Additive polynomials and the elementary properties of $\mathbb{F}_p((t))$

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$\mathbb{F}_p((t))$ denotes the field of formal Laurent series over the field with p elements; it carries the t-adic valuation v_t . A well known open problem in model theoretic algebra is to find a complete recursive axiom system for the elementary theory of $\mathbb{F}_p((t))$. This would yield that it is decidable. A similar result was shown by Ax–Kochen and Ershov for the p-adics $\mathbb{Q}_p, v_p$. An adaptation of their axiom system to the case of $\mathbb{F}_p((t))$ is:

$(*)$ "henselian defectless valued field of characteristic p with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$".

Theorem 1 (K., 1989)
This axiom system is not complete. While

$$K \;=\; \mathcal{O}_K \;+\; \wp(K) \;+\; tK^p \;+\; \ldots \;+\; t^{p-1}K^p \tag{1}$$

(with $\mathcal{O}_K$ the valuation ring and $\wp(X) = X^p - X$) holds for $K = \mathbb{F}_p((t))$, there is an extension of $\mathbb{F}_p((t))$ of transcendence degree 1, with $v(t)$ still the least positive element in the value group, which satisfies $(*)$ but not (1).

As $\wp(K)$, tK^p, … , $t^{p-1}K^p$ are the images of K under the additive polynomials $\wp(X)$, tX^p, … , $t^{p-1}X^p$, the question arises what one can say elementarily about subgroups of $(K, +)$ of the form

$$f_1(K) \;+\; \ldots \;+\; f_n(K) \tag{2}$$

for any additive polynomials $f_1, \ldots, f_n$ (a polynomial f is called additive if $f(a + b) = f(a) + f(b)$ for all a, b).

Definition: Let (K, v) be a valued field and S a subset of K. We say that S has the **optimal approximation property (OA)** if each $x \in K$ has an optimal approximation from S w.r.t. the v-metric, that is, there is some (in general non-unique) closest point in S. This property is elementary if S is definable.

Theorem 2 (K., 1998)
There is an elementary condition for the coefficients of $f_1, \ldots, f_n$ such that (2) has (OA) in **every** *maximally valued field (K, v) which contains the coefficients.*

Using the local compactness of $\mathbb{F}_p((t))$, we can do better:

Theorem 3 (van den Dries and K., 1999)
In $\mathbb{F}_p((t))$, (2) has (OA) for **all** *choices of $f_1, \ldots, f_n$.*

Quantifying over the coefficients of all additive polynomials, we obtain a recursive elementary axiom scheme (S) which holds in $\mathbb{F}_p((t))$.

Question: Is $(*)$ + (S) complete?