ), a given subset is quantifier free definable without constants, then it is a finite boolean combination of convex sets $f_i^{-1}(G^{>0})$ and $f_i^{-1}(\{0\})$, where the f_i are χ-polynomials. Hence, if (G, χ) is a divisible centripetal contraction group, then every subset definable without constants is a finite union of convex subsets of G.*

Let us also mention the following corollary to Lemma 3.2:

Corollary 3.4 *For a divisible centripetal contraction group (G, χ), every nonzero term in $\mathcal{L}_{\mathrm{cg}}(x)$ is a surjective map from G onto G.*

Proof: In view of part a) of Lemma 3.2, we have to show our assertion only for a nonzero χ-polynomial. Let f and m be as in that lemma. Given $d \in G$, we want to show the existence of some $a \in G$ such that $f(a) = d$. First, since χ is surjective, the same holds for χ^m in the place of f. Since G is divisible, it also holds for $z_m \chi^m$. So there is some $b \in G$ such that $z_m \chi^m b = d$. And there is also some $a \in G$ such that $z_m \chi^m a = d - \sum_{i=m+1}^n z_i \chi^i b$. From the proof of Lemma 3.2 we already know that $v z_i \chi^i b > v z_m \chi^m b = vd$ for all $i > m$. By (CV3) it follows that $z_i \chi^i a = z_i \chi^{i-m}(\chi^m a) = z_i \chi^{i-m} z_m^{-1}(d - \sum_{i=m+1}^n z_i \chi^i b) = z_i \chi^{i-m} z_m^{-1} d = z_i \chi^{i-m}(\chi^m b) = z_i \chi^i b$ for all $i > m$. Hence, $z_m \chi^m a = d - \sum_{i=m+1}^n z_i \chi^i a$, showing that $f(a) = d$. $\square$

Now we turn to the subsets of G which are definable with constants. First, we observe that terms containing nonzero constants need not be monotone or surjective. Indeed, consider the map defined by the term

$$t(x) = \chi(x + a) - \chi(x - a)$$

for $0 \neq a \in G$. It is immediately seen to be symmetrical: $t(-g) = t(g)$. For $vg < va$, we have $t(g) = \chi g - \chi g = 0$ by (CV3). For $vg > va$, we compute $t(g) = \chi a - \chi(-a) = 2\chi a$, again by (CV3). For $vg = va$, we have $v(g + a) \geq va$ and $v(g - a) \geq va$, hence $|\chi(g + a)| \leq |\chi a|$ and $|\chi(g - a)| \leq |\chi a|$ by (CV5), showing that $|t(g)| \leq 2|\chi a|$. So t is neither monotone nor surjective. We have to adjust our demands.

A partition of G into finitely many convex subsets will be called a **finite convex partition**. We will say that a map $t: G \to G$ is **piecewise monotone** if there is a finite convex partition of G such that on every single partition set, the map t is monotone. Then it will follow that the preimage of a convex set, if intersected with one of the partition sets, is empty or convex. In particular, the sets $t^{-1}(G^{>0})$ and $t^{-1}(\{0\})$ will be empty or finite unions of convex sets (each partition set containing at most one of them). So in view of Lemma 3.1, the weak o-minimality of the theory of divisible centripetal contraction groups will be proved if we are able to show that every divisible centripetal contraction group (G, χ) has the following property:

(PM) *for every term $t(x)$ in the language $\mathcal{L}_{\mathrm{cg}}(G, x)$, the map $x \mapsto t(x)$ is piecewise monotone on G.*

As we did above, we will again determine a class of "polynomials" which represent other terms as maps from suitable subsets of G into G. Since now constants come into the game, it cannot be expected that χ-polynomials will suffice. So we have to introduce generalized χ-polynomials. First, we will recursively define **monic generalized χ-monomials** together with their **characteristic domains** :

- x and every $c \in G$ are monic generalized χ-monomials, and G is their characteristic domain;
- for every $a \in G$, the term $\chi[a](x) := \chi(x - a)$ is a monic generalized χ-monomial, and $G[a] := \{g \in G \mid v(g - a) > va \text{ if } a \neq 0\}$ is its characteristic domain;
- for $n > 1$ and all $a_1, \ldots, a_n \in G$, $\chi[a_1, \ldots, a_n](x) := \chi(\chi[a_1, \ldots, a_{n-1}](x) - a_n)$ is a monic generalized χ-monomial, provided that its characteristic domain

$$G[a_1, \ldots, a_n] := \{g \in G[a_1, \ldots, a_{n-1}] \mid v(\chi[a_1, \ldots, a_{n-1}](g) - a_n) > va_n \text{ if } a_n \neq 0\}$$

is nonempty. Let us note that $G[0] = G$ and for $n > 1$,

$$G[a_1, \ldots, a_n] = \begin{cases} G[a_1, \ldots, a_{n-1}] & \text{if } a_n = 0 \\ G[a_1, \ldots, a_n] \cap \chi[a_1, \ldots, a_{n-1}]^{-1}(a_n + \mathcal{M}_G^{va_n}) & \text{if } a_n \neq 0. \end{cases} \tag{2}$$

A term is called a **generalized χ-monomial** if it is of the form $zt(x)$ with $t(x)$ a generalized χ-monomial and $0 \neq z \in \mathbb{Z}$, and we define its characteristic domain to be the same as that of $t(x)$.

Given elements $a_1, \ldots, a_n \in G$, we consider

$$f(x) = \sum_{i=1}^{n} z_i \chi[a_1, \ldots, a_i](x) + z_0 x + c \tag{3}$$

with $z_0, \ldots, z_n \in \mathbb{Z}$, $z_n \neq 0$ and $c \in G$. Such a term will be called **generalized χ-polynomial**, provided that $G[a_1, \ldots, a_n]$, called its **characteristic domain**, is nonempty. The sequence $a_1, \ldots, a_n$ will be called the **characteristic sequence** of $f(x)$. The meaning of the characteristic domain is that it is the set of elements where the χ-polynomial, viewed as a map from G into G, cannot be represented in a simpler form. For instance, if $a, g \in G$ such that $v(g - a) = va$, then by (CV2),

$\chi[a](g) = \chi(g-a) = \pm\chi(a)$; up to the sign (which depends only on the sign of $g-a$), this is a constant not depending on g. If $v(g-a) < va$, then $v(g-a) = vg$, and we have that $\chi[a](g) = \chi(g-a) = \chi(g) = \chi[0](g)$ by virtue of (CV3).

We see at once from the definition that we run into difficulties if we try to add generalized χ-polynomials whose characteristic sequences are different. We will discuss this point later. Beforehand, we have to determine the basic properties of generalized χ-monomials and characteristic domains. We also study preimages of generalized χ-monomials and their intersections with characteristic domains. For the proof of Theorem 1.3, we will need the information that such intersections are empty or infinite or consist of just one constant which appears as a coefficient in the generalized χ-monomial. For this purpose, we need to keep track of open convex sets. Their intersections are always infinite, if nonempty. Note that by our general assumption that (G,χ) be a nontrivial centripetal precontraction group, the value set vG has no maximal element, according to Lemma 2.1 (note that $\infty \notin vG$ by definition). Hence if $d \neq 0$, then the convex subgroup $\mathcal{M}_G^{vd}$ is nontrivial and thus, every coset $c + \mathcal{M}_G^{vd}$ is open convex.

Lemma 3.5 *As a map from G into G, every generalized χ-monomial $z\chi[a_1,\ldots,a_n]$ is monotone; if it is monic, then it preserves $\leq$. The preimage $(z\chi[a_1,\ldots,a_n])^{-1}(S)$ of a convex set S is convex. If (G,χ) is a centripetal contraction group and $t(x)$ is a monic generalized χ-monomial which is not a constant, then $t(x)$ is surjective and the preimage of an open convex set is again an open convex set. The same holds for every generalized χ-monomial if (G,χ) is a divisible centripetal contraction group.*

Proof: For constants, there is nothing to show. Otherwise, our assertion follows by induction: x (the identity map) preserves $\leq$ and is surjective. If $t(x)$ preserves $\leq$, then the same is true for $t(x)-a$ and $\chi(t(x)-a)$, for arbitrary $a \in G$. If $t(x)$ is monotone, then the same is true for $zt(x)$, for every $z \in \mathbb{Z}$. If $t(x)$ is surjective and if (G,χ) is a centripetal contraction group, i.e. χ is surjective, then the same is true for $t(x)-a$ and $\chi(t(x)-a)$, for arbitrary $a \in G$. If in addition G is divisible, then also $zt(x)$ is surjective. The other assertions now follow from Lemma 2.3. □

Lemma 3.6 *Let $c, a_1, \ldots, a_n \in G$. Assume that $a_1 \neq c$ and set $d = c - a_1$. If $c \in G[a_1,\ldots,a_n]$, then all monomials $\chi[a_1,\ldots,a_i](x)$, $1 \leq i \leq n$, are constant on $c + \mathcal{M}_G^{vd}$. If $c + \mathcal{M}_G^{vd} \cap G[a_1,\ldots,a_n] \neq \emptyset$, then $c + \mathcal{M}_G^{vd} \subseteq G[a_1,\ldots,a_n]$.*

Proof: First, suppose that $c \in G[a_1,\ldots,a_n]$ and $c' \in c + \mathcal{M}_G^{vd}$. Then $v(c'-c) > vd = v(c-a_1)$. Writing $c'-a_1 = (c-a_1)+(c'-c)$, we thus find $\chi(c'-a_1) = \chi(c-a_1)$ by (CV3). Hence, also $\chi[a_1,\ldots,a_i](c') = \chi[a_1,\ldots,a_i](c)$ for $1 \leq i \leq n$. This also yields $c' \in G[a_1,\ldots,a_n]$, showing that $c + \mathcal{M}_G^{vd} \subseteq G[a_1,\ldots,a_n]$.

Now suppose that $c + \mathcal{M}_G^{vd} \cap G[a_1,\ldots,a_n] \neq \emptyset$ and that $\tilde{c}$ is an element of this intersection. Then by part a) of Lemma 2.2, $c + \mathcal{M}_G^{vd} = \tilde{c} + \mathcal{M}_G^{vd}$, and by what we have already proved, we obtain that $c + \mathcal{M}_G^{vd} \subseteq G[a_1,\ldots,a_n]$. □

From this lemma, we see that the coefficient a_1 plays a special role. For example, we have that $\chi[a_1]^{-1}(\{0\}) = \{a_1\}$. This set is convex but not open. The next lemma shows that only a_1 prevents a preimage from being open convex:

Lemma 3.7 *For every convex subset $S \subseteq G$, the set $\chi[a_1,\ldots,a_n]^{-1}(S)\setminus\{a_1\}$ is the union of at most two open convex subsets of G. Consequently, its intersection with any open convex set not containing a_1 is again an open convex set. Also $G[a_1,\ldots,a_n]$ is an open convex set, and if there is a nonzero element among $a_2,\ldots,a_n$, then $a_1 \notin G[a_1,\ldots,a_n]$.*

Proof: If $S \subseteq G$ is any convex subset of G, then in view of Lemma 3.5, the same holds for its preimage $\chi[a_1,\ldots,a_n]^{-1}(S)$. If $c \neq a_1$ is any element of this preimage, then by Lemma 3.6, also the open convex coset $c + \mathcal{M}_G^{v(c-a_1)}$ is contained in it. Consequently, $\chi[a_1,\ldots,a_n]^{-1}(S) \setminus \{a_1\}$ is the union of at most two open convex sets.

Now we consider the sets $G[a_1,\ldots,a_n]$. Trivially, G itself is open convex. Let us consider $G[a]$ for $a \in G$. If $a = 0$, then $G[a] = G$ and there is nothing to show. If $a \neq 0$, then

$$G[a] = \{g \in G \mid v(g-a) > va\} = a + \mathcal{M}_G^{va}$$

which is open convex. Now let $n > 1$ and $a_1,\ldots,a_n \in G$, and assume that we have already shown that $G[a_1,\ldots,a_{n-1}]$ is open convex. If $a_n = 0$, then $G[a_1,\ldots,a_{n-1},0] = G[a_1,\ldots,a_{n-1}]$ by (2) and there is nothing to show. If $a_n \neq 0$, then by (2),

$$G[a_1,\ldots,a_n] = G[a_1,\ldots,a_{n-1}] \cap \chi[a_1,\ldots,a_{n-1}]^{-1}(a_n + \mathcal{M}_G^{va_n})\ .$$

Since $a_n + \mathcal{M}_G^{va_n}$ is convex, both sets on the right hand side are convex, and consequently, $G[a_1,\ldots,a_n]$ is convex. Suppose that $i > 1$ is the minimal index such that $a_i \neq 0$ (it exists since we have assumed $a_n \neq 0$). Then $v(\chi[a_1,\ldots,a_{i-1}](a_1) - a_i) = v(\chi^{i-1}(0) - a_i) = va_i$, showing that a_1 does not lie in $G[a_1,\ldots,a_n]$.

Since $G[a_1,\ldots,a_n]$ is convex, we see that $G[a_1,\ldots,a_{n-1}]$ can only intersect with one of the possibly two open convex subsets which $\chi[a_1,\ldots,a_{n-1}]^{-1}(a_n + \mathcal{M}_G^{va_n}) \setminus \{a_1\}$ consists of, according to our above consideration. Since the intersection of open convex subsets of G is again an open convex subset of G, we have now proved that $G[a_1,\ldots,a_n]$ is open convex. □

Lemma 3.8 *For all $g \in G[a_1,\ldots,a_n]$,*

$$v\chi[a_1,\ldots,a_n](g) > v\chi[a_1,\ldots,a_{n-1}](g) > \ldots > v\chi[a_1](g) > v(g-a_1) \geq vg\ .$$

(Read "$\infty > \infty$" as "$\infty \geq \infty$", so that we do not have to exclude the case where some of the monomials are 0.)

Proof: Let $g \in G[a_1,\ldots,a_n]$. By definition of this set, we have $v(g-a_1) \geq vg$ and $v(\chi[a_1,\ldots,a_i](g) - a_{i+1}) \geq v\chi[a_1,\ldots,a_i](g)$ for $1 \leq i < n$. Since χ is centripetal, it follows by Lemma 2.1 that $v\chi[a_1,\ldots,a_{i+1}](g) = v\chi(\chi[a_1,\ldots,a_i](g) - a_{i+1}) > v(\chi[a_1,\ldots,a_i](g) - a_{i+1}) \geq v\chi[a_1,\ldots,a_i](g)$. □

Unlike in the case without constants, a term $t(x)$ in the language $\mathcal{L}_{\rm cg}(G,x)$ may not equal a generalized χ-polynomial as a map on all of G. We have to use suitable partitions of G. Given a finite convex partition P of G, we will say that $t(x)$ is P-**representable** if on every set M in P, $t(x)$ equals some generalized χ-polynomial f_M as a map from M to G, and M is contained in the characteristic domain of f_M. Our goal is to show that for every term $t(x)$ in the language $\mathcal{L}_{\mathrm{cg}}(G,x)$ there exists a finite convex partition P of G such that $t(x)$ is P-representable. This will be done by induction on the complexity of the terms built up in the language $\mathcal{L}_{\mathrm{cg}}(G,x)$. We have to consider addition and the application of "$-$" and "χ".

Let us first consider addition. Assume that t, t' are the terms which we have to add, and that they equal the generalized χ-polynomial f_M , $f'_{M'}$ on the partition sets M, M' respectively. If $M \cap M'$ is nonempty, then the task is to find a finite convex partition of $M \cap M'$ such that on every of the partition sets, $f_M + f'_{M'}$ equals a generalized χ-polynomial. But $M \cap M'$ is a subset of the intersection of the characteristic domains of f_M and $f'_{M'}$. So first we give such a partition for this intersection and then intersect the partition sets with $M \cap M'$ to obtain a finite convex partition of $M \cap M'$ (note that the latter set is itself convex). Thus, we can now forget about the sets M, M' and just find the suitable partition of the intersection of the characteristic domains (if this is nonempty). So we have to compute such intersections. Let $a, b \in G$ with $a < b$. If $a \neq 0 \neq b$, then by part b) of Lemma 2.2,

$$a + \mathcal{M}_G^{va} \cap b + \mathcal{M}_G^{vb} = \begin{cases} \emptyset & \text{if } v(a-b) = \min\{va, vb\} \\ a + \mathcal{M}_G^{va} = b + \mathcal{M}_G^{vb} & \text{if } v(a-b) > \min\{va, vb\}\ . \end{cases}$$

Hence, $G[a] \cap G[b] = G[a]$ if $v(a-b) > \min\{va, vb\}$. The same holds if $b = 0$ since then, $G[b] = G$. In both cases, (CV3), (CV1) and (C-) show that the maps $\chi(x-a)$ and $\chi(x-b)$ behave in the following way:

$\chi(x-b) = \chi(a-b)$ is constant for $x \in a + \mathcal{M}_G^{v(a-b)}$

$\chi(x-a) = \chi(b-a)$ is constant for $x \in b + \mathcal{M}_G^{v(a-b)}$

$\chi(x-a) = \chi(x-b)$ for $x \in a + \mathcal{M}_G^{va}$ with $x < a + \mathcal{M}_G^{v(a-b)}$ or $x > b + \mathcal{M}_G^{v(a-b)}$,

$\chi(x-a) = -\chi(x-b)$ for $x \in a + \mathcal{M}_G^{va}$ with $a + \mathcal{M}_G^{v(a-b)} < x < b + \mathcal{M}_G^{v(a-b)}$.

Note that $a + \mathcal{M}_G^{v(a-b)} < b + \mathcal{M}_G^{v(a-b)}$. The case $a = 0 < b$ is similar.

More generally, let also $a_1, \ldots, a_m \in G$. If $a \neq 0 \neq b$, then in view of (2) and part b) of Lemma 2.2, we obtain: If $v(a-b) = \min\{va, vb\}$, then $a + \mathcal{M}_G^{va} \cap b + \mathcal{M}_G^{vb} = \emptyset$, hence also $\chi[a_1, \ldots, a_m]^{-1}(a + \mathcal{M}_G^{va}) \cap \chi[a_1, \ldots, a_m]^{-1}(b + \mathcal{M}_G^{vb}) = \emptyset$ and thus, $G[a_1, \ldots, a_m, a] \cap G[a_1, \ldots, a_m, b] = \emptyset$. We can neglect this case since we only have to work with *nonempty* intersections of characteristic domains. If $v(a-b) > \min\{va, vb\}$, then we have $a + \mathcal{M}_G^{va} = b + \mathcal{M}_G^{vb}$ and $G[a_1, \ldots, a_m, a] \cap G[a_1, \ldots, a_m, b] = G[a_1, \ldots, a_m, a]$. This also holds if $b = 0$. For both cases, we define a finite convex partition of $G[a_1, \ldots, a_m, a]$ as follows (the case $a = 0$ is treated similarly). Let

$$\begin{aligned} M_2 &:= G[a_1, \ldots, a_m] \cap \chi[a_1, \ldots, a_m]^{-1}(a + \mathcal{M}_G^{v(a-b)})\ , \\ M_4 &:= G[a_1, \ldots, a_m] \cap \chi[a_1, \ldots, a_m]^{-1}(b + \mathcal{M}_G^{v(a-b)})\ . \end{aligned}$$

Since $a + \mathcal{M}_G^{v(a-b)} \subseteq G[a]$ and $b + \mathcal{M}_G^{v(a-b)} \subseteq G[b]$, we have that $M_2 \subseteq G[a_1, \ldots, a_m, a]$ and $M_4 \subseteq G[a_1, \ldots, a_m, a]$. Since $a + \mathcal{M}_G^{v(a-b)}$ and $b + \mathcal{M}_G^{v(a-b)}$ are convex subsets of G satisfying $a+\mathcal{M}_G^{v(a-b)} < b+\mathcal{M}_G^{v(a-b)}$, Lemma 3.7 shows that M_2 and M_4 are convex subsets of G satisfying $M_2 < M_4$. Now we set

$$\begin{aligned} M_1 &:= \{g\in G[a_1,\ldots,a_m,a] \mid g<M_2\}\;,\\ M_3 &:= \{g\in G[a_1,\ldots,a_m,a] \mid M_2<g<M_4\}\;,\\ M_5 &:= \{g\in G[a_1,\ldots,a_m,a] \mid g>M_4\}\;. \end{aligned}$$

Note that $M_1 < M_2 < M_3 < M_4 < M_5$. By Lemma 3.5, the map $\chi[a_1,\ldots,a_m]$ preserves $\leq$. This gives

$$M_1 = G[a_1,\ldots,a_m]\cap\chi[a_1,\ldots,a_m]^{-1}(\{g\in G\mid g<a+\mathcal{M}_G^{v(a-b)}\}\;,$$
$$M_3 = G[a_1,\ldots,a_m]\cap\chi[a_1,\ldots,a_m]^{-1}(\{g\in G\mid a+\mathcal{M}_G^{v(a-b)}<g<b+\mathcal{M}_G^{v(a-b)}\}\;,$$
$$M_5 = G[a_1,\ldots,a_m]\cap\chi[a_1,\ldots,a_m]^{-1}(\{g\in G\mid \{g\in G\mid g>b+\mathcal{M}_G^{v(a-b)}\}\;.$$

Since $a\neq 0$, we can infer from Lemma 3.7 that $a_1\notin G[a_1,\ldots,a_m]$ and that M_1,M_2,M_3,M_4,M_5 are open convex subsets. Thus, $P=\{M_1,M_2,M_3,M_4,M_5\}$ is a finite convex partition of $G[a_1,\ldots,a_m,a]$.

From our initial discussion of the maps $\chi(x-a)$ and $\chi(x-b)$, we now see that the maps $\chi[a_1,\ldots,a_n,a](x)$ and $\chi[a_1,\ldots,a_m,b](x)$ behave in the following way:

$$\begin{array}{ll} \chi[a_1,\ldots,a_m,a](x)=\chi[a_1,\ldots,a_m,b](x) & \text{for } x\in M_1\\ \chi[a_1,\ldots,a_m,b](x)=\chi(a-b)\ \text{ is constant} & \text{for } x\in M_2\\ \chi[a_1,\ldots,a_m,a](x)=-\chi[a_1,\ldots,a_m,b](x) & \text{for } x\in M_3\\ \chi[a_1,\ldots,a_m,a](x)=\chi(b-a)\ \text{ is constant} & \text{for } x\in M_4\\ \chi[a_1,\ldots,a_m,a](x)=\chi[a_1,\ldots,a_m,b](x) & \text{for } x\in M_5\,. \end{array}$$

Now assume that we are given two generalized χ-polynomials. If the characteristic sequence of one is an initial part of that of the other, then we have no difficulty to add the two polynomials: we only have to add the two constant terms and the coefficients of corresponding monomials. Addition works even in the sense of adding the polynomials as terms. Assume now that we have to add two generalized χ-polynomials $f(x)$, $h(x)$ with different characteristic sequences on the *nonempty* intersection of their characteristic domains. Our goal is to find a finite convex partition of M such that on each partition set, the sum of $f(x)$ and $h(x)$ as maps equals another generalized χ-polynomial. We consider the smallest index where the sequences differ: let $m\geq 0$, $a<b$ and $a_1,\ldots,a_m,a$ resp. $a_1,\ldots,a_m,b$ initial parts of the characteristic sequences of $f(x)$ resp. $h(x)$. We assume that $a\neq 0$ (the case $a=0$ is similar to the case $b=0$). Then we have shown above that $G[a_1,\ldots,a_m,a]\cap G[a_1,\ldots,a_m,b]=G[a_1,\ldots,a_m,a]$ and that this set splits into five open convex sets. Let us see what happens on these sets.

On M_2, $h(x)$ has the property that all monomials $\chi[a_1,\ldots,a_i](x)$ are constant for $i>m$. That is, as a map, $h(x)$ equals a generalized χ-polynomial $\tilde{h}(x)$ with characteristic sequence $a_1,\ldots,a_m$, and we may add $f(x)$ and $\tilde{h}(x)$ to obtain a map which equals the map $f+h$. Note that the characteristic domain of the generalized χ-polynomial $f+\tilde{h}$ is that of f, containing M_2.

On M_4, now $f(x)$ has the property that $h(x)$ had on M_2, and the situation is symmetrical. The characteristic domain will be that of h, containing M_4.

On M_1 and M_5, we can just replace b by a in $h(x)$ without changing its behaviour as a map. On M_3 we have to be a bit more carefull. The computation

$$\begin{aligned}\chi[a_1,\ldots,a_m,b,b'](x) &= \chi(\chi[a_1,\ldots,a_m,b](x)-b') = \chi(-\chi[a_1,\ldots,a_m,a](x)-b')\\ &= -\chi[a_1,\ldots,a_m,a,-b'](x)\end{aligned}$$

shows that we have to replace every b_i by $-b_i$ for $i > m+1$, and the coefficients z_i in $h(x)$ by $-z_i$ for $i > m$.

On the three sets M_1 , M_3 and M_5 we now have turned $h(x)$ into a generalized χ-polynomial whose characteristic sequence coincides with that of $f(x)$ on the first $m+1$ entries. So we proceed by induction on n, again partitioning the sets M_1 , M_3 and M_5 according to the rules that we have just established. Our induction will stop when the smaller length of the characteristic sequences of f and h is reached (it may as well stop earlier). Note that in every induction step, the characteristic domains obtained always include the intersection of the characteristic domains of f and h and thus all constructed partition sets.

We have proved:

Lemma 3.9 *Let $t_1(x)$ and $t_2(x)$ be terms in $\mathcal{L}_{\rm cg}(G,x)$ and P_1, P_2 finite convex partitions of G such that $t_i(x)$ is P_i-representable, for $i = 1,2$. Then there is a finite convex partition P of G which is finer than P_1 and P_2 such that the the term $t_1(x)+t_2(x)$ is P-representable.*

Since the application of the function "$-$" turns a generalized χ-polynomial into a generalized χ-polynomial (with the same characteristic domain and characteristic sequence), it now just remains to consider the application of χ. In the sequel, for $n = 0$ we set $G[a_1,\ldots,a_n] = G$ and $\chi[a_1,\ldots,a_n](x) = x$.

Lemma 3.10 *a) Let $f(x)$ be a nonzero generalized χ-polynomial of the form (3). Assume that m is the minimal index such that $z_m \neq 0$. Then for every g in the set $\chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n])$ (which contains the characteristic domain $G[a_1,\ldots,a_n]$), the values of all summands $z_i\chi[a_1,\ldots,a_i](g)$ in $f(g)$ with $m+1 \leq i \leq n$ are bigger than $vz_m\chi[a_1,\ldots,a_m](g)$ (or equal if $\chi[a_1,\ldots,a_m](g) = 0$). That gives*

$$\begin{aligned} v(f(g)-c) &= v(z_m\chi[a_1,\ldots,a_i](g))\\ \chi(f(g)-c) &= \chi(z_m\chi[a_1,\ldots,a_i](g)) \;=\; \mathrm{sign}(z_m)\cdot\chi[a_1,\ldots,a_i,0](g)\end{aligned}$$

for all $g \in \chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n])$.
b) As a map from its characteristic domain into G, every generalized χ-polynomial is monotone. More precisely, if f and m are as in a), then f is monotonically increasing if $z_m > 0$ and monotonically decreasing if $z_m < 0$. If $m = 0$ then f is strictly monotone. In any case, if S is a convex subset of G then also $G[a_1,\ldots,a_n]\cap f^{-1}(S)$ is convex.

Proof: a): If $g \in \chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n])$, then $\chi[a_1,\ldots,a_m](g) \in G[a_{m+1},\ldots,a_n]$ and by virtue of Lemma 3.8, all values $v\chi[a_1,\ldots,a_i](g)$, for $m+1 \leq$

$i \leq n$, are bigger than $v\chi[a_1, \ldots, a_m](g)$ (or equal if $\chi[a_1, \ldots, a_m](g) = 0$). Since multiplication by $z \neq 0$ does not change the value and multiplication by $z = 0$ lifts it to ∞, this proves our first assertion. The formula for $v(f(g) - c)$ follows by (V3), and the formula for $\chi(f(g) - c)$ follows by (NV3).

b): We have already shown in Lemma 3.5 that every generalized χ-monomial is monotone. In view of the assertion on the values of the χ-monomials that we have already proved in part a), the proof for generalized χ-polynomials is a straightforward adaptation of the proof of part b) of Lemma 3.2 (now working with $a, b \in G[a_1, \ldots, a_n]$). We leave it to the reader. Finally, the convexity follows from the monotonicity together with Lemma 2.3 and from the fact that $G[a_1, \ldots, a_n]$ is convex by virtue of Lemma 3.7. □

Let f and m be as in the lemma. We will compute $\chi f(g)$ for $g \in G[a_1, \ldots, a_n]$. If the constant term c of f is zero, then the solution is already given by part a) of the lemma. It remains to treat the case where $c \neq 0$. If $f(x) = zx - c$ with c not divisible by z in G, then we cannot expect that χf is a generalized χ-polynomial as defined above. We would have to choose a more general definition such that also $\chi(zx - c)$ is a generalized χ-polynomial. Everything would also work with this definition, but for the sake of simplicity we rather prefer to **require in the following that G is divisible**. So we can put $\tilde{c} := -z_m^{-1} c$.

If $m = n$, that is, if $f - c$ is a monomial, then (using (CZ) to remove z_m):

$$\chi f(g) = \mathrm{sign}(z_m) \cdot \chi(\chi[a_1, \ldots, a_m](g) - \tilde{c}) = \mathrm{sign}(z_m) \cdot \chi[a_1, \ldots, a_m, \tilde{c}](g)$$

for all $g \in G[a_1, \ldots, a_m]$, and we are done. If $m < n$, then we proceed as follows. We substitute

$$y(x) := \chi[a_1, \ldots, a_m](x) - a_{m+1}$$

and obtain

$$\left. \begin{array}{r} f(x) = F(y) := \sum_{i=m+2}^{n} z_i \chi[a_{m+2}, \ldots, a_i](\chi y) + z_{m+1} \chi y + z_m y - z_m d \\ \text{with} \quad d := -a_{m+1} - z_m^{-1} c = \tilde{c} - a_{m+1}\,. \end{array} \right\} \qquad (4)$$

We have that $vd = v(z_m a_{m+1} + c)$.

If $vy < vd$, then by part a) of the foregoing lemma, $z_m y$ is the summand of least value in $F(y)$, and by (CV3), we get $\chi F(y) = \chi(z_m y) = \mathrm{sign}(z_m) \cdot \chi y$. So we obtain

$$\chi f(g) = \mathrm{sign}(z_m) \cdot \chi[a_1, \ldots, a_{m+1}](g) \quad \text{for } g \in G[a_1, \ldots, a_n] \cap y^{-1}(G \setminus \mathcal{O}_G^{vd})\,. \quad (5)$$

If $vd = \infty$, i.e. $d = 0$, then $\mathcal{O}_G^{vd} = \{0\}$. In this case, we have that $y(g) = 0$ and $\chi[a_1, \ldots, a_{m+1}](g) = \chi y(g) = 0$ for all $g \in y^{-1}(\mathcal{O}_G^{vd})$. Lemma 3.8 tells us that then also the higher monomials are zero, so that we find $f(g) = 0$. Hence, the equation of (5) holds for all $g \in G$ in this special case, and we are done.

So let us assume from now on that $d \neq 0$. We are going to split up $G[a_1, \ldots, a_n]$ into five convex subsets. Since $\mathcal{O}_G^{vd}$ is a convex subgroup of G, the set $G \setminus \mathcal{O}_G^{vd}$ is the union of the two open convex subsets $G^{<0} \setminus \mathcal{O}_G^{vd}$ and $G^{>0} \setminus \mathcal{O}_G^{vd}$ of G (one initial and one cofinal). Also the sets $a_{m+1} + (G^{<0} \setminus \mathcal{O}_G^{vd})$ and $a_{m+1} + (G^{>0} \setminus \mathcal{O}_G^{vd})$ are open convex. By virtue of Lemma 3.5 and Lemma 3.7, also

$$\begin{aligned} M_1 &:= G[a_1,\ldots,a_n]\cap y^{-1}(G^{<0}\setminus \mathcal{O}_G^{vd}) \\ M_5 &:= G[a_1,\ldots,a_n]\cap y^{-1}(G^{>0}\setminus \mathcal{O}_G^{vd}) \end{aligned}$$

are convex, and if (G,χ) is a contraction group, then both sets are open convex. Moreover,

$$M' := G[a_1,\ldots,a_n]\cap y^{-1}(\mathcal{O}_G^{vd})$$

is convex, and we have that $\{M_1, M', M_5\}$ is a finite convex partition of $G[a_1,\ldots,a_n]$ such that $M_1 < M' < M_5$.

If $vy = v(y-d) = vd$, then by part a) of the foregoing lemma,

$$\chi F(y) = \chi(z_m y + z_m a_{m+1} + c) = \operatorname{sign}(z_m)\cdot\chi(y-d) = \operatorname{sign}(z_m)\cdot(\pm\chi d)\ ,$$

and with $\epsilon := \operatorname{sign}(z_m)\cdot\chi|d|$, we find that

$$\chi f(g) = \begin{cases} -\epsilon & \text{on } M_2 := G[a_1,\ldots,a_n]\cap y^{-1}(\{g\in\mathcal{O}_G^{vd}\mid g < d+\mathcal{M}_G^{vd}\}) \\ \epsilon & \text{on } M_4 := G[a_1,\ldots,a_n]\cap y^{-1}(\{g\in\mathcal{O}_G^{vd}\mid g > d+\mathcal{M}_G^{vd}\})\ . \end{cases} \tag{6}$$

Let us set

$$M_3 := G[a_1,\ldots,a_n]\cap y^{-1}(d+\mathcal{M}_G^{vd}) = G[a_1,\ldots,a_n]\cap\chi[a_1,\ldots,a_m]^{-1}(\tilde{c}+\mathcal{M}_G^{vd})\ .$$

Since

$$\{\,\{g\in\mathcal{O}_G^{vd}\mid g<d+\mathcal{M}_G^{vd}\}\ ,\ d+\mathcal{M}_G^{vd}\ ,\ \{g\in\mathcal{O}_G^{vd}\mid g>d+\mathcal{M}_G^{vd}\}\,\}$$

is a convex partition of $\mathcal{O}_G^{vd}$, the same arguments as before show $\{M_2, M_3, M_4\}$ to be a convex partition of M' satisfying $M_2 < M_3 < M_4$, and that these sets are open convex if (G,χ) is a contraction group. So, $\{M_1, M_2, M_3, M_4, M_5\}$ is a convex partition of $G[a_1,\ldots,a_n]$, satisfying $M_1 < M_2 < M_3 < M_4 < M_5$. On the sets M_1 and M_5 , $\chi f(g)$ equals the generalized χ-monomial $\operatorname{sign}(z_m)\cdot\chi[a_1,\ldots,a_{m+1}](x)$, according to (5). Note that M_1 and M_5 are contained in $G[a_1,\ldots,a_n]$, which in turn is a subset of the characteristic domain $G[a_1,\ldots,a_{m+1}]$ of this χ-monomial. On the sets M_2 and M_4 , $\chi f(g)$ equals the constant $-\epsilon$ resp. ϵ, according to (6). Note that M_2 and M_4 are subsets of the characteristic domain G of the constants $-\epsilon$ and ϵ.

Finally, it remains to analyze the behaviour of χf on M_3, or in other words, for $y\in d+\mathcal{M}_G^{vd}$. Fortunately, in this case we know the behaviour of the higher monomials very well: by virtue of (NV3), we have $\chi y = \chi d = \chi[a_{m+1}](-z_m^{-1}c) = \chi[a_{m+1}](\tilde{c})$ and thus also

$$\chi[a_{m+2},\ldots,a_i](\chi y) = \chi[a_{m+2},\ldots,a_i](\chi d) = \chi[a_{m+1},\ldots,a_i](\tilde{c})$$

for $m+2\le i\le n$. That is, if we set

$$c' = f^{[m]}(\tilde{c}) \quad \text{with} \quad f^{[m]}(x) := x - z_m^{-1}\cdot\sum_{i=m+1}^{n} z_i\chi[a_{m+1},\ldots,a_i](x)\ , \tag{7}$$

then for all $g\in M_3$ we have

$$\begin{aligned} f(g) &= z_m\cdot(y+a_{m+1}-c')\;,\\ \chi f(g) &= \mathrm{sign}(z_m)\cdot\chi(y+a_{m+1}-c')=\mathrm{sign}(z_m)\cdot\chi[a_1,\ldots,a_m,c'](g)\;. \end{aligned}$$

Note that by part b) of Lemma 3.10, the generalized χ-polynomial $f^{[m]}(x)$ is strictly monotone on its characteristic domain $G[a_{m+1},\ldots,a_n]$.

Now it remains to show that the partition set M_3 is contained in the characteristic domain $G[a_1,\ldots,a_m,c']$ of the χ-monomial $\mathrm{sign}(z_m)\cdot\chi[a_1,\ldots,a_m,c'](g)$. For later use, let us prove a little bit more. Assume that

$$\tilde{c}+\mathcal{M}_G^{vd}\cap G[a_{m+1},\ldots,a_n]\neq\emptyset\;. \tag{8}$$

Then

$$v(\tilde{c}-c')>vd=v(\tilde{c}-a_{m+1})>va_{m+1}=v\tilde{c}=vc'\;,$$

and it follows that $\tilde{c},c'\in G[a_{m+1},\ldots,a_n]$. Indeed, let us assume (8). Then by Lemma 3.6, $\tilde{c}+\mathcal{M}_G^{vd}\subseteq G[a_{m+1},\ldots,a_n]$. In particular, $\tilde{c}\in G[a_{m+1},\ldots,a_n]$, and from Lemma 3.8 we deduce that $v\chi[a_{m+1},\ldots,a_i](\tilde{c})>v\chi(\tilde{c}-a_{m+1})>v(\tilde{c}-a_{m+1})=vd>va_{m+1}=v\tilde{c}$ for $m+1\leq i\leq n$. Since $v\tilde{c}=vc$ and multiplication by z_m does not change the value, it follows from the definition of c' that $v(\tilde{c}-c')>vd$. Hence, $vc'=v\tilde{c}$ and $c'+\mathcal{M}_G^{vd}=\tilde{c}+\mathcal{M}_G^{vd}$, showing that $c'\in G[a_{m+1},\ldots,a_n]$.

As a special case, we obtain the fact that $\tilde{c}\in G[a_{m+1},\ldots,a_n]$ implies that $c'\in G[a_{m+1},\ldots,a_n]$. That is, $f^{[m]}$ maps $G[a_{m+1},\ldots,a_n]$ into $G[a_{m+1},\ldots,a_n]$.

Further, if $M_3\neq\emptyset$, then (8) holds, and by what we have just proved, we get

$$\begin{aligned} M_3 &= G[a_1,\ldots,a_n]\cap\chi[a_1,\ldots,a_m]^{-1}(\tilde{c}+\mathcal{M}_G^{vd})\\ &= G[a_1,\ldots,a_n]\cap\chi[a_1,\ldots,a_m]^{-1}(c'+\mathcal{M}_G^{vd})\\ &\subseteq G[a_1,\ldots,a_n]\cap\chi[a_1,\ldots,a_m]^{-1}(c'+\mathcal{M}_G^{vc'})\subseteq G[a_1,\ldots,a_m,c']\;. \end{aligned}$$

We have shown that on all sets of the convex partition $\{M_1,M_2,M_3,M_4,M_5\}$ of $G[a_1,\ldots,a_n]$, the map χf equals a suitable χ-monomial (which may even be a constant), such that the set is contained in the characteristic domain of the χ-monomial. As before in the case of addition, we have to intersect the above partition with the convex set $M\subseteq G[a_1,\ldots,a_n]$ on which f was given, obtaining a new convex partition. If M is open convex and if (G,χ) is a contraction group, then also the intersections will be open convex. We have proved:

Lemma 3.11 *Let $t(x)$ be a term in $\mathcal{L}_{\mathrm{cg}}(G,x)$ and P a finite convex partition of G such that $t(x)$ is P-representable. Then there is a finite convex partition P' of G which is finer than P such that the term $\chi t(x)$ is P'-representable.*

In view of Lemma 3.1, the following lemma proves Theorem 1.2.

Lemma 3.12 *For every centripetal precontraction group (G,χ) and every term $t(x)$ in the language $\mathcal{L}_{\mathrm{cg}}(G,x)$, there is a finite convex partition P of G such that $t(x)$ is P-representable. Every centripetal precontraction group has the property (PM).*

Proof: The proof proceeds by induction on the complexity of the terms built up in the language $\mathcal{L}_{\mathrm{cg}}(G,x)$. With $P=\{G\}$, our assertion is true for all terms $zx+c$ with $z\in\mathbb{Z}$ and $c\in G$. Next, changing t to $-t$ needs no change of the partition. The addition of two terms was treated in Lemma 3.9. Finally, the application of χ to a term was treated in Lemma 3.11.

The second assertion follows from the first by virtue of Lemma 2.3 and part b) of Lemma 3.10. □

From our observations about the partition sets during the above considerations, we obtain the following addition to the foregoing lemma:

Lemma 3.13 *If (G,χ) is a contraction group, then the partition P mentioned in Lemma 3.12 can be chosen so as to contain only open convex sets.*

Let us describe the sets of roots of generalized χ-polynomials f in a divisible centripetal contraction group (G,χ). We start with two special cases that we derive from our above considerations. If $f-c$ is a generalized χ-monomial, say $z_n\chi[a_1,\ldots,a_n](x)$, then its set of roots is precisely

$$\chi[a_1,\ldots,a_n]^{-1}(\tilde{c})\,,\quad \text{where } \tilde{c}=-z_n^{-1}c\;.$$

Now let f be represented in the form (4) with $m<n$. In the special case $d=0$, which means that $\tilde{c}=a_{m+1}$, we have seen that f is zero on the set

$$y^{-1}(\{0\})=\chi[a_1,\ldots,a_m]^{-1}(a_{m+1})\;. \tag{9}$$

From part a) of Lemma 3.10 we conclude that every root

$$g\in\chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n])$$

must satisfy $y(g)=0$. Consequently, every such root lies in the set (9). This shows that there is a root in $\chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n])$ if and only if

$$\tilde{c}\in G[a_{m+1},\ldots,a_n]\;.$$

Note that $\chi[a_1,\ldots,a_m]$ is surjective because we have assumed (G,χ) to be a contraction group. Further, we see that if c' is defined as in (7), then $c'=\tilde{c}=a_{m+1}$ because all higher monomials are zero. So we can write the set (9) as

$$\chi[a_1,\ldots,a_m]^{-1}(c')\;.$$

Formally, one obtains the same for the case where $f-c$ is a generalized χ-monomial since then, the sum over the higher monomials in (7) is empty. With the convention that $G[a_{n+1},\ldots,a_n]=G$, the above condition on $\tilde{c}$ for f to have a root also holds in this case, because $\tilde{c}\in G$ always holds and f always has a root.

The following lemma shows that our results also hold in general.

Lemma 3.14 *Let (G,χ) be a divisible centripetal contraction group, and let f and m be as in Lemma 3.10. Then the equation $f(x)=0$ has a solution in the set*

$$\chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n]) \tag{10}$$

if and only if $\tilde{c} = -z_m^{-1}c \in G[a_{m+1},\ldots,a_n]$. If this is the case, then with c' as defined in (7), the set of solutions in the set (10) is precisely the nonempty convex set

$$\chi[a_1,\ldots,a_m]^{-1}(c')\ .$$

Proof: We can assume that $m<n$ and $c\neq -z_m a_{m+1}$ because we have already treated the remaining cases. Let f be represented in the form (4). By our assumption, $d\neq 0$.

Suppose that $\tilde{c}\in G[a_{m+1},\ldots,a_n]$ and let $d=\tilde{c}-a_{m+1}$. We have already shown that this implies $c'\in\tilde{c}+\mathcal{M}_G^{vd}$ and $c'\in G[a_{m+1},\ldots,a_n]$. But if this holds, then there is some a in the set (10) such that $\chi[a_1,\ldots,a_m](a)=c'$. Indeed, since we have assumed (G,χ) to be a divisible centripetal contraction group, we know by Lemma 3.5 that every generalized χ-monomial is surjective (if it is not a constant). Using the fact that the monomials $\chi[a_{m+1},\ldots,a_i](x)$ are constant on $\tilde{c}+\mathcal{M}_G^{vd}$, we obtain:

$$\begin{aligned}
f(a) &= c+z_m\chi[a_1,\ldots,a_m](a)+\sum_{i=m+1}^{n} z_i\chi[a_1,\ldots,a_i](a)\\
&= c+z_m c'+\sum_{i=m+1}^{n} z_i\chi[a_{m+1},\ldots,a_i](c') \qquad (11)\\
&= c+z_m c'+\sum_{i=m+1}^{n} z_i\chi[a_{m+1},\ldots,a_i](\tilde{c}) \;=\; c+z_m c'-z_m c'-c \;=\; 0\ .
\end{aligned}$$

We have thereby shown that every $a\in\chi[a_1,\ldots,a_m]^{-1}(c')$ satisfies $f(a)=0$. Since $c'\in G[a_{m+1},\ldots,a_n]$, the set $\chi[a_1,\ldots,a_m]^{-1}(c')$ is a subset of (10). It is nonempty: Indeed, since we assume that (G,χ) is a contraction group, $\chi[a_1,\ldots,a_m]$ is surjective.

For the converse, assume that there is some a in the set (10) such that $f(a)=0$. Set $c_a:=\chi[a_1,\ldots,a_m](a)\in G[a_{m+1},\ldots,a_n]$. Then in view of Lemma 3.8,

$$v(z_m c_a+c)=v\left(\sum_{i=m+1}^{n} z_i\chi[a_{m+1},\ldots,a_i](c_a)\right)\geq v\chi[a_{m+1}](c_a)>v(c_a-a_{m+1})$$

and thus, $v(c_a-\tilde{c})=v(c_a+z_m^{-1}c)>v(c_a-a_{m+1})=v(\tilde{c}-a_{m+1})$. This shows that $c_a\in\tilde{c}+\mathcal{M}_G^{vd}$. By Lemma 3.6 it also follows that $\tilde{c}\in G[a_{m+1},\ldots,a_n]$. It remains to show that $a\in\chi[a_1,\ldots,a_m]^{-1}(c')$. Using again the fact that the monomials $\chi[a_{m+1},\ldots,a_i](x)$ are constant on $\tilde{c}+\mathcal{M}_G^{vd}$, we compute:

$$0\;=\;f(a)\;=\;c+z_m\chi[a_1,\ldots,a_m](a)+\sum_{i=m+1}^{n} z_i\chi[a_1,\ldots,a_i](a)$$

$$= c + z_m c_a + \sum_{i=m+1}^{n} z_i \chi[a_{m+1}, \dots, a_i](c_a)$$
$$= c + z_m c_a + \sum_{i=m+1}^{n} z_i \chi[a_{m+1}, \dots, a_i](\tilde{c}) \ .$$

It follows that

$$c_a \ = \ \tilde{c} - z_m^{-1} \sum_{i=m+1}^{n} z_i \chi[a_{m+1}, \dots, a_i](\tilde{c}) \ = \ c' \ ,$$

showing that $a \in \chi[a_1, \dots, a_m]^{-1}(c')$. We have now shown that $\chi[a_1, \dots, a_m]^{-1}(c')$ is the set of roots of f in the set (10), provided that there is any root in this set. $\square$

Assume f and m to be as in Lemma 3.10. Given $b \in G$, the foregoing lemma applied to $f - b$ helps us to determine the preimage $f^{-1}(b) \cap G[a_1, \dots, a_n]$. Indeed, it is the convex set

$$G_b := G[a_1, \dots, a_n] \cap \chi[a_1, \dots, a_m]^{-1}(c'_b)$$

with c'_b determined by (7), applied to $f - b$ in the place of f. By Lemma 3.7, $G[a_1, \dots, a_n]$ is open convex, and $\chi[a_1, \dots, a_m]^{-1}(c') \setminus \{a_1\}$ is the union of at most two open convex sets. Consequently, also $G_b \setminus \{a_1\}$ is the union of at most two open convex sets. In particular, G_b is infinite if it is not empty or equal to $\{a_1\}$.

Now we can give the

Proof Theorem 1.3. Recall that an element a is called definable if the singleton $\{a\}$ is definable, and that the definable closure of a set A is the set of all elements which are definable with constants from A.

Let the term $t \in \mathcal{L}_{\rm cg}(G, x)$ be represented by f on a convex partition set $M \subseteq G[a_1, \dots, a_n]$. In view of Lemma 3.13 we can assume that M is open convex. Let S be any convex subset of G. Now $M \cap f^{-1}(S)$ is the union of sets of the form $M \cap G_b$ which like G_b are infinite if they are not empty or equal to $\{a_1\}$. This proves: $M \cap f^{-1}(S)$ is infinite if it is not empty or equal to $\{a_1\}$. We conclude that in a divisible centripetal contraction group (G, χ), generalized χ-polynomials in which χ appears can only serve to define infinite preimages, apart from the coefficient a_1. On the other hand, the terms in the simple language of ordered groups are able to define more ample finite sets: with given constants in a subset $A \subseteq G$, precisely every element of the divisible hull of the subgroup generated by A can be defined in (G, χ). Altogether, we find that the set of all elements definable by terms in the language $\mathcal{L}_{\rm cg}(A, x)$ is precisely the divisible hull of the precontraction group generated by A. Since for ordered structures, definable closure is the same as algebraic closure, this proves Theorem 1.3. $\square$

Our remark succeeding to Theorem 1.3 is proved as follows. Since (G, χ) is centripetal, $\chi b = a$ implies that $vb < va$. For the same reason, all elements in the precontraction group H generated by a have value $\geq va$, and the same is true for its divisible hull $\tilde{H}$ because it has the same value set as H. By definiton of v, the same is also true for the convex hull of $\tilde{H}$ in G. Consequently, b is not an element of this convex hull.

For the conclusion of this paper, let us derive two corollaries from Lemma 3.14. The first is a criterion for roots which resembles a bit the Hensel-Rychlik criterion for roots of polynomials over henselian valued fields (because the bound does not depend on the approximative root).

Corollary 3.15 *Let (G,χ) be a divisible centripetal contraction group, and let f and m be as in Lemma 3.10. Assume that $m<n$ and $c\neq -z_m a_{m+1}$. Then the equation $f(x)=0$ has a solution in the set (10) if and only if*

$$\exists x\in\chi[a_1,\ldots,a_m]^{-1}(G[a_{m+1},\ldots,a_n]):\ vf(x)>v(\tilde{c}-a_{m+1})\ .$$

Proof: If $f(a)=0$ then trivially, $vf(a)>v(\tilde{c}-a_{m+1})$. For the converse, suppose that $vf(b)>v(\tilde{c}-a_{m+1})$ for some b in the set (10). We set $c_b:=\chi[a_1,\ldots,a_m](b)$. If $vf(b)=v(c_b-\tilde{c})$ then we obtain that $v(c_b-\tilde{c})>v(\tilde{c}-a_{m+1})=v(c_b-a_{m+1})$. On the other hand, if $vf(b)\neq v(c_b-\tilde{c})$ then there must be some $i>m$ such that $v(c_b-\tilde{c})\geq vz_i\chi[a_{m+1},\ldots,a_i](c_b)$ which in view of $c_b\in G[a_{m+1},\ldots,a_n]$ and Lemma 3.8 yields that $v(c_b-\tilde{c})>v(c_b-a_{m+1})$. We have shown that the latter holds in every case. It follows from Lemma 3.6 that $\tilde{c}\in G[a_{m+1},\ldots,a_n]$, which by Lemma 3.14 yields the existence of the required root. □

Finally, let us give an analogue to Corollary 3.4:

Corollary 3.16 *Let (G,χ) be a divisible centripetal contraction group. If f is a generalized χ-polynomial and M is a convex subset of its characteristic domain, then $f(M)$ is a convex subset of G.*

Proof: Let f and m be as in Lemma 3.10. Suppose that there are $g_1,g_2\in M$ and $b\in G$ such that $f(g_1)<b<f(g_2)$. We want to show that there is $g_b\in M$ such that $f(g_b)=b$. Since $f-f(g_i)$, $i=1,2$, have a zero in M, and since M is a subset of $G[a_1,\ldots,a_n]$ and hence also of the set (10), Lemma 3.14 shows that $\tilde{c}_i:=-z_m^{-1}(c-f(g_i))\in G[a_{m+1},\ldots,a_n]$ and that $g_i\in\chi[a_1,\ldots,a_m]^{-1}(c'_i)$, where $c'_i:=f^{[m]}(\tilde{c}_i)$ as defined in (7). The element $\tilde{c}_b:=-z_m^{-1}(c-b)$ lies properly between $\tilde{c}_1$ and $\tilde{c}_2$ and since $G[a_{m+1},\ldots,a_n]$ is convex, it is also contained in $G[a_{m+1},\ldots,a_n]$. Thus, setting $c'_b:=f^{[m]}(\tilde{c}_b)$, we deduce from Lemma 3.14 that all elements of the set $\chi[a_1,\ldots,a_m]^{-1}(c'_b)$ are solutions for $f(x)-b=0$. We also find that c'_b lies properly between c'_1 and c'_2 since $f^{[m]}$ is strictly monotone. Since $\chi[a_1,\ldots,a_m]$ is monotone, it follows that all elements of $\chi[a_1,\ldots,a_m]^{-1}(c'_b)$ lie between g_1 and g_2 and are thus contained in M because M was assumed to be convex. We have shown that the equation $f(x)=b$ admits a solution in M. □

References

[D–M–M] van den Dries, L. – Macintyre, A. – Marker, D.: *The elementary theory of restricted analytic functions with exponentiation*, Annals of Math. **140** (1994), 183–205

[K] Kuhlmann, F.-V.: *Abelian groups with contractions I*, in: Proceedings of the Oberwolfach Conference on Abelian Groups 1993, Amer. Math. Soc. Contemporary Mathematics **171** (1994), 217–241

[KS] Kuhlmann, S.: *On the structure of nonarchimedean exponential fields I*, Archive for Math. Logic **34** (1995), 145–182

[K–K1] Kuhlmann, F.-V. – Kuhlmann, S.: *On the structure of nonarchimedean exponential fields II*, Comm. in Algebra **22(12)** (1994), 5079–5103

[K–K2] Kuhlmann, F.-V. – Kuhlmann, S.: *Residue fields of arbitrary convex valuations on restricted analytic fields with exponentiation I*, The Fields Institute Preprint Series, Toronto, (1996); also in: Structures Algébriques Ordonnées, Séminaire Paris VII (1997)

[K–K–S] Kuhlmann, F.-V. – Kuhlmann, S. – Shelah, S.: *Exponentiation in power series fields*, Proc. Amer. Math. Soc. **125** (1997), 3177–3183

[MP–M–S] Macpherson, D. – Marker, D. – Steinhorn, Ch.: *Weakly o-minimal structures and real closed fields*, preprint (1994)

[P–S] Pillay, A. – Steinhorn, Ch.: *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295** (1986), 565–592

[W] Wilkie, A. J.: *Model completeness results for expansions of the real field*, J. Amer. Math. Soc. **9** (1996), 1051–1094