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VALUATION THEORY

OF FIELDS, ABELIAN GROUPS AND MODULES

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Chapter 1

1.1 Introduction

Since the work of Hensel, valuation theory is an important tool in number theory, and when it came into existence, it was itself the subject of deep mathematical research. Then it was W. Krull who introduced the *general valuation theory*, studying valuations of arbitrary rank. For this reason, the valuations that we are studying are commonly known as **Krull valuations**. Enabled by his work, valuation theory witnessed its second heyday when it was fruitfully applied in algebraic geometry and field theory. The book of O. Zariski [ZA–SA2] represents this aspect, and it is certainly one of the few books that is till the present day quite useful as an introduction to the theory of valuations of arbitrary rank. Later, valuation theory was replaced by other tools in algebraic geometry, and there are famous geometers who tell that they don't know anything about it. In contrast to this, it has maintained its importance in number theory. But although there are books which present interesting new aspects in the exposition of valuation theory (for instance, J. Neukirch's book [NEU]), valuation theory is not anymore the main subject of research.

The appearance of model theoretic algebra initiated a third period in which valuations themselves were the subject of research. The work of J. Ax and S. Kochen as well as that of Y. Ershov brought main impulses to this research. Ax and Kochen applied the model theory of valued fields of residue characteristic 0 to number theory (cf. Section 14.9), which generated a widespread interest. Since then, model theorists have successfully treated various classes of valued fields, and some branches of field theory have been fruitfully "supported" by model theoretic results for valued fields. Let us just mention the books on formally real fields [PR1] by A. Prestel and on formally p-adic fields [PR–ROQ] by Prestel and P. Roquette, which both contain a reasonable portion of model theory. Model theorist's work on valuations has also brought quite interesting purely valuation theoretical results. As examples we would like to mention the work of F. Delon, Y. Ershov and S. Basarab. While valuation theory still persists implicitly in the works of model theorists, it has ceased to be a main subject. From a certain point on, the remaining open problems had turned out to be too complex to be handled with the commonly known tools of valuation theory. The most striking open question is whether the power series field $\mathbb{F}_p((t))$ (which carries a canonical valuation) has a decidable theory. Although this appears to be a very nice field, it turns out that we know little about its valuation theoretical properties. At this point, we want to emphasize that the model theoretic question is just an incentive to think about these properties and that even mathematicians who do not believe in model theory should be able to concede some importance to the purely field theoretical questions about $\mathbb{F}_p((t))$.

There are only a few books that concentrate on valuation theory. From the books of O. F. G. Schilling [SCHI], P. Ribenboim [RIB12] and O. Endler [END2], generations of students have learnt valuation theory. While the books of Ribenboim and Endler show a more modern and nicely developed state of the theory, they are not easily accessable or not quite easy to read. It seems that the lack of modern books on valuation theory is caused by the fact that valuation theory is not any more a center of research interest. On the other hand, it is frequently used in recent research, for instance by field theorists, and this has also led to new valuation theoretical results and approaches which can not be found in any book. The only exception seems to be the recent book of S. Warner [WA3] on topological fields, which contains many interesting details on valuation theory, together with comprehensive exercises and historical remarks.

The present work intends to fill the gap and revitalize the interest in valuation theory and its open questions. It is planned to combine it with some chapters from the author's thesis [KU1] and to publish it as a book. We have omitted these chapters here since they will be taken over without essential changes. We have also omitted details in some other sections (like the one on the composition of valuations). We have cited some results of the omitted chapters in Section 12.7. The present exposition comprises in some sense the complement to these chapters. It consists of

• introductions to several topics of general valuation theory,

• a thorough study of selected notions and phenomena of valuation theory,

• an approach to main open problems (which can also be understood as a sketch of a working plan),

• an introduction to basic notions of model theoretic algebra and a deduction of main results on the model theory of valued fields.

Let us describe some main points of this exposition. We emphasize phenomena that are characteristical for the borderline between classical valuation theory and open problems. The main example for this is the defect of finite extensions of valued fields. Roughly speaking, whereever a nontrivial defect appeared and one could not get rid of it, classical valuation gave up and model theoretic algebra had no chance. But it will be made very clear in our exposition that without mastering the defect, the open problems will not be accessible. So we have devoted a whole chapter to the defect (Chapter 8), giving detailed examples and showing how to measure it. Many notions in that chapter (like **valuation disjoint extensions**) were introduced in [KU1] and are due to P. Roquette. While being a more or less neglected tool in [KU1], they are now treated systematically, thereby facilitating the original proofs.

Originally, the interest in the defect originates from the above mentioned model theoretical problem. An important impact of model theory is that it forces us to consider the most general valuations (e.g., there is no restriction to finite rank). Further, the open problems force us to study valuations of positive residue characteristic and their bad behaviour, namely the defect. So the present work can also be seen as an exposition of general valuation theory which opens up the classical approach towards the consideration of these unsufficiently studied phenomena. We should mention that there is some evidence that the theory of valued fields of positive characteristic may regain a certain importance for the algebraic geometry of positive characteristic and its open questions.

On the other hand, the study of these phenomena has also led to an introduction of and

an emphasis on new notions even in the more classical part of valuation theory. Throughout this exposition, we will be concerned over and over again with the complementary notions of immediate extensions and extensions admitting a valuation basis (or a valuation transcendence basis). These notions have turned out to play a key role in many respects. For instance, in good situations one can split up a given extension into two extensions, one immediate, one having a valuation basis (or a valuation transcendence basis). Then one can study them separately. Their meaning is that the former has the least relation to the respective value groups and residue fields, while the latter is in the strongest possible way determined by them. This approach of separating the two types of extensions is also crucial for our treatment to the model theory of valued fields. For this treatment and many other questions in valuation theory, a further new notion is of great importance. It is the notion of a **tame extension**, which we introduce in Chapter 12. The complementary notion of **purely wild extension** encodes all the bad behaviour of valuations of residue characteristic $p > 0$. The study of the relation between tame and purely wild extensions is a further step on our way to master the defect. The notion of a **tame field** (a valued field whose algebraic extensions are all tame) plays a key role in our exposition of the model theory of valued fields since it allows to unify the treatment of various different classes of valued fields. Tame fields have a good model theory, and in some respect they represent all classes of valued fields whose model theory is known. (The class of finitely ramified fields is derived from tame fields by a composition with a very special valuation of rank one which is not tame but nevertheless well-behaved).

A second important impact of model theory is that it forces us to work as much as possible with **elementary** notions. Hence, instead of working with completions or maximal fields, we rather use the notion of a henselian field. For this reason, we also study the different equivalent forms of this notion in detail (cf. Section 9.1, Section 9.2, Section 9.3, Section 10.1). In this connection, we point out that the notion of a **complete field** has undergone a remarkable shift of its importance in valuation theory. While it does not play a role anymore in the theory of extensions of valuations in our exposition, it becomes reasonably important when different sorts of defects have to be classified.

For the study of immediate extensions, we have developed a new tool, called **approximation types**. They admit a more precise description than pseudo Cauchy sequences do. But as it is also the case for the notion of immediate extensions, it can already be introduced for less complex structures, namely for ultrametric spaces and for valued abelian groups. These structures are underlying structures of valued fields, but they also have their independent interest. At this point, it is high time to emphasize that valuation theory is not restricted to field theory. It witnessed interesting applications in the theory of abelian groups, and also ultrametric spaces are studied independently. However, a detailed exposition of the common points in these different branches does not exist to our knowledge. One reason for writing such an exposition is the fact that valuation theory provides a common way of thinking in all of these branches, and this can be made very clear by such notions as immediate extensions and valuation bases. Common is also the goal to study the structure with respect to invariants which are given by value group and residue field resp. by the skeleton. But even more important, it has turned out that valuation theory of fields is not as separated from the valuation theory of abelian groups as it is commonly assumed by field theorists. In many situations, extensions of valued fields have been studied by considering the valued vector spaces connected with them. A more recent and rather

convincing example is provided by exponential fields, and for this reason, we give a brief introduction to their theory and more generally, to the theory of the natural valuation of nonarchimedean ordered fields and abelian groups. In the situation of exponential fields, we have two ordered groups, the additive group of the field and the multiplicative group of positive elements, and both are isomorphic (as ordered groups). So here we have a field theoretical situation where we are really studying ordered abelian groups, and if the field is nonarchimeden ordered, then we can study them by considering their natural valuations. It was the idea of S. Kuhlmann to study them by determining their skeletons. This has led to criteria for the existence of exponentials on given nonarchimedean fields and to further interesting results (cf. [KUS1,2], [KU–KUS1,2]). Although we are in a field theoretical situation, these results are not obtained by a purely field theoretical valuation theory. Moreover, the valuation theoretical study of the value groups of nonarchimedean exponential fields has led to a comprehensive description of their model theory (cf. [KU4,5]), which would have been rather inaccessible without the notion of the natural valuation of an ordered abelian group.

The second main example is dominating a large part of our exposition. Following a suggestion by L. v. d. Dries, the open questions about the field $\mathbb{F}_p((t))$ can be studied by considering it as a valued module admitting multiplication by t and application of the Frobenius. There is some evidence that the main problems are encoded in this structure. We have made the idea more precise. Our version is to ask for the structure of $\mathbb{F}_p((t))$ as a module over a ring of additive polynomials with coefficients in some subfield which contains t. To introduce this structure, we have devoted a chapter to additive polynomials (Chapter 11). They also have important applications for other results, and this actually confirms their importance for valued fields of positive characteristic. In that chapter, we also show why these rings of additive polynomials are very complex (or actually, misbehaved) in the case of the field $\mathbb{F}_p((t))$.

To be able to work out this approach, we have to develop a suitable notion of **valued modules**. This is done in Chapter 4. It has turned out that the structure in question can not be grasped by any classical notion of a valued module. So we had to start with the following rather un-classical definition: *a valued module is a valued abelian group with a module structure.* Nothing is said about the compatibility between valuation and module structure. This gives rise to the question for a classification of the possible forms of compatibility and indeed, different classical notions can be derived in this way. In the same chapter, we treat the theory of valued vector spaces (in their different forms, one of them being of importance for exponential fields). Here, we also introduce the notion of a **valuation basis**, and we show the connection to the notion of a **p-basis** for abelian p-groups.

We also study immediate extensions of valued modules, following the approach that I. Kaplansky used for valued fields (cf. Section 4.8 and Section 7.10). It is amazing to see that although the structure of valued modules over rings of additive polynomials is quite complex and still unknown (in the interesting cases), the theory of their immediate extensions is rather accessible. But we have to take into the bargain that we must deal with a rather clumsy looking notion of valued modules, the **finitely exceptional** ones. Everything would have been easier with the **ordinary** ones. But the generalization to finitely exceptional modules is not due to the author's addiction to generalizing everything that can be generalized. It is really necessary because the valued modules over rings of

additive polynomials *are* finitely exceptional and *not* ordinary (for quite natural reasons). And in fact, the notion of finitely exceptional turns out to be quite appropriate for certain valuation theoretical questions.

A further strong argument for considering the theory of valued abelian groups is the observation that the main principle behind the classical proofs of Hensel's Lemma can actually be formulated already in this setting. We prove a generalized Hensel's Lemma and several related results in the last sections of Chapter 3. Again, this is not just for generalizing the generalizable. The Hensel's lemma at the level of groups has also natural applications to valued fields. It is in fact stronger than the usual Hensel's Lemma as it can be applied also to power series (this result is not contained in our exposition) and to additive polynomials. A further advantage is the fact that a finite product of valued abelian groups carries an induced group valuation, so we can directly apply our generalized Hensel's Lemma to prove a multidimensional Hensel's Lemma.

The results of Section 3.8 are even stronger than the generalized Hensel's Lemma. Again, they have been proved in view of our general problem concerning the structure of $\mathbb{F}_p((t))$. They lead to a structure theorem (Theorem 11.19) for maximal valued (i.e., spherically complete) fields of positive characteristic that can not be deduced from the simple fact that they are henselian. The latter assertion is made precise in Section 14.11. The theorem gives some important information on the structure and model theory of $\mathbb{F}_p((t))$, as described in Section 11.5 and Section 14.11. This leads to the formulation of three open problems, which can be understood to sketch a program for the further investigation of the structure and the model theory of power series fields (and more generally, maximal valued fields) in positive characteristic.

Let us also mention that these last results are based on the new notion of **spherically continuous maps** which we have introduced in Section 2.2. This notion seems to be quite appropriate for dealing with spherically complete valued structures and for such a setting as used in the generalized Hensel's Lemma. It should be mentioned that this notion is not of topological nature.

The reader who is only interested in the more classical part of valuation theory should not hesitate to start his reading with Chapter 5 (and to omit the paragraphs which refer too much to earlier chapters). Most of the exposition of the classical results is independent of the earlier material.

We have tried to put together the easiest known proofs (for example, for the fact that the henselization is an immediate extension). For the convenience of the reader, we have included a chapter (Chapter 6) which summarizes the basic results from algebra that we are using, such as infinite Galois theory, separable and inseparable field extensions and the Taylor expansion for polynomials in arbitrary characteristic. The introduction to model theoretical algebra (Chapter 13) is written especially for algebraists, giving many algebraic examples, reducing formalism as far as possible and using a basically algebraic approach.

We beg the pardon of the reader for every mistake and incompatibility contained in this rather long exposition. After all...

...nobody is defectless.

Heidelberg, 31. 12. 1994 F.–V. Kuhlmann

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1.3 Some notation, prerequisites and conventions

The reader should be familiar with the very basic facts from algebra such as abelian groups, p-groups, modules and vector spaces, fields, algebraic and transcendental field extensions, degree and transcendence degree, separable, purely inseparable, normal and Galois extensions, primitive elements, algebraic closures, the characteristic, perfect fields, the fundamentals of Galois theory. All this can be found e.g. in [LANG3], [BOU], [JAC], [ZA–SA1]. Suitable books in german are e.g. [LOR1,2], [FI–SR], [VDW1,2].

For the algebraic structures that we will introduce in this book (like ultrametric spaces, ordered and valued groups, valued fields), we will not bother to tell at every point what the substructures, extensions, isomorphisms and embeddings are. There is a general concept for these notions, cf. section 13.1. In most cases, their meaning is anyhow obvious. We will frequently identify isomorphic objects without mentioning, wherever we do not have to consider the isomorphism itself. So, embedded structures will be addressed as substructures. Whenever we have a structure S (a set, group, field, etc.) and a substructure S_0 (a subset, subgroup, subfield, etc.), we will call S_0 **proper** if $S_0 \neq S$. An ideal is proper if it is not the whole ring, and it is **nontrivial** if it contains $\{0\}$ as a proper subset. We will use the inclusion symbol $\subset$ in the sense of $\subseteq$. If we wish to indicate a proper inclusion, we will use $\subsetneq$. Also the symbol $\lhd$ will not exclude equality.

If $\sim$ is an equivalence relation on a set S, then $S/\sim$ denotes the set of equivalence classes. If T is introduced as a subset of S, then by $T/\sim$ we will mean the subset of $S/\sim$ of all equivalence classes of elements in T, and if $s \in S$, then $s/\sim$ will denote the equivalence class of s (that is, we will use "$/\sim$" like the corresponding map which associates to an element its equivalence class). If we are dealing with quotients G/H of groups or modules, then we will use "$/H$" analogously, if there is no danger of confusion.

For a set S, its cardinality is denoted by $|S|$, and for any cardinal κ, the **successor cardinal** of κ (i.e., the smallest cardinal bigger than κ) is denoted by κ^+. We will frequently use sequences (a_ν) which are indexed by ordinals $\nu < \lambda$. By a **subsequence** we then mean a sequence (a_{ν_ρ}) such that for every $\mu < \lambda$ there is some ρ such that $\mu \leq \nu_\rho$ (that is, the indices ν_ρ form a cofinal sequence in λ).

If a degree or a group index happens to be infinite, then they will implicitly be understood as supernatural numbers, if appropriate.

We will not distinguish between inner and outer products, and we will frequently drop some other formalities, hoping that the reader will get it straight (whatever that means).

Whenever we talk of **ordered** structures in this book, we will mean **totally ordered** structures, that is, every two elements are comparable. Ordered abelian groups will always be written additively (except for the ordered multiplicative group of positive elements of an ordered field).

We will not bother too much about equivalence of valuations and places. The reader may think of considering equivalence classes of valuations instead of single valuations.

Part I

From ultrametric spaces to valued fields

Chapter 2

Ultrametric spaces

2.1 Ordered sets

Let T be a set and $<$ a binary relation on T. If for all $x \in T$,

(OR$\not$) $\quad x \not< x\,,$

(OT) $\quad x < y \wedge y < z \implies x < z\,,$

(OC) $\quad x < y \vee x = y \vee y < x\,,$

then $(T, <)$ is called an **ordered set** and $<$ is called an **ordering of** T. Axiom (OR$\not$) says that $<$ is anti-reflexive, axiom (OT) states transitivity and axiom (OC) declares that every two elements are comparable. In the presence of axiom (OC), one also speaks of a **total ordering**, but we will omit "total" since we will only consider total orders in this book. In the literature, a totally ordered set is also called a **fully ordered set**, a **linearly ordered set** or a **chain**. In contrast to this, $(T, <)$ is called a **partially ordered set** as soon as (OR$\not$) and (OT) are satisfied.

Let $(T, <)$ be an ordered set. If $a, b \in T$, then $a \leq b$ stands for $a < b \vee a = b$, as usual. If S_1, S_2 are nonempty subsets of T and $a \in T$, we will write $a < S_2$ if $a < b$ for all $b \in S_2$, and further, we will write $S_1 < S_2$ if $a < S_2$ for all $a \in S_1$. Similarly, we use the relations $>$, $\leq$ and $\geq$.

If $(T', <')$ is another ordered set and f is a map from T into T', then f is said to **preserve the ordering** (or to be **order preserving**) if for all $a, b \in T$, $a < b$ implies $f(a) <' f(b)$. It then follows that f is injective. If f is also onto, then it is called an **order isomorphism**. The **order type of** $(T, <)$ is the class of all ordered sets which admit an order isomorphism onto $(T, <)$. The finite ordered sets can be represented by the number of their elements. The above map f is said to **reverse the ordering** (or to be **order reversing**) if for all $a, b \in T$, $a < b$ implies $f(b) <' f(a)$.

In the following, let S be a subset of T. Then S is said to be **dense in** $(T, <)$ if for all elements $\alpha, \beta \in T$ with $\alpha < \beta$, there exists $\gamma \in S$ such that $\alpha < \gamma < \beta$. T is said to be **dense** if it is dense in itself, that is, if for every two unequal elements of T there is a third element of T strictly between them. T is called **discretely ordered**, if for every $\alpha \in T$, the set $\{\beta \in T \mid \beta > \alpha\}$, if nonempty, admits a minimal element, and the set $\{\gamma \in T \mid \gamma < \alpha\}$, if nonempty, admits a maximal element. That means, if α is not the smallest element in T, then it admits an immediate predecessor $\beta < \alpha$ such that no element of T lies properly inbetween β and α, and if α is not the maximal element in T, then it admits an immediate

successor γ such that no element of T lies properly inbetween α and γ. The properties "dense" and "discretely ordered" are mutually exclusive. An element $\alpha \in T$ is called an **endpoint of** T if it is the maximal or the minimal element of T. For example, $(\mathbb{Q}, <)$ is a densely ordered set without endpoints, and $(\mathbb{Z}, <)$ is a discretely ordered set without endpoints (while $(\mathbb{N}, <)$ has one endpoint).

Let $(T_1, <)$ and $(T_2, <)$ be two ordered sets. Their **lexicographic product**, denoted by $T_1 \amalg T_2$, is the cartesian product $T_1 \times T_2$ endowed with the ordering given by

(LEX) $$(x, y) < (x', y') \iff x < x' \vee (x = x' \wedge y < y') \ .$$

If T_2 is densely ordered without endpoints, then so is $T_1 \amalg T_2$. If T_2 is discretely ordered without endpoints, then so is $T_1 \amalg T_2$. The **sum of** $(T_1, <)$ **and** $(T_2, <)$, denoted by $T_1 + T_2$, is the disjoint union of T_1 and T_2, endowed with an extension of the orders of T_1 and T_2 such that every element of T_1 is smaller than every element of T_2. This sum can be viewed as the subset $\{(i, \alpha) \mid \alpha \in T_i \wedge i \in \{1, 2\}\}$ of the lexicographic product $\{1, 2\} \amalg (T_1 \cup T_2)$, endowed with the restriction of the lexicographic ordering. We will most often use the sum in the case where we have to add a **last element** to an ordered set T. Indeed, in the sum $T + \{\infty\}$ the element ∞ is bigger than every element of T. Instead of $T + \{\infty\}$ we will also write $T\infty$.

S is called **convex in** $(T, <)$ if for every two elements $\alpha, \beta \in S$ and every $\gamma \in T$ such that $\alpha \leq \gamma \leq \beta$, it follows that $\gamma \in S$. The **convex hull of** S **in** T is the set

$$\{\gamma \in T \mid \exists \alpha, \beta \in S : \ \alpha \leq \gamma \leq \beta\} \ .$$

A subset Λ of T is an **initial segment of** T if for every $\alpha \in \Lambda$ and every $\gamma \in T$ with $\gamma \leq \alpha$, it follows that $\gamma \in \Lambda$. Symmetrically, Λ is a **final segment of** T if for every $\alpha \in \Lambda$ and every $\gamma \in T$ with $\gamma \geq \alpha$, it follows that $\gamma \in \Lambda$. Consequently, Λ is an initial segment of T if and only if it is a final segment of T with respect to the reversed ordering. Note also that Λ is an initial segment of T if and only if Λ is convex and $\Lambda < T \setminus \Lambda$ (for the initial segment $\emptyset$, let it be understood that $\emptyset < T$).

S is called **coinitial in** T if its convex hull is an initial segment of T. Symmetrically, S is called **cofinal in** T if its convex hull is a final segment of T. Consequently, S is cofinal in T if and only if for every $\beta \in T$ there is some $\alpha \in S$ such that $\alpha \geq \beta$. The **coinitiality type** of $T \neq \emptyset$ is the minimal ordinal λ such that there is a coinitial subset S in T which admits an order reversing bijection onto λ. Similarly, the **cofinality type** of $T \neq \emptyset$ is the minimal ordinal λ such that there is a cofinal subset S in T of order type λ. Note that λ is 1 or a limit ordinal.

Example 2.1 The ordered set $(\mathbb{Z}, <)$ is discretely ordered, but coinitial and cofinal in $(\mathbb{Q}, <)$. Hence, the convex hull of $(\mathbb{Z}, <)$ in $(\mathbb{Q}, <)$ is $\mathbb{Q}$. The negative integers form an initial segment of $(\mathbb{Z}, <)$, and the positive integers form a final segment. $(\mathbb{Q}, <)$ is dense, and it is dense in $(\mathbb{R}, <)$. The lexicographic product $\mathbb{Q} \amalg \mathbb{Z}$ is discretely ordered, while $\mathbb{Z} \amalg \mathbb{Q}$ is densely ordered. $(\mathbb{Z}, <)$ is the sum $(-\mathbb{N}, <) + \{0\} + (\mathbb{N}, <)$. The sum $(\mathbb{Z}, <) + (\mathbb{Q}, <)$ is neither densely nor discretely ordered. $\diamondsuit$

Lemma 2.2 *Let S be a subset of the ordered set T. If S is dense in T, then for every element $\alpha \in T$, the initial segment $\{\beta \in S \mid \beta < \alpha\}$ of S is cofinal in the initial segment $\{\beta \in T \mid \beta < \alpha\}$ of T. The converse holds if T is dense.*

Proof: Assume that S is dense in T and that $\alpha \in T$. We have to show that $\{\beta \in S \mid \beta < \alpha\}$ is cofinal in $\{\beta \in T \mid \beta < \alpha\}$. This is equivalent to: for every $\beta \in T$, $\beta < \alpha$, there is some $\gamma \in S$ such that $\beta \leq \gamma < \alpha$. But this follows from the density of S in T.

For the converse, assume that T is dense and that for every $\alpha \in T$, the set $\{\beta \in S \mid \beta < \alpha\}$ is cofinal in $\{\beta \in T \mid \beta < \alpha\}$. Given $\alpha, \beta \in T$ with $\beta < \alpha$, we have to show that there exists $\gamma \in S$ such that $\beta < \gamma < \alpha$. By the density of T, there exists $\delta \in T$ such that $\beta < \delta < \alpha$, hence $\delta \in \{\beta \in T \mid \beta < \alpha\}$. Now the existence of γ follows by the cofinality of $\{\beta \in S \mid \beta < \alpha\}$ in $\{\beta \in T \mid \beta < \alpha\}$. □

The initial segments of T are ordered by inclusion. Hence, if Λ_1, Λ_2 are initial segments of T, then Λ_1 is smaller than Λ_2 if and only if Λ_2 contains an element α such that $\alpha > \Lambda_1$. Every element of $\alpha \in T$ can be identified with the smallest initial segment in which it is contained, that is $\{\beta \in T \mid \beta \leq \alpha\}$; in this way, T is an ordered subset of the ordered set of all its initial segments. We may thus use a symbol γ for initial segments as well as for elements of T (but if we say $\gamma \in T$, we will always mean an element). An initial segment of T is representing an element of T if and only if it contains a maximal element. Note that in writing $\gamma_1 < \gamma_2$ for initial segments of T, we interpret them as elements of T and not as sets since by our general convention on the the meaning of $S_1 < S_2$, the interpretation as sets would give a completely different meaning (which actually is impossible for initial segments).

Let S be a subset of the ordered set T. A **cut in** $(S, <)$ is a partition (Λ, Λ') of S into two convex subsets Λ, Λ' where Λ is empty or an initial segment of S. Similarly, if Λ' is nonempty, then it is a final segment of S. (Note that in contrast to the notion of **Dedekind cut**, we admit the cases where $\Lambda = \emptyset$ or $\Lambda' = \emptyset$. Our notion is sometimes called **Cuesta Dutari cut** in the literature.) Since $\Lambda' = S \setminus \Lambda$, the cut is uniquely determined by the initial segment Λ. We say that the cut (Λ, Λ') is **realized** (**in** T) **by the element** $\alpha \in T$ if $\Lambda \leq \alpha \leq \Lambda'$. Further, the cut

$$(\{\beta \in S \mid \beta \leq \alpha\}, \{\beta \in S \mid \beta > \alpha\})$$

is called the **cut induced by** α **in** S; this cut is realized by α. We have:

Lemma 2.3 *If two elements α_1 and α_2 in two extensions of $(S, <)$ induce the same cut in $(S, <)$, and if this cut is not realized by any element of S, then $(S \cup \{\alpha_1\}, <)$ and $(S \cup \{\alpha_2\}, <)$ are order isomorphic over S.*

Proof: By our condition that the cut is not realized by any element of S, we know that $\alpha_1 \notin S$ and $\alpha_2 \notin S$. Thus for every $\beta \in S$, $\alpha_1 > \beta$ or $\alpha_1 < \beta$. Since α_1 and α_2 realize the same cut (Λ, Λ') in $(S, <)$, we have $\alpha_1 > \beta \Leftrightarrow \beta \in \Lambda \Leftrightarrow \alpha_2 > \beta$ and $\alpha_1 < \beta \Leftrightarrow \beta \in \Lambda' \Leftrightarrow \alpha_2 < \beta$. This shows that $\alpha_1 \mapsto \alpha_2$ induces an order preserving bijection between $(S \cup \{\alpha_1\}, <)$ and $(S \cup \{\alpha_2\}, <)$ over S. □

Every cut in S is realized by at most two elements of S. If $\alpha < \alpha'$ are two elements of S realizing (Λ, Λ'), then α is the maximal element of Λ and α' is the minimal element of Λ' and there is no element of S between α and α'. Consequently, this situation can not appear if S is dense. If α is the only element realizing (Λ, Λ'), then either α is the maximal element of Λ and Λ' has no minimal element, or α is the minimal element of Λ' and Λ has no maximal element. On the other hand, every $\alpha \in S$ realizes exactly two cuts in S, namely,

$$(\{\beta \in S \mid \beta < \alpha\}, \{\beta' \in S \mid \beta' \geq \alpha\}) \text{ and } (\{\beta \in S \mid \beta \leq \alpha\}, \{\beta' \in S \mid \beta' > \alpha\}) \qquad (2.1)$$

This shows that the cuts realized in S are not in one to one correspondence with the elements of S. But there is a surjective map sending every realized cut in S to the least of the at most two elements which realize it. If δ is this element, then it is called the **supremum of** Λ and denoted by $\sup_S \Lambda$.

We would like to have a supremum for every subset of an ordered set S, but there are certainly ordered sets in which not every cut is realized (e.g., the sum $\mathbb{Z} + \mathbb{Z}$). To overcome the lack of suprema, one passes to the **completion of** S. It can be constructed as follows. On the set of all cuts in S, we introduce an equivalence relation. Two cuts (Λ, Λ') and (Λ_1, Λ_1') are **equivalent** if they are equal or both realized in S and satisfy $\sup_S \Lambda = \sup_S \Lambda_1$. If a cut is not realized in S, then it is equivalent only to itself. Every equivalence class contains at most two cuts. If it contains two cuts, then both are realized by exactly one (and the same) element α, and they are as described in (2.1). But note that in a discrete ordering, the two cuts of (2.1) are not equivalent if α is not the minimal element of the ordering. Now the set of all equivalence classes is the completion of S. An element of the completion will be called a **distance** (**in** S). The equivalence class of (Λ, Λ') will be denoted by $\mathrm{dist}_S \Lambda$. The notion "distance" is chosen in view of our later applications, cf. the distance of an approximation type.

If (Λ, Λ') and (Λ_1, Λ_1') are two cuts, then

$$(\Lambda, \Lambda') < (\Lambda_1, \Lambda_1')$$

will indicate that $\Lambda_1 \setminus \Lambda \neq \emptyset$. Note that $(\Lambda, \Lambda') \leq (\Lambda_1, \Lambda_1')$ is equivalent to $\Lambda \subseteq \Lambda_1$. If $(\Lambda, \Lambda') < (\Lambda_1, \Lambda_1')$, and they are realized by $\alpha \in S$ and $\alpha_1 \in S$ respectively, then it will follow that $\alpha \leq \alpha_1$. Equivalent cuts are consecutive; hence the ordering of cuts induces an ordering of distances. The ordered set of distances contains S as an ordered subset via the embedding

$$S \ni \alpha \mapsto \mathrm{dist}_S\{\beta \in S \mid \beta \leq \alpha\} \; . \tag{2.2}$$

We will identify α with its image under this embedding. Frequently, we will use "δ" to denote an arbitrary distance. We leave it to the reader to show that the completion of S is **complete**, i.e., that every subset of this completion has a supremum (observe that we also view $(S, \emptyset)$ as a cut in S). For further information on the completion of ordered sets, we refer the reader to J. Rosenstein [ROS], Chapter 2, §4.

Embedding (2.2) determines the order between an element α of S and an arbitrary distance δ in S. Namely, if δ is the distance of the cut (Λ, Λ'), then

$$\alpha \geq \delta \iff \alpha \geq \Lambda \tag{2.3}$$

$$\alpha \leq \delta \iff \alpha \leq \Lambda' \; . \tag{2.4}$$

If in this situation, Λ has no maximal element, then (2.3) improves to

$$\alpha \geq \delta \Longleftrightarrow \alpha \geq \Lambda \Longleftrightarrow \alpha > \Lambda \Longleftrightarrow \alpha \in \Lambda' \; , \tag{2.5}$$

and if Λ' has no minimal element, then (2.4) improves to

$$\alpha \leq \delta \Longleftrightarrow \alpha \leq \Lambda' \Longleftrightarrow \alpha < \Lambda' \Longleftrightarrow \alpha \in \Lambda \; . \tag{2.6}$$

Example 2.4 In the discretely ordered set $(\mathbb{Z}, <)$, there are the following cuts:

a) For every $n \in \mathbb{Z}$, the two cuts $(\{m \in \mathbb{Z} \mid m < n\}, \{m \in \mathbb{Z} \mid m \geq n\})$ and $(\{m \in \mathbb{Z} \mid m \leq n\}, \{m \in \mathbb{Z} \mid m > n\})$. These both are not equivalent, since the smallest element which realizes the first one is $n-1$, while the smallest element realizing the second one is n. Hence, $\mathrm{dist}_{\mathbb{Z}}\{m \in \mathbb{Z} \mid m < n\} = \mathrm{dist}_{\mathbb{Z}}\{m \in \mathbb{Z} \mid m \leq n-1\} = n-1$ and $\mathrm{dist}_{\mathbb{Z}}\{m \in \mathbb{Z} \mid m \leq n\} = n$. We see that these cuts are equivalent only to themselves.

b) The two cuts $(\emptyset, \mathbb{Z})$ and $(\mathbb{Z}, \emptyset)$. These are the only cuts which are not realized in $\mathbb{Z}$. If $\mathbb{Z}$ appears as a value set of a valuation, then it will be augmented by a maximal element ∞. In $\mathbb{Z}\infty$, instead of the latter cut we will have the following two equivalent cuts which are realized by ∞:

$$(\mathbb{Z}, \{\infty\}) \quad \text{and} \quad (\mathbb{Z} \cup \{\infty\}, \emptyset)\ .$$

Via the embedding (2.2), the distance of these cuts is identified with the element ∞. We write $\mathrm{dist}_{\mathbb{Z}\infty}\mathbb{Z} = \mathrm{dist}_{\mathbb{Z}\infty}\mathbb{Z} \cup \infty = \infty$. Note that $\mathbb{Z}\infty$ is an example of an ordered set with maximal element. If we replace $\mathbb{Z}$ by $\mathbb{N}$, then we have an example for an ordered set with a smallest element, namely 1. In this case, we have two equivalent cuts

$$(\emptyset, \mathbb{N}) \quad \text{and} \quad (\{1\}, \{n \in \mathbb{N} \mid n \geq 2\})\ ,$$

whose distance is identified with the element 1. If we replace $\mathbb{Z}$ by $-\mathbb{N}$, then we have another example of an ordered set with maximal element. Observe that $(-\mathbb{N})\infty$ is rather different from $\mathbb{Z}\infty$: we have $\mathrm{dist}_{(-\mathbb{N})\infty}(-\mathbb{N})\infty = \infty$, but $\mathrm{dist}_{(-\mathbb{N})\infty} - \mathbb{N} = 0$.

Now consider the ordered set $(\mathbb{Q}, <)$. In contrast to our foregoing examples, there are two equivalent cuts induced by every element $q \in \mathbb{Q}$, namely, $(\{r \in \mathbb{Q} \mid r < q\}, \{r \in \mathbb{Q} \mid r \geq q\})$ and $(\{r \in \mathbb{Q} \mid r \leq q\}, \{r \in \mathbb{Q} \mid r > q\})$. This is because both cuts are realized only by the element q. Actually, this is a consequence of the fact that the ordered abelian group $\mathbb{Q}$ is dense. The distance of the above cuts is identified with the element q.

Concerning the two cuts mentioned under b) above, $\mathbb{Q}$ behaves like $\mathbb{Z}$. But there is a further main difference between both examples: we have not yet indicated all possible cuts in $\mathbb{Q}$. Indeed, every irrational real number s induces a single cut in $\mathbb{Q}$, namely, $(\{r \in \mathbb{Q} \mid r < s\}, \{r \in \mathbb{Q} \mid r > s\})$. Every such cut is not realized in $\mathbb{Q}$. ♢

Exercise 2.1 *Let T be an ordered set without endpoints (an* **endpoint** *is a maximal or minimal element of the ordered set). Prove: a subset S is dense in T if and only if the convex hull of S in T is equal to T. Adapt this assertion for the case of an ordered set T with endpoints.*

Exercise 2.2 *Show that in a discrete ordering without endpoints, the two cuts given in (2.1) will not be equivalent. Give an example of two unequal but equivalent cuts in some discrete ordering with endpoints.*

2.2 Ultrametric spaces

Remark 2.5 Most of the valuation theorists who have dealt with Krull valuations, and in particular with those of rank > 1, have preferred the additive notation where the value group is written additively. Not necessarily, but usually connected with this notation is the "philosophy" of taking the value of 0 to be an element bigger than all other, and to take the valuation ideal to be the set of all elements having positive value. In this philosophy, the world appears somewhat inverted: $v(a-b)$ is large if the elements a and b are near to each other; like gravitation, $v(a-b)$ measures proximity and not remoteness. Although it may be unfamiliar to many readers, we also prefer to work with this philosophy throughout the book, even when considering ultrametric spaces. So let us introduce the following mnemonic:

It is of high value to be near to each other.

Let Γ be an ordered set. By $\Gamma\infty$ we will denote the ordered set that we obtain from Γ by adding a new element ∞ and extending the ordering of Γ such that ∞ becomes the last element of $\Gamma\infty$. Written as a sum of two ordered sets, $\Gamma\infty = \Gamma + \{\infty\}$.

A set X together with a surjective map $u : X \times X \to \Gamma\infty$ is called an **ultrametric space** and u is called an **ultrametric** if the following axioms are satisfied, for all $x, y, z \in X$:

(UM 0) $\quad u(x,y) = \infty \iff x = y\,,$

(UMT) $\quad u(x,y) \geq \min\{u(x,z), u(z,y)\}\,,$

(UMS) $\quad u(x,y) = u(y,x)\,.$

Axiom (UMT) is called the **ultrametric triangle law**, and (UMS) expresses the symmetry of the ultrametric. We will call $uX := \Gamma$ the **value set of** (X,u). Hence, $uX\infty = uX + \{\infty\} = \Gamma\infty$. Note that we exclude ∞ from uX, so we have that $uX = \{u(x,y) \mid x, y \in X \wedge x \neq y\}$. Although it does not fit too well in our philosophy, it seems unavoidable to call $u(x,y)$ the **distance between** x **and** y.

Lemma 2.6 *Axioms (UMT) and (UMS) imply*

(UM$\neq$) $\quad u(x,z) \neq u(z,y) \Longrightarrow u(x,y) = \min\{u(x,z), u(z,y)\}\,,$

which in view of (UMT) can also be written as

(UM=) $\quad u(x,y) > \min\{u(x,z), u(z,y)\} \Longrightarrow u(x,z) = u(z,y)\,.$

Proof: Assume $u(x,z) < u(z,y)$ so that $u(x,z) = \min\{u(x,z), u(z,y)\}$; we have to show that $u(x,y) = u(x,z)$. If this were not the case, then $u(x,y) > u(x,z)$, by virtue of (UMT). Applying (UMT) again, we find $u(x,z) \geq \min\{u(x,y), u(y,z)\}$. By (UMS), $u(y,z) = u(z,y)$, and we can now deduce $u(x,z) \geq \min\{u(x,y), u(z,y)\} > u(x,z)$, a contradiction. □

By induction, axiom (UMT) yields

$$\forall x_1, \ldots, x_n \in X : \; u(x_1, x_n) \geq \min_{1 \leq i \leq n-1} u(x_i, x_{i+1})\,. \tag{2.7}$$

For a given $x \in X$ and $\alpha \in uX\infty$, we define the following **ultrametric balls of radius** α **around** x:

$$B_\alpha(x) := \{y \in X \mid u(x,y) \geq \alpha\}\,, \quad B^\circ_\alpha(x) := \{y \in X \mid u(x,y) > \alpha\}\,.$$

One might think of "closed" and "open" balls, but one should note that in the topology induced by an ultrametric, these balls are open and closed at the same time. (As a consequence, (X,u) is totally disconnected.) We will also write $B_\alpha(X,x)$, $B^\circ_\alpha(X,x)$ to indicate the space in which we are working. Note that $B_\infty(x) = \{x\}$ and $B^\circ_\infty(x) = \emptyset$. Further, we define:

$$\mathbf{B}_\alpha(X) := \{B_\alpha(x) \mid x \in X\}\,, \qquad \mathbf{B}^\circ_\alpha(X) := \{B^\circ_\alpha(x) \mid x \in X\}$$
$$\mathbf{B}(X) := \bigcup_{\alpha \in uX\infty} \mathbf{B}_\alpha(X)\,, \qquad \mathbf{B}^\circ(X) := \bigcup_{\alpha \in uX\infty} \mathbf{B}^\circ_\alpha(X)\,.$$

Lemma 2.7 *Let (X,u) be an ultrametric space. Then for all $\alpha,\beta \in uX$ and all $x,y \in X$,*

$$\alpha \geq \beta \ \wedge\ B_\alpha(x) \cap B_\beta(y) \neq \emptyset \Longrightarrow B_\alpha(x) \subset B_\beta(y)\ . \tag{2.8}$$

Consequently, $B_\alpha(x) = B_\alpha(y)$ whenever $y \in B_\alpha(x)$; this shows that every point in a ball is a "center" of the ball. The same assertions hold for the balls B°. Further,

$$\begin{aligned} \alpha \geq \beta \ \wedge\ B^\circ_\alpha(x) \cap B_\beta(y) \neq \emptyset \ &\Longrightarrow\ B^\circ_\alpha(x) \subset B_\beta(y) \\ \alpha > \beta \ \wedge\ B_\alpha(x) \cap B^\circ_\beta(y) \neq \emptyset \ &\Longrightarrow\ B_\alpha(x) \subset B^\circ_\beta(y)\ . \end{aligned}$$

Proof: Suppose that $z \in B_\alpha(x) \cap B_\beta(y)$; then $u(x,z) \geq \alpha$ and $u(y,z) \geq \beta$. If $x' \in B_\alpha(x)$ then $u(x,x') \geq \alpha$ and thus, $u(y,x') \geq \min\{u(y,z), u(z,x), u(x,x')\} \geq \min\{\alpha,\beta\} = \beta$, hence $x' \in B_\beta(y)$. We have thus proved assertion (2.8). The proof of the further assertions is left to the reader. □

The sets $\mathbf{B}(X)$ and $\mathbf{B}^\circ(X)$ are partially ordered by inclusion. The first assertion of the lemma states that every two balls with a nonempty intersection are comparable. The same can be deduced for the set $\mathbf{B}(X) \cup \mathbf{B}^\circ(X)$. Thus, a set of balls will be totally ordered as soon as every two balls in the set have a nonempty intersection; in this case, it is called a **nest of balls**. The ultrametric space (X,u) is called **spherically complete** if for every nest $\mathbf{B} \subset \mathbf{B}(X)$ of balls, their intersection $\bigcap \mathbf{B} = \bigcap_{B\in\mathbf{B}} B$ is nonempty. In the literature, also the names **spherically compact** and **ultracomplete** is used for the same property. A nest $\{B_\alpha(x_\alpha) \in \mathbf{B}(X) \mid \alpha \in S\}$ of balls will be called a **completion nest** if S is a cofinal subset of uX. Spherical completeness is stronger than the following property: (X,u) is called **complete** if for every completion nest $\mathbf{B} \subset \mathbf{B}(X)$, the intersection $\bigcap \mathbf{B}$ is nonempty.

Given a nest of balls $\mathbf{B} \subset \mathbf{B}(X)$, it can be written as $\{B_\alpha(x_\alpha) \mid \alpha \in S\}$ with S a subset of uX and x_α elements of X. Let Λ be the minimal initial segment of uX containing S. We call Λ the **support segment of $\mathbf{B}$** and denote it by $\Lambda(\mathbf{B})$. If S is not itself an initial segment of uX, that is, if $S \neq \Lambda$, then we can **fill up $\mathbf{B}$** to a nest of balls $\mathbf{B}' = \{B_\alpha(x_\alpha) \mid \alpha \in \Lambda\}$ such that $\mathbf{B} \subset \mathbf{B}'$ and $\mathbf{B}'$ has the same intersection as $\mathbf{B}$. Indeed, if $\alpha \in \Lambda \setminus S$ then we can pick $\beta \in S$ such that $\beta > \alpha$ and we can set $x_\alpha = x_\beta$. Note that $B_\alpha(x_\alpha)$ does not depend on the actual choice of β: If also $\gamma \in S$ such that $\gamma > \alpha$, then the balls $B_\beta(x_\beta)$ and $B_\gamma(x_\gamma)$ have a nonempty intersection since $\mathbf{B}$ is a nest of balls. Consequently, $u(x_\beta, x_\gamma) \geq \min\{\beta,\gamma\} > \alpha$, showing that $B_\alpha(x_\beta) = B_\alpha(x_\gamma)$. Since $\mathbf{B} \subset \mathbf{B}'$ and every $B_\alpha(x_\alpha) \in \mathbf{B}'$ contains some $B_\beta(x_\beta) \in \mathbf{B}$, we have $\bigcap \mathbf{B}' = \bigcap \mathbf{B}$.

A finite nest of balls always contains a minimal ball. Hence, every ultrametric space with a finite value set is spherically complete. More generally, if the value set uX is a reversed wellordering, then every ascending chain of values in uX is finite, which yields that all nests of balls contain a minimal ball, and again, (X,u) is spherically complete. Similarly, if uX contains a maximal element $\alpha_{\max}$, then (X,u) is complete since every cofinal subset in uX must contain $\alpha_{\max}$.

Let us see how maps between ultrametric spaces preserve the property of being spherically complete. Let (X,u) and (X',u') be ultrametric spaces and $f : X \to X'$ an arbitrary map. We will say that f is **spherically continuous** if for every nest of balls $\mathbf{B}' \subset \mathbf{B}(f(X))$ in $(f(X),u')$, there is a nest of balls $\mathbf{B} \subset \mathbf{B}(X)$ such that for every ball $B' \in \mathbf{B}'$ there is a ball $B \in \mathbf{B}$ with $f(B) \subset B'$. It then follows that $f(\bigcap \mathbf{B}) \subset \bigcap_{B\in\mathbf{B}} f(B) \subset \bigcap \mathbf{B}'$. If (X,u) is spherically complete, then $\bigcap \mathbf{B}$ is nonempty and thus, also $\bigcap \mathbf{B}'$ is nonempty. This proves:

Lemma 2.8 *Let (X,u) and (X',u') be ultrametric spaces and $f: X \to X'$ a spherically continuous map. If (X,u) is spherically complete, then so is $(f(X),u')$.*

The reader should well note that we have admitted balls of radius ∞ to be used in nests of balls. That is, $\{B_\infty(x)\}$ is itself a nest of balls, and $f(B_\infty(x)) = \{f(x)\}$ is contained in every ball which contains $f(x)$. This is the reason for the following fact: *If $(f(X),u')$ is spherically complete, then f is spherically continuous.* Hence with our definition, spherical continuity does not imply continuity with respect to the topology defined by the basic open sets $B_\alpha(x)$, $\infty \neq \alpha$, $x \in X$. (If we would exclude balls of radius ∞ from the nests of balls in our definition, then indeed, spherical continuity would imply continuity with respect to that topology.) Conversely, continuity does not imply spherical continuity (see Exercise 2.8).

Which maps are spherically continuous? Here is a criterion:

Lemma 2.9 *Let (X,u) and (X',u') be ultrametric spaces and $f: X \to X'$ a map.*
a) Suppose that for all $x,y,z \in X$,

$$u(x,y) \geq u(x,z) \implies u'(f(x),f(y)) \geq u'(f(x),f(z))\ . \tag{2.9}$$

Then f is spherically continuous. Hence if (X,u) is spherically complete, then so is $(f(X),u')$.
b) Suppose that for all $x,y,z \in X$,

$$u(x,y) > u(x,z) \implies u'(f(x),f(y)) > u'(f(x),f(z))\ . \tag{2.10}$$

Then f is injective and f^{-1} is spherically continuous. Hence if $(f(X),u')$ is spherically complete, then so is (X,u).

Proof: a): Let $\mathbf{B}' \subset \mathbf{B}(f(X))$ be a nest of balls in $(f(X),u')$. Since it is totally ordered by inclusion, we can pick a coinitial strictly descending chain of balls $B'_\nu := B_{\alpha_\nu}(f(x_\nu)) \in \mathbf{B}'$, $\nu < \lambda$, where λ is an ordinal. For every $\nu < \lambda$, we pick some $x'_\nu \in X$ such that $f(x'_\nu) \in B'_\nu \setminus B'_{\nu+1}$. Now we set $\beta_\nu := \min\{u(x'_\nu,x_\nu), u(x'_\nu,x_{\nu+1})\}$ and $B_\nu := B_{\beta_\nu}(x_\nu) = B_{\beta_\nu}(x'_\nu)$. Then (2.9) shows that $x \in B_\nu$ implies $u'(f(x'_\nu),f(x)) \geq \min\{u'(f(x'_\nu),f(x_\nu)), u'(f(x'_\nu),f(x_{\nu+1}))\}$. By virtue of $f(x_\nu), f(x'_\nu), f(x_{\nu+1}) \in B'_\nu$, this implies $f(x) \in B'_\nu$. That is, $f(B_\nu) \subset B'_\nu$. Hence, the coinitiality of $\{B'_\nu \mid \nu < \lambda\}$ in $\mathbf{B}'$ yields that for every $B' \in \mathbf{B}'$ there is some $\nu < \lambda$ such that $f(B_\nu) \subset B'_\nu \subset B'$.

Now let $\nu < \mu < \lambda$. Then $f(x_\mu) \in B'_{\nu+1}$, and $u'(f(x_\mu),f(x_{\nu+1})) > u'(f(x'_\nu),f(x_{\nu+1}))$ by our choice of x'_ν. By contraposition of (2.9), we obtain that $u(x_\mu,x_{\nu+1}) > u(x'_\nu,x_{\nu+1}) \geq \beta_\nu$, which implies that $x_\mu \in B_\nu$. Hence, every two balls B_μ and B_ν have a nonempty intersection. This shows that $\mathbf{B} = \{B_\nu \mid \nu < \lambda\}$ is a nest of balls. We have proved that f is spherically continuous. The second assertion now follows from the foregoing lemma.

b): Assume that f satisfies (2.10), and let $x,z \in X$, $x \neq z$. Then $u(x,x) = \infty > u(x,z)$. By (2.10), $\infty = u'(f(x),f(x)) > u'(f(x),f(z))$, that is, $f(x) \neq f(z)$. This proves f to be injective. By contraposition, (2.10) yields that (2.9) holds for f^{-1} in the place of f. The assertion now follows from part a) of our lemma. □

Let (X_i,u_i), $i \in I$, be ultrametric spaces whose value sets u_iX_i are all contained in a common ordered set, and assume that I is finite or $\bigcup_{i\in I} u_iX_i$ is wellordered. Then their

product will be the cartesian product $\prod_{i\in I} X_i$ equipped with the ultrametric $\prod_{i\in I} u_i : \prod_{i\in I} X_i \times \prod_{i\in I} X_i \to (\bigcup_{i\in I} u_i X_i)\infty$ defined by

$$(\prod_{i\in I} u_i)\,((x_i)_{i\in I},(y_i)_{i\in I}) := \min_{i\in I} u_i(x_i,y_i)\ .$$

We leave it to the reader to verify that this map satisfies (UM 0), (UMT) and (UMS). Note that indeed every element of $\bigcup_{i\in I} u_i X_i$ appears as the distance of two suitably chosen elements of $\prod_{i\in I} X_i$. In case of two ultrametric spaces, we write $(X_1 \times X_2, u_1 \times u_2)$. Examples for products of ultrametric spaces are provided by Lemma 4.12 below.

Lemma 2.10 *If the product $(\prod_{i\in I} X_i\,,\,\prod_{i\in I} u_i)$ of ultrametric spaces is spherically complete, then the same holds for all (X_i,u_i), $i\in I$. If for $1\le i\le n$, the ultrametric spaces (X_i,u_i) are spherically complete (resp. complete), then the same holds for the finite product $(\prod_{i=1}^n X_i\,,\,\prod_{i=1}^n u_i)$.*

Proof: Assume that the product $(X,u) := (\prod_{i\in I} X_i\,,\,\prod_{i\in I} u_i)$ is spherically complete and that $\mathbf{B}\subset\mathbf{B}(X)$ is a nest of balls in (X_j,u_j) for some fixed $j\in I$. Let us write $\mathbf{B} = \{B_\alpha(X_j,x_\alpha)\mid \alpha\in S\}$ with S a subset of u_jX_j and x_α elements in X_j. Pick some elements $x_i\in X_i$, $i\in I$, and set $x_{j,\alpha}=x_j$ and $x_{i,\alpha}=x_i$ for $i\neq j$. Define $\underline{x}_\alpha := (x_{i,\alpha})_{i\in I}\in\prod_{i\in I}X_i$ for every $\alpha\in S$. Then $\{B_\alpha(X,\underline{x}_\alpha)\mid\alpha\in S\}$ is a nest of balls in the product. By assumption, its intersection contains an element $\underline{y}$. In view of the definition of $\prod_{i\in I}u_i$ it follows that its j-th coordinate $y_j\in X_j$ lies in the intersection of $\mathbf{B}$.

For the converse, assume that the finitely many ultrametric spaces (X_i,u_i), $1\le i\le n$, are spherically complete. Let $\mathbf{B}=\{B_\alpha(X,\underline{x}_\alpha)\mid\alpha\in S\}$ be a nest of balls in the product $(X,u) := (\prod_{i=1}^n X_i\,,\,\prod_{i=1}^n u_i)$, with $S\subset\bigcup_{i=1}^n u_iX_i$. We have to show that the intersection of $\mathbf{B}$ is nonempty, so we can assume w.l.o.g. that S has no maximal element. We write $\underline{x}_\alpha = (x_{i,\alpha})_{1\le i\le n}$. Although $\alpha\in S$ may not lie in u_jX_j, the notation $B_\alpha(X_j,x_{j,\alpha})$ still makes sense since by assumption, $\bigcup_{i=1}^n u_iX_i$ is an ordered set containing all ordered sets u_iX_i. By definition, $\prod_{i=1}^n u_i(\underline{x},\underline{y})\ge\alpha$ implies that $u_i(x_i,y_i)\ge\alpha$ for all i. This yields that every two sets in $\{B_\alpha(X_j,x_{j,\alpha})\mid\alpha\in S\}$ have nonempty intersection since the same is true for the sets in $\mathbf{B}$. However, $B_\alpha(X_j,x_{j,\alpha})$ is not a ball in the original sense. But we can extract a single element or a nest of balls $\mathbf{B}_j$ in (X_j,u_j) from $\{B_\alpha(X_j,x_{j,\alpha})\mid\alpha\in S\}$ as follows. If $u_jX_j\cap S$ is not cofinal in S then choose $\alpha\in S$ such that $\alpha > u_jX_j\cap S$ and define $y_j := x_{j,\alpha}$. It will follow that for every $\beta\in S$ with $\beta\ge\alpha$ we have $u_j(y_j,x_{j,\beta})>S$. Now assume that $u_jX_j\cap S$ is cofinal in S. By virtue of our filling up procedure for nests of balls, we can assume w.l.o.g. that S is an initial segment of $\bigcup_{i=1}^n u_iX_i$. So $u_jX_j\cap S$ will be an initial segment of u_jX_j which is cofinal in S, and $\mathbf{B}_j := \{B_\alpha(X_j,x_{j,\alpha})\mid\alpha\in u_jX_j\cap S\}$ is a set of balls which has the same intersection as $\{B_\alpha(X_j,x_{j,\alpha})\mid\alpha\in S\}$. Since every two elements in the latter have nonempty intersection, it follows that this also holds for $\mathbf{B}_j$, which is consequently a nest of balls in (X_j,u_j). By assumption on (X_j,u_j), we find that $\bigcap\mathbf{B}_j$ contains an element y_j. By construction of the elements y_j for $1\le j\le n$, it follows that the element $\underline{y} := (y_1,\ldots,y_n)$ lies in the intersection of $\mathbf{B}$.

Now assume that the finitely many ultrametric spaces (X_i,u_i), $1\le i\le n$, are complete; we have to show that their product is complete. We proceed as before with the same $\mathbf{B}$, but assuming that S is cofinal in $\bigcup_{i=1}^n u_iX_i$. Assuming again that S is an initial segment, we find that $S=\bigcup_{i=1}^n u_iX_i$ and consequently, $u_jX_j\cap S = u_jX_j$ for every j. So for the construction of the elements y_j it suffices that every (X_i,u_i) be complete. □

For an application of this lemma, see the proof of Theorem 8.26.

We shall now prove an Ultrametric Fixed Point Theorem which is due to S. Prie-Crampe (cf. [PC2]). A generalization to ultrametric spaces whose value sets are not totally ordered, was given by S. Prie-Crampe and P. Ribenboim (cf. [PC–RIB1]). In the latter paper, it was remarked that the Fixed Point Theorem holds for a slightly more general class of maps than that which are commonly called **contractive**. This generalization, however, is of significant importance for some of our applications, so we will make it available by the following definition. A map $\Xi : X \to X$ will be called **self-contractive** if it satisfies

(SC1) $\quad x \neq \Xi x \;\Rightarrow\; u(\Xi x, \Xi(\Xi x)) > u(x, \Xi x)\,,$

(SC2) $\quad u(\Xi x, \Xi y) \geq u(x, y)$

for all $x, y \in X$. The map Ξ is called **contractive** if it satisfies $u(\Xi x, \Xi y) > u(x, y)$ for all $x, y \in X$, $x \neq y$. Hence, every contractive map is self-contractive. An element x is called a fixed point of Ξ if $\Xi x = x$. Note that a contractive map Ξ has at most one fixed point; indeed, if $x \neq y$ were fixed points of Ξ, then $u(x, y) = u(\Xi x, \Xi y) > u(x, y)$, a contradiction.

Theorem 2.11 (Ultrametric Fixed Point Theorem)
The ultrametric space (X, u) is spherically complete if and only if every self-contractive map $\Xi : X \to X$ admits a fixed point.

Proof: Assume first that (X, u) is spherically complete and that $\Xi : X \to X$ is a self-contractive map. Let $\alpha_x := u(x, \Xi x)$ and $B_x := B_{\alpha_x}(x)$. The set $S = \{B_x \mid x \in X\}$ is partially ordered by inclusion. Let T be a maximal totally ordered subset of S. Note that no ball is empty, and thus, the intersection of any two balls in T is nonempty. Since (X, u) is spherically complete, there is some $z \in \bigcap_{B_x \in T} B_x$. Pick an arbitrary $B_x \in T$; we wish to show that $B_z \subset B_x$. By (SC2) we know that $u(\Xi z, \Xi x) \geq u(z, x)$. Since $z \in B_x$, we have $u(z, x) \geq u(x, \Xi x)$. Altogether, $u(z, \Xi z) \geq \min\{u(z, x), u(x, \Xi x), u(\Xi x, \Xi z)\} = u(x, \Xi x)$. By (2.8) it now follows that $B_z \subset B_x$. We find that B_z must be the smallest element of T. We claim that z is a fixed point of Ξz. Otherwise, $u(\Xi z, \Xi(\Xi z)) > u(z, \Xi z)$ by assumption on Ξ. This yields that $z \notin B_{\Xi z}$ and in view of (2.8), $B_{\Xi z} \subsetneq B_z$. But this contradicts the maximality of T.

For the converse, suppose that $\mathbf{B} \subset \mathbf{B}(X)$ is a nest of balls. Then (2.8) yields that inclusion imposes a total ordering on $\mathbf{B}$. Let $B_\nu = B_{\alpha_\nu}(x_\nu)$, $\nu < \lambda$, be a coinitial descending sequence in $\mathbf{B}$, where λ is an ordinal. Let us assume that $\bigcap_{B \in \mathbf{B}} B = \emptyset$. Then also $\bigcap_{\nu < \lambda} B_\nu = \emptyset$ since the sequence B_ν is coinitial in $\mathbf{B}$. Hence, for every $x \in X$, there is a minimal ordinal ν such that $x \notin B_\nu$. If B_ν was given as $B_{\alpha_\nu}(x_\nu)$, then we set $\Xi x := x_\nu$. Since $x_\nu \in B_\nu$ and $x \notin B_\nu$, we have $x \neq \Xi x$, showing that Ξ admits no fixed point. Let us show that Ξ is contractive. Let $x, y \in X$ such that $x \neq y$ and $\Xi x = x_\nu$, $\Xi y = x_\mu$. If $x_\nu = x_\mu$ then $u(\Xi x, \Xi y) = \infty > u(x, y)$. Otherwise, assume w.l.o.g. that $\mu < \nu$, that is, $B_\nu \subset B_\mu$. Then by our definition of Ξ, $x \in B_\mu$ but $y \notin B_\mu$. It follows that $u(x, y) < \alpha_\mu$ and further, $u(\Xi x, \Xi y) = u(x_\mu, x_\nu) \geq \alpha_\mu > u(x, y)$, as required. We have shown the existence of a contractive map $\Xi : X \to X$ without a fixed point. □

The first direction of the above proof is taken from the paper [PC–RIB1]; it also works for ultrametric spaces with partially ordered value sets. The proof shows that in the assertion of our theorem, we could replace "self-contractive" by "contractive". Strengthening the properties of the map Ξ, we can weaken the assumptions on the ultrametric space:

Lemma 2.12 *Let (X,u) be a complete ultrametric space. Assume that the self-contractive map $\Xi : X \to X$ satisfies in addition that for every $x \in X$ and $\alpha \in uX$ there is some $n \in \mathbb{N}$ such that $u(\Xi^{n+1}x, \Xi^n x) > \alpha$. Then Ξ admits a fixed point.*

Proof: By a modification of the first part of the proof of the preceding theorem. Choose an arbitrary $x \in X$. By (SC1) and (2.8), the balls $B_{\Xi^n x}$ form a totally ordered subset T' of S. Let T be a maximal totally ordered subset of S containing T'. Then by our additional assumption on Ξ, T contains nonempty balls of arbitrarily high radius. Now the existence of z is guaranteed by our assumption that (X,u) be complete. □

Let (X,u) and (X',u') be ultrametric spaces. An injective map $\iota : X \to X'$ is called an **embedding of** (X,u) **in** (X',u') if it satisfies

$$u(x,y) < u(z,t) \iff u(\iota x, \iota y) < u(\iota z, \iota t) \tag{2.11}$$

for all $x,y,z,t \in X$. If in addition ι is onto, then it is called an **isomorphism of** (X,u) **onto** (X',u'). Since we have assumed $u : X \times X \to uX$ to be onto for every ultrametric space (X,u), it follows that an embedding ι of (X,u) in (X',u') induces an order preserving injective map $\rho : uX \to u'X'$ such that $u'(\iota x, \iota y) = \rho u(x,y)$ for all $x,y \in X$. If ι is an isomorphism, then also ρ is an order isomorphism.

We will now ask for invariants of the ultrametric space (X,u). The value set is one, describing the "vertical structure" of the space. It remains to find invariants for the "horizontal structure" (cf. Remark 3.27 below for an explanation of these terms). An **equilateral polygon of distance** α is a set P_α of elements of X every two of which have distance α. It is called **maximal** if it can not be enlarged, that is, for every $x \in X$ there is some $y \in P_\alpha$ such that $u(x,y) \neq \alpha$. For every $\alpha \in uX$, every $x \in X$ belongs to some maximal equilateral polygon of distance α. But it may belong to more than one: if $y, y' \in X$ such that $u(x,y) = \alpha$ and $u(y,y') > \alpha$, then $\{x,y\}$ is a subset of at least one maximal equilateral polygon of distance α, which can not contain y', and $\{x,y'\}$ is a subset of at least one maximal equilateral polygon of distance α, which can not contain y. Conversely, two maximal equilateral polygons P_α and P'_α of distance α both containing x together with at least one other element, will admit elements $y \in P_\alpha$ and $y' \in P'_\alpha$ such that $u(y,y') > \alpha$. Indeed, for all $y \in P_\alpha$ and $y' \in P'_\alpha$ we have $u(y,y') \geq \min\{u(y,x), u(x,y')\} = \alpha$ by (UMT). If equality would always hold, then $P_\alpha \cup P'_\alpha$ would be an equilateral polygon of distance α properly containing P_α or P'_α in contradiction to their maximality. Let us find suitable objects to represent maximal equilateral polygons, avoiding the nonuniqueness phenomenon.

For a given initial segment γ of uX, let $x \sim_\gamma y$ stand for $u(x,y) > \gamma$. Then $\sim_\gamma$ is an equivalence relation (cf. Exercise 2.4). On the set $X/\!\sim_\gamma$, we define $u/\!\sim_\gamma$ by

$$u/\!\sim_\gamma (x/\!\sim_\gamma, y/\!\sim_\gamma) \;=\; u(x,y) \quad \text{if } \; x/\!\sim_\gamma \neq \; y/\!\sim_\gamma ;$$

it follows from (UM$\neq$) that this is well defined. The reader may show that we have obtained an ultrametric space again; we denote it by $(X,u)/\!\sim_\gamma$. Its value set is just γ. Note that by part a) of Lemma 2.9, the canonical epimorphism $(X,u) \to (X,u)/\!\sim_\gamma$ is spherically continuous. If $\gamma = \alpha \in uX$, then α is the maximal element in $u/\!\sim_\alpha$ $(X/\!\sim_\alpha)$. Consequently, in this space every element belongs to just one maximal equilateral polygon of distance α. In other words, the space $(X,u)/\!\sim_\alpha$ admits a unique partition into disjoint maximal

equilateral polygons of distance α. The reduction of an equilateral polygon P_α modulo $\sim_\alpha$ is an equilateral polygon of distance α in $(X,u)/\sim_\alpha$ and will be denoted by $\overline{P_\alpha}$. Two maximal equilateral polygons P_α and P'_α of distance α reduce to the same polygon modulo $\sim_\alpha$ if and only if there are $y \in P_\alpha$ and $y' \in P'_\alpha$ such that $u(y,y') \geq \alpha$ (cf. Exercise 2.5). Further, P_α is maximal if and only if $\overline{P_\alpha}$ is; in this case, $\overline{P_\alpha}$ will be called an **α-polygon of** (X,u). The reader may show that

$$x \in P_\alpha \ \wedge \ P_\alpha \text{ maximal} \implies \overline{P_\alpha} = B_\alpha(x)/\sim_\alpha$$

(cf. Exercise 2.5).

For $x \in X$, we define $\text{u-co}_\alpha x := x/\sim_\alpha$. The maps so obtained for all $\alpha \in uX$ will be called **ultrametric coefficient maps**. For all $x, y \in X$, they satisfy

(UCO) $\quad u(x,y) > \alpha \iff \text{u-co}_\alpha x = \text{u-co}_\alpha y \qquad$ (for all $\alpha \in uX$).

By construction, $\text{u-co}_\alpha x$ and $\text{u-co}_\alpha y$ lie in the same α-polygon if and only if $u(x,y) = \alpha$.

The α-polygons are invariants of the ultrametric space (X,u) for every $\alpha \in uX$, since the set of α-polygons is preserved under isomorphisms.

Now we are able to give an estimation of the cardinality of the space in terms of the size of the α-polygons and of the value set:

Lemma 2.13 *Let κ be a cardinal such that $|\overline{P_\alpha}| \leq \kappa$ for all α-polygons $\overline{P_\alpha}$ of the ultrametric space (X,u) and for all $\alpha \in uX$. Then*

$$|X| \leq \kappa^{|uX|} \ .$$

Proof: Let us choose a bijection $\iota : uX \to |uX|$. By hypothesis, for every $\alpha \in uX$ and every α-polygon $\overline{P_\alpha}$ we can choose an isomorphism φ onto some cardinal $\leq \kappa$. Hence, we may think of a map φ sending every such polygon into κ and being injective on each single polygon. Thus, to every element $x \in X$ we have associated a map

$$|uX| \to \kappa \ , \quad \iota\alpha \mapsto \varphi(\text{u-co}_\alpha x) \ .$$

It just remains to show that for every two elements $x \neq y$ from X we obtain different maps. Let $u(x,y) = \alpha$ (this is in uX since $x \neq y$). Then $\text{u-co}_\alpha x$ and $\text{u-co}_\alpha y$ lie in the same α-polygon, but $\text{u-co}_\alpha x \neq \text{u-co}_\alpha y$ by (UCO). Hence, $\varphi(\text{u-co}_\alpha x) \neq \varphi(\text{u-co}_\alpha y)$. □

In general, in an ultrametric space (X,u) there may be elements x for which the map $u(\,.\,,x) : X \to uX$ is not surjective. If α is not in the image, then the α-polygon containing $\text{u-co}_\alpha x$ is a singleton, that is, it just consists of the element $\text{u-co}_\alpha x$. On the other hand, there must exist an α-polygon with at least two elements since $u : X \times X \to uX$ is surjective by definition. We will later see that ultrametric spaces which are underlying valued groups or fields do not show the described pathology, that is, no α-polygon is a singleton. Even more, they satisfy the following homogeneity condition introduced by S. Prie-Crampe and P. Ribenboim:

(UMH) for every $\alpha \in uX$, all α-polygons have the same cardinality.

In this case, we can pick one α-polygon and identify with it all other α-polygons via a bijection. In the proof of the foregoing lemma, we can replace the embeddings in κ

by these bijections to obtain unified coefficient sets. (In this way, a good notion of the **skeleton of an ultrametric space** can be defined and a Hahn-Embedding-type theorem can be proved, cf. [PC–RIB2] and Exercise 2.6). The name **coefficient** has its origin in the coefficients of power series (representing elements in valued fields); see our discussion of coefficient maps in section 5.3.

Given a nest $\mathbf{B} \subset \mathbf{B}(X)$ of balls, we can associate with it the minimal initial segment $\gamma = \gamma_{\mathbf{B}}$ of uX containing the radii of all balls in $\mathbf{B}$. Then reducing modulo $\sim_\gamma$, $\mathbf{B}$ will become a nest $\overline{\mathbf{B}}$ of balls in $(X,u)/\!\sim_\gamma$ whose radii form a cofinal set in $\gamma = u/\!\sim_\gamma (X/\!\sim_\gamma)$. An element $x \in X$ will lie in $\bigcap \mathbf{B}$ if and only if $x/\!\sim_\gamma$ lies in $\bigcap \overline{\mathbf{B}}$. This proves:

Lemma 2.14 *The ultrametric space (X,u) is spherically complete if and only if for every initial segment γ of uX, the ultrametric space $(X,u)/\!\sim_\gamma$ is complete.*

Let $(X,u) \subset (Y,u')$ be an **extension of ultrametric spaces**, that is, $X \subset Y$ and u' restricted to $X \times X$ coincides with u. Then we will also just write $(X,u) \subset (Y,u)$. Now (X,u) will be called a **nice ultrametric subspace of** (Y,u), or just **nice in** (Y,u), if for every $y \in Y$ the set $u(y,X) := \{u(y,x) \mid x \in X\}$ admits a maximum (i.e., for every element in Y there is a nearest element in X; however, it may not be uniquely determined). Note that if $y \in X$, then $\max\{u(y,x)\} = \infty$. On the other hand, $(X,u) \subset (Y,u)$ is said to be an **immediate extension** if for all $y \in Y$ and all $x \in X \setminus \{y\}$ there is some $x' \in X$ such that $u(y,x') > u(y,x)$. The properties "nice ultrametric subspace" and "immediate extension" are mutually exclusive. Both properties are transitive:

Lemma 2.15 (Transitivity.) *Let $(X,u) \subset (X_1,u) \subset (Y,u)$ be ultrametric spaces.*
a) If (X,u) is nice in (X_1,u) and (X_1,u) is nice in (Y,u), then (X,u) is nice in (Y,u).
b) If $(X,u) \subset (X_1,u)$ and $(X_1,u) \subset (Y,u)$ are immediate, then so is $(X,u) \subset (Y,u)$.

Proof: a): Let $y \in Y \setminus X$; we will show that $u(y,X)$ admits a maximum. From our hypothesis we know that $u(y,X_1)$ admits a maximum, say $u(y,x_1)$ with $x_1 \in X_1$, and that also $u(x_1,X)$ admits a maximum, say $u(x_1,x_0)$ with $x_0 \in X$. Note that $u(y,X) \subset u(y,X_1)$. If $u(y,x_1) \in u(y,X)$ then $u(y,x_1) = \max u(y,X)$, and we are done. Otherwise, we have $u(y,x_1) > u(y,X)$. This yields

$$\forall x \in X:\ u(y,x) = \min\{u(y,x_1), u(x_1,x)\} = u(x_1,x)\ ,$$

which shows that $u(y,x_0) = u(x_1,x_0) = \max u(y,X)$.
b): Let $y \in Y$ and $x \in X \setminus \{y\}$. We show the existence of $x' \in X$ satisfying $u(y,x') > u(y,x)$. By assumption, there is $x_1' \in X_1$ such that $u(y,x_1') > u(y,x)$. Hence, $u(x,x_1') = \min\{u(y,x_1'), u(y,x)\} = u(y,x)$. Again by assumption, there is $x' \in X$ such that $u(x_1',x') > u(x_1',x) = u(y,x)$. It follows that $u(y,x') \geq \min\{u(y,x_1'), u(x_1',x')\} > u(y,x)$, as desired. □

Conversely, it follows directly from the definition that if (X,u) is nice in (Y,u), then (X,u) is nice in (X_1,u), and if $(X,u) \subset (Y,u)$ is immediate, then so is $(X,u) \subset (X_1,u)$; but the respective properties do not necessarily follow for $(X_1,u) \subset (Y,u)$ (cf. Exercise 2.6).

An ultrametric space will be called **maximal** if it does not admit proper immediate extensions. Further, (X,u) will be called **spherically closed** in (Y,u), if every nest of balls $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$, $S \subset uX$, in (X,u) has a nonempty intersection whenever the associated nest of balls $\{B_\alpha(Y,x_\alpha) \mid \alpha \in S\}$ in (Y,u) has a nonempty intersection.

Lemma 2.16 *(X,u) is nice in (Y,u) if and only if it is spherically closed in (Y,u). In particular, if (X,u) is spherically complete then it is nice in every ultrametric space extension and consequently, it is maximal.*

Proof: Suppose that (X,u) is spherically closed in (Y,u); we wish to show that (X,u) is nice in (Y,u). Let $y \in Y$. Define $B_\alpha := \{x \in X \mid u(x,y) \geq \alpha\}$ for all $\alpha \in uX$. Consider all nonempty ones among these balls. They form a nest $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ of balls since by definition, $B_\alpha \subset B_\beta$ for $\beta \leq \alpha$. By construction, the corresponding nest $\{B_\alpha(Y,x_\alpha) \mid \alpha \in S\}$ in (Y,u) contains y in its intersection. Hence by hypothesis, the intersection of $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ is nonempty; let x' be an element of this intersection. If $u(y,x')$ is the maximum of $u(y,X)$ then we are done. If there exists some $x_0 \in X$ such that $u(y,x') < u(y,x_0)$, we wish to show that $u(y,x_0)$ is the maximum of $u(y,X)$. Suppose there is $x_1 \in X$ such that $u(y,x_0) < u(y,x_1)$. Then $u(y,x') < u(y,x_0) = u(x_0,x_1)$ by (UM=), showing that $x' \notin B_\alpha$ for $\alpha = u(x_0,x_1) \in uX$. This contradicts our choice of x'.

Now suppose that (X,u) is nice in (Y,u). Let $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ be a nest of balls in (X,u) such that $\{B_\alpha(Y,x_\alpha) \mid \alpha \in S\}$ in (Y,u) admits an element $y \in Y$ in its intersection. Suppose that $x' \in X$ such that $u(y,x')$ is maximal. Then $u(y,x') \geq u(y,x_\alpha)$ for all $\alpha \in S$. This shows that x' lies in all $B_\alpha(X,x_\alpha)$ and thus in their intersection. We have proved that (X,u) is spherically closed in (Y,u).

Finally, if (X,u) is spherically complete then every nest of balls has a nonempty intersection and thus, (X,u) is spherically closed in every ultrametric space extension. Now the last assertion follows from what we have already proved and the fact that an ultrametric space which is nice in every extension can not admit proper immediate extensions. □

Suppose that (X,u) is not spherically complete; does there exist an extension in which (X,u) is not nice? Let $\mathbf{B} \subset \mathbf{B}(X)$ be a nest of balls in (X,u) having an empty intersection. Let $y \notin X$. We define a extension of u to $Y := X \cup \{y\}$ as follows. First, we set $u(y,y) = \infty$. Given $x \in X$, there is some nonempty ball in $\mathbf{B}$ which does not contain x. Choose some x' in that ball and set $u(x,y) = u(y,x) := u(x',x)$. This is welldefined: suppose x'' is in some other ball which does not contain x; we know that one of the balls is contained in the other, so x' and x'' are contained in a common ball which does not contain x, that is, $u(x'',x') > u(x,x')$, which yields $u(x,x') = u(x,x'')$. By our definition, for $x \neq y$ we have $u(y,x) \neq \infty$; this shows that (Y,u) satisfies (UM 0). It also satisfies symmetry by definition. It remains to show (UMT). The only nontrivial case is where precisely one of the appearing elements is our new element y. Note that it suffices to show $u(x,z) \geq \min\{u(x,y),u(y,z)\}$ since this implies that $u(x,y) \geq \min\{u(x,z),u(z,y)\}$. So let x' and z' such that by our definition, $u(y,x) = u(x',x)$ and $u(y,z) = u(z',z)$. Then there is a ball containing x' and z' such that x or z does not lie in this ball, that is, $u(x',z') > u(x',x)$ or $u(x',z') > u(z',z)$. In both cases, we obtain $u(x,z) \geq \min\{u(x,x'),u(x',z'),u(z',z)\} = \min\{u(x,x'),u(z',z)\} = \min\{u(x,y),u(y,z)\}$.

We have now proved that $(X,u) \subset (Y,u)$ is an extension of ultrametric spaces. Let us show that $u(y,X)$ does not admit a maximum. Let $x \in X$ and x' in a ball B' so that $u(y,x) = u(x',x)$ by our definition. Now let x'' be an element in a ball B'' not containing x' so that we have $u(y,x') = u(x'',x')$. Since one of the balls contains the other and since $x' \notin B''$, we conclude that $x'' \in B'$. Since $x \notin B'$, we have $u(x',x) < u(x'',x')$ showing that $u(y,x) < u(y,x')$. Since $x \in X$ was arbitrary, this proves our claim. Since $Y \setminus X$ only consists of the element y, we have even proved that $(X,u) \subset (Y,u)$ is an immediate extension. By our construction and the foregoing lemma, we have now proved:

Theorem 2.17 *An ultrametric space is spherically complete if and only if it is nice in every ultrametric space extension, and this is the case if and only if it is maximal.*

Remark 2.18 The basic properties have been studied by M. Guennoun [GUE]. Later, F. Delon [DEL1], [DEL3] studied the model theory of ultrametric spaces, thereby introducing some nice concepts for the description of their structure. We have learnt the use of polygons from [DEL3].

Recently, research is done on ultrametric spaces and valued structures whose value sets are only partially ordered. There is a series of papers of P. Ribenboim on **generalized power series fields** (cf. [RIB15] and the references given there). This notion triggered the interest in ultrametric spaces with partially ordered value sets, leading to the Ultrametric Fixed Point Theorem as proved in the paper of S. Prie-Crampe and P. Ribenboim [PC–RIB1]. As an immediate consequence, the question for a Hahn Embedding Theorem arose. It turned out that the key problem was to develop a sound notion of a **skeleton**. Although this problem is solved, the Hahn Embedding Theorem is not as nice as in the case of valued abelian groups with totally ordered value sets. In this connection, the homogeneity (UMH) is a significant condition enabling better results. See [PC–RIB2].

The assertion of Theorem 2.17 that an ultrametric space is spherically complete if and only if it is maximal, was proved by E. Schörner [SCHÖ], a student of S. Prie-Crampe. He also proved the existence of maximal immediate extensions of ultrametric spaces. Unfortunately, uniqueness fails for the maximal immediate extensions. In fact, every non-maximal ultrametric space admits maximal immediate extensions of arbitrary cardinality. (However, it can be shown that for every ultrametric space satisfying (UMH) there is a maximal immediate extension again satisfying (UMH) and having the same value set; the cardinality of such a maximal immediate extension is then bounded by the estimate given in Lemma 2.13.) On the other hand, Schörner shows the existence of maximal immediate extensions with the universal property of being embeddable in every other spherically complete extension. We should point out here that Schörner uses a stronger notion of "immediate" since he adds the condition that $uX = uY$. With respect to this notion, an ultrametric space has a unique maximal immediate extension if and only if its completion is maximal.

Exercise 2.3 *Show that the first assertion of Lemma 2.10 does not hold for "complete" in the place of "spherically complete", and that its second assertion does not hold for the product of infinitely many ultrametric spaces.*

Exercise 2.4 *Show that $\sim_\gamma$ is an equivalence relation and that $(X,u)/\sim_\gamma$ is an ultrametric space.*

Exercise 2.5 *Let P_α and P'_α be equilateral polygons of distance α.*

a) Assume $y \in P_\alpha$ and $y' \in P'_\alpha$ such that $u(y,y') \geq \alpha$. Show that $P_\alpha \cup P'_\alpha \subset B_\alpha(y)$.

b) Assume in addition that one of the two polygons is maximal and deduce that $u(z,y') > \alpha$ for some $z \in P_\alpha$.

c) Assume in addition that both polygons are maximal and show that for every $z \in P_\alpha$ there is precisely one $z' \in P'_\alpha$ such that $u(z,z') > \alpha$.

d) Prove that two maximal equilateral polygons P_α and P'_α of distance α reduce to the same polygon modulo $\sim_\alpha$ if and only if there are $y \in P_\alpha$ and $y' \in P'_\alpha$ such that $u(y,y') \geq \alpha$. Conclude that $\overline{P_\alpha} = B_\alpha(x)/\sim_\alpha$.

Exercise 2.6 *Let $(X,u) \subset (Y,u)$ be an extension of ultrametric spaces.*

a) Let $y \in Y$ and show that $u(X \cup \{y\}) \setminus uX$ contains at most one element.

b) Show that every α-polygon of X has a natural embedding in a unique α-polygon of Y. Prove that $(X,u) \subset (Y,u)$ is immediate if and only if the α-polygons of X are not enlarged in Y, and that this is the case if and only if every maximal equilateral polygon of X remains maximal in Y.

c) Give examples of immediate extensions $(X,u) \subset (Y,u)$ where $uX \subsetneq uY$ or/and (Y,u) admits a new α-polygon for some $\alpha \in uX$. By choosing this new α-polygon to consist of at least two elements, show

that a proper immediate subextension $(X,u) \subset (X_1,u)$ can be obtained such that $(X_1,u) \subset (Y,u)$ is not immediate.

d) After a sneak preview of the next sections, try to define the skeleton of an ultrametric space and to use it to characterize immediate extensions of ultrametric spaces. Any problems? Give a better definition of the skeleton under the assumption that the ultrametric space satisfies the homogeneity condition (UMH). Show that still, an immediate extension of such a space need not satisfy (UMH). Looking again at the case of valued groups below, conclude that the obstruction lies in the lack of a zero. Define a zero in every α-polygon (in a compatible way). Now you are prepared to prove a Hahn Embedding Theorem for ultrametric spaces. If you don't succeed, just hope that [PC–RIB2] will appear soon.

Exercise 2.7 *Show that every extension of a spherically complete ultrametric space by finitely many points is spherically complete again.*

Exercise 2.8 *Construct a map of ultrametric spaces which is not spherically continuous, but continuous with respect to the topology given by the basic open sets $B_\alpha(x)$, $\infty \neq \alpha$, $x \in X$. (Hint: Take a suitable space (X,u) with value set $uX = \omega + 1$. Let $\gamma = \{\alpha \mid \alpha < \omega\}$, the set of finite ordinals, and consider the space $(X,u)/\!\sim_\gamma$. Construct some map $\overline{f}$ on this space which is not spherically continuous, and lift it to a continuous map f on (X,u).)*

Exercise 2.9 *a) Show that the main properties of ultrametric spaces, like spherical completeness, maximality, homogeneity, are preserved under isomorphisms of ultrametric spaces. State and prove a similar assertion for the main properties of extensions of ultrametric spaces.*

b) What happens if we replace condition (2.11) by "$u(x,y) < u(y,z) \Leftrightarrow u(\iota x, \iota y) < u(\iota y, \iota z)$"? Do such embeddings also induce order embeddings $\rho : uX \to u'X'$, and do they also preserve the above properties?

2.3 Approximation types: basic definitions

At the end of the last section, we have already treated some properties of extensions of ultrametric spaces. The question arises to give a criterion when two extensions of the same ultrametric space (X,u) are isomorphic over X. The simplest version of this question is the case where the extensions contain just one new element y resp. y'. In this case, an isomorphism over X can only send y to y'. The answer is as simple as the question: it is necessary and sufficient that $u(y,x) = u(y',x)$ for all $x \in X$. If $u(X \cup \{y\}) = uX = u(X \cup \{y'\})$, our answer is immediately seen to be true since it contains all information that has to be checked for an isomorphism of the ultrametric spaces $(X \cup \{y\}, u)$ and $(X \cup \{y'\}, u)$. If $u(X \cup \{y\}) \neq uX$, then we also have to construct the isomorphism $\rho : u(X \cup \{y\}) \to u(X \cup \{y'\})$ over uX. But $u(X \cup \{y\}) \setminus uX$ can only contain one element $u(y,x_0)$ (cf. Exercise 2.6), and if we understand the assertion "$u(y,x_0) = u(y',x_0)$" as to say that also $u(y',x_0) \notin uX$ and that both $u(y,x_0)$ and $u(y',x_0)$ induce the same cut in uX, then Lemma 2.3 gives the required isomorphism of the value sets, and we are done.

Less easy to answer is the question how the information about the values can be encoded in the ultrametric space (X,u) itself, without using the symbols y and y'. We are looking for a structure induced by y in (X,u) which if it is equal to that induced by y', tells us the existence of the isomorphism that we have asked for. For instance, for ordered sets the adequate structure is the cut induced by an element of an arbitrary extension, as Lemma 2.3 shows. A second role of such intrinsic structures is that they may tell us what sort of extensions the ultrametric would admit. For example, if there are nests of balls with empty intersection, then the space admits proper immediate extensions, and vice versa, according to Theorem 2.17. But nests of balls in (X,u) will in general not suffice to describe extensions which are not immediate:

Example 2.19 Let $(X \cup \{y\}, u)$ be such that there is some $x_0 \in X$ with $u(y, x_0) = \max_{x \in X} u(y, x)$ and $\alpha := u(y, x_0) \in uX$. On the other hand, let $(X \cup \{y'\}, u)$ be such that $u(y', x_0) = \max_{x \in X} u(y', x)$ with $u(y', x_0) > \alpha$, but $u(y', x_0) < \beta$ for every $\beta \in uX$, $\beta > \alpha$. Then for all $\beta \in uX$, the balls $B_\beta(X \cup \{y\}, y) \cap X$ and $B_\beta(X \cup \{y'\}, y') \cap X$ are equal. Indeed, for $\beta > \alpha$, they are both empty, and for $\beta \leq \alpha$, they are both equal to $B_\beta(X, x_0)$. On the other hand, $B^\circ_\alpha(X \cup \{y\}, y) \cap X = \emptyset$ whereas $B^\circ_\alpha(X \cup \{y'\}, y') \cap X = B^\circ_\alpha(X, x_0)$.

This example shows that the required structure should also pay regard to the balls B°. Further, the value $u(y', x_0) \notin uX$ induces in uX the cut whose initial segment is $\{\beta \mid \beta \leq \alpha\}$, which is the set of radii β for which the balls $B_\beta(X \cup \{y'\}, y') \cap X$ are nonempty. ♢

We will now introduce the required structure called approximation type. It will play an important role in our investigation of immediate extensions of valued fields, groups and modules.

In the following, let (X, u) be an ultrametric space with value set uX. Recall that we always exclude ∞ from uX, and that we write $uX\infty$ for $uX \cup \{\infty\}$. We will now introduce a modification of the notion "nest of balls". For this, let $\mathbf{B}(X)$, $\mathbf{B}^\circ(X)$. $\mathbf{B}_\alpha(X)$ and $\mathbf{B}^\circ_\alpha(X)$, $\alpha \in uX$, the sets of balls as introduced in section 2.2. A map

$$\begin{array}{rrcl} \mathbf{A}: & uX\infty & \to & \mathbf{B}(X) \cup \{\emptyset\} \times \mathbf{B}^\circ(X) \cup \{\emptyset\} \\ & uX\infty \ni \alpha & \mapsto & (\mathbf{A}_\alpha, \mathbf{A}^\circ_\alpha) \in \mathbf{B}_\alpha \cup \{\emptyset\} \times \mathbf{B}^\circ_\alpha \cup \{\emptyset\} \end{array}$$

will be called an **approximation type over** (X, u) if the balls $\mathbf{A}_\alpha, \mathbf{A}^\circ_\alpha$ form a totally ordered subset of $\mathbf{B}(X) \cup \mathbf{B}^\circ(X)$ (and hence a nest of balls), which means that they satisfy:

(AT) $\quad \alpha > \beta \implies \mathbf{A}^\circ_\alpha \subset \mathbf{A}_\alpha \subset \mathbf{A}^\circ_\beta \subset \mathbf{A}_\beta\,.$

In particular, the set

$$\Lambda(\mathbf{A}) = \{\alpha \in uX\infty \mid \mathbf{A}_\alpha \neq \emptyset\}$$

is an initial segment of $uX\infty$ (it may be empty). It will be called the **support of A**.

If $x \in \mathbf{A}_\alpha$ (resp. $x \in \mathbf{A}^\circ_\alpha$), then $\mathbf{A}_\alpha$ (resp. $\mathbf{A}^\circ_\alpha$) is nonempty and thus a ball of radius α, and it follows from Lemma 2.7 that $\mathbf{A}_\alpha = B_\alpha(x)$ (resp. $\mathbf{A}^\circ_\alpha = B^\circ_\alpha(x)$). In view of (AT), this shows:

$$x \in \mathbf{A}_\alpha \implies \mathbf{A}_\alpha = B_\alpha(x) \wedge \forall \beta < \alpha:\ \mathbf{A}_\beta = B_\beta(x) \wedge \mathbf{A}^\circ_\beta = B^\circ_\beta(x) \tag{2.12}$$

$$x \in \mathbf{A}^\circ_\alpha \implies \forall \beta \leq \alpha:\ \mathbf{A}_\beta = B_\beta(x) \wedge \mathbf{A}^\circ_\beta = B^\circ_\beta(x) \tag{2.13}$$

As direct consequences of our definition and the properties of balls, we note, for every $x, y \in X$ and $\alpha, \beta \in uX\infty$:

(at 1) $\mathbf{A}^\circ_\infty = \emptyset$

(at 2) $x, y \in \mathbf{A}_\alpha \implies u(x, y) \geq \alpha$ (at 3) $(u(x, y) \geq \alpha \wedge x \in \mathbf{A}_\alpha) \implies y \in \mathbf{A}_\alpha$

(at 2°) $x, y \in \mathbf{A}^\circ_\alpha \implies u(x, y) > \alpha$ (at 3°) $(u(x, y) > \alpha \wedge x \in \mathbf{A}^\circ_\alpha) \implies y \in \mathbf{A}^\circ_\alpha$

An approximation type will be called **value-immediate** if it satisfies

(ATVI) $\quad \forall x \in X\ \exists \alpha \in uX\infty:\ x \in \mathbf{A}_\alpha \setminus \mathbf{A}^\circ_\alpha\,,$

and **residue-immediate** if it satisfies

(ATRI) $\forall \alpha \in uX : \ \mathbf{A}_\alpha \neq \emptyset \Rightarrow \mathbf{A}^\circ_\alpha \neq \emptyset$.

An approximation type will be called **immediate approximation type** if it is value-immediate and residue-immediate. For the motivation of these notions, see (2.14), (2.15) and Lemma 2.21 below. Observe that every residue-immediate approximation type is already determined by its support and the balls of the form $\mathbf{A}^\circ_\alpha$: for every $\alpha \in \Lambda(\mathbf{A})$, there is some $x \in \mathbf{A}^\circ_\alpha$ by virtue of (ATRI), and we have $\mathbf{A}_\alpha = B_\alpha(x)$ by (2.13). On the other hand, every value-immediate and thus also every immediate approximation type is already determined by its support and the balls of the form $\mathbf{A}_\alpha$. This is seen as follows. For every $\beta \in \Lambda(\mathbf{A})$ such that $\mathbf{A}^\circ_\beta \neq \emptyset$, we can pick some $x \in \mathbf{A}^\circ_\beta$. Then by (ATVI) there is some $\alpha \in \Lambda(\mathbf{A})$ such that $x \in \mathbf{A}_\alpha \setminus \mathbf{A}^\circ_\alpha$. In view of (AT), we must have $\beta < \alpha$. Hence, $\mathbf{A}^\circ_\beta = B^\circ_\beta(x)$ by (2.12). **In particular, when we are working with immediate approximation types, we may forget about the balls $\mathbf{A}^\circ_\alpha$.**

Now let $(X, u) \subset (Y, u)$ be an extension of ultrametric spaces. For $y \in Y$, the map

$$uX\infty \ni \alpha \mapsto (\mathrm{appr}(y, X)_\alpha, \mathrm{appr}(y, X)^\circ_\alpha)$$

where

$$\begin{aligned} \mathrm{appr}(y, X)_\alpha &:= \{x \in X \mid u(y, x) \geq \alpha\} = B_\alpha(Y, y) \cap X \ , \\ \mathrm{appr}(y, X)^\circ_\alpha &:= \{x \in X \mid u(y, x) > \alpha\} = B^\circ_\alpha(Y, y) \cap X \ , \end{aligned}$$

will be called the **approximation type of y over X** and will be denoted by

$$\mathrm{appr}(y, X)$$

if it is clear which ultrametric is refered to. Otherwise, we will use "appr_u" (resp. "appr_v" if we are dealing with a valuation v). The above map is indeed an approximation type in the already defined sense; the proof is straightforward and thus left to the reader.

Conversely, if $\mathbf{A}$ is an approximation type over (X, u) and there exists an element $y \in Y$ such that $\mathbf{A} = \mathrm{appr}(y, X)$, then we say that y **realizes** $\mathbf{A}$ (in (Y, u)).

Lemma 2.20 *Let $\mathbf{A}$ is an approximation type over (X, u) and $(X, u) \subset (Y, u)$ an extension of ultrametric spaces. Then $y \in Y$ realizes $\mathbf{A}$ if and only if for all $\alpha \in uX\infty$, we have that $u(y, x) \geq \alpha$ for all $x \in \mathbf{A}_\alpha$ and $u(y, x) > \alpha$ for all $x \in \mathbf{A}^\circ_\alpha$. On the other hand, if $y \in Y$ does not realize $\mathbf{A}$ then there is some $\alpha \in uX\infty$ and a nonempty ball $\mathbf{A}_\alpha$ such that $u(y, x) < \alpha$ for all $x \in \mathbf{A}_\alpha$, or a nonempty ball $\mathbf{A}^\circ_\alpha$ such that $u(y, x) \leq \alpha$ for all $x \in \mathbf{A}^\circ_\alpha$.*

Proof: The first assertion follows directly from the definitions. The second assertion follows from the first and the following fact: If there is $z \in \mathbf{A}_\alpha$ such that $u(y, z) < \alpha$, then it follows from (at 2) and the triangle inequality that $u(y, x) < \alpha$ for all $x \in \mathbf{A}_\alpha$. Similarly, if there is $z \in \mathbf{A}^\circ_\alpha$ such that $u(y, z) \leq \alpha$, then it follows from (at 2°) and the triangle inequality that $u(y, x) \leq \alpha$ for all $x \in \mathbf{A}^\circ_\alpha$. □

In section 13.14, we will show that every approximation type $\mathbf{A}$ is realized by some element in some extension of the ultrametric space (X, u). Hence, we can always represent an approximation type in the form $\mathrm{appr}(y, X)$. If $\mathbf{A}$ is realized by some $x \in X$, then $\mathbf{A}$ will be called a **trivial approximation type**.

The approximation type $\mathrm{appr}(y, X)$ tells us how the element y can be approximated by elements from X. For an arbitrary approximation type $\mathbf{A}$,

$$\mathrm{dist}\mathbf{A} := \mathrm{dist}_{uX\infty}\Lambda(\mathbf{A})$$

will be called the **distance of A**. If $\mathbf{A} = \mathrm{appr}(y, X)$, then we will write

$$\mathrm{dist}(y, X) \quad \text{and} \quad \Lambda(y, X)$$

in the place of "dist$\mathbf{A}$" and "$\Lambda(\mathbf{A})$". Again, we will use indices if we have to indicate which ultrametric (resp. valuation) we are dealing with. The distance measures how "well" y can be approximated. If the distance is ∞, which means that $uX \subset \Lambda(y, X)$, then y can be approximated arbitrarily well. We say that the **dist A is assumed** (**by an element of** X) if $\Lambda(\mathbf{A})$ admits a maximal element (that is, among the nonempty balls $\mathbf{A}_\alpha, \mathbf{A}^\circ_\alpha$, there is a smallest one). If this element is not ∞ (that is, the smallest nonempty ball is not of radius ∞), then **dist A is finitely assumed**. Note that in contrast to the value sets of ultrametric spaces, the support of an approximation type $\mathbf{A}$ may contain ∞ by our definition. We will see in Lemma 2.27 below that this is the case if and only if $\mathbf{A}$ is trivial.

Let us translate (ATVI) and (ATRI) to the case $\mathbf{A} = \mathrm{appr}(y, X)$. By the definition of $\mathrm{appr}(y, X)$, $\mathbf{A}$ is value-immediate if and only if

$$\forall x \in X : \; u(y, x) \in uX\infty \; . \tag{2.14}$$

Further, $\mathbf{A}$ is residue-immediate if and only if

$$\forall x \in X : \; [u(y, x) \in uX \; \Rightarrow \; \exists x' \in X : \; u(y, x') > u(y, x)] \; . \tag{2.15}$$

This means that for every $\alpha \in \Lambda(y, X)$, the α-polygons of $(X \cup \{y\}, u)$ and (X, u) coincide. For $\alpha \in uX \setminus \Lambda(y, X)$, the element y will reduce to an α-polygon with only one point which is not appearing as an α-polygon of (X, u) (cf. Exercises 2.10 and 2.14). In the case of valued fields, (2.15) implies that y does not have a new residue. The following lemma gives the motivation for the name "immediate approximation type":

Lemma 2.21 *Let $(X, u) \subset (Y, u)$ be an extension of ultrametric spaces and $y \in Y$. Then* $\mathrm{appr}\,(y, X)$ *is immediate if and only if for every $x \in X$, $x \neq y$, there is some $x' \in X$ such that $u(y, x') > u(y, x)$. The extension $(X, u) \subset (Y, u)$ is immediate if and only if for every $y \in Y$, the approximation type* $\mathrm{appr}\,(y, X)$ *is immediate.*

Proof: Suppose that $\mathrm{appr}\,(y, X)$ is immediate and that x is an arbitrary element of X. By (2.14), $u(y, x) \in uX\infty$. Hence $y = x$, or $u(y, x) \in uX$ and by (2.15) there is some $x' \in X$ such that $u(y, x') > u(y, x)$. This proves the "$\Rightarrow$"-part of the first and the "$\Leftarrow$"-part of the second assertion.

Now let $y \in Y$ and suppose that for arbitrary $x \in X$ with $y \neq x$, there is $x' \in X$ such that $u(y, x') > u(y, x)$ (by definition, this is always the case if the extension $(X, u) \subset (Y, u)$ is immediate). Then by (2.15), $\mathrm{appr}\,(y, X)$ is residue-immediate. On the other hand, we obtain $u(y, x) = u(x, x') \in uX$ by virtue of (UM=). By (2.14), this shows that $\mathrm{appr}\,(y, X)$ is value-immediate. □

From the first assertion of this lemma it follows that the properties "nice in" and "immediate" exclude each other even on the level of approximation types.

Corollary 2.22 *The ultrametric space (X,u) is nice in (Y,u) if and only if for every $y \in Y \setminus X$, the approximation type* $\operatorname{appr}(y,X)$ *is not immediate.*

Proof: Assume that (X,u) is nice in (Y,u). Then the set $\{u(y,x) \mid x \in X\}$ admits a maximum for every $y \in Y$. Hence by the foregoing lemma, $\operatorname{appr}(y,K)$ can not be immediate, unless this maximum is assumed for $x = y$, that is, $y \in X$.

For the converse, suppose that for every $y \in Y \setminus X$, the approximation type $\operatorname{appr}(y,X)$ is not immediate. Then by the foregoing lemma, every set $\{u(y,x) \mid x \in X\}$ admits a maximum. If $y \in X$, then $\max\{u(y,x)\} = \infty$. Hence, (X,u) is nice in (Y,u). □

Examples 2.23 Let $(X,u) \subset (Y,u)$ be an extension of ultrametric spaces. Suppose that $y \in Y$ and $x \in X$ such that $u(y,x) \notin uX$. For every $x' \in X$ we will then have $u(y,x') \in uX \cup \{u(y,x)\}$ since otherwise we would obtain $u(x,x') = \min\{u(y,x), u(y,x')\} \notin uX$ by (UMT). Now $\operatorname{appr}(y,X)_\alpha \neq \emptyset$ if and only if $\alpha < u(y,x)$. Moreover, x is contained in $\operatorname{appr}(y,X)_\alpha$ and $\operatorname{appr}(y,X)^\circ_\alpha$ for all of these α. That is, $\operatorname{appr}(y,X)$ is residue-immediate but not value-immediate. Furthermore, we see that $\Lambda(y,X)$ is equal to $\{\alpha \in uX \mid \alpha < u(y,x)\}$. If $u(y,x) > uX$, then the distance of $\operatorname{appr}(y,X)$ is the maximal element of uX if there is one, and it is ∞ otherwise.

Now assume that there is $\alpha \in uX$ such that there is some maximal equilateral polygon of distance α which can be enlarged by y in (Y,u). This means that there is no $x' \in X$ such that $u(y,x') > \alpha$, showing that $\operatorname{appr}(y,X)_\beta = \emptyset$ for every $\beta > \alpha$ and $\operatorname{appr}(y,X)^\circ_\beta = \emptyset$ for every $\beta \geq \alpha$. Let $x \in X$ be any element of the mentioned α-polygon. Then $u(y,x) = \alpha$ and thus, x lies in every $\operatorname{appr}(y,X)_\beta$ for $\beta \leq \alpha$ and in every $\operatorname{appr}(y,X)^\circ_\beta$ for $\beta < \alpha$. Hence, x lies in every nonempty $\operatorname{appr}(y,X)_\beta$. Since $\operatorname{appr}(y,X)_\alpha \neq \emptyset$ while $\operatorname{appr}(y,X)^\circ_\alpha = \emptyset$ we see that $\operatorname{appr}(y,X)$ is not residue-immediate. On the other hand, we will see in the next section that $\operatorname{appr}(y,X)$ is value-immediate. The distance of $\operatorname{appr}(y,X)$ is just α. ♢

2.4 Approximation types: basic properties

We will now constitute some basic properties of approximation types. Throughout this section, let $\mathbf{A}$ be an approximation type over the ultrametric space (X,u).

Lemma 2.24 *a) If the* $\operatorname{dist}\mathbf{A}$ *is not finitely assumed, then $\mathbf{A}$ is residue-immediate.*
b) If there is some γ such that $\mathbf{A}_\gamma \neq \emptyset$ whereas $\mathbf{A}^\circ_\gamma = \emptyset$, then $\mathbf{A}$ is value-immediate.
c) Suppose that $\operatorname{dist}\mathbf{A}$ *is finitely assumed, that is, $\Lambda(\mathbf{A})$ admits a maximal element $\gamma \neq \infty$. Then $\mathbf{A}$ is residue-immediate if and only if $\mathbf{A}^\circ_\gamma \neq \emptyset$, and $\mathbf{A}$ is value-immediate if and only if $\mathbf{A}^\circ_\gamma = \emptyset$. Consequently, $\mathbf{A}$ is not immediate.*

Proof: a): Suppose that $\operatorname{dist}\mathbf{A}$ is not finitely assumed and let $\alpha \in \Lambda(\mathbf{A})$. Then there exists $\beta > \alpha$ such that $\mathbf{A}_\beta \neq \emptyset$. Now $\mathbf{A}_\beta \subset \mathbf{A}^\circ_\alpha$, hence $\mathbf{A}^\circ_\alpha \neq \emptyset$. Since $\alpha \in \Lambda(\mathbf{A})$ was arbitrary, our first assertion is proved.

b): Let $x_0 \in \mathbf{A}_\gamma$. For $x \in X$, let $\beta = u(x_0,x)$. If $\beta \geq \gamma$, then $x \in \mathbf{A}_\gamma$, but $x \notin \mathbf{A}^\circ_\gamma$ since we have assumed $\mathbf{A}^\circ_\gamma = \emptyset$. If $\beta < \gamma$, then $x_0 \in \mathbf{A}_\beta$ and $x_0 \in \mathbf{A}^\circ_\beta$ by (AT). In view of $\beta = u(x_0,x)$, it follows that $x \in \mathbf{A}_\beta \setminus \mathbf{A}^\circ_\beta$. So we have proved that for every $x \in X$ there exists a value $\beta \in uX\infty$ with $x \in \mathbf{A}_\beta \setminus \mathbf{A}^\circ_\beta$, that is, $\mathbf{A}$ is value-immediate.

c): Suppose that $\Lambda(\mathbf{A})$ admits a maximal element $\gamma \neq \infty$. Let $\mathbf{A}^\circ_\gamma \neq \emptyset$. Then for all $\alpha \leq \gamma$, $\mathbf{A}^\circ_\alpha \neq \emptyset$. Since $\mathbf{A}_\alpha \neq \emptyset$ implies that $\alpha \leq \gamma$, this proves that $\mathbf{A}$ is residue-immediate.

On the other hand, for $x_0 \in \mathbf{A}^\circ_\gamma$ there is no β such that $x_0 \in \mathbf{A}_\beta \setminus \mathbf{A}^\circ_\beta$. Hence, $\mathbf{A}$ is not value-immediate. Now let $\mathbf{A}^\circ_\gamma = \emptyset$. Then $\mathbf{A}$ is not residue-immediate. On the other hand, $\mathbf{A}$ is value-immediate by part b). □

By part a) of this lemma, if $\mathbf{A}$ is not residue-immediate, then dist $\mathbf{A}$ is finitely assumed, hence in view of part c) of the lemma, $\mathbf{A}$ must be value-immediate. Thus, we obtain

Corollary 2.25 *Every approximation type is value-immediate or residue-immediate.*

Assume that $\mathbf{A}$ is not residue-immediate. Then dist $\mathbf{A}$ is finitely assumed by some element $\gamma \in uX$. By part c) of Lemma 2.24, we have that $\mathbf{A}^\circ_\gamma = \emptyset$. By (AT), the latter implies that $\mathbf{A}^\circ_\alpha = \emptyset = \mathbf{A}_\alpha$ for all $\alpha > \gamma$. By (2.12), $\mathbf{A}_\gamma$ determines the balls $\mathbf{A}_\alpha$ and $\mathbf{A}^\circ_\alpha$ for all $\alpha < \gamma$. This proves:

Corollary 2.26 *Assume that the approximation type $\mathbf{A}$ is not residue-immediate. Then $\gamma = \operatorname{dist}\mathbf{A}$, and $\mathbf{A}$ is uniquely determined by the ball $\mathbf{A}_\gamma$ and the information that $\mathbf{A}^\circ_\gamma = \emptyset$.*

The following lemma characterizes the trivial approximation types.

Lemma 2.27 *The approximation type $\mathbf{A}$ is trivial if and only if $\mathbf{A}_\infty \neq \emptyset$. More precisely, $\mathbf{A}$ is of the form*

$$\mathbf{A} = \operatorname{appr}(x_0, X) \quad \textit{with} \quad x_0 \in X \tag{2.16}$$

if and only if $x_0 \in \mathbf{A}_\infty$. Consequently, $\operatorname{appr}(x_0, X) = \operatorname{appr}(z, X)$ implies that $x_0 = z$.

If $\mathbf{A}$ satisfies (2.16), then it can be described as follows: for all $\alpha \in uX$, we have that $\mathbf{A}_\alpha = B_\alpha(x_0)$ and $\mathbf{A}^\circ_\alpha = B^\circ_\alpha(x_0)$. For every $x \neq x_0$,

$$x \in \mathbf{A}_\beta \setminus \mathbf{A}^\circ_\beta \quad \textit{for} \quad \beta = u(x_0, x)\ . \tag{2.17}$$

Every trivial approximation type is immediate.

Proof: If $\mathbf{A}$ satisfies (2.16), then $x_0 \in \mathbf{A}_\infty$ because of $u(x_0, x_0) = \infty$. For the converse, assume $x_0 \in \mathbf{A}_\infty$. By part a) of Lemma 2.24, $\mathbf{A}$ is residue-immediate, and by part b) of the same lemma, it is value-immediate. Hence, $\mathbf{A}$ is immediate. Further, $x_0 \in \mathbf{A}_\alpha$ and $x_0 \in \mathbf{A}^\circ_\alpha$ for all $\alpha \in uX\infty$. Since every element of a ball is a center of that ball, this implies that $\mathbf{A}_\alpha = B_\alpha(x_0) = \operatorname{appr}(x_0, X)_\alpha$ and $\mathbf{A}^\circ_\alpha = B^\circ_\alpha(x_0) = \operatorname{appr}(x_0, X)^\circ_\alpha$. This gives $\mathbf{A} = \operatorname{appr}(x_0, X)$. Now, (2.17) follows directly from the definition of the balls. Further, $\operatorname{appr}(x_0, X) = \operatorname{appr}(z, X)$ with $x_0 \in X$ implies that $x_0 \in \operatorname{appr}(x_0, X)_\infty = \operatorname{appr}(z, X)_\infty$. This means that $u(z, x_0) = \infty$, showing that $z = x_0$. □

This lemma together with Lemma 2.21 yields:

Corollary 2.28 *If $(X, u) \subset (Y, u)$ is an immediate extension of ultrametric spaces and $y \in Y \setminus X$, then $\operatorname{appr}(y, X)$ is a nontrivial immediate approximation type.*

Lemma 2.24 has shown that $\mathbf{A}$ can not be immediate if its distance is finitely assumed. If dist $\mathbf{A}$ is assumed but not finitely assumed, it means that $\mathbf{A}_\infty \neq \emptyset$; in this case, Lemma 2.27 shows that $\mathbf{A}$ is trivial. We have proved:

Corollary 2.29 *Let $\mathbf{A}$ be an immediate approximation type. Then $\operatorname{dist}\mathbf{A}$ is not finitely assumed (that is, $\Lambda(\mathbf{A})$ has no maximal element $\neq \infty$). If in addition $\mathbf{A}$ is nontrivial, then $\operatorname{dist}\mathbf{A}$ is not assumed (that is, $\Lambda(\mathbf{A})$ has no maximal element).*

An element $x_0 \in X$ will be called a **universal element of** the approximation type $\mathbf{A}$ if $x_0 \in \mathbf{A}_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$. From Lemma 2.27 we know that every trivial approximation type has a unique universal element.

Lemma 2.30 *Let $\mathbf{A}$ be an approximation type over (X, u). Then $\mathbf{A}$ is nontrivial and immediate if and only if it has no universal element. If for $x_0 \in X$ there is no $\alpha \in uX\infty$ such that $x_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$ (and thus, $\mathbf{A}$ is not value-immediate), then x_0 is a universal element of $\mathbf{A}$. If $\mathbf{A}_\gamma \neq \emptyset$ and $\mathbf{A}_\gamma^\circ = \emptyset$ for some $\gamma \in uX\infty$ (and thus, $\mathbf{A}$ is trivial or not residue-immediate), then every element of $\mathbf{A}_\gamma$ is a universal element of $\mathbf{A}$.*

If $x_0 \in X$ is a universal element of $\mathbf{A}$, then every nonempty ball $\mathbf{A}_\alpha$ is equal to $B_\alpha(x_0)$, and every nonempty ball $\mathbf{A}_\alpha^\circ$ is equal to $B_\alpha^\circ(x_0)$.

Proof: Assume that $\mathbf{A}$ admits a universal element x_0, and suppose that $\mathbf{A}$ is immediate. We have to show that $\mathbf{A}$ is trivial. By (ATVI), there is some $\alpha \in uX\infty : x_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$. But Corollary 2.29 says that dist$\mathbf{A}$ is not finitely assumed. Hence, $\alpha = \infty$ and by Lemma 2.27, $\mathbf{A}$ is trivial.

For the converse, we have to show that $\mathbf{A}$ admits a universal element if it is trivial or not immediate. The case of a trivial approximation type is settled by Lemma 2.27. So assume that $\mathbf{A}$ is not immediate. Assume first that $\mathbf{A}$ is not value-immediate, and let $x_0 \in X$ be such that there is no $\alpha \in uX\infty$ satisfying $x_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$. Suppose that there is some nonempty ball $\mathbf{A}_\beta$ not containing x_0. Let $x' \in \mathbf{A}_\beta$. Then (at 3) shows that $\alpha := u(x', x_0) < \beta$. By (AT), $x' \in \mathbf{A}_\beta \subset \mathbf{A}_\alpha^\circ \subset \mathbf{A}_\alpha$. By (at 2°) and (at 3), we obtain that $x_0 \notin \mathbf{A}_\alpha^\circ$, but $x_0 \in \mathbf{A}_\alpha$. Since this contradicts our choice of x_0, we find that x_0 is contained in every nonempty ball $\mathbf{A}_\beta$, i.e., x_0 is a universal element of $\mathbf{A}$.

Now assume that $\mathbf{A}$ is not residue-immediate, and let $\gamma \in uX$ such that $\mathbf{A}_\gamma \neq \emptyset$ and $\mathbf{A}_\gamma^\circ = \emptyset$. Then by (AT), $\alpha \in \Lambda(\mathbf{A})$ implies that $\alpha \leq \gamma$, and again by (AT) that $\mathbf{A}_\gamma \subset \mathbf{A}_\alpha$. This shows that every element of $\mathbf{A}_\gamma$ is a universal element of $\mathbf{A}$.

The proof of the last assertion of the lemma is an easy exercise. □

Corollary 2.31 *Assume that the approximation type $\mathbf{A}$ is not immediate, and choose x_0 according to the foregoing lemma. Then $\mathbf{A}$ is uniquely determined by $\Lambda(\mathbf{A})$, x_0 and the information whether $\mathbf{A}$ is value-immediate or residue-immediate.*

Proof: If $\alpha \notin \Lambda(\mathbf{A})$, then $\mathbf{A}_\alpha = \emptyset = \mathbf{A}_\alpha^\circ$ by definition of $\Lambda(\mathbf{A})$. By the foregoing lemma, $\mathbf{A}_\alpha = B_\alpha(x_0)$ for every $\alpha \in \Lambda(\mathbf{A})$, and if $\mathbf{A}_\alpha^\circ$ is nonempty, then $\mathbf{A}_\alpha^\circ = B_\alpha^\circ(x_0)$. If α is not the maximal element of $\Lambda(\mathbf{A})$, then $\mathbf{A}_\alpha^\circ$ is nonempty by virtue of (AT). If α is the maximal element of $\Lambda(\mathbf{A})$, then it follows from part c) of Lemma 2.24 that $\mathbf{A}_\alpha^\circ$ is nonempty if $\mathbf{A}$ is residue-immediate, and that $\mathbf{A}_\alpha^\circ = \emptyset$ if $\mathbf{A}$ is value-immediate. □

If an approximation type is not value-immediate, then it corresponds to a cut in $uX\infty$, as the next lemma will show. In the case of an approximation type appr(y, X) which is not value-immediate, (2.14) must be injured. We will see that the element x_0 of Lemma 2.30 is the malefactor.

Lemma 2.32 *The approximation type $\mathbf{A}$ over (X, u) is not value-immediate if and only if there is some $x_0 \in X$ such that*

$$(\, \{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha^\circ\}\,,\, \{\alpha \in uX\infty \mid x_0 \notin \mathbf{A}_\alpha\}\,) \tag{2.18}$$

is a cut in $uX\infty$ with ∞ appearing in the upper set. In this case,

a) $\mathbf{A}$ *is uniquely determined by* x_0 *and the cut (2.18),*

b) x_0 *is a universal element for* $\mathbf{A}$*,*

c) the cut (2.18) is equal to the cut

$$(\Lambda(y, X), uX \setminus \Lambda(y, X))$$

d) if an element y in some extension of (X, u) realizes $\mathbf{A}$, then $u(y, x_0) \notin uX$ and $u(y, x_0)$ realizes the cut (2.18).

Proof: If $\mathbf{A}$ is not value-immediate then by Lemma 2.30 there is x_0 which lies in every nonempty ball of $\mathbf{A}$. That is, $\Lambda(\mathbf{A}) = \{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha\}$. Since $\mathbf{A}$ is residue-immediate by Lemma 2.25, $\{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha\} = \{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha^\circ\}$. Hence, (2.18) is just the cut $(\Lambda(\mathbf{A}), uX \setminus \Lambda(\mathbf{A}))$ (which proves assertion c)). Since $\mathbf{A}$ is not value-immediate, Lemma 2.27 shows that $\mathbf{A}$ is not trivial and that $x_0 \notin \mathbf{A}_\infty$. That is, ∞ lies in the upper set of the cut (2.18).

For the converse, assume that (2.18) is a cut with ∞ in the upper set. From the latter, it follows that $x_0 \notin \mathbf{A}_\infty$. Suppose that there exists $\alpha \in uX\infty$ such that $x_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$. Then $\alpha < \infty$. It follows that α lies neither in the upper nor in the lower set of the cut (2.18). This contradiction shows that there does not exist an α such that $x_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$. That is, $\mathbf{A}$ is not value-immediate. To show that $\mathbf{A}$ is uniquely determined by x_0 and the cut (2.18), we conclude further that $\mathbf{A}$ is residue-immediate, according to Lemma 2.25. Moreover, from Lemma 2.30 we can infer that x_0 is a universal element of $\mathbf{A}$ (which is assertion b)). We thus obtain that $\Lambda(\mathbf{A}) = \{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha\} = \{\alpha \in uX\infty \mid x_0 \in \mathbf{A}_\alpha^\circ\}$. This proves that $\Lambda(\mathbf{A})$ and the fact that $\mathbf{A}$ is residue-immediate follow from the cut (2.18). By the foregoing corollary, we now obtain that $\mathbf{A}$ is uniquely determined by x_0 and the cut (2.18), which is assertion a).

To prove assertion d), assume that $\mathbf{A} = \operatorname{appr}(y, X)$. Then $u(y, x_0) = \alpha \in uX$ would yield that $x_0 \in \operatorname{appr}(y, X)_\alpha \setminus \operatorname{appr}(y, X)_\alpha^\circ$, which we already have shown to be impossible. Hence, $u(y, x_0) \notin uX$. Now let α be any value in uX. If $\alpha < u(y, x_0)$ then $x_0 \in \operatorname{appr}(y, X)_\alpha = \mathbf{A}_\alpha$, showing that $\alpha \in \Lambda(y, X)$. If $\alpha > u(y, x_0)$ then $x_0 \notin \operatorname{appr}(y, X)_\alpha$. Since we know already that x_0 is a universal element of $\mathbf{A}$, we obtain that $\alpha \in uX \setminus \Lambda(y, X)$. In view of assertion c), we have now proved assertion d). □

As a further consequence of Lemma 2.30, we find that immediate approximation types are already determined by the balls $\mathbf{A}_\alpha$, in the following sense:

Corollary 2.33 *Let $\mathbf{A}$ and $\mathbf{A}'$ be two approximation types over (X, u) and assume that $\mathbf{A}$ is immediate. Then $\mathbf{A} = \mathbf{A}'$ if and only if $\mathbf{A}_\alpha = \mathbf{A}'_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$.*

Proof: The implication "⇒" is trivial. To show the other implication, let us assume that $\mathbf{A}_\alpha = \mathbf{A}'_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$. First suppose that $\infty \in \Lambda(\mathbf{A})$. Then $x_0 \in \mathbf{A}_\infty = \mathbf{A}'_\infty$ implies that $\mathbf{A} = \operatorname{appr}(x_0, X) = \mathbf{A}'$ by virtue of Lemma 2.27.

Now suppose that $\infty \notin \Lambda(\mathbf{A})$; hence by Lemma 2.27, $\mathbf{A}$ is nontrivial. Since $\mathbf{A}$ is assumed to be immediate, we know from Lemma 2.30 that for every $x \in X$ there is some $\alpha \in \Lambda(\mathbf{A})$ such that $x \notin \mathbf{A}_\alpha = \mathbf{A}'_\alpha$. Applying Lemma 2.30 again, we find that also $\mathbf{A}'$ is immediate and nontrivial. Since the intersection $\bigcap_{\alpha \in \Lambda(\mathbf{A})} \mathbf{A}'_\alpha$ is empty, we know by (AT) that $\mathbf{A}'_\beta = \emptyset$ for every $\beta > \Lambda(\mathbf{A})$. On the other hand, $\Lambda(\mathbf{A})$ is an initial segment without

maximal element (Corollary 2.29), so we find $\mathbf{A}'_\beta = \emptyset$ for every $\beta \notin \Lambda(\mathbf{A})$. Consequently, $\mathbf{A}_\alpha = \mathbf{A}'_\alpha$ for all $\alpha \in uX$. Now $\mathbf{A}'$ being an immediate approximation type, we know already that it is completely determined by its balls of the form $\mathbf{A}_\alpha$. Thus, we have proved $\mathbf{A} = \mathbf{A}'$. □

We leave it to the reader to show that immediate approximation types are as well determined by the balls $\mathbf{A}^\circ_\alpha$, in the sense of this lemma. The foregoing corollary implies the following analogue of Lemma 2.20:

Corollary 2.34 *Let* $\mathbf{A}$ *is an immediate approximation type over* (X,u) *and* $(X,u) \subset (Y,u)$ *an extension of ultrametric spaces. Then* $y \in Y$ *realizes* $\mathbf{A}$ *if and only if for all* $\alpha \in uX\infty$*, we have that* $u(y,x) \geq \alpha$ *for all* $x \in \mathbf{A}_\alpha$*. On the other hand, if* $y \in Y$ *does not realize* $\mathbf{A}$ *then there is some* $\alpha \in \Lambda(\mathbf{A})$ *such that* $u(y,x) < \alpha$ *for all* $x \in \mathbf{A}_\alpha$ *.*

A further corollary to Lemma 2.30 deals with the behaviour of immediate approximation types under ultrametric space extensions.

Corollary 2.35 *Let* $(X,u) \subset (Y',u) \subset (Y,u)$ *be extensions of ultrametric spaces and let* $y \in Y$ *with nontrivial immediate approximation type* $\operatorname{appr}(y,X)$*. If* $\operatorname{appr}(y,Y')$ *is not nontrivial and immediate, then there exists* $y' \in Y'$ *such that* $\operatorname{appr}(y',X) = \operatorname{appr}(y,X)$*.*

Proof: If $\operatorname{appr}(y,Y')$ is not nontrivial and immediate, then by Lemma 2.30 there is some $y' \in Y'$ such that $y' \in \operatorname{appr}(y,Y')_\alpha$ and thus $\operatorname{appr}(y,Y')_\alpha = B_\alpha(Y',y')$ for all $\alpha \in \Lambda(y,Y')$. It follows that $\operatorname{appr}(y,X)_\alpha = \operatorname{appr}(y,Y')_\alpha \cap X = B_\alpha(Y',y') \cap X = \operatorname{appr}(y',X)_\alpha$ for every $\alpha \in \Lambda(y,Y)$. From the foregoing corollary we can now infer that $\operatorname{appr}(y',X) = \operatorname{appr}(y,X)$. □

Which elements in an ultrametric space extension of (X,u) have equal approximation types over (X,u)? The foregoing corollary helps to answer this question:

Lemma 2.36 *Let* $(X,u) \subset (Y,u)$ *be an extension of ultrametric spaces and* $y,y' \in Y$*.*

a) For every $\alpha \in \Lambda(y,X)$*,* $\operatorname{appr}(y,X)_\alpha = \operatorname{appr}(y',X)_\alpha$ *holds if and only if* $u(y,y') \geq \alpha$*.*

b) If $\operatorname{appr}(y,X)$ *is immediate, then*

$$\operatorname{appr}(y,X) = \operatorname{appr}(y',X) \iff u(y,y') \geq \Lambda(y,X) \ .$$

Proof: a): Let $\alpha \in uX$. If $u(y,y') \geq \alpha$, then $B_\alpha(Y,y) = B_\alpha(Y,y')$, which yields that $\operatorname{appr}(y,X)_\alpha = B_\alpha(Y,y) \cap X = B_\alpha(Y,y') \cap X = \operatorname{appr}(y',X)_\alpha$. If $u(y,y') < \alpha$, then $B_\alpha(Y,y) \cap B_\alpha(Y,y') = \emptyset$, whence $\operatorname{appr}(y,X)_\alpha \cap \operatorname{appr}(y',X)_\alpha = \emptyset$; for $\alpha \in \Lambda(y,X)$, this yields that $\operatorname{appr}(y,X)_\alpha \neq \operatorname{appr}(y',X)_\alpha$ since then, the former set is nonempty.

b): If $u(y,y') \geq \Lambda(y,X)$ does not hold, then there is some $\alpha \in \Lambda(y,X)$ such that $\alpha > u(y,y')$. By part a), it follows that $\operatorname{appr}(y,X)_\alpha \neq \operatorname{appr}(y',X)_\alpha$. This proves implication "⇒".

If $u(y,y') \geq \Lambda(y,X)$ holds, then $u(y,y') \geq \alpha$ for all $\alpha \in \Lambda(y,X)$. Again by part a), it follows that $\operatorname{appr}(y,X)_\alpha = \operatorname{appr}(y',X)_\alpha$ for all $\alpha \in \Lambda(y,X)$. Since $\operatorname{appr}(y,X)$ is assumed to be immediate, the foregoing corollary now shows that $\operatorname{appr}(y,X) = \operatorname{appr}(y',X)$. □

Corollary 2.37 *Let* $(X,u) \subset (Y,u)$ *be an extension of ultrametric spaces and* $y,y' \in Y$*. If* $\operatorname{appr}(y,X)$ *is immediate, then there exists* $\alpha \in \Lambda(y,X)$ *such that* $u(y,y') \geq u(x,y')$ *for all* $x \in \operatorname{appr}(y,X)_\alpha$ *.*

Proof: If $u(y,y') \geq \Lambda(y,X)$ then $\operatorname{appr}(y,X) = \operatorname{appr}(y',X)$ by the foregoing lemma. Then it follows that $u(y,y') \geq u(y,x) = u(y',x)$ for all $x \in X$. Now assume that $u(y,y') \geq \Lambda(y,X)$ does not hold. Then $u(y,y') < \infty$ and there is some $\alpha \in \Lambda(y,X)$ such that $u(y,y') < \alpha$, because $\Lambda(y,X)$ has no maximal element $\neq \infty$ by Corollary 2.29. Hence, $u(y,y') = \min\{u(y,x), u(x,y')\} = u(x,y')$ for all $x \in \operatorname{appr}(y,X)_\alpha$. □

We will now come back to the question that has led to the introduction of approximation types. We will show how approximation types can serve to establish isomorphisms of ultrametric spaces. In the course of this book, this will gain even more importance when isomorphisms of richer ultrametric structures like valued groups and fields can be traced back to approximation types.

Lemma 2.38 *Let (Y,u) and (Y',u') be extensions of the ultrametric space (X,u). Further, let $y \in Y$ and $y' \in Y'$. Then there is an isomorphism $(X \cup \{y\}, u) \to (X \cup \{y'\}, u')$ over (X,u) if and only if* $\operatorname{appr}(y,X) = \operatorname{appr}(y',X)$.

Proof: Assume that $\operatorname{appr}(y,X)$ is value-immediate and that $\operatorname{appr}(y,X) = \operatorname{appr}(y',X)$. Let x be an arbitrary element of X. Then by (ATVI), there exists $\alpha \in uX$ such that $x \in \operatorname{appr}(y,X)_\alpha \setminus \operatorname{appr}(y,X)^\circ_\alpha = \operatorname{appr}(y',X)_\alpha \setminus \operatorname{appr}(y',X)^\circ_\alpha$, that is, $u(y,x) = \alpha = u(y',x)$. This shows that $u(X \cup \{y\}) = uX = u(X \cup \{y'\})$ and that $y \mapsto y'$ induces an isomorphism $(X \cup \{y\}, u) \to (X \cup \{y'\}, u')$ over (X,u). The proof of the converse is left to the reader.

If $\operatorname{appr}(y,X)$ is not value-immediate, then it is residue-immediate by Corollary 2.25, and by Lemma 2.30 there is $x_0 \in X$ which lies in every nonempty ball in $\operatorname{appr}(y,X)$. By Lemma 2.32 we know that both $u(y,x_0)$ and $u(y',x_0)$ realize the cut $(\Lambda(y,X), uX \setminus \Lambda(y,X))$ in uX. Hence, the assignment $u(y,x_0) \mapsto u(y',x_0)$ constitutes an isomorphism ρ between the ordered sets $uX \cup \{u(y,x_0)\}$ and $uX \cup \{u(y',x_0)\}$ over uX.

Now let x be an arbitrary element of X; we have to show that $\rho u(y,x) = u(y',x)$. If $u(x_0,x) \in \Lambda(y,X)$, then $u(x_0,x) < u(y,x_0)$ and $u(x_0,x) < u(y',x_0)$ and by (UMT), $u(y,x) = \min\{u(y,x_0), u(x_0,x)\} = u(x_0,x) = \min\{u(y',x_0), u(x_0,x)\} = u(y',x)$. If $u(x_0,x) \notin \Lambda(y,X)$, then $u(x_0,x) > u(y,x_0)$ and $u(x_0,x) > u(y',x_0)$ and by (UMT), $u(y,x) = \min\{u(y,x_0), u(x_0,x)\} = u(y,x_0)$ and $u(y',x) = \min\{u(y',x_0), u(x_0,x)\} = u(y',x_0) = \rho u(y,x)$. □

To conclude this section, we will show the connection between approximation types and the notions "spherically complete" and "complete". To every nest $\mathbf{B} \subset \mathbf{B}(X)$ of balls, we can associate an immediate approximation type by the following refinement of our filling up procedure for nests of balls. If the intersection of all balls in $\mathbf{B}$ contains an element $x \in X$, then we set $\operatorname{appr}(\mathbf{B}) := \operatorname{appr}(x,X)$. Since this approximation type is trivial, it is immediate. If the intersection of all balls in $\mathbf{B}$ is empty, then we proceed as follows. Assume that $\mathbf{B} = \{B_\alpha(x_\alpha) \mid \alpha \in S\}$ for some subset $S \subset uX$. For every $\beta > S$, we let $\operatorname{appr}(\mathbf{B})_\beta := \operatorname{appr}(\mathbf{B})^\circ_\beta := \emptyset$. For $\beta \not> S$, there is some $\alpha' \in S$ such that $\alpha' \geq \beta$. There must be some $\alpha \in S$ such that $\alpha > \alpha' \geq \beta$ since otherwise, $\mathbf{B}$ would contain a smallest ball and there would be an element of X in the intersection. Now we set $\operatorname{appr}(\mathbf{B})_\beta := B_\beta(x_\alpha)$ and $\operatorname{appr}(\mathbf{B})^\circ_\beta := B^\circ_\beta(x_\alpha)$. This definition does not depend on the actual choice of α. Indeed, if also $\gamma \in S$ such that $\gamma > \beta$, then the balls $B_\alpha(x_\alpha)$ and $B_\gamma(x_\gamma)$ have a nonempty intersection since $\mathbf{B}$ is a nest of balls. Consequently, $u(x_\alpha, x_\gamma) \geq \min\{\alpha,\gamma\} > \beta$, showing that $B_\beta(x_\alpha) = B_\beta(x_\gamma)$ and $B^\circ_\beta(x_\alpha) = B^\circ_\beta(x_\gamma)$. Note that $B_\alpha(x_\alpha) \in \mathbf{B}$ implies that $\operatorname{appr}(\mathbf{B})_\alpha = B_\alpha(x_\alpha)$. By this construction, $\operatorname{appr}(\mathbf{B})$ satisfies (AT) and is thus an approximation type. Since the intersection of all balls in $\mathbf{B}$ was

assumed to be empty, the same is consequently true for the intersection of all nonempty balls of appr $(\mathbf{B})$. In view of Lemma 2.30 we conclude that appr $(\mathbf{B})$ is immediate and nontrivial.

We will call appr $(\mathbf{B})$ an **immediate approximation type associated with B**. By our construction, it is uniquely determined if the intersection of $\bigcap \mathbf{B}$ is empty or contains at most one element. If $\bigcap \mathbf{B}$ is empty, then $\Lambda(\text{appr}\,(\mathbf{B})) = \Lambda(\mathbf{B})$; in particular, $\Lambda(\text{appr}\,(\mathbf{B})) = uX$ if and only if $\mathbf{B}$ is of the form $\{B_\alpha(x_\alpha) \mid \alpha \in S\}$ with S cofinal in uX. Let us call an approximation type a **completion type** if it is immediate and its distance is ∞. Recall that the latter is the case if its support is equal to $uX\infty$ or to uX, depending on whether the approximation type is trivial or not.

If $\bigcap \mathbf{B}$ is nonempty, but S is cofinal in uX and uX has no maximal element, then the intersection contains at most one element, showing that $\mathbf{B}$ admits a unique associated immediate approximation type. Indeed, if x and y are two distinct elements of X then $u(x, y) < \infty$, so by hypothesis there is some $\alpha \in S$, $\alpha > u(x, y)$; but then, x and y can not be both contained in $B_\alpha(x_\alpha)$.

Conversely, if $\mathbf{A}$ is a nontrivial immediate approximation type over (X, u), then $\infty \notin \Lambda(\mathbf{A})$, and the balls $\mathbf{A}_\alpha$, $\alpha \in \Lambda(\mathbf{A})$ form a nest whose intersection is empty, according to Lemma 2.30.

Lemma 2.39 *Let $\mathbf{B} \subset \mathbf{B}(X)$ be a nest of balls with empty intersection $\bigcap \mathbf{B}$. Then there is a uniquely determined approximation type* appr $(\mathbf{B})$ *such that $B_\alpha(x_\alpha) =$* appr $(\mathbf{B})_\alpha$ *for every ball $B_\alpha(x_\alpha) \in \mathbf{B}$. Further,* appr $(\mathbf{B})$ *is nontrivial and immediate, and $\Lambda(\mathbf{B}) = \Lambda(\text{appr}\,(\mathbf{B}))$. In particular,* appr $(\mathbf{B})$ *is a completion type if and only if $\mathbf{B}$ is a completion nest.*

Conversely, if $\mathbf{A}$ is a nontrivial immediate approximation type over (X, u), then $\{\mathbf{A}_\alpha \mid \alpha \in \Lambda(\mathbf{A})\}$ is a nest of balls whose intersection is empty. This nest is a completion nest if and only if $\mathbf{A}$ is a completion type.

From this lemma, we deduce:

Lemma 2.40 *The ultrametric space (X, u) is spherically complete if and only if every immediate approximation type over (X, u) is trivial. Further, (X, u) is complete if and only if every completion type over (X, u) is trivial.*

What happens to immediate approximation types over (X, u) in a spherically complete extension of (X, u)? To answer this question, we need:

Lemma 2.41 *Let $\mathbf{A}$ be a immediate approximation type over (X, u) and (Y, u) an extension of (X, u). If $y \in Y$ is such that $y \in B_\alpha(Y, x_\alpha)$ for $x_\alpha \in \mathbf{A}_\alpha$ and all $\alpha \in \Lambda(\mathbf{A})$, then $\mathbf{A} =$* appr (y, X).

Proof: The assumption yields that for all $\alpha \in \Lambda(\mathbf{A})$, we have $B_\alpha(Y, x_\alpha) = B_\alpha(Y, y)$, whence $\mathbf{A}_\alpha = B_\alpha(Y, x_\alpha) \cap X = B_\alpha(Y, y) \cap X$ for all $\alpha \in \Lambda(\mathbf{A})$. Since already these balls determine the immediate approximation type $\mathbf{A}$ (cf. Corollary 2.33), we have proved that $\mathbf{A} =$ appr (y, X). □

Lemma 2.42 *If the extension (Y, u) of (X, u) is spherically complete, then every immediate approximation type over (X, u) is realized by some element of (Y, u).*

Proof: Let $\mathbf{A}$ be an immediate approximation type over (X,u). For every $\alpha \in \Lambda(\mathbf{A})$, let $x_\alpha \in \mathbf{A}_\alpha$. Then $B_\alpha(Y,x_\alpha)$ is a nest of balls in (Y,u), which by assumption must have a nonempty intersection. If $y \in Y$ lies in this intersetion, then it satisfies the assumption of the foregoing lemma. This lemma shows that $\mathbf{A} = \operatorname{appr}(y,X)$. □

Exercise 2.10 *Compare the α-polygons of $(X \cup \{y\},u)$ and (X,u) for every $\alpha \in u(X \cup \{y\})$ under each of the three assumptions that $\operatorname{appr}(y,X)$ be value-immediate, residue-immediate, immediate.*

Exercise 2.11 *Let (Y,u) and (Y',u') be two extensions of (X,u) and $y \in Y$, $y' \in Y'$. Suppose that $(X,u) \subset (Y,u)$ is immediate. Show that $\operatorname{appr}(y,X) = \operatorname{appr}(y',X)$ if and only if $u(y,x) = u(y',x)$ for all $x \in X$.*

Exercise 2.12 *Give criteria for the equality of arbitrary approximation types, in the spirit of part b) of Lemma 2.36. Discuss to which extent the distance can be employed in such criteria.*

Exercise 2.13 *For an arbitrary ultrametric space (X,u), define a relation $U(x_1,x_2,x_3) :\Leftrightarrow u(x_1,x_2) > u(x_2,x_3)$. Show that Lemma 2.38 holds without a condition on the approximation type $\operatorname{appr}(y,X)$ if the ultrametric u is replaced by the corresponding relation U in each of the three spaces.*

2.5 Completion types

Let us return to the notion of complete ultrametric spaces. If a given space (X,u) is not complete, the question arises whether there is an extension which is complete and which is as small as possible, in some sense. We will construct this extension, called the **completion of** (X,u) and denoted by $(X,u)^c$ or (X^c,u). By Lemma 2.40 we know that there exists a nontrivial completion type over (X,u), if (X,u) is not complete.

We let X^c be the set of all completion types over (X,u). By Lemma 2.27, $\operatorname{appr}(x,X)$ is a completion type for every $x \in X$, and if $x \neq x' \in X$, then $\operatorname{appr}(x,X) \neq \operatorname{appr}(x',X)$. We can thus view X as a subset of X^c. The extension of the ultrametric u from X to X^c is given as follows. Let $\mathbf{A}$ and $\mathbf{A}'$ be two completion types over (X,u). If $\mathbf{A} = \mathbf{A}'$ then $u(\mathbf{A},\mathbf{A}') = \infty$. If $\mathbf{A} \neq \mathbf{A}'$ then for some $\alpha \in uX$ we must have $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$, and we set $u(\mathbf{A},\mathbf{A}') = u(x_\alpha,x'_\alpha)$ for arbitrary $x_\alpha \in \mathbf{A}_\alpha$, $x'_\alpha \in \mathbf{A}'_\alpha$. Then $u(x_\alpha,x'_\alpha) < \alpha$ since otherwise, the balls would coincide by virtue of Lemma 2.7. Using this fact, the reader may show that our definition does not depend on the choice of α and the elements x_α, x'_α of the respective balls, and that the so defined map coincides with u on X. It evidently satisfies (UM 0) and (UMS). For the proof of (UMT), note the following: if we have $u(\mathbf{A},\mathbf{A}') = u(x_\alpha,x'_\alpha)$ by our above definition, then also $u(\mathbf{A},\mathbf{A}') = u(x_\gamma,x'_\gamma)$ for every $\gamma \geq \alpha$ and $x_\gamma \in \mathbf{A}_\gamma$, $x'_\gamma \in \mathbf{A}'_\gamma$. So if we have to compare the distance between three completion types, then we can choose γ large enough as to represent these distances as distances between three elements $x_\gamma, x'_\gamma, x''_\gamma \in X$, and the ultrametric triangle law on X^c will follow from that on X.

As a direct consequence of our definition, we have $uX^c = uX$. Furthermore, let us observe the following: if $\mathbf{A}_\alpha = B_\alpha(x_\alpha)$ for all $\alpha \in uX$, then for arbitrary $\beta \in uX$, we can choose $\mathbf{A}' = \operatorname{appr}(x_\beta,X)$ to obtain $u(\mathbf{A},\mathbf{A}') > \beta$. Indeed, for $\alpha \leq \beta$, we have $\mathbf{A}_\alpha = B_\alpha(x_\beta) = \mathbf{A}'_\alpha$. Consequently, if α is chosen such that $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$, then $\alpha > \beta$. For $x_\alpha \in \mathbf{A}_\alpha$, this yields that $x_\alpha \in \mathbf{A}_\beta(x_\beta)$, that is, $u(\mathbf{A},\mathbf{A}') = u(x_\alpha,x_\beta) \geq \beta$. If equality holds, then we replace x_β by x_α and set $\mathbf{A}' = \operatorname{appr}(x_\alpha,X)$. Then we obtain $u(\mathbf{A},\mathbf{A}') > \beta$. We have proved:

$$\forall y \in X^c \, \forall \beta \in uX^c \, \exists x \in X : \; u(y,x) > \beta \; . \tag{2.19}$$

This gives rise to the following definition: for an extension $(X,u) \subset (Y,u)$, it will be said that (X,u) **is dense in** (Y,u) if for all $y \in Y$ and all $\beta \in uY$ there is some $x \in X$ such that $u(y,x) > \beta$. This means that every element in Y can be approximated arbitrarily well by elements from X. A weaker property is that the extension $(X,u) \subset (Y,u)$ be immediate. It means that for no $y \in Y \setminus X$ there is a best approximation by elements from X; however, $u(y,X) \cap uX$ may be bounded from above by some element in uX. We will consider these properties in Lemma 2.43 below.

We have not yet shown that (X^c,u) is complete. So let $\{B_\alpha(X^c,y_\alpha) \mid \alpha \in S\}$ be a nest of balls in (X^c,u), with S a cofinal subset of $uX = uX^c$. By (2.19), for every $\alpha \in S$ there is some $x_\alpha \in X$ such that $u(x_\alpha,y_\alpha) > \alpha$, whence $B_\alpha(X^c,y_\alpha) = B_\alpha(X^c,x_\alpha)$. So $B_\alpha(X^c,y_\alpha) \cap X = B_\alpha(X,x_\alpha) \neq \emptyset$ for every $\alpha \in uX$. It follows that $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ is a nest of balls in (X,u). Its associated immediate approximation type $\mathbf{A}$ is a completion type in (X,u) and hence an element of X^c; it is uniquely determined since by assumption, uX admits no maximal element. By construction, $\mathbf{A}$ satisfies $u(y_\alpha,\mathbf{A}) \geq \min\{u(y_\alpha,x_\alpha),u(x_\alpha,\mathbf{A})\} \geq \alpha$ for every α. That is, $\mathbf{A}$ is an element of $\bigcap\{B_\alpha(X^c,y_\alpha) \mid \alpha \in uX^c\}$. This proves that (X^c,u) is complete.

The following lemma gives a characterization of density:

Lemma 2.43 *Let $(X,u) \subset (Y,u)$ be an extension of ultrametric spaces. If (X,u) is dense in (Y,u), then $(X,u) \subset (Y,u)$ is immediate and $uX = uY$. Further, (X,u) is dense in (Y,u) if and only if uX is cofinal in uY and for every $y \in Y$, $\operatorname{appr}(y,X)$ is a completion type. In particular, if (X,u) is complete and dense in (Y,u), then $X = Y$.*

Proof: Suppose that (X,u) is dense in (Y,u). Then for every $y \in Y$ and $x \in X$, there is some $x' \in X$ such that $u(y,x') > \beta := u(y,x)$. This proves that $(X,u) \subset (Y,u)$ is immediate. Now let $y,y' \in Y$ and choose $x,x' \in X$ such that $u(y,x) > u(y,y')$ and $u(y',x') > u(y,y')$. It follows that $u(x,x') = \min\{u(y,x),u(y,y'),u(y',x')\} = u(y,y')$. This proves that $uY = uX$.

Suppose that uX is cofinal in uY and that for every $y \in Y$, $\operatorname{appr}(y,X)$ is a completion type. Then for every $\beta \in uY$ there is some $\alpha \in uX$ such that $\alpha \geq \beta$. On the other hand, for every $\alpha \in uX$ the ball $\operatorname{appr}(y,X)_\alpha$ is nonempty. By (ATRI), the same is true for $\operatorname{appr}(y,X)^\circ_\alpha$, that is, there is some $x \in X$ such that $u(y,x) > \alpha \geq \beta$. This proves that (X,u) is dense in (Y,u).

For the converse, suppose that (X,u) is dense in (Y,u). We have already shown that this implies that $(X,u) \subset (Y,u)$ is immediate and $uX = uY$. From Lemma 2.21 we infer that $\operatorname{appr}(y,X)$ is immediate for every $y \in Y$. If $\alpha \in uX$, then choose $x' \in X$ such that $u(y,x') > \alpha$. Then either $x' = y$ and $\operatorname{appr}(y,X) = \operatorname{appr}(x',X)$ has distance ∞, or $u(y,x') \in uY$. In the latter case, choose $x \in X$ such that $u(y,x) > u(y,x')$. Then by (UM=), $u(y,x') = u(x',x) \in uX$ showing that $\alpha < u(x',x) \in \Lambda(y,X)$. We have proved that $\operatorname{dist}(y,X)$ is bigger than every element of uX, proving that this distance is ∞ and that $\operatorname{appr}(y,X)$ is a completion type.

If (X,u) is complete, then by Lemma 2.40, every completion type over (X,u) is trivial. If in addition, (X,u) is dense in (Y,u) and $y \in Y$, then by what we have already shown, $\operatorname{appr}(y,X)$ is a completion type and thus trivial. By Lemma 2.27 this means $y \in X$. It follows that $Y = X$. □

Corollary 2.44 *Let $(X,u) \subset (Y,u)$ be an extension of ultrametric spaces and $y \in Y$ such that uX is cofinal in $u(X \cup \{y\})$. Then* $\operatorname{appr}(y,X)$ *is a completion type if and only if* $\operatorname{dist}(y,X) = \infty$.

Proof: Implication "⇒" follows directly from the definition of completion types. Now assume that $\operatorname{dist}(y,X) = \infty$. Let $y \neq x \in X$, hence $u(y,x) < \infty$. Since we have assumed uX to be cofinal in $u(X \cup \{y\})$, it follows that there is some $\alpha \in uX$ such that $\alpha \geq u(y,x)$. Since $\operatorname{dist}(y,X) = \infty$, there must exist some $x' \in X$ such that $u(y,x') > \alpha \geq u(y,x)$. This proves that (X,u) is dense in $(X \cup \{y\},u)$. From the foregoing lemma we can now infer that $\operatorname{appr}(y,X)$ is a completion type. □

How many different elements can realize a completion type?

Lemma 2.45 *Let $(X,u) \subset (Y,u)$ be an extension of ultrametric spaces such that uX is cofinal in uY. Then every completion type over (X,u) is realized by at most one element of (Y,u).*

Proof: Let $\mathbf{A}$ be a completion type over (X,u) which is realized by the elements $y, y' \in Y$. That is, $\operatorname{appr}(y,X) = \mathbf{A} = \operatorname{appr}(y',X)$. Since every completion type is an immediate approximation type, it follows from Lemma 2.36 that $u(y,y') \geq \Lambda(y,X)$. But $uX \subset \Lambda(y,X)$ since $\operatorname{appr}(y,X)$ is a completion type. Since uX is cofinal in uY by assumption, it follows that $u(y,y') = \infty$ or $u(y,y')$ is the maximal element of uX. The latter is impossible since if uX has a maximal element, then the condition $\operatorname{dist}\mathbf{A} = \infty$ implies that $\infty \in \Lambda(y,X)$, which yields that $u(y,y') = \infty$. So we find that $y = y'$. □

The following lemma is an analogue to Lemma 2.42:

Lemma 2.46 *If the extension (Y,u) of (X,u) is complete and if uX is cofinal in uY, then every completion type over (X,u) is realized by a unique element of (Y,u).*

Proof: Let $\mathbf{A}$ be a completion type over (X,u). For every $\alpha \in \Lambda(\mathbf{A}) = uX$, let $x_\alpha \in \mathbf{A}_\alpha$. Then $B_\alpha(Y,x_\alpha)$ is a nest of balls in (Y,u) whose radii form a cofinal subset of uY since uX is assumed to be cofinal in uY. Hence, the balls must have a nonempty intersection in the complete ultrametric space (Y,u). If $y \in Y$ lies in this intersection, then $\mathbf{A} = \operatorname{appr}(y,X)$ by Lemma 2.41. The uniqueness follows from the foregoing lemma. □

Note that if (X,u) is dense in (Y,u) and if $(X,u) \subset (X_1,u) \subset (X_2,u) \subset (Y,u)$, then (X_1,u) is dense in (X_2,u). The following lemma summarizes the basic properties of the completion:

Lemma 2.47 *(X,u) is dense in $(X,u)^c$ and $(X,u) \subset (X,u)^c$ is an immediate extension. If (X,u) is dense in (Y,u), then (Y,u) admits a unique embedding in $(X,u)^c$ over (X,u); if in addition (Y,u) is complete, then the embedding is onto.*

Proof: The first two assertions follow from (2.19) and Lemma 2.43. Now assume that (X,u) is dense in (Y,u). By the standard argument using Zorn's Lemma, there is a maximal subextension $(X,u) \subset (X_1,u)$ of $(X,u) \subset (Y,u)$ which admits a unique embedding in $(X,u)^c$ over (X,u). We can identify (X_1,u) with its image in $(X,u)^c$. We know that also (X_1,u) is dense in (Y,u). If $(X_1,u) \neq (Y,u)$, then there is some $y \in Y \setminus X_1$. By Lemma 2.43, $\operatorname{appr}(y,X)$ is a completion type over (X_1,u). By the foregoing lemma, it is realized by a unique element $y' \in X^c$. Now Lemma 2.38 shows that there is a (uniquely

determined) isomorphism $(X_1 \cup \{y\}, u) \to (X_1 \cup \{y'\}, u')$ over (X_1, u), contrary to our maximality assumption for (X_1, u). Consequently, $X_1 = Y$ showing that (Y, u) can be embedded in $(X, u)^c$ over (X, u). If in addition (Y, u) is complete, then the same holds for its image in $(X, u)^c$, and since this is dense in $(X, u)^c$, it follows from Lemma 2.43 that the embedding is onto X^c. □

Finally, let us note:

Lemma 2.48 *Suppose that $(X, u) \subset (Y, u)$ is an extension of ultrametric spaces such that uX is cofinal in uY. Then if $y, y' \in Y$ such that $\mathrm{appr}\,(y, X)$ is a completion type and $\mathrm{appr}\,(y, X) = \mathrm{appr}\,(y', X)$, then $y = y'$. In particular, if (X, u) is dense in (Y, u) and $y, y' \in Y$, then $\mathrm{appr}\,(y, X) = \mathrm{appr}\,(y', X)$ implies that $y = y'$.*

Proof: Assume that $(X, u) \subset (Y, u)$ with uX cofinal in uY, and let $y, y' \in Y$. Suppose that $y \neq y'$, that is, $u(y, y') < \infty$. Since $u(y, y') \in uY$ and uX is cofinal in uY, there exists some $\alpha \in uX$ such that $\alpha \geq u(y, y')$. It follows that $\mathrm{appr}\,(y, X)^\circ_\alpha \neq \mathrm{appr}\,(y', X)^\circ_\alpha$ and thus $\mathrm{appr}\,(y, X) \neq \mathrm{appr}\,(y', X)$. This proves our first assertion. The second assertion follows from the first by virtue of Lemma 2.43. □

Exercise 2.14 *Show: if (X, u) is dense in (Y, u), then $uX = uY$ and all α-polygons of (Y, u) are already α-polygons of (X, u).*

Exercise 2.15 *Show that under the assumptions of Lemma 2.48, y and y' are equal already if there is a completion nest in (X, u) which is realized by both y and y', provided that uY has no maximal element. If the latter does not hold, give counterexamples to this assertion. Conclude that completion types carry more information than completion nests in $\mathbf{B}(X)$.*

2.6 Pseudo Cauchy sequences

Pseudo Cauchy sequences have played an important role in the development of valuation theory. We will thus introduce them here, although we prefer to work with approximation types since they express more information. However, there are some situations where it is more convenient to work with pseudo Cauchy sequences. For instance, certain maps may send pseudo Cauchy sequences into pseudo Cauchy sequences while it is not easy to describe how the approximation types are transformed. Another example is the task of constructing immediate extensions with certain properties. In this case, pseudo Cauchy sequences can represent a natural approach.

In what follows, let (X, u) always be an ultrametric space. A sequence $(x_\nu)_{\nu<\lambda}$ of elements in X, indexed by a limit ordinal λ, is called a **pseudo Cauchy sequence** if there exists some $\nu_0 < \lambda$ such that

(PCS) $\quad u(x_\nu, x_{\nu+1}) < u(x_{\nu+1}, x_\mu)$ or $x_\nu = x_{\nu+1} = x_\mu$ whenever $\nu_0 \leq \nu < \mu < \lambda$.

Note that in contrast to the definition in [KAP1], we admit pseudo Cauchy sequences to be **eventually constant**, that is, there is some ν_0 such that $\nu_0 < \lambda$ and $x_\nu = x_{\nu_0}$ whenever $\nu_0 \leq \nu < \lambda$. However, if we work with a pseudo Cauchy sequence $(x_\nu)_{\nu<\lambda}$ without mentioning ν_0, then it will implicitly be understood that $\nu_0 = 0$.

Let us state some properties of pseudo Cauchy sequences.

Lemma 2.49 *Let $(x_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in (X,u). Then*

$$u(x_\nu,x_\mu)=u(x_\nu,x_{\nu+1}) \ \text{ whenever } \nu_0\le\nu<\mu<\lambda \tag{2.20}$$

and

$$\begin{aligned} u(x_\nu,x_\mu)<u(x_{\nu'},x_{\mu'}) \ \text{ or } \ x_\nu=x_\mu=x_{\nu'}=x_{\mu'} \\ \text{whenever } \nu_0\le\nu<\mu<\lambda \text{ and } \nu<\nu'<\mu'<\lambda\,. \end{aligned} \tag{2.21}$$

Further, if $y\in X$, then either

$$u(y,x_\nu)<u(y,x_\mu) \ \text{ whenever } \nu_0\le\nu<\mu<\lambda\,, \tag{2.22}$$

or there is μ_0 such that $\nu_0\le\mu_0<\lambda$ and

$$u(y,x_\mu)=u(y,x_{\mu_0}) \text{ whenever } \mu_0\le\mu<\lambda\,.$$

Property (2.22) is equivalent to

$$u(y,x_\nu)=u(x_\nu,x_{\nu+1}) \ \text{ whenever } \nu_0\le\nu<\lambda\,. \tag{2.23}$$

Proof: Let $\nu_0<\nu<\mu<\lambda$. By definition, $u(x_\nu,x_{\nu+1})<u(x_{\nu+1},x_\mu)$ or $x_\nu=x_{\nu+1}=x_\mu$. In the first case, $u(x_\nu,x_\mu)=\min\{u(x_\nu,x_{\nu+1}),u(x_{\nu+1},x_\mu)\}=u(x_\nu,x_{\nu+1})$ by (UM$\neq$). In the second case, the assertion is trivial. Having proved this, we let $\nu_0\le\nu<\mu<\lambda$ and $\nu<\nu'<\mu'<\lambda$, hence $\nu+1<\mu'$, and we compute $u(x_{\nu'},x_{\mu'})\ge\min\{u(x_{\nu'},x_{\nu+1}),u(x_{\nu+1},x_{\mu'})\}=u(x_{\nu+1},x_{\nu+2})>u(x_\nu,x_{\nu+1})=u(x_\nu,x_\mu)$.

Now let $y\in X$ and suppose that (2.22) does not hold. Then there are ν and μ_0 such that $\nu_0\le\nu<\mu_0<\lambda$ and $u(y,x_\nu)\ge u(y,x_{\mu_0})$. Let $\mu_0<\mu<\lambda$, and assume that $u(y,x_{\mu_0})\neq u(y,x_\mu)$. Then $u(x_\nu,x_{\mu_0})\ge\min\{u(y,x_\nu),u(y,x_{\mu_0})\}\ge u(y,x_{\mu_0})\ge\min\{y,u(x_{\mu_0}),u(y,x_\mu)\}=u(x_{\mu_0},x_\mu)$, where the last equality holds by (UM$\neq$). But this is a contradiction to assertion (2.21), which we have already proved above.

Suppose that y satisfies (2.22). Then $u(y,x_\nu)<u(y,x_{\nu+1})$ and thus, $u(x_\nu,x_{\nu+1})=\min\{u(y,x_\nu),u(y,x_{\nu+1})\}=u(y,x_\nu)$ whenever $\nu_0\le\nu<\lambda$. Hence, y satisfies (2.23). Conversely, (2.22) follows from (2.23) and assertion (2.21). □

Let $\mathbf{S}=(x_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence. For every $\nu<\lambda$, we set

$$\gamma_\nu:=u(x_\nu,x_{\nu+1})\,.$$

Then (2.21) tells us that either $(\gamma_\nu)_{\nu_0<\nu<\lambda}$ is a strictly increasing sequence in uX, or γ_ν eventually becomes equal to ∞. Hence, if uX has a maximal element, then every pseudo Cauchy sequence in (X,u) is eventually constant. By (2.20), we have that $\gamma_\nu=u(x_\nu,x_\mu)$ whenever $\nu_0\le\nu<\mu<\lambda$. An element $y\in X$ is called a **limit of S** if it satisfies (2.22) or equivalently, (2.23). Since $u(y,x_{\nu+1})\ge\gamma_{\nu+1}>\gamma_\nu$ implies that $u(y,x_\nu)=\min\{\gamma_\nu,u(y,x_{\nu+1})\}=\gamma_\nu$, both conditions are equivalent to

$$u(y,x_\nu)\ge\gamma_\nu \ \text{ whenever } \nu_0\le\nu<\lambda\,.$$

Assume that $y\in X$ is not a limit of $\mathbf{S}$. Then by the foregoing lemma, there is μ_0 such that $\nu_0\le\mu_0<\lambda$ and $u(y,x_\mu)=u(y,x_{\mu_0})$ whenever $\mu_0\le\mu<\lambda$. It follows that $\gamma_\mu>\gamma_{\mu_0}=u(x_{\mu_0},x_{\mu_0+1})\ge\min\{u(y,x_{\mu_0}),u(y,x_{\mu_0+1})\}=u(y,x_{\mu_0})=u(y,x_\mu)$ whenever $\mu_0<\mu<\lambda$. Conversely, if there is μ_0 such that $\nu_0<\mu_0<\lambda$ and $\gamma_\mu>u(y,x_\mu)$, then in view of (2.23), y can not be a limit of $\mathbf{S}$. We have proved:

Lemma 2.50 *An element $y \in X$ is no limit of the pseudo Cauchy sequence $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ if and only if $u(y, x_\nu)$ is constant for large enough $\nu < \lambda$, and this is the case if and only if there is some $\mu_0 < \lambda$ such that*

$$u(y, x_\nu) < \gamma_\mu \quad \textit{whenever } \mu_0 < \nu < \lambda \; . \tag{2.24}$$

A pseudo Cauchy sequence may admit more than one limit. The following lemma describes the set of all limits.

Lemma 2.51 *Let $y \in X$ be a limit of $\mathbf{S}$. Then $z \in X$ is also a limit of $\mathbf{S}$ if and only if*

$$u(y, z) > \gamma_\nu \quad \textit{whenever } \nu_0 \leq \nu < \lambda \; .$$

Proof: Since y is a limit of $\mathbf{S}$, we have $u(y, x_\nu) = \gamma_\nu$ for $\nu_0 \leq \nu < \lambda$. If $u(y, z) > \gamma_\nu$, then $u(z, x_\nu) = \min\{u(y, z), u(z, x_\nu), \} = \min\{\gamma_\nu, u(y, z)\} = \gamma_\nu$ for $\nu_0 \leq \nu < \lambda$, showing that also z is a limit of $\mathbf{S}$. Conversely, suppose that z is a limit of $\mathbf{S}$, that is, $u(z, x_\nu) = \gamma_\nu$ for $\nu_0 \leq \nu < \lambda$. Then $u(y, z) \geq \min\{u(y, x_{\nu+1}), u(x_{\nu+1}, z)\} \geq \gamma_{\nu+1} > \gamma_\nu$ for $\nu_0 \leq \nu < \lambda$. □

The least initial segment of $uX\infty$ containing all γ_ν for $\nu_0 \leq \nu < \lambda$ will be called the **support segment of S** and denoted by $\Lambda(\mathbf{S})$. So the assertion of the foregoing lemma can be expressed as follows. If y is a limit of $\mathbf{S}$, then z is a limit of $\mathbf{S}$ if and only if

$$u(y, z) \geq \Lambda(\mathbf{S}) \; .$$

Note that this is equivalent to "$u(y, z) > \Lambda(\mathbf{S})$" if $\Lambda(\mathbf{S})$ has no maximal element, and that $\Lambda(\mathbf{S})$ has a maximal element, namely ∞, if and only if $\mathbf{S}$ is eventually constant. The pseudo Cauchy sequence $\mathbf{S}$ is called a **Cauchy sequence** if for every $\alpha \in uX$ there is some ν such that $\nu_0 < \nu < \lambda$ and $\gamma_\nu > \alpha$, that is, if $\Lambda(\mathbf{S})$ contains uX. Consequently, the foregoing lemma shows that *every Cauchy sequence admits at most one limit.* Every constant pseudo Cauchy sequence $(x)_{\nu<\lambda}$ is a Cauchy sequence and admits only x as a limit.

We shall now describe the relation between pseudo Cauchy sequences, nests of balls and approximation types. Let $\mathbf{B} \subset \mathbf{B}(X)$ be a nest of balls in (X, u). We want to associate a pseudo Cauchy sequence $\mathbf{S_B}$ to $\mathbf{B}$. If $\mathbf{B}$ has a nonempty intersection, then we pick some element $y \in \bigcap \mathbf{B}$ and let $\mathbf{S_B}$ be the constant sequence $(y)_{\nu<\lambda}$, where λ is some limit ordinal, e.g. the coinitiality type of the set $\mathbf{B}$, ordered by inclusion. Now assume that $\bigcap \mathbf{B}$ is empty. In particular, there is no smallest ball in $\mathbf{B}$. We pick an arbitrary ball $B_0 = B_{\alpha_0}(x_0) \in \mathbf{B}$. Suppose that we have already chosen balls $B_\nu = B_{\alpha_\nu}(x_\nu) \in \mathbf{B}$ for all $\nu < \mu$ such that $\alpha_\nu \leq u(x_\nu, x_{\nu'}) < \alpha_{\nu'}$ whenever $\nu < \nu' < \mu$. First consider the case of $\mu = \mu' + 1$ a successor ordinal. Since $\bigcap \mathbf{B} = \emptyset$, there are balls in $\mathbf{B}$ which do not contain $x_{\mu'}$; we choose $B_\mu = B_{\alpha_\mu}(x_\mu)$ to be such a ball. Then $u(x_{\mu'}, x_\mu) < \alpha_\mu$, and since $\mathbf{B}$ is ordered by inclusion, we must have that $B_\mu \subset B_{\mu'}$. Further, $u(x_\nu, x_\mu) = \min\{u(x_\nu, x_{\mu'}), u(x_{\mu'}, x_\mu)\} = u(x_\nu, x_{\mu'}) < \alpha_{\mu'} < \alpha_\mu$. Now consider the case of μ a limit ordinal. If $\bigcap_{\nu<\mu} B_\nu = \emptyset$, then we set $\lambda := \mu$, and our construction is finished. If the intersection is nonempty, then $\mathbf{B}$ contains a ball $B_\mu = B_{\alpha_\mu}(x_\mu)$ which is contained in this intersection. Then for every $\nu < \mu$, we have that $\alpha_\mu \geq u(x_{\nu+1}, x_\mu) \geq \alpha_{\nu+1} > u(x_\nu, x_{\nu+1})$ and thus, $u(x_\nu, x_\mu) = \min\{u(x_\nu, x_{\nu+1}), u(x_{\nu+1}, x_\mu)\} = u(x_\nu, x_{\nu+1}) < \alpha_\mu$.

After a number of steps which is limited by the cardinality of $\mathbf{B}$, we will arrive at a monotonically decreasing sequence $(B_\nu)_{\nu<\lambda}$ of balls in $\mathbf{B}$ such that the intersection $\bigcap_{\nu<\lambda} B_\nu$

is empty. Then this sequence must be coinitial in $\mathbf{B}$. By a slight refinement of our construction, it can be shown that λ may be taken to be the coinitiality type of $\mathbf{B}$. The sequence $\mathbf{S_B} := (x_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence. Indeed, for $\nu < \mu < \lambda$, we know that x_μ lies in $B_{\nu+1}$, hence $u(x_\nu, x_{\nu+1}) < \alpha_{\nu+1} \leq (x_{\nu+1}, x_\mu)$. By construction, $\Lambda(\mathbf{B}) = \Lambda(\mathbf{S_B})$.

Let (Y, u) be an extension of (X, u). If $\mathbf{S}$ is a pseudo Cauchy sequence in (X, u), then it is also a pseudo Cauchy sequence in (Y, u). Suppose that $y \in Y$ is limit of $\mathbf{S_B}$. Then $u(y, x_\nu) = u(x_\nu, x_{\nu+1}) \geq \alpha_\nu$ for all $\nu < \lambda$, which shows that $y \in \bigcap_{\nu<\lambda} B_{\alpha_\nu}(Y, x_\nu)$. Conversely, if the latter holds, then $u(y, x_{\nu+1}) \geq \alpha_{\nu+1} > u(x_\nu, x_{\nu+1})$. This yields that $u(y, x_\nu) = \min\{u(y, x_{\nu+1}), u(x_{\nu+1}, x_\nu)\} = u(x_\nu, x_{\nu+1})$ for all $\nu < \lambda$, showing that y is a limit of $\mathbf{S_B}$.

Now assume that $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence without a limit in (X, u). As before, we let $\gamma_\nu = u(x_\nu, x_{\nu+1})$. Then $\mathbf{B_S} := \{B_{\gamma_\nu}(x_\nu) \mid \nu_0 < \nu < \lambda\}$ is a nest of balls because every ball $B_{\gamma_\nu}(x_\nu)$ contains all balls $B_{\gamma_\mu}(x_\mu)$ for $\nu < \mu < \lambda$. Its intersection is empty; indeed, an element x in the intersection would satisfy $u(x_\nu, x) \geq \gamma_\nu$ for $\nu_0 < \nu < \lambda$ and would thus be a limit of $\mathbf{S}$. Our above construction applied to $\mathbf{B_S}$ will (essentially) give back $\mathbf{S}$. We have now proved:

Lemma 2.52 *Let (X, u) be an ultrametric space. Then for every nest of balls $\mathbf{B} \subset \mathbf{B}(X)$ in (X, u) with $\bigcap \mathbf{B} = \emptyset$, there is a pseudo Cauchy sequence $\mathbf{S_B}$ without a limit in (X, u), such that $\Lambda(\mathbf{B}) = \Lambda(\mathbf{S_B})$ and that for every extension $(X, u) \subset (Y, u)$ of ultrametric spaces, an element $y \in Y$ lies in $\bigcap\{B_\alpha(Y, x_\alpha) \mid B_\alpha(x_\alpha) \in \mathbf{B}\}$ if and only if it is a limit of $\mathbf{S_B}$. In particular, $\mathbf{A}$ is a completion nest if and only if $\mathbf{S_B}$ is a Cauchy sequence.*

Conversely, for every pseudo Cauchy sequence $\mathbf{S}$ without a limit in (X, u), there exists a nest $\mathbf{B_S}$ of balls in (X, u) with $\bigcap \mathbf{B_S} = \emptyset$, such that the relation between $\mathbf{S}$ and $\mathbf{B_S}$ is the same as between $\mathbf{B}$ and $\mathbf{S_B}$ that we have described above.

From this lemma together with Lemma 2.39, we obtain an analogous lemma which describes the relation between approximation types and pseudo Cauchy sequences. It is also capable of relating trivial approximation types to certain pseudo Cauchy sequences. The proof is an easy exercise which we leave to the reader.

Lemma 2.53 *Let (X, u) be an ultrametric space. Then for every immediate approximation type $\mathbf{A}$ in (X, u), there is a pseudo Cauchy sequence $\mathbf{S_A}$ in (X, u) such that $\Lambda(\mathbf{A}) = \Lambda(\mathbf{S_A})$ and that for every extension $(X, u) \subset (Y, u)$ of ultrametric spaces, an element $y \in Y$ realizes $\mathbf{A}$ if and only if it is a limit of $\mathbf{S_A}$. In particular, $\mathbf{A}$ is a completion type if and only if $\mathbf{S_A}$ is a Cauchy sequence. Further, $\mathbf{A}$ is trivial if and only if $\mathbf{S_A}$ admits a (unique) limit in X (in which case we may even choose $\mathbf{S_A}$ to be constant).*

Conversely, for every pseudo Cauchy sequence $\mathbf{S}$ with at most one limit in (X, u), there exists a unique immediate approximation type $\mathbf{A_S}$ in (X, u) such that the relation between $\mathbf{S}$ and $\mathbf{A_S}$ is the same as between $\mathbf{A}$ and $\mathbf{S_A}$ that we have described above.

We obtain the following characterization of spherically complete ultrametric spaces which is analogous to Lemma 2.40:

Corollary 2.54 *The ultrametric space (X, u) is spherically complete if and only if every pseudo Cauchy sequence in (X, u) has a limit in X. Further, (X, u) is complete if and only if every Cauchy sequence in (X, u) has a limit in X.*

Exercise 2.16 *Let $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence. Show that y is a limit of $\mathbf{S}$ if and only if $u(y, x_\nu) < u(y, x_{\nu+1})$ whenever $\nu_0 \leq \nu < \lambda$.*

Exercise 2.17 *Assume that every pseudo Cauchy sequence in (X, u) admits a limit in some extension (Y, u) of (X, u) (which is true; cf. Lemma 13.85). Prove: if $\mathbf{B}$ is a nest of balls and $\mathbf{S}$ is a pseudo Cauchy sequence such that for every extension (Y, u) of (X, u), the intersection $\bigcap\{B_\alpha(Y, x_\alpha) \mid B_\alpha(x_\alpha) \in \mathbf{B}\}$ is precisely the set of all limits of $\mathbf{S}$ in Y, then it follows that $\Lambda(\mathbf{B}) = \Lambda(\mathbf{S})$.*

Chapter 3

Valued abelian groups

3.1 Definition and basic properties

Let G be an abelian group and $\Gamma\infty$ as defined in section 2.1. A map

$$v : G \ni x \mapsto vx \in \Gamma\infty$$

from G onto $\Gamma\infty$ is called a **valuation of** G if for all $x, y \in G$,

(V 0) $\quad vx = \infty \iff x = 0\,,$

(VT) $\quad v(x-y) \geq \min\{vx, vy\}\,.$

Axiom (VT) can be viewed as the **ultrametric triangle inequality** for valued abelian groups. If we set $x = 0$ in (VT) and then replace y once by x and once by $-x$, we obtain $v(-x) \geq vx \geq v(-x)$, whence we obtain the "symmetry":

(VS) $\quad v(-x) = vx$

for all $x \in G$. Using this and axiom (VT), the following axiom scheme can be deduced by induction:

(VZ) $\quad v(nx) \geq vx \qquad\qquad (0 \neq n \in \mathbb{Z})\,.$

We will frequently omit the brackets and write "vnx". We will always omit brackets if the range of the symbol "v" is sufficiently clear.

If v is a valuation on the abelian group G, then we will write (G, v) and call it a **valued abelian group**, or shorter a **valued group**. Also, we will use $\mathbf{G}$ to denote a valued group, if the valuation was already specified before and there is no danger of confusion. We set $vG := \Gamma$ and call it the **value set of** G. Note that by our definition, vG does not contain the element ∞; by axiom (V 0), $vG = \{va \mid 0 \neq a \in G\}$. Following our above definition, we write $vG\infty$ for the ordered set $\{va \mid a \in G\} = vG \cup \{\infty\}$.

For torsion elements, we can improve (VZ) as follows:

Lemma 3.1 *Let (G, v) be a valued abelian group and a a torsion elements of G. Let $m \in \mathbb{N}$ be such that $ma = 0$ and assume $n \in \mathbb{N}$ to be prime to m. Then $vna = va$.*

Proof: Since n is prime to m, we can find integers r, s such that $1 = rn + sm$. It follows that $va = v(rn + sm)a = v(rna + sma) = vrna \geq vna \geq va$. So equality must hold everywhere, and we obtain that $vna = va$. □

Two valuations $v : G \to \Gamma\infty$ and $v' : G \to \Gamma'\infty$ (resp. ultrametrics $u : X \times X \to \Gamma\infty$ and $u' : X \times X \to \Gamma'\infty$) are called **equivalent** if there is an order isomorphism $\rho : \Gamma \to \Gamma'$ such that $\rho \circ v = v'$ (resp. $\rho \circ u = u'$). See our remark in section 1.3. We will say that v' is **coarser than** v or a **coarsening of** v if $v'a \geq v'b$ implies that $va \geq vb$, for all $a, b \in G$. In this case, v is said to be **finer than** v'.

Lemma 3.2 *Let G be an abelian group. If $v : G \to \Gamma\infty$ is a valuation on G, then $u(x, y) := v(x - y)$ is an ultrametric on G, satisfying $u(x + z, y + z) = u(x, y)$ for all $x, y, z \in G$. Conversely, if $u : G \times G \to \Gamma\infty$ is an ultrametric on G satisfying*

(UMG) $\quad u(x - y, 0) = u(x, y)$

for all $x, y \in G$, then $vx := u(x, 0)$ is a valuation on G.

Proof: Let v be a valuation on G and set $u(x, y) := v(x - y)$. Then (UM 0) follows immediately from (V 0), and (UMS) follows from (VS). Further, (VT) and (VS) give

$$v(x - y) = v((x - z) - (y - z)) \geq \min\{v(x - z), v(y - z)\} = \min\{v(x - z), v(z - y)\} ,$$

which translates to (UMT). Finally, $u(x + z, y + z) = u(x, y)$ is a trivial consequence of $u(x, y) = v(x - y)$.

For the converse, let u be an ultrametric on G satisfying $u(x - y, 0) = u(x, y)$, and set $vx := u(x, 0)$. Then (V 0) follows immediately from (UM 0). Further, $v(x - y) = u(x - y, 0) = u(x, y) \geq \min\{u(x, 0), u(0, y)\} = \min\{u(x, 0), u(y, 0)\} = \min\{vx, vy\}$, which is (VT). □

One may call the axiom (UMG) the **compatibility of the ultrametric with the group structure**. As a consequence of this lemma and Lemma 2.6, where we set $z = 0$, we obtain:

(V≠) $\quad vx \neq vy \Longrightarrow v(x - y) = \min\{vx, vy\}$,

(V=) $\quad v(x - y) > \min\{vx, vy\} \Longrightarrow vx = vy$.

From (V≠), replacing y by $-y$ and using (VS), we deduce by induction:

(VM) $\quad v(\sum_{1 \leq i \leq n} x_i) = \min_{1 \leq i \leq n} vx_i$ if all nonzero x_i have distinct values.

Axiom (UMG) can be equivalently interpreted as to say that addition by an arbitrary element of G is a automorphism of the ultrametric space (G, u), that is, it preserves the ultrametric: $u(x, y) = u(x + z, y + z)$.

Further, (UMG) implies the homogeneity (UMH). Indeed, since addition by an arbitrary element of G preserves the ultrametric, it shifts maximal equilateral polygons onto maximal equilateral polygons of the same distance. If $\alpha \in vG$ and x_1 and x_2 are members of two maximal equilateral polygons which reduce to different α-polygons, then addition of $x_2 - x_1$ shifts the first onto the second polygon, showing that the respective α-polygons are of equal cardinality. Consequently, in view of the above lemma, *every valuation of an abelian group has the homogeneity property* (UMH).

A subset $\mathcal{S}$ of a valued group $\mathbf{G} = (G, v)$ will be called **v-convex** if it satisfies

(vCONV) $\quad x, y \in \mathcal{S} \wedge v(x-y) \leq v(x-z) \implies z \in \mathcal{S}$.

Viewing $\mathbf{G}$ as an ultrametric space as we have done above, we see that all balls in $\mathbf{G}$ are v-convex. A subset $\mathcal{S}$ of $\mathbf{G}$ is v-convex if and only if for all $a, b \in \mathcal{S}$, the ball $B_{v(a-b)}(a)$ is contained in $\mathcal{S}$. Given a subgroup H of G, the reader may show that H is a **v-convex subgroup of $\mathbf{G}$** if H is the foreimage $v^{-1}(S)$ of a final segment S of $vG\infty$. Let γ be an initial segment of $vG\infty$ (recall that we identify an element γ of vG with the initial segment having γ as its maximal element, cf. section 2.1). We will consider

$$\mathcal{O}^\gamma_{\mathbf{G}} := \{a \in G \mid va \geq \gamma\} \quad \text{and} \quad \mathcal{M}^\gamma_{\mathbf{G}} := \{a \in G \mid va > \gamma\} \; . \tag{3.1}$$

which are v-convex subgroups of G, except for $\mathcal{M}^\infty_{\mathbf{G}} = \emptyset$. (The definition works equally well for any subset of vG in the place of γ.) If γ does not represent an element of vG, then it contains no maximal value, implying that every element of $\mathcal{O}^\gamma_{\mathbf{G}}$ has value $> \gamma$ and consequently, $\mathcal{O}^\gamma_{\mathbf{G}} = \mathcal{M}^\gamma_{\mathbf{G}}$. If γ is an element of $vG\infty$, then we can take γ' to be the initial segment $\{\alpha \in vG \mid \alpha < \gamma\}$ and obtain $\mathcal{O}^\gamma_{\mathbf{G}} = \mathcal{M}^{\gamma'}_{\mathbf{G}}$. More generally, this works for every v-convex subgroup (H, v): if we take $\gamma' = vG \setminus vH$, then we get $H = \mathcal{M}^{\gamma'}_{\mathbf{G}}$. We have thus shown that all v-convex subgroups can be represented in the form $\mathcal{M}^\gamma_{\mathbf{G}}$. Nevertheless, also the notation $\mathcal{O}^\gamma_{\mathbf{G}}$ will be usefull. If $a \in G$, then we set $\mathcal{O}_{\mathbf{G}}(a) := \mathcal{O}^{va}_{\mathbf{G}}$ and call it a **principal subgroup**. Similarly, we set $\mathcal{M}_{\mathbf{G}}(a) := \mathcal{M}^{va}_{\mathbf{G}}$. In particular, $\mathcal{O}_{\mathbf{G}}(0) = \{0\}$ and $\mathcal{M}_{\mathbf{G}}(0) = \emptyset$.

Lemma 3.3 *The set v-Conv($\mathbf{G}$) of all v-convex subgroups of $\mathbf{G}$ is ordered by inclusion. The map $\gamma \mapsto \mathcal{M}^\gamma_{\mathbf{G}}$ is an order reversing bijection between the ordered set of initial segments of vG and v-Conv($\mathbf{G}$), with inverse $H \mapsto vG \setminus vH$. In particular, it induces an order reversing bijection between initial segments of the form $\{\alpha \in vG \mid \alpha < \gamma\}$ with $\gamma \in vG$ and the principal subgroups of $\mathbf{G}$.*

The proof is straightforward. Just recall that for initial segments γ_1, γ_2 of T, $\gamma_1 < \gamma_2$ implies that γ_2 contains an element α such that $\alpha > \gamma_1$ and thus, $\mathcal{M}^{\gamma_1}_{\mathbf{G}}$ contains all elements of value α while $\mathcal{M}^{\gamma_2}_{\mathbf{G}}$ does not. This shows the injectivity of the map.

The balls with center 0 in $\mathbf{G}$ appear as v-convex subgroups: For $\gamma \in vG$, $B_\gamma(0) = \mathcal{O}^\gamma_{\mathbf{G}}$ and $B^\gamma(0) = \mathcal{M}^\gamma_{\mathbf{G}}$. By virtue of addition, we actually know all balls: For every $a \in G$,

$$B_\gamma(a) = a + \mathcal{O}^\gamma_{\mathbf{G}} \quad \text{and} \quad B^\gamma(a) = a + \mathcal{M}^\gamma_{\mathbf{G}}$$

(and $B_\infty(a) = a + \mathcal{O}^\infty_{\mathbf{G}} = \{a\}$), which are cosets modulo the respective v-convex subgroups. Indeed, $b \in B_\gamma(a) \Leftrightarrow u(b,a) = u(a,b) \geq \gamma \Leftrightarrow b - a \in \mathcal{O}^\gamma_{\mathbf{G}} \Leftrightarrow b \in a + \mathcal{O}^\gamma_{\mathbf{G}}$, and similarly for the balls $B^\gamma(a)$. Further, the equivalence relation $\sim_\gamma$ now has an easy interpretation: $a \sim_\gamma b \Leftrightarrow a - b \in \mathcal{M}^\gamma_{\mathbf{G}}$.

Lemma 3.4 *Let γ be an initial segment of vG. The group $G/\sim_\gamma = G/\mathcal{M}^\gamma_{\mathbf{G}}$ carries a valuation $v/\gamma : G/\sim_\gamma \to \gamma$ which for all $a \in G$ satisfies*

$$va \in \gamma \implies v/\gamma\,(a/\sim_\gamma) = va \; . \tag{3.2}$$

If $u(a,b) = v(a-b)$ is the ultrametric associated with v, then $v/\gamma\, c = u/\sim_\gamma (c, 0)$ is the valuation associated with the ultrametric $u/\sim_\gamma$.

In particular, the value set of $(\mathcal{O}^\gamma_{\mathbf{G}}/\mathcal{M}^\gamma_{\mathbf{G}}, v/\gamma)$ consists just of the one element γ if $\gamma \in vG$; the group is trivial if $\gamma \notin vG$.

Proof: Let v/γ on $G/\sim_\gamma$ be given as the valuation associated with $u/\sim_\gamma$. We know that u satisfies axiom (UMG). Now $u/\sim_\gamma(a/\sim_\gamma - b/\sim_\gamma, 0) = u/\sim_\gamma((a-b)/\sim_\gamma, 0) = u(a-b,0) = u(a,b) = u/\sim_\gamma(a/\sim_\gamma, b/\sim_\gamma)$, showing that also $u/\sim_\gamma$ satisfies axiom (UMG). Hence by Lemma 3.2, v/γ is a valuation. Assertion (3.2) follows directly from our construction.

Now consider the subgroup $\mathcal{O}_{\mathbf{G}}^\gamma$ of G. Every element of it has value $\geq \gamma$. If γ does not represent a value in vG, then it contains no maximal value, implying that every element of $\mathcal{O}_{\mathbf{G}}^\gamma$ has value $> \gamma$; in this case, $\mathcal{O}_{\mathbf{G}}^\gamma = \mathcal{M}_{\mathbf{G}}^\gamma$ and the factor group is trivial. If on the other hand, $\gamma \in vG$, then the elements in $\mathcal{O}_{\mathbf{G}}^\gamma$ which are not equivalent to zero modulo $\sim_\gamma$ are precisely those of value γ. By the first part of the lemma, this yields that γ is the only element in $v/\gamma\,(\mathcal{O}_{\mathbf{G}}^\gamma/\mathcal{M}_{\mathbf{G}}^\gamma)$. □

3.2 The skeleton

Let us have a look at the polygons in $\mathbf{G}$. By virtue of addition, we can shift them around without loosing the property of being equilateral (since $v((a+c)-(b+c)) = v(a-b)$). In particular, if $\alpha \in vG$ and P_α is an equilateral polygon of distance α, then for every $a \in P_\alpha$, the set $P_\alpha - a$ is an equilateral polygon of distance α whose nonzero elements all have value α. Moreover, $P_\alpha - a$ is maximal if and only if P_α is. That is, every maximal equilateral polygon of distance α can be shifted onto some maximal equilateral polygon P_α^0 of distance α which is entirely contained in $\mathcal{O}_{\mathbf{G}}^\alpha$. Its reduction $\overline{P_\alpha^0}$ modulo $\sim_\alpha$ (that is, modulo $\mathcal{M}_{\mathbf{G}}^\alpha$) is a maximal equilateral polygon in $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$. But the last lemma shows that the value set of this group contains but one element, which yields that every maximal equilateral polygon in the group is equal to the group itself. It follows that P_α^0 is a set of representatives for the factor group $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$, and that any maximal equilateral polygon of distance α containing x is of the form $x+\mathcal{R}$ where $\mathcal{R}$ is some set of representatives. We see that in our search for "horizontal" invariants, we can now replace the α-polygons by the factor group $\mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha$. This leads to the following definitions.

Let $\alpha \in vG$. The α–**component of** (G,v) is the factor group

$$\mathrm{C}^\alpha\mathbf{G} := \mathcal{O}_{\mathbf{G}}^\alpha/\mathcal{M}_{\mathbf{G}}^\alpha\ ,$$

which again is a nontrivial abelian group. Further, we define $\mathrm{C}^\infty\mathbf{G} := \{0\}$. For $a \in G$, the va-component $\mathrm{C}_{\mathbf{G}}(a) := \mathrm{C}^{va}\mathbf{G}$ will be called the **component of** a. The natural epimorphism $\mathcal{O}_{\mathbf{G}}^\alpha \to \mathrm{C}^\alpha\mathbf{G}$ will be denoted by co_α and called the **α-coefficient map of $\mathbf{G}$**. If γ is an initial segment of vG not representing an element of vG, then $\mathcal{O}_{\mathbf{G}}^\gamma/\mathcal{M}_{\mathbf{G}}^\gamma$ is the trivial group and co_γ the trivial epimorphism. A direct consequence of our definition is:

$$\text{if } va = vb = \alpha < \infty\,,\ \text{then} \quad \mathrm{co}_\alpha a = \mathrm{co}_\alpha b \iff v(b-a) > \alpha\ . \tag{3.3}$$

For simplicity, we set $\mathrm{co}\,a := \mathrm{co}_{va}a$ and call co the **coefficient map of $\mathbf{G} = (G,v)$**. It satisfies

(CO<) $\quad vx < vy \implies \mathrm{co}\,(x+y) = \mathrm{co}\,x\,,$

(CO+) $\quad vx = vy = v(x+y) \implies \mathrm{co}\,(x+y) = \mathrm{co}\,x + \mathrm{co}\,y\,.$

The pair $(va, \mathrm{co}\,a)$ will be called the **bone of** a. In view of (3.3) and (V=), we have the following criterion for the equality of bones:

$$b \neq 0 \ \wedge\ (va, \mathrm{co}\,a) = (vb, \mathrm{co}\,b) \iff v(b-a) > vb\ . \tag{3.4}$$

The collection

$$\operatorname{sk}\mathbf{G} := \bigcup_{\alpha\in vG\infty} \{\alpha\}\times \mathrm{C}^\alpha\mathbf{G}$$

of all bones of $\mathbf{G}$ is called the **skeleton of G** and is also denoted by $\operatorname{sk}(G,v)$. We use $\operatorname{sk}G$ if there is no danger of confusion. In general, $\bigcup_{\gamma\in\Gamma}\{\gamma\}\times C_\gamma$ will be called a **skeleton** if Γ is an ordered set and all C_γ are abelian groups. We write shortly $\operatorname{sk}_{\gamma\in\Gamma}B_\gamma$ for such a skeleton. In the literature, the reader will also find notations like B_γ for the components and $[\Gamma\,;\,B_\gamma\,,\,\gamma\in\Gamma]$ for the skeleton. We will not use "B_γ" in order not to get confused with balls.

The sum of two bones (γ,ζ) and (γ',ζ') is defined whenever $\gamma'=\gamma$ and is then equal to $(\gamma,\zeta+\zeta')$, according to (CO+). We will frequently identify $\{\gamma\}\times\mathrm{C}^\gamma\mathbf{G}$ with the group $\mathrm{C}^\gamma\mathbf{G}$ and call it the **γ-component of the skeleton**. Note that the ∞–component is the trivial group, it consists of the single element $(\infty,0)$.

Let $\mathcal{S}=\operatorname{sk}_{\gamma\in\Gamma}C_\gamma$ and $\mathcal{S}'=\operatorname{sk}_{\gamma\in\Gamma'}B'_\gamma$ be two skeletons. If $\Gamma\subset\Gamma'$ and for every $\gamma\in\Gamma$, the γ–component C_γ is a subgroup of the γ–component B'_γ, then we say that $\mathcal{S}$ is a **subskeleton of $\mathcal{S}'$** and write $\mathcal{S}\subset\mathcal{S}'$. If $\gamma\in\Gamma'\setminus\Gamma$ then we define the γ-component of $\mathcal{S}$ to be the trivial group 0. In this way, we obtain that C_γ is a subgroup of B'_γ for every $\gamma\in\Gamma'$. An **isomorphism of $\mathcal{S}$** onto $\mathcal{S}'$ is a system $\sigma=(\sigma_\Gamma,\{\sigma_\gamma\mid\gamma\in\Gamma\})$ such that $\sigma_\Gamma:\Gamma\to\Gamma'$ is an isomorphism of ordered sets and $\sigma_\gamma: C_\gamma\to B'_{\sigma_\Gamma\gamma}$ is an isomorphism of abelian groups for every $\gamma\in\Gamma$. An **embedding of $\mathcal{S}$** in $\mathcal{S}'$ is an isomorphism of $\mathcal{S}$ onto a subskeleton of $\mathcal{S}'$.

At this point, we should discuss the possible definitions for what we will call an **embedding** resp. **isomorphism of valued groups**. The meaning of "**preserve the valuation**" actually depends on the language that we use for valued groups (cf. 13.15). Let us use a binary predicate which expresses the relation $vx<vy$. Then $\iota:\ (G,v)\to(G',v')$ is an embedding of valued abelian groups if it is an embedding of G in G' and satisfies

$$vx<vy\ \Leftrightarrow\ v'\iota x<v'\iota y\ .$$

The reader may show that such an embedding induces an embedding $\sigma:\operatorname{sk}\mathbf{G}\to\operatorname{sk}\mathbf{G}'$ satisfying

$$(v\iota a,\operatorname{co}\iota a)=\sigma(va,\operatorname{co}a)\quad\text{for all } a\in G\ .$$

If this holds, then we say that ι **preserves** σ (or **induces** σ). If ι is an isomorphism, then so is σ. An embedding (resp. isomorphism) of valued abelian groups is at the same time an embedding (resp. isomorphism) of the underlying ultrametric spaces.

On the other hand, we could take v as a map. In this case, we would work with embeddings of structures of the form (G,vG,v) or, if we add the coefficient map, of the form $(G,\operatorname{sk}G,(v,\operatorname{co}))$. But our above consideration shows that this is only a formal difference; in all cases, we obtain the same class of group embeddings.

The following lemma determines the skeleton of a v-convex subgroup and the corresponding valued factor group:

Lemma 3.5 *Let $\mathbf{H}$ be a v-convex subgroup of $\mathbf{G}$. Then $\operatorname{sk}\mathbf{H}=\operatorname{sk}_{\alpha\in vH\infty}\mathrm{C}^\alpha\mathbf{G}$. Up to isomorphism, the skeleton of the factor group G/H endowed with the valuation induced by v, is the subskeleton $\operatorname{sk}_{\alpha\in(vG\setminus vH)\infty}\mathrm{C}^\alpha\mathbf{G}$ of $\operatorname{sk}\mathbf{G}$.*

Proof: Let us first consider $\mathbf{H}$. For every $\alpha \in vH$, we have $\mathcal{O}^\alpha_{\mathbf{G}} = \mathcal{O}^\alpha_{\mathbf{H}}$ and $\mathcal{M}^\alpha_{\mathbf{G}} = \mathcal{M}^\alpha_{\mathbf{H}}$ and consequently, $\mathrm{C}^\alpha\mathbf{G} = \mathrm{C}^\alpha\mathbf{H}$. This proves our first assertion.

Now we compute the skeleton of G/H. By Lemma 3.3, we can write $H = \mathcal{M}^\gamma_{\mathbf{G}}$ for some initial segment γ of vG. Then the valuation induced by v on G/H is v/γ. We know already from Lemma 3.4 that $v/\gamma\,(G/H) = \gamma = vG \setminus vH$. To determine the components of $(G/H, v/\gamma)$, let $\alpha \in vG \setminus vH = \gamma$. Then $\mathcal{M}^\gamma_{\mathbf{G}} \subset \mathcal{M}^\alpha_{\mathbf{G}} \subset \mathcal{O}^\alpha_{\mathbf{G}}$. Since the α-component of $(G/H, v/\gamma)$ is just $(\mathcal{O}^\alpha_{\mathbf{G}}/\mathcal{M}^\gamma_{\mathbf{G}})/\,(\mathcal{M}^\alpha_{\mathbf{G}}/\mathcal{M}^\gamma_{\mathbf{G}})$, it is thus equal to $\mathrm{C}^\alpha\mathbf{G} = \mathcal{O}^\alpha_{\mathbf{G}}/\mathcal{M}^\alpha_{\mathbf{G}}$ by the isomorphism theorem. □

The valued factor group $(G/H, v/\gamma)$ may also be denoted by $\mathbf{G}/\mathbf{H}$. Let H, H' be subgroups of (the additively written abelian group) G. Then H' is called a **group complement** for H in G if G is the direct sum of H and H', or equivalently, if $G = H + H'$ and $H \cap H' = \{0\}$. The reader may show the following: *If the v-convex subgroup $\mathbf{H}$ of $\mathbf{G}$ admits a group complement H' in G, then with the restriction of the valuation from G to H', the latter is isomorphic to $\mathbf{G}/\mathbf{H}$.* Given two valued groups (H, v) and (H', v'), we let $(H', v') \amalg (H, v)$ be the direct sum of H' and H with the uniquely determined valuation w having value set $v'H' + vH$ and satisfying $wa' = v'a' < va = wa$ for all $a' \in H'$ and $a \in H$. Then (H, v) is a v-convex subgroup of $(H', v') \amalg (H, v)$. The α-component of $(H', v') \amalg (H, v)$ is that of (H', v') if $\alpha \in v'H'$, and that of (H, v) if $\alpha \in vH$. We leave it to the reader to show: *If the v-convex subgroup $\mathbf{H}$ of $\mathbf{G}$ admits a group complement H' in G, then with the restriction of the valuation v from G to H', we have $\mathbf{G} \cong (H', v) \amalg (H, v)$.*

Let $\mathbf{H} \subset \mathbf{G}$ be an extension of valued groups. Then $vH \subset vG$ and for every $\alpha \in vH$, we have $\mathcal{O}^\alpha_{\mathbf{H}} \subset \mathcal{O}^\alpha_{\mathbf{G}}$ and $\mathcal{M}^\alpha_{\mathbf{G}} \cap H = \mathcal{M}^\alpha_{\mathbf{H}}$, showing that we have a canonical embedding

$$\mathrm{C}^\alpha\mathbf{H} \longrightarrow \mathrm{C}^\alpha\mathbf{G}\,, \qquad a + \mathcal{M}^\alpha_{\mathbf{H}} \mapsto a + \mathcal{M}^\alpha_{\mathbf{G}} \tag{3.5}$$

of groups. We have proved that $\mathrm{sk}\,\mathbf{H} \subset \mathrm{sk}\,\mathbf{G}$. Observe that because of $vH \subset vG$ we can form $\mathcal{O}^\alpha_{\mathbf{H}}$ and $\mathcal{M}^\alpha_{\mathbf{H}}$ also for all $\alpha \in vG$, and if $\alpha \in vG \setminus vH$ then $\mathcal{O}^\alpha_{\mathbf{H}} = \mathcal{M}^\alpha_{\mathbf{H}}$, which gives that $\mathrm{C}^\alpha\mathbf{H} = 0$, in accordance with our convention just introduced for abstract skeletons.

Given an arbitrary extension $\mathbf{H} \subset \mathbf{G}$ of valued groups, the question arises whether we can give an estimation for $(G : H)$ in terms of the indices of the corresponding components.

Lemma 3.6 *Let $\mathbf{H} \subset \mathbf{G}$ be an extension of valued groups. Reading the indices as finite or infinite cardinals,*

$$(G : H) \;\geq\; \sum_{\alpha \in vG} (\mathrm{C}^\alpha\mathbf{G} : \mathrm{C}^\alpha\mathbf{H})\,.$$

Proof: For every $\alpha \in vG$, take representatives for the different cosets of $\mathrm{C}^\alpha\mathbf{G}$ modulo $\mathrm{C}^\alpha\mathbf{H}$; we have to show that the so obtained representatives all belong to different cosets of G modulo H. It suffices to show the following: if $a, b \in G$ satisfy $\mathrm{co}\,a \notin \mathrm{C}^{va}\mathbf{H}$ and $\mathrm{co}\,b \notin \mathrm{C}^{vb}\mathbf{H}$ and $va = vb \;\Rightarrow\; \mathrm{co}_{va}a - \mathrm{co}_{va}b \notin \mathrm{C}^{va}\mathbf{H}$, then $a - b \notin H$. But this is almost trivial: if $va \neq vb$, say $va < vb$, then $(v(a-b), \mathrm{co}\,(a-b)) = (va, \mathrm{co}\,a)$ and $a - b \notin H$ because of $\mathrm{co}\,a \notin \mathrm{C}^{va}\mathbf{H}$; if $va = vb$, then $a - b \notin H$ because of $\mathrm{co}\,(a-b) = \mathrm{co}_{va}a - \mathrm{co}_{va}b \notin \mathrm{C}^{va}\mathbf{H}$. □

An extension $H \subset G$ of abelian groups will be called **algebraic** if G/H is a torsion group. The extension is called **finite** if the index $(G : H)$ is finite. A finite extension is algebraic. An extension is algebraic if and only if it is the union of finite extensions. By the foregoing lemma, all extensions $\mathrm{C}^\alpha\mathbf{H} \subset \mathrm{C}^\alpha\mathbf{G}$ are finite if $(G : H)$ is finite. If G is the

union of finite extensions of H then for every $\alpha \in vG$, the component $\mathrm{C}^\alpha\mathbf{G}$ is the union of finite extensions of $\mathrm{C}^\alpha\mathbf{H}$. We have thus proved:

Corollary 3.7 *If $\mathbf{H} \subset \mathbf{G}$ is an algebraic extension of valued groups, then $\mathrm{C}^\alpha\mathbf{H} \subset \mathrm{C}^\alpha\mathbf{G}$ is an algebraic extension of abelian groups, for every $\alpha \in vG$.*

Exercise 3.1 *Discuss the possible definitions for "homomorphism of valued groups".*

3.3 Immediate extensions and maximal groups

The extension $\mathbf{H} \subset \mathbf{G}$ is called **immediate** if for every $\alpha \in vG$, the canonical embedding (3.5) is onto, that is, if the induced embedding $\mathrm{sk}\,\mathbf{H} \subset \mathrm{sk}\,\mathbf{G}$ is actually an isomorphism. Loosely speaking, "immediate" means that the skeletons are equal. At this point, we have to distinguish between "isomorphic" and "equal"; non-immediate extensions may still have isomorphic skeletons, the isomorphism *not* being the induced embedding (see Exercise 3.9). If $\mathbf{H} \subset \mathbf{G}$ is an immediate extension, then for every $\alpha \in vG$, the component $\mathrm{C}^\alpha\mathbf{H} = \mathrm{C}^\alpha\mathbf{G}$, which says that $\alpha \in vH$. Hence, $vH = vG$ for every immediate extension $\mathbf{H} \subset \mathbf{G}$. As a direct consequence of our definition, we obtain:

Lemma 3.8 *Let $\mathbf{G} \subset \mathbf{G}_1 \subset \mathbf{H}$ be valued groups. Then $\mathbf{G} \subset \mathbf{H}$ is immediate if and only if $\mathbf{G} \subset \mathbf{G}_1$ and $\mathbf{G}_1 \subset \mathbf{H}$ are.*

Observe that this result is stronger than the corresponding result for ultrametric spaces. (It is actually axiom (UMG) that gives this stronger result). Since the above definition of immediate extensions is quite different from that of immediate extensions of ultrametric spaces, we will now show the relation between them. Let $b \in G$. Assume that there is some $a \in H$ such that $v(b-a) > vb$. Then by (3.4), the bone of b is equal to the bone of a and thus lies in $\mathrm{sk}\,\mathbf{H}$. Conversely, if the latter holds, then there is some $a \in H$ such that a and b have equal bones. Again by (3.4), this yields that $v(b-a) > vb$. We have shown:

$$\forall b \in G \setminus \{0\}: \quad (vb, \mathrm{co}\, b) \in \mathrm{sk}\,\mathbf{H} \iff \exists a \in H:\ v(b-a) > vb\ . \tag{3.6}$$

This gives the following important characterization of immediate extensions:

Lemma 3.9 *The extension $\mathbf{H} \subset \mathbf{G}$ is immediate if and only if for every $b \in G \setminus \{0\}$ there is some $a \in H$ such that $v(b-a) > vb$.*

A valued group is called **maximal** or **maximally valued** if it does not admit a proper immediate extension. It is called a **spherically complete group** if its underlying ultrametric space is spherically complete.

Corollary 3.10 *The extension $\mathbf{H} \subset \mathbf{G}$ is immediate if and only if the underlying extension of ultrametric spaces is immediate. Consequently, a spherically complete valued group is maximal.*

Proof: Assume $y \in G$ and $x' \in H$ and set $b := y - x'$. If $a \in H$ is chosen such that $v(b-a) > vb$, then for $x := x' + a$ we have that $v(y-x) = v(b-a) > vb = v(y-x')$. For the converse, given $b \in G$, then set $y := b$ and $x' = 0$. If $x \in H$ is chosen such that $v(y-x) > v(y-x')$, then for $a := x$ we have that $v(b-a) = v(y-x) > v(y-x') = vb$.

From Lemma 2.16 it follows that a spherically complete ultrametric space does not admit proper immediate extensions. We conclude that a spherically complete valued group does not admit proper immediate valued group extensions and is thus a maximal group. □

By an **approximation type over G** we mean an approximation type over the underlying ultrametric space. We know from section 3.1 that in the underlying ultrametric space of a valued group, the balls are cosets with respect to suitable v-convex subgroups. We find that for an approximation type $\mathbf{A}$ over $\mathbf{G}$, every nonempty $\mathbf{A}_\alpha$ can just be presented as the coset $\mathbf{A}_\alpha = c_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha$ for an arbitrary $c_\alpha \in \mathbf{A}_\alpha$, and every nonempty $\mathbf{A}_\alpha^\circ$ can be presented as the coset $\mathbf{A}_\alpha^\circ = c_\alpha^\circ + \mathcal{M}_{\mathbf{G}}^\alpha$ for an arbitrary $c_\alpha^\circ \in \mathbf{A}_\alpha^\circ$.

The following lemma is just the transposition of (ATVI) and (ATRI) to the case where $\mathbf{A} = \mathrm{appr}(x, G)$ is an approximation type over the valued group $\mathbf{G}$.

Lemma 3.11 *Let* $\mathbf{A} = \mathrm{appr}(x, G)$. $\mathbf{A}$ *is value-immediate if and only if*

$$\forall c \in G:\ v(x-c) \in vG\infty\ .$$

$\mathbf{A}$ *is residue-immediate if and only if*

$$\forall c \in G:\quad v(x-c) \in vG \Rightarrow \exists c' \in G:\ v(x-c') > v(x-c)\ .$$

This in turn is equivalent to

$$\forall c \in G:\quad v(x-c) = \alpha \in vG \Rightarrow \mathrm{co}\,(x-c) \in C_\alpha G\ .$$

Consequently, $\mathbf{A}$ *is immediate if and only if*

$$\forall c \in G:\quad \exists c' \in G:\ v(x-c') > v(x-c)\ .$$

This lemma together with Lemma 3.9 shows that $\mathbf{H} \subset \mathbf{G}$ is immediate if and only if every element in G has an immediate approximation type over H. But this is already a consequence of Lemma 2.21, which is the corresponding assertion for ultrametric spaces.

As we are dealing with the ultrametric space underlying a valued group (G, v), the question arises whether its completion also carries the structure of a valued group. To show this, we will employ the following simple observation: if $B_\alpha(x_\alpha)$ and $B_\alpha(x'_\alpha)$ are balls in a valued group, then the same holds for the sum $B_\alpha(x_\alpha) + B_\alpha(x'_\alpha)$ since it is equal to $x_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha + x'_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha = x_\alpha + x'_\alpha + \mathcal{O}_{\mathbf{G}}^\alpha = B_\alpha(x_\alpha + x'_\alpha)$. The same holds for balls of the form B_α°. Given two completion types $\mathbf{A}$ and $\mathbf{A}'$, we define the sum $\mathbf{A} + \mathbf{A}'$ by setting $(\mathbf{A} + \mathbf{A}')_\alpha := \mathbf{A}_\alpha + \mathbf{A}'_\alpha$ and $(\mathbf{A} + \mathbf{A}')_\alpha^\circ := \mathbf{A}_\alpha^\circ + \mathbf{A}'^\circ_\alpha$ (if B is some ball, then $B + \emptyset = \emptyset$). Now $\mathbf{A} + \mathbf{A}'$ is again an approximation type. Since $\mathbf{A}$ and $\mathbf{A}'$ are completion types, all values in vG appear as radii of balls in $\mathbf{A} + \mathbf{A}'$, which proves that it is also a completion type. We leave it to the reader to verify that G^c with this addition is an abelian group extension of G.

Now recall how we extend the ultrametric to the completion: we set $u(\mathbf{A}, \mathbf{A}') = u(x_\alpha, x'_\alpha)$ if $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$ and $x_\alpha \in \mathbf{A}_\alpha$, $x'_\alpha \in \mathbf{A}'_\alpha$ (cf. page 43). But $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$ is equivalent to $B_\alpha(x_\alpha - x'_\alpha) \neq B_\alpha(0)$, that is, $(\mathbf{A} - \mathbf{A}')_\alpha \neq \mathrm{appr}\,(0, X)_\alpha$. Moreover, $u(x_\alpha, x'_\alpha) = u(x_\alpha - x'_\alpha, 0)$ since the ultrametric is induced by the group valuation v. This shows that the extension of u to G^c again satisfies (UMG), proving that $vx := u(x, 0)$ is a group valuation on G^c

which extends the valuation of G. The valued group (G^c, v) is called the **completion of** (G, v) and is also denoted by $(G, v)^c$. Furthermore, (G, v) is dense in $(G, v)^c$ in the sense of ultrametric spaces, which is equivalent to $\forall y \in G^c\, \forall \beta \in vG^c\, \exists x \in G :\ v(x - y) \geq \beta$. In particular, by the previous corollary, (G^c, v) is an immediate extension of (G, v).

A maximal immediate extension of a valued group is maximal. Let us ask for their existence. The idea is to apply Zorn's Lemma. Indeed, if we have an increasing chain of valued abelian groups which are immediate extensions of $\mathbf{G}$, with index set an ordinal, then the union is again a valued group which is an immediate extension of $\mathbf{G}$ (this follows from Lemma 13.7 since valued abelian groups can be axiomatized by universal axioms). To obtain that *every* increasing chain of immediate extensions of $\mathbf{G}$ admits a union (which is a set and not a proper class), we just need to know that the cardinality of the extension groups is bounded from above. As a corollary to Lemma 2.13, we get such a bound depending on $|vG|$ and the size of the polygons in $\mathbf{G}$. But we have obtained by our above arguments that for every $\alpha \in vG$, all α-polygons of $\mathbf{G}$ are of equal size, namely $|\mathrm{C}^\alpha \mathbf{G}|$. This shows:

Lemma 3.12 *Let κ be a cardinal which is an upper bound for the cardinality of all components of the valued group $\mathbf{G}$. Then*

$$|G| \leq \kappa^{|vG|} .$$

Since immediate extensions do not enlarge value set and components, this lemma yields:

Corollary 3.13 *For every valued group there exists a maximal immediate extension.*

Given an ordered index set Γ and for every $\gamma \in \Gamma$ an arbitrary abelian group C_γ, we can form the **Hahn sum** $\coprod_{\gamma\in\Gamma} C_\gamma$. As an abelian group, this is the direct sum of the groups C_γ, represented as the set of all tuples $(\zeta_\gamma)_{\gamma\in\Gamma}$ with only finitely many of the $\zeta_\gamma \in C_\gamma$ nonzero. For a given $0 \neq c = (\zeta_\gamma)_{\gamma\in\Gamma}$, let γ_0 be the smallest index such that $\zeta_{\gamma_0} \neq 0$; we let $vc = \gamma_0$. This defines the valuation on $\coprod_{\gamma\in\Gamma} C_\gamma$.

In a similar way, we define a group called the **Hahn product**, denoted by $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. Consider the product $\prod_{\gamma\in\Gamma} C_\gamma$ and an element $c = (\zeta_\gamma)_{\gamma\in\Gamma}$ of this group. Then the **support of** c is the set of all $\gamma \in \Gamma$ for which $\zeta_\gamma \neq 0$. As a set, the Hahn product is the subset of $\prod_{\gamma\in\Gamma} C_\gamma$ containing all elements whose support is a well-ordered subset of Γ. In particular, the support of every element c in the Hahn product has a minimal element γ_0, which again enables us to define $vc = \gamma_0$. Finally, the Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two wellordered sets is again wellordered.

Let us take $\mathbf{G}$ to be the Hahn sum or the Hahn product as defined above. In both cases, for arbitrary $\gamma \in \Gamma$, the group $\mathcal{O}_\mathbf{G}^\gamma$ consists of all elements whose minimal element in the support is γ, whereas $\mathcal{M}_\mathbf{G}^\gamma$ contains all elements whose support "starts later". At γ, the tuples assume values in C_γ. This shows that $\mathcal{O}_\mathbf{G}^\gamma / \mathcal{M}_\mathbf{G}^\gamma$ is isomorphic to C_γ. Consequently, the skeleton of the Hahn sum $\coprod_{\gamma\in\Gamma} C_\gamma$ and of the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$ (and of every group in between) is just $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$. Consequently, they are also called the **Hahn sum** (resp. **Hahn product**) **over the skeleton** $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$. Note that we have just constructed an example of an immediate extension. (It may happen that Hahn sum and Hahn product coincide, e.g. if Γ is finite, but in general, the extension is nontrivial.)

Lemma 3.14 *Every Hahn product is spherically complete (as an ultrametric space). Consequently, it is a maximal group.*

Proof: Let $\mathbf{B}$ be a nest of balls in the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. We have to show that its intersection is nonempty. Since $\mathbf{B}$ is ordered by inclusion, we can choose a coinitial decreasing sequence $(B_\nu)_{\nu<\lambda}$ of balls $B_\nu = B_{\alpha_\nu}(a_\nu) \in \mathbf{B}$. Then $(\alpha_\nu)_{\nu<\lambda}$ will be a cofinal increasing sequence in $\Lambda(\mathbf{B})$. Let us write $a_\nu = (\zeta_{\nu,\gamma})_{\gamma\in\Gamma}$ with $\zeta_{\nu,\gamma} \in C_\gamma$.

Suppose that $\nu < \mu < \lambda$. Since $B_\mu \subset B_\nu$, we have $v(a_\nu - a_\mu) \geq \alpha_\nu$. That is, $\zeta_{\nu,\gamma} = \zeta_{\mu,\gamma}$ for all $\gamma < \alpha_\nu$. So we can define an element $a = (a_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} C_\gamma$ by setting $a_\gamma = \zeta_{\nu,\gamma}$ if there exists $\nu < \lambda$ with $\gamma < \alpha_\nu$, and setting $a_\gamma = 0$ otherwise. The latter guarantees that every subset of the support of a which is (strictly) bounded from above by some α_ν, is already a subset of the support of $a_\nu \in \mathbf{H}_{\gamma\in\Gamma} C_\gamma$. This shows that the support of a is wellordered and $a \in \mathbf{H}_{\gamma\in\Gamma} C_\gamma$. By our construction, $v(a - a_\nu) \geq \alpha_\nu$ for every $\nu < \lambda$ and thus, $a \in B_\nu$. Consequently, a is an element of $\bigcap \mathbf{B} = \bigcap_\nu B_\nu$. □

This lemma shows that the Hahn product is a maximal immediate extension of every valued group which lies between the Hahn sum and the Hahn product over the same skeleton. Using these twins of valued groups, we can give some examples for immediate approximation types and immediate extensions. In particular, we are now able to show that the maximal immediate extensions of valued groups are not necessarily unique up to isomorphism.

Example 3.15 Consider the Hahn sum $\coprod_{i\in\mathbb{N}} C_i$ and the Hahn product $\mathbf{H}_{i\in\mathbb{N}} C_i$ where all C_i are equal to $\mathbb{Z}$. We will write shorter: $\coprod_{\mathbb{N}} \mathbb{Z}$ and $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$. Since every subset of $\mathbb{N}$ is well-ordered, $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$ is the full product over the C_i, or in other words, it consists of all maps from $\mathbb{N}$ into $\mathbb{Z}$. Let us denote by b the element in $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$ which has a 1 at every entry; it is not an element of $\coprod_{\mathbb{N}} \mathbb{Z}$. Further, let b_i be the element of $\coprod_{\mathbb{N}} \mathbb{Z}$ which has a 1 at the first i entries and 0 everywhere else. We have $\coprod_{\mathbb{N}} \mathbb{Z} \subset \mathbf{H}_{\mathbb{N}} \mathbb{Z}$, so we can consider the approximation type of b over $\coprod_{\mathbb{N}} \mathbb{Z}$. Since $v(b-b_i) = i+1$, we have that b_i is a center of every ball $\operatorname{appr}(b, \coprod_{\mathbb{N}} \mathbb{Z})_j$ for $j \leq i+1$, but it is not contained in the balls $\operatorname{appr}(b, \coprod_{\mathbb{N}} \mathbb{Z})^\circ_{i+1}$ and $\operatorname{appr}(b, \coprod_{\mathbb{N}} \mathbb{Z})_{i+2}$. We find that $\operatorname{dist}(b, \coprod_{\mathbb{N}} \mathbb{Z})$ is ∞; it is not assumed since $b \notin \coprod_{\mathbb{N}} \mathbb{Z}$. Further, for every element $c \in \coprod_{\mathbb{N}} \mathbb{Z}$ there is some j such that c is not contained in $\operatorname{appr}(b, \coprod_{\mathbb{N}} \mathbb{Z})_j$: we only have to choose j so big that the finitely many nonzero entries of c lie below $j-1$, then it will follow that $v(b-c) \leq j-1$. Hence by Lemma 2.30, $\operatorname{appr}(b, \coprod_{\mathbb{N}} \mathbb{Z})$ is a nontrivial approximation type. Its distance being ∞, it is even a nontrivial completion type. In the same way, replacing b by any element of $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$, it is seen that $\coprod_{\mathbb{N}} \mathbb{Z}$ is dense in $\mathbf{H}_{\mathbb{N}} \mathbb{Z}$. If we replace the index set $\mathbb{N}$ by the ordinal $\omega+1$ and define b as above, then the approximation type of b will still be a nontrivial immediate approximation type, but its distance is not any longer ∞ (in fact, it is ω); so it is not a completion type.

All this remains true if we replace the components $\mathbb{Z}$ by, say, $\mathbb{Z}/2\mathbb{Z}$. The element b as defined above now satisfies $2b = 0$ like any other element in the group. But we can also construct elements which realize the same immediate approximation type as b but do not have the same torsion. Indeed, let us take index set $\omega+1$; note that $\mathbf{H}_{\omega+1} \mathbb{Z}/2\mathbb{Z} \cong (\mathbf{H}_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/2\mathbb{Z}$. Let a be a generator of $\mathbb{Z}/4\mathbb{Z}$. We can embed $\mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$ by sending the 1 of $\mathbb{Z}/2\mathbb{Z}$ to $2a$. This induces an embedding of $(\mathbf{H}_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/2\mathbb{Z}$ over $\mathbf{H}_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ in $(\mathbf{H}_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}) \amalg \mathbb{Z}/4\mathbb{Z}$. In the latter group, we define b' to be the element which has a (or $3a$) at the last entry and 1 everywhere else (so $b' = b - a'$ or $b' = b + a'$ for $a' = (0, \ldots, 0, \ldots, a)$). Now $\operatorname{appr}(b', \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z}) = \operatorname{appr}(b, \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z})$; this can be verified by

a direct computation, but it also follows from the fact that appr $(b, \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z})$ is immediate and $v(b-b') = \omega > \Lambda(b, \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z})$ (cf. Lemma 2.36). Since $2b' = 2a' \in \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z}$, the fact that appr $(b', \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z})$ is immediate implies that for $\mathbf{H} := \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z}$ and $\mathbf{G} := \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}b'$, the extension $\mathbf{H} \subset \mathbf{G}$ is immediate (cf. Exercise 3.3). Despite of this, $\mathbf{G}$ has an element of order 4 whereas $\mathbf{H}$ does not.

We know already that $\mathbf{H}_{\omega+1} \mathbb{Z}/2\mathbb{Z}$ is an immediate extension of $\mathbf{H} = \coprod_{\omega+1} \mathbb{Z}/2\mathbb{Z}$. By the last lemma, this group is maximal, and so it is a maximal immediate extension of $\mathbf{H}$. On the other hand, Corollary 3.13 asserts the existence of a maximal immediate extension $\mathbf{G}'$ of $\mathbf{G}$, which is consequently also a maximal immediate extension of $\mathbf{H}$. While $\mathbf{G}'$ contains an element of order 4, the group $\mathbf{H}_{\omega+1} \mathbb{Z}/2\mathbb{Z}$ does not, and it is even a vector space over the field $\mathbb{F}_2$ of two elements. So these two maximal immediate extensions of $\mathbf{H}$ are not isomorphic at all, even if one drops the condition that the isomorphism fixes the subgroup $\mathbf{H}$. In some sense, $\mathbf{H}_{\omega+1} \mathbb{Z}/2\mathbb{Z}$ is the "sound" maximal immediate extension of $\mathbf{H}$, regarding the additional structure of $\mathbf{H}$ being a vector space. ♢

If $\mathcal{S}$ is a v-convex subset of the valued group $\mathbf{G}$, then every ball in $\mathcal{S}$ is a ball in $\mathbf{G}$. This implies:

Lemma 3.16 *Every v-convex subset $\mathcal{S}$ of a spherically complete group $\mathbf{G}$ is spherically complete.*

For the conclusion of this section, we note the following facts whose proof we leave to the reader:

Lemma 3.17 *a) Let $\mathbf{H}$ be a v-convex subgroup of the valued group $\mathbf{G}$. Then $\mathbf{G}$ is spherically complete if and only if $\mathbf{H}$ and $\mathbf{G}/\mathbf{H}$ are spherically complete.*
b) Let $\mathbf{H}$ and $\mathbf{H}'$ be two valued groups. Then $\mathbf{H}' \amalg \mathbf{H}$ is maximal if and only if $\mathbf{H}$ and $\mathbf{H}'$ are, and it is spherically complete if and only if $\mathbf{H}$ and $\mathbf{H}'$ are.

Exercise 3.2 *Under which condition is the completion of a Hahn sum equal to the Hahn product?*

Exercise 3.3 *Let $\mathbf{H} \subset \mathbf{G}$ be an extension of valued groups. Given $b \in G$ and* appr $(b, \mathbf{H})$*, compute* appr $(y+a, \mathbf{H})$ *for $a \in H$. Show that* appr $(y+a, \mathbf{H})$ *is immediate if and only if* appr $(b, \mathbf{H})$ *is immediate. Conclude that the extension $\mathbf{H} \subset \mathbf{G}$ is immediate if in a given set of representatives for the cosets of G modulo H every element has an immediate approximation type over $\mathbf{H}$.*

3.4 p-valuations of abelian groups

For the field theorist, examples of valued abelian groups are easily obtained by taking valued fields (see Chapter 4) and forgetting the multiplication. But this is not only a procedure to produce examples. It will be one aim of this book to show its various useful applications.

For the abelian group theorist, the first and main example is that of a p-valuation, where p is a prime. A map v from the abelian group G onto $\Gamma\infty$ is called a **p-valuation** if Γ is a set of ordinals and v satisfies (VT) together with

(pV) $\qquad vx \neq \infty \implies vpx > vx\,.$

A **valuated group** is an abelian group together with a p-valuation for every prime p. The prototype of a p-valuation is the p-**height function**. By induction (possibly transfinite), we define the following **filtration of** G. Let $p^0G := G$ and $pG := \{pa \mid a \in G\}$; this is a subgroup of G. Having defined $p^\mu G$ for an ordinal μ, we set $p^{\mu+1}G := p(p^\mu G)$. If λ is a limit ordinal and we have defined $p^\mu G$ for all $\mu < \lambda$, we set $p^\lambda G := \bigcap_{\mu<\lambda} p^\mu G$. This definition establishes a descending chain of subgroups, which must eventually become constant (see Exercise 3.6). That is, there is some minimal ordinal ν such that $p(p^\nu G) = p^\nu G$. Hence, $H := p^\nu G$ is p-**divisible**, i.e., for every $a \in H$ there is some $b \in H$ such that $a = pb$. In fact, H is the maximal p-divisible subgroup of G. If G is a p-**group**, i.e., the order of every element is a power of p, then H is even **divisible**, i.e., for every $a \in H$ and every $n \in \mathbb{N}$, there is some $b \in H$ such that $a = nb$ (see Exercise 3.5). A p-group is called **reduced** if its maximal divisible subgroup is $\{0\}$. The ordinal ν is called the p-**height of** G.

The p-height function is introduced as follows. Given an element $a \in G \setminus p^\nu G$, there is a minimal ordinal $\alpha < \nu$ such that $a \in p^\alpha G \setminus p^{\alpha+1}G$; we set $h_{G,p}(a) = \alpha$ (we may say that a is divisible by p^α but not by $p^{\alpha+1}$; in the case of α being a natural number, this indeed holds in the usual sense). For $a \in p^\nu G$ we set $h_{G,p}(a) = \infty$. Then $h_{G,p} : G \to \nu\infty$ is onto (here, the ordinal ν is interpreted as the set of all ordinals $\mu < \nu$). We leave it to the reader to verify (VT) and (pV). Further, we see that (V 0) holds if and only if $p^\nu G = \{0\}$. In particular, we find that the height function in a reduced p-group G is a valuation of G.

Now p-valuations appear as the restrictions of height functions to subgroups; in that sense, they are a generalization of height functions. Indeed, if the height function is restricted to a subgroup, it may not coincide with the height function of that subgroup since the height of an element in a subgroup H of G may be lower than in G. An element which is divisible by p in G may not be so in H. We only can say that $h_{G,p}(a) \geq h_{H,p}(a)$ for all $a \in H$. The following properties of the height function may be lost in a subgroup:

(pH1) $\quad h_{G,p}(x) > h_{G,p}(y) \wedge x \neq 0 \implies \exists z : \; h_{G,p}(y) \leq h_{G,p}(z) < h_{G,p}(x) \wedge pz = x \; ,$

(pH2) $\quad h_{G,p}(x) = h_{G,p}(z) + 1 \implies \exists z : \; h_{G,p}(y) = h_{G,p}(z) \wedge pz = x \; .$

Nevertheless, the restriction of $h_{G,p}$ still satisfies (VT) and (pV) (since these axioms are universal). Conversely, it was shown that every p-valuation arises in this way: every abelian group H with a p-valuation v can be embedded in an abelian group G in a way that v coincides with the restriction of $h_{G,p}$ to H (cf. [RIC–WA]).

Remark 3.18 For the development of the theory of valuated groups see [RIC], [RIC–WA], [HU–WA]. A slightly different definition of a p-valuation was given in [HI–ME].

As we have seen, the p-height function $h_{G,p}$ is a valuation if and only if G is reduced. If G is not reduced, then it does not satisfy (V0). This is bad in view of the notion of valuation independence that will be developed in the next chapter (if (V0) does not hold, then we loose the fact that valuation independent elements are $\mathbb{Z}$-independent). So let us modify the definition of the height function in order to obtain a valuation v. Suppose that the value set of the p-height function of G is $\nu\infty$, that is, ν is the height of G. For $a \in G$, we set $va = h_{G,p}(a)$ if $h_{G,p}(a) < \infty$. If $a \neq 0$ and $h_{G,p}(a) = \infty$, then we set $va = \nu$. Finally, we let $v0 = \infty$. Then v is a valuation on G, and we will call it the p-**height valuation of** G. Its value set is still ν if G is reduced, and it is $\nu + 1$ if G is not reduced. In the latter case, v does not satisfy (pV) since then there are divisible elements $a \neq 0$ such that $pa \neq 0$. But v will satisfy the modified axiom

(pV′) $vx \neq \nu, \infty \implies vpx > vx$.

Since our valuation v on G is derived from the p-height function on G, it satisfies

(pH1′) $\infty > vx > vy \implies \exists z : vy \leq vz < vx \wedge pz = x$,

(pH2′) $vx = vy + 1 \implies \exists z : vy = vz \wedge pz = x$.

Since descending sequences in ordinals are finite, axiom (pH1′) yields that for every element a of value $va < \nu$ there is some element b of value $vb = 0$ (this is, $h_{G,p}(b) = 0$) and some $n \in \mathbb{N}$ such that $a = p^n b$. Further, if vx is a limit ordinal and $vy < vx$, then there exists some y' such that $vy < vy' < vx$ and it follows from (pH1′) that there exists z such that $vy < vz < vx$ and $pz = x$.

There is a further p-valuation on a p-group. It is induced by the **order** $\mathrm{ord}(a)$ of the elements $a \in G$ in the following way. We set $va := -\mathrm{ord}(a)$ for $a \neq 0$, and $v0 = \infty$. Then (V0) and (VT) hold. For convenience, we identify ∞ with the integer 0. Then the following stronger form of (pV) is true:

$$x \neq 0 \implies vpx = vx + 1 \ .$$

In view of this, all components of (G, v) are $\mathbb{F}_p$-vector spaces. The value set of v is either $-\mathbb{N}$ or $\{m \in -\mathbb{N} \mid -n \leq m < 0\}$ for some natural number n. We will call v the **order valuation of** G.

In section 4.5, we will discuss applications of the p-height valuation and the order valuation of a p-group.

Exercise 3.4 *Prove (pH1) and (pH2). Give examples where these properties of the p-height function are not inherited by subgroups. For which p-groups can the conditions "$x \neq 0$" in (pH1) and "$\infty > vx$" in (pH1′) be omitted?*

Exercise 3.5 *Show that a p-group is q-divisible for every prime $q \neq p$ (hint: write $1 = rp^n + sq$ with $r, s \in \mathbb{Z}$). Conclude that a p-divisible p-group is divisible. For a p-valuation v, prove that $v(nx) = vx$ whenever $n \neq 0$ is not divisible by p (same hint).*

Exercise 3.6 *Show that the height of a p-group G is smaller than $|G|^+$.*

3.5 Ordered abelian groups and their natural valuations

For the ordered mathematician who is considering ordered abelian groups and ordered fields, the main example of a valuation is the "natural" one, which represents the archimedean classes.

An **ordered abelian group** is an abelian group $(G, +)$ equipped with an ordering $<$ which is compatible with the addition, that is, it satisfies

(OG) $x < y \implies x + z < y + z$

for all $x, y, z \in G$. We will write $(G, <)$ to indicate an ordered abelian group. From (OG) it follows that $a < b \Leftrightarrow 0 < b - a$. Hence, the ordering on G is already determined by the set $\mathbf{P} = \{a \in G \mid 0 \leq a\}$, which we call the **positive cone of** $(G, <)$. The positive cone satisfies

(PC+) $\mathbf{P} + \mathbf{P} \subset \mathbf{P}$

(PC∩) $\mathbf{P} \cap -\mathbf{P} = \{0\}$

(PC∪) $\mathbf{P} \cup -\mathbf{P} = G\,.$

Conversely, if a subset $\mathbf{P}$ of an abelian groups G satisfies these axioms, then it is called a **positive cone in** G and indeed, it is the positive cone of an ordering on G which is defined by $a \leq b \;\Leftrightarrow\; b - a \in \mathbf{P}$ (cf. Exercise 3.7).

We set $\mathrm{sign}(0) = 0$ and for $a \in G$, we set $\mathrm{sign}(a) = 1$ if $a > 0$, and $\mathrm{sign}(a) = -1$ if $a < 0$. If we use the sign with respect to the positive cone $\mathbf{P}$, then we will also write $\mathrm{sign}_{\mathbf{P}}$. Further, we set $|a| := \max\{a, -a\} = \mathrm{sign}(a) \cdot a$. Two elements $a, b \in G$ are called **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. The reader may show that this is indeed an equivalence relation. The ordered group $(G, <)$ is called **archimedean** if all nonzero elements are archimedean equivalent. Let va denote the equivalence class of a. The set of equivalence classes is ordered as follows: $va < vb$ if and only if $|a| > |b|$ and a and b are not archimedean equivalent; in this case, we will say that b is **archimedean smaller than** a. We write $\infty := v0$; this is the maximal element in the ordered set of equivalence classes. The map $a \mapsto va$ is a valuation on G, called the **natural valuation** (cf. Exercise 3.8). Here, the elements of the value set have a concrete interpretation as archimedean classes, but modulo order isomorphism, it looses this meaning. Since we are identifying equivalent valuations, we should better think of defining the natural valuation in the following way: "$va = vb$ if a and b are archimedean equivalent, and $va < vb$ if b is archimedean smaller than a". With this definition, it is immediately seen that the restriction of the natural valuation v to a subgroup H of G is (equivalent to) the natural valuation of $(H, <)$. (Although the equivalence class of $x \in H$ may be smaller in H than in G, there is an order isomorphism between $v_{|H}(H)$ and the value set of the natural valuation of (H, v).)

Alternatively, we define the natural valuation by the axiom

(NV) $\qquad vx \leq vy \iff \exists n \in \mathbb{N} :\; n|x| \geq |y|\,.$

This actually defines the valuation in terms of a relation rather than a map. See the remarks in section 13.1. We leave it to the reader to verify that a map $v : G \to vG\infty$ is the natural valuation if and only if it satisfies (NV); the only thing to prove is that the right side of (NV) expresses the fact that y is archimedean equivalent or archimedean smaller than x. Note that for every $a \in G$ and every $n \in \mathbb{Z} \setminus \{0\}$, the element $na \in G$ is archimedean equivalent to a and so, the natural valuation satisfies the axiom scheme

(NVZ) $\qquad v(nx) = vx \qquad\qquad (0 \neq n \in \mathbb{Z})\,.$

In particular, this shows that every ordered abelian group is torsion free. Further, it follows from (NV) that the natural valuation satisfies

$$\forall x, y :\; (vx < vy \Rightarrow |x| > |y|) \;\wedge\; (|x| > |y| \Rightarrow vx \leq vy)\,. \tag{3.7}$$

Observe that a valuation v of G is coarser than the natural valuation if and only if it satisfies the implication "$\Rightarrow$" of (NV). In this case, v is called **compatible with the ordering** or a **convex valuation of** $(G, <)$, and $<$ will be called a **v-compatible ordering**. Hence, the convex valuations of $(G, <)$ are precisely the coarsenings of its

natural valuation. The reader may show that a valuation v of $(G,<)$ is convex if and only if for every $\gamma \in vG$, the subset $\{a \in G \mid a > 0 \wedge va = \gamma\}$ of the positive cone is convex.

In the following, let v be the natural valuation of the ordered group G. Then the order type of vG will be called the **principal rank of the ordered group** G. In particular, G is archimedean if and only if its principal rank is 1, that is, vG consists of just one element.

Let $(H,<)$ be a subgroup of $(G,<)$. It will be called a **convex subgroup** if it is convex as a subset of the ordered set G. We leave it to the reader to show that this is equivalent to: $\forall x \in H\, \forall y \in G:\ |x| \geq |y| \Rightarrow y \in H$. Here, we can replace "$|x| \geq |y|$" by "$\exists n \in \mathbb{N}:\ n|x| \geq |y|$". Indeed, since H is assumed to be a subgroup, $x \in H$ implies that $nx \in H$; moreover, $n|x| = |nx|$. By virtue of (NV), it now follows that the condition is equivalent to

$$\forall x \in H\, \forall y \in G:\ vx \leq vy \Rightarrow y \in H\ , \tag{3.8}$$

that is, (H,v) is a v-convex subgroup of (G,v). On the one hand, this shows that for every initial segment γ of vG, the subgroups $\mathcal{O}_{\mathbf{G}}^{\gamma}$ and $\mathcal{M}_{\mathbf{G}}^{\gamma}$ of G are convex subgroups. On the other hand, we deduce that every convex subgroup $(H,<)$ of $(G,<)$ is of the form $\mathcal{M}_{\mathbf{G}}^{\gamma}$ for some initial segment γ of vG. The reader may show that a valuation w of $(G,<)$ is convex if and only if for every $\gamma \in wG$, the subgroup $\mathcal{O}_w^{\gamma}$ is convex.

Let a be an element in the ordered abelian group $(G,<)$ with natural valuation v. Then $\mathcal{O}_{\mathbf{G}}(a)$ is called a **principal convex subgroup**; it is the set of all elements in $(G,<)$ which are archimedean smaller or equivalent to a. It is contained in every convex subgroup that contains a and thus, it is equal to the intersection of all convex subgroups containing a. Similarly, $\mathcal{M}_{\mathbf{G}}(a)$ is the set of all elements which are archimedean smaller than a, and it is equal to the union of all convex subgroups not containing a. The following lemma is a consequence of Lemma 3.3:

Lemma 3.19 *The set* $\mathrm{Conv}(G,<)$ *of proper convex subgroups of* $(G,<)$ *is ordered by inclusion. The map* $\gamma \mapsto \mathcal{M}_{\mathbf{G}}^{\gamma}$ *induces an order reversing bijection between the ordered set of nonempty initial segments of* vG *and* $\mathrm{Conv}(G,<)$. *Similarly, the map* $\alpha \mapsto \mathcal{O}_{\mathbf{G}}^{\alpha}$ *induces an order reversing bijection between* $vG\infty$ *and the set of principal convex subgroups of* $(G,<)$.

The order type of $\mathrm{Conv}(G,<)$ is called the **rank of** $(G,<)$ and denoted by $\mathrm{rk}\,(G,<)$. If $\mathrm{Conv}(G,<)$ is finite, then the rank is equal to the number of elements of $\mathrm{Conv}(G,<)$. The lemma shows that the principal rank is equal to the reversed order type of the nonzero principal convex subgroups of $(G,<)$. If this is finite, then vG is finite and every convex subgroup is already principal convex. Hence if rank or principal rank are finite, then they are both equal to $|vG|$.

Let $(G,<)$ and $(G',<)$ be two ordered groups and $\rho : G \to G'$ a group homomorphism. Then we say that ρ **preserves** $\leq$ if $g \leq h$ implies $\rho g \leq \rho h$, for all $g,h \in G$. This holds if and only if ρ maps the positive cone of $(G,<)$ into the positive cone of $(G',<)$. Note that a group monomorphism preserving $\leq$ also preserves the order. If ρ preserves $\leq$ and g and h lie in the kernel of ρ and if $a \in G$ such that $g \leq a \leq h$, then $0 = \rho g \leq \rho a \leq \rho h = 0$, showing that also a lies in the kernel of ρ. This proves:

Lemma 3.20 *If* $\rho : G \to G'$ *is a group homomorphism of ordered groups* $(G,<)$ *and* $(G',<)$ *and if* ρ *preserves* $\leq$, *then its kernel* $\ker\rho$ *is a convex subgroup of* $(G,<)$.

Every convex subgroup is the kernel of a homomorphism which preserves $\leq$. Indeed, as an analogue to Lemma 3.4, we have

Lemma 3.21 *Let $(H,<)$ be a convex subgroup of $(G,<)$ and let $\mathbf{P}$ be the positive cone of $(G,<)$. Then $<$ induces an ordering on the factor group G/H whose positive cone is $\mathbf{P}/H$. The canonical epimorphism $G \to G/H$ preserves $\leq$, that is,*

(OFG) $\qquad x \leq y \implies x+H \leq y+H$.

Note that this ordering on G/H is in fact the one that you obtain when you read $x+H$, $y+H$ as subsets of the ordered set G and interpret $x+H < y+H$ in the usual way.

If $a,b \in G \setminus H$, then a and b are archimedean equivalent in G if and only if $a+H$ and $b+H$ are archimedean equivalent in G/H.

Proof: We have $\mathbf{P}/H+\mathbf{P}/H = (\mathbf{P}+\mathbf{P})/H \subset \mathbf{P}/H$ and $\mathbf{P}/H \cup -\mathbf{P}/H = (\mathbf{P}\cup-\mathbf{P})/H = G/H$. So it remains to prove property (PC$\cap$) for $\mathbf{P}/H$. Assume that $a+H$ is an element of $\mathbf{P}/H \cap -\mathbf{P}/H$, that is, there is $b \in -P$ such that $a+H = b+H$. But then $a-b \in H$, and since $a, -b \in P$, we have $0 \leq a \leq a-b$. By the convexity of H, we obtain $a \in H$ showing that $a+H$ is the zero in G/H, as required.

Further, we have $a \leq b \Rightarrow b-a \in \mathbf{P} \Rightarrow b-a+H \in \mathbf{P}/H \Rightarrow a+H \leq b+H$, which is (OFG). Since H is convex, so are the subsets $a+H$ and $b+H$ of G. If they are not equal, they are disjoint and by convexity, it follows that $a+H < b+H$ or $a+H > b+H$ as sets; but if $a+H < b+H$ holds, then by (OFG), we have $a+H < b+H$ as elements of $(G/H,<)$. Conversely, if $a+H < b+H$ does not hold as sets, then by what we have shown, we have $a+H = b+H$ or $a+H > b+H$ as sets and thus, $a+H = b+H$ or $a+H > b+H$ as elements of $(G/H,<)$. This shows that both interpretations of "$<$" coincide.

For our final assertion, note that a and b are archimedean equivalent in G if and only if $n|a|-|b| \in \mathbf{P}$ and $n|b|-|a| \in \mathbf{P}$. This in turn holds if and only if $n|a|-|b|+H \in \mathbf{P}/H$ and $n|b|-|a|+H \in \mathbf{P}/H$. Since $na+H = n(a+H)$ and $|a|+H = |a+H|$ (interpreting $a+H$ as an element of G/H, not as a set), the latter is equivalent to $n|a+H|-|b+H| \in \mathbf{P}/H$ and $n|b+H|-|a+H| \in \mathbf{P}/H$, which expresses that $a+H$ and $b+H$ are archimedean equivalent in G/H. □

The factor group G/H with its induced ordering will be denoted by $(G,<)/H$.

Corollary 3.22 *Let $(G,<)$ and $(G',<)$ be two ordered groups and $\rho : G \to G'$ a group homomorphism which preserves $\leq$. Then the embedding $G/\ker\rho \to G'$ induced by ρ is order preserving, that is, an embedding $(G,<)/\ker\rho \to (G',<)$.*

Proof: If η denotes the canonical epimorphism $G \to G/\ker\rho$, then the embedding $\overline{\rho} : G/\ker\rho \to G'$ may be written as $\rho \circ \eta^{-1}$. This maps the positive cone of $(G,<)/\ker\rho$ into that of $(G',<)$, hence $\overline{\rho}$ preserves $\leq$. Since $\overline{\rho}$ is injective, it thus preserves the order. □

We can determine the natural valuation of the factor group:

Corollary 3.23 *If we write $H = \mathcal{M}^{\gamma}_{\mathbf{G}}$ according to Lemma 3.3, then the natural valuation of $(G/H,<)$ is v/γ. If $a \in G \setminus H$, then the component of a in G is isomorphic to that of $a+H$ in G/H, as ordered abelian groups.*

Proof: The fact that v/γ is the natural valuation of $(G/H,<)$ follows from the last assertion of Lemma 3.21 and formula (3.2) of Lemma 3.4. The isomorphism between the components follows from Lemma 3.5; we only have to show that it preserves the ordering.

Let $\mathbf{P}_\alpha$ be the positive cone of $\mathcal{O}_\mathbf{G}^\alpha$. Applying Lemma 3.21 several times, we find that $(\mathbf{P}_\alpha/\mathcal{M}_\mathbf{G}^\gamma)/(\mathcal{M}_\mathbf{G}^\alpha/\mathcal{M}_\mathbf{G}^\gamma)$ is the positive cone of $(\mathcal{O}_\mathbf{G}^\alpha/\mathcal{M}_\mathbf{G}^\gamma)/(\mathcal{M}_\mathbf{G}^\alpha/\mathcal{M}_\mathbf{G}^\gamma)$, and that $\mathbf{P}_\alpha/\mathcal{M}_\mathbf{G}^\alpha$ is the positive cone of $\mathcal{O}_\mathbf{G}^\alpha/\mathcal{M}_\mathbf{G}^\alpha$. Both sets are sent onto each other by the isomorphism theorem, showing that the isomorphism preserves the ordering. □

Let $\alpha \in vG$. Then all elements in $\mathcal{O}_\mathbf{G}^\alpha \setminus \mathcal{M}_\mathbf{G}^\alpha$ have value α and are thus archimedean equivalent. Since they are precisely those elements which are not reducing to 0 modulo $\mathcal{M}_\mathbf{G}^\alpha$, we can deduce from the foregoing lemma:

Corollary 3.24 *Every component of an ordered abelian group carries an induced ordering, and this ordering is archimedean.*

This fact gives rise to the following definition. The skeleton $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$ is called an **ordered skeleton** if every component C_γ is an ordered abelian group. An isomorphism (resp. embedding) $\{\sigma_\Gamma, \sigma_\gamma \mid \gamma \in \Gamma\}$ of skeletons is called **isomorphism** (resp. **embedding**) **of ordered skeletons** if σ_γ preserves the ordering of C_γ for every $\gamma \in \Gamma$. The **natural skeleton of an ordered abelian group** $(G, <)$ is the skeleton of G with respect to the natural valuation where the components $\mathrm{C}^\alpha\mathbf{G}$ carry the induced ordering, and these components are called the **archimedean components of** $(G, <)$. So the natural skeleton is an ordered skeleton, and if we say that natural skeletons are isomorphic we will mean that they are isomorphic as ordered skeletons. Every isomorphism of ordered groups induces an isomorphism of their natural skeletons.

As a consequence of the definition of the induced ordering, we obtain that the ordering of a group G is uniquely determined by its ordered skeleton:

Corollary 3.25 *Let (G, v) be a valued abelian group whose skeleton $\mathrm{sk}_{\alpha\in vG}\mathrm{C}^\alpha\mathbf{G}$ is an ordered skeleton with archimedean groups $\mathrm{C}^\alpha\mathbf{G}$. Then there is a unique ordering on G whose natural valuation is v and which satisfies, for all $a \in G$,*

$$a > 0 \iff \mathrm{co}\, a > 0\ .$$

In particular, there is a unique extension of this ordering to every immediate extension of (G, v).

Corollary 3.26 *Let $(G, <)$ and $(G', <)$ be ordered abelian groups and let v be their natural valuation. Assume σ to be an isomorphism of their natural skeletons and ι to be an isomorphism of the valued groups $\mathbf{G}$ and $\mathbf{G}'$ which preserves σ. Then ι also preserves the ordering.*

Proof: Since ι preserves σ, which is an isomorphism of ordered skeletons, we have $(v\iota a, \mathrm{co}\,\iota a) = (\sigma_{vG} va, \sigma_{va}\mathrm{co}\, a)$. Now we compute: $a > 0 \Leftrightarrow \mathrm{co}\, a > 0 \Leftrightarrow \sigma_{va}\mathrm{co}\, a > 0 \Leftrightarrow \mathrm{co}\,\iota a > 0 \Leftrightarrow \iota a > 0$. □

Remark 3.27 The value set of the natural valuation reflects the archimedean incomparability of elements, while the components reflect the ordering between archimedean equivalent elements. Thinking of the "size" of elements (for example "nonstandard big", "standard", "infinitesimally small") we may thus say that the value set encodes the "vertical structure" of the ordered group, while the components encode its "horizontal structure".

Remark 3.28 The above systematic treatment of ordered abelian groups with their natural valuation the author has learned from S. Kuhlmann (cf. [KUS1], [KUS2]). In particular, her notion of an "ordered skeleton" and her characterization of order preserving isomorphisms (Corollary 3.26) turned out to be very useful.

The theory of ordered groups with their natural valuations and ordered skeletons was extensively exploited by S. Kuhlmann in the investigation of the structure of non-archimedean ordered exponential fields (cf. [KUS1], [KUS2], [KU-KUS]). Some of the results will be presented in section 5.4 below.

Given two ordered abelian groups $\mathbf{G}_1$ and $\mathbf{G}_2$, the **lexicographic product** of $\mathbf{G}_1$ and $\mathbf{G}_2$, denoted by $\mathbf{G}_1 \amalg \mathbf{G}_2$, is the direct sum $G_1 \oplus G_2$, endowed with the lexicographic product of their orderings. We leave it to the reader to show that this ordering is compatible with the addition of the product group. Then $\mathbf{G}_2$ is a convex subgroup of $\mathbf{G}_1 \amalg \mathbf{G}_2$, and $\mathbf{G}_1$ is isomorphic to the ordered group $(\mathbf{G}_1 \amalg \mathbf{G}_2)/\mathbf{G}_2$. For the groups endowed with their natural valuations, $\mathbf{G}_1 \amalg \mathbf{G}_2$ coincides with the definition that we have already given for valued groups. Hence, the value set of $\mathbf{G}_1 \amalg \mathbf{G}_2$ is the sum $vG_1 + vG_2$ of ordered sets, and the α-component of $\mathbf{G}_1 \amalg \mathbf{G}_2$ is that of $\mathbf{G}_1$ if $\alpha \in vG_1$ and that of $\mathbf{G}_2$ if $\alpha \in vG_2$. (We may call $\mathrm{sk}\,\mathbf{G}_1 \amalg \mathbf{G}_2$ the **sum** of $\mathrm{sk}\,\mathbf{G}_1$ and $\mathrm{sk}\,\mathbf{G}_2$.) The reader may prove the following: *If the convex subgroup* $\mathbf{G}_2$ *of an ordered group* $\mathbf{G}$ *admits a group complement* G_1 *in* G *(which is always the case if* G *is divisible), then with the induced ordering on* G_1*, we have* $\mathbf{G} = \mathbf{G}_1 \amalg \mathbf{G}_2$.

Example 3.29 The groups $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, endowed with the usual ordering are archimedean ordered groups. Every archimedean ordered group admits an order preserving embedding in $\mathbb{R}$ (cf. Lemma 4.56). The lexicographic products $\mathbb{Z} \amalg \mathbb{Q}$, $\mathbb{Q} \amalg \mathbb{Z}$ and $\mathbb{R} \amalg \mathbb{R}$ are ordered groups of rank 2. In the densely ordered group $\mathbb{Z} \amalg \mathbb{Q}$, $\mathbb{Q}$ is a proper convex subgroup. In the discretely ordered group $\mathbb{Q} \amalg \mathbb{Z}$, $\mathbb{Z}$ is a proper convex subgroup.

By induction, we form lexicographical products $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$. The two latter groups are divisible ordered abelian groups. Groups of the form $\mathbb{Z}^n$ are sometimes called **discrete groups** or **generalized discrete groups** (cf. [Z–S]). ♢

More generally, we can form the **ordered Hahn sum**, also called **lexicographical sum**, again denoted by $\coprod_{\gamma\in\Gamma} C_\gamma$, if the components C_γ are arbitrary (not necessarily archimedean) ordered abelian groups. Similarly, we obtain the **ordered Hahn product**, also called **lexicographical product**, again denoted by $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$. As abelian groups, these are the Hahn sum and the Hahn product as introduced in section 3.3. The ordering is then defined as follows. Given an element $c = (\zeta_\gamma)_{\gamma\in\Gamma}$, let γ_0 be the minimal element of its support. Then we let $c > 0$ if and only if $\zeta_{\gamma_0} > 0$ (which is the same as defining $\mathrm{sign}(c) := \mathrm{sign}(\zeta_{\gamma_0})$). If all C_γ are archimedean ordered, then the valuation v of the Hahn sum (resp. the Hahn product), as defined in section 3.3, is indeed the natural valuation of the ordered Hahn sum (resp. ordered Hahn product). But if there are non-archimedean ordered groups among the C_γ, then v will be a proper coarsening of the natural valuation. In this case, the natural skeleton will be obtained by replacing every non-archimedean ordered component C_γ in $\mathrm{sk}_{\gamma\in\Gamma} C_\gamma$ by the natural skeleton of C_γ (where we omit the ∞-component).

Every torsion free abelian group G admits a divisible extension group $\tilde{G}$ which has the universal property that it admits a unique embedding in every other divisible extension group. It is consequently unique up to isomorphism and is called the **divisible hull of** G.

It is an algebraic extension of G. It can be represented as the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Like every torsion free divisible abelian group, it is a $\mathbb{Q}$-vector space, and its $\mathbb{Q}$-dimension is called the **rational rank of** G (which can be a finite or infinite cardinal), denoted by $\operatorname{rr} G$. It is equal to the cardinality of every maximal set of rationally independent elements in G. Recall that elements a_i, $i \in I$, in an abelian group G are called **rationally independent**, if for every choice of integers n_i, only finitely many of them nonzero, $\sum n_i a_i = 0$ implies that all n_i are zero. If H is a subgroup of G, then a_i, $i \in I$, are called **rationally independent over** H, if for every choice of integers n_i, only finitely many of them nonzero, and every $a \in H$, we have that $a + \sum n_i a_i = 0$ implies that a and all n_i are zero. We will see later (cf. Corollary 4.30) that the cardinality of the rank of an ordered abelian group can not exceed its rational rank.

Example 3.30 The ordered Hahn sum $\coprod_{\mathbb{N}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$ have rank and principal rank ω, which is the order type of $\mathbb{N}$; all proper convex subgroups are principal. Both ordered groups have no smallest nonzero convex subgroup. The rational rank of $\coprod_{\mathbb{N}} \mathbb{Q}$ is the countable cardinal $\aleph_0$, and the rational rank of $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$ is $2^{\aleph_0}$. If we replace $\mathbb{N}$ by any ordinal ν then still, all proper convex subgroups are principal (and the group admits a smallest nonzero convex subgroup if and only if ν is a successor ordinal). In contrast to this, the ordered Hahn sum $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$ both contain precisely one proper non-principal convex subgroup, which is of the form $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$ resp. $\mathbf{H}_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q}$. Their principal rank is $\mathbb{Z}+\mathbb{Z}$, while their rank is $\mathbb{Z}+1+\mathbb{Z}$. In the ordered Hahn sum $\coprod_{\mathbb{R}} \mathbb{Q}$ and the ordered Hahn product $\mathbf{H}_{\mathbb{R}} \mathbb{Q}$, for Λ a cofinal segment of $\mathbb{R}$ the convex subgroups of the form $\coprod_{\Lambda} \mathbb{Q}$ resp. $\mathbf{H}_{\Lambda} \mathbb{Q}$ are non-principal if and only if Λ has no smallest element.

There are natural order preserving isomorphisms $\coprod_{\mathbb{Z}+\mathbb{Z}} \mathbb{Q} \cong \coprod_{\mathbb{Z}} \mathbb{Q} \amalg \coprod_{\mathbb{Z}} \mathbb{Q}$ and $\coprod_{\mathbb{Z}}(\mathbb{Q} \amalg \mathbb{Q}) \cong \coprod_{\mathbb{Z}} \mathbb{Q}$. In a natural way, $\coprod_{\mathbb{Z}} \mathbb{Q}$ is an ordered subgroup of the ordered groups $\coprod_{\mathbb{Z}} \mathbb{R}$, $\coprod_{\mathbb{Q}} \mathbb{Q}$, $\coprod_{\mathbb{Q}} \mathbb{R}$ and $\coprod_{\mathbb{R}} \mathbb{R}$. The same holds for the respective Hahn products. The reader should determine in which sense these isomorphisms and embeddings also preserve the natural valuation. Observe that on the valued Hahn sum $\coprod_{\mathbb{Z}}(\mathbb{Q} \amalg \mathbb{Q})$ whose components are all equal to the group $\mathbb{Q} \amalg \mathbb{Q}$ of rank 2, the valuation is a proper coarsening of the natural valuation. ♢

At present, we are interested in the extension of the ordering to the divisible hull.

Lemma 3.31 *Every ordered group $(G, <)$ admits a unique extension of the ordering to its divisible hull $\tilde{G}$. Under this ordering, every element of $\tilde{G}$ is archimedean equivalent to some element of G and thus, $vG = v\tilde{G}$ for the natural valuation v. Moreover, the α-component of $\tilde{G}$ is precisely the divisible hull of the α-component of G, for every $\alpha \in vG$.*

Proof: Let $\mathbf{P}$ be the positive cone of $(G, <)$. We leave it to the reader to show that $\overline{\mathbf{P}} := \{a \in \tilde{G} \mid \exists n \in \mathbb{N} : \ na \in \mathbf{P}\}$ is a positive cone in $\tilde{G}$ and that in view of (PC+), it is the only one containing $\mathbf{P}$. Given $a \in \tilde{G}$, there is an integer $n \neq 0$ such that $na \in G$, and a and na are archimedean equivalent; in particular, $va = v(na)$.

Now consider an element in the α-component of $\tilde{G}$; we can represent it as $\operatorname{co}_\alpha a$ with suitable $a \in \tilde{G}$. Let $n \neq 0$ such that $na \in G$; then we have $n \operatorname{co}_\alpha a = \operatorname{co}_\alpha na \in \mathrm{C}^\alpha \mathbf{G}$ showing that $\operatorname{co}_\alpha a$ lies in the divisible hull of $\mathrm{C}^\alpha \mathbf{G}$. Conversely, since $\tilde{G}$ is divisible, the same holds for the subgroup $\mathcal{M}^\alpha_{\mathbf{G}}$ and its factor group $\mathrm{C}^\alpha \mathbf{G}$. □

Using that a and na are archimedean equivalent for $n \neq 0$, we can also deduce that every convex subgroup H of G is **pure in** G, that is, for every $a \in G$, $na \in H$ implies that $a \in H$. The converse is certainly not true: $\mathbb{Q}$ is pure but not convex in $\mathbb{R}$. For every subgroup H of G, $\{a \in G \mid \exists n \neq 0 : na \in H\}$ is a pure subgroup of G, called the **relative divisible closure of** H **in** G. Since it is a subgroup of the divisible hull of H, it also has the same natural value set as H, and its α-component is contained in the divisible hull of the α-component of H. However, it may be smaller than the relative divisible closure of the α-component of H in the α-component of G:

Example 3.32 Let r be a real number $\notin \mathbb{Q}$. Consider $H = \mathbb{Z} \amalg \mathbb{Z}$ and $G = H + \mathbb{Z}(\frac{1}{2}, r) \subset \mathbb{R} \amalg \mathbb{R}$. Then for $\alpha = v(1,0)$, we have that $\mathrm{C}^\alpha H = \mathbb{Z}$ is not pure in $\mathrm{C}^\alpha G = \frac{1}{2}\mathbb{Z}$, although H is pure in G. $\diamondsuit$

The relative divisible closure is an algebraic extension. By the previous lemma, every algebraic extension of an ordered abelian group admits a unique extension of the ordering.

While $\mathbb{Z}$ is an ordered subgroup of every nontrivial ordered abelian group, it may not be a convex subgroup, as already the examples $\mathbb{Q}$ and $\mathbb{R}$ show. In fact, $\mathbb{Z}$ is a convex subgroup of the ordered group $\mathbf{G}$ if and only if $\mathbf{G}$ admits a **smallest positive element** (here, we mean "positive" in the sense "> 0"), and this is the case if and only if $\mathbf{G}$ is discretely ordered. Indeed, if $g \in G$ is the smallest positive element, then $\mathbb{Z}g$ (the cyclic group generated by g) is a convex subgroup of $\mathbf{G}$. If moreover, the factor group $G/\mathbb{Z}g$ is divisible, then $\mathbf{G}$ is called a **$\mathbb{Z}$-group**. In some sense that will be made precise by model theory in section 13.11, the $\mathbb{Z}$-groups are the groups which are similar to $\mathbb{Z}$ (cf. Theorem 13.69). The prototype of a $\mathbb{Z}$-group of rank 2 is $\mathbb{Q} \amalg \mathbb{Z}$. It is the smallest $\mathbb{Z}$-group of rank 2 since its rational rank is 2. The largest $\mathbb{Z}$-group of rank 2 is $\mathbb{R} \amalg \mathbb{Z}$. But note that not in every $\mathbb{Z}$-group, the convex subgroup $\mathbb{Z}$ is a direct summand. The reason for this is the fact that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is nontrivial and that every extension of $\mathbb{Z}$ by $\mathbb{Q}$ can be ordered such that it is a $\mathbb{Z}$-group; see [HI–MO], Theorem 3.1 and Proposition 3.2.

Let v be the natural valuation of the $\mathbb{Z}$-group $\mathbf{G}$. Then vG contains a largest element, namely the archimedean class vg of g. Further, co_{vg} is an isomorphism of $\mathbb{Z}g$ onto the vg-component $\mathrm{C}^{vg}\mathbf{G}$ (as ordered groups).

The relative divisible closure of an ordered abelian group in a divisible ordered abelian group is again a divisible ordered abelian group. $\mathbb{Z}$-groups have a similar property:

Lemma 3.33 *Let $\mathbf{G}$ be a $\mathbb{Z}$-group with least positive element g. Then for every $n \in \mathbb{N}$ and $a \in G$ there is some $m \in \mathbb{N}$ such that $a + mg$ is divisible by n in G. In particular, if $\mathbf{H}$ is a subgroup of $\mathbf{G}$ containing g, then the relative divisible closure of $\mathbf{H}$ in $\mathbf{G}$ is again a $\mathbb{Z}$-group. On the other hand, if $\mathbf{H} \subset \mathbf{G}$ is an extension of $\mathbb{Z}$-groups, both having the same least positive element, then G/H is torsion free.*

Proof: Let $n \in \mathbb{N}$ and $a \in G$. Since $G/\mathbb{Z}g$ is divisible, there is some $b \in G$ such that $nb + \mathbb{Z}g = a + \mathbb{Z}g$. Hence, $nb = a + mg \in G$ for some $m \in \mathbb{Z}$, showing that $a + mg$ is divisible by n in G. Writing $m = m' + kn$ with $m', k \in \mathbb{Z}$ and $0 \leq m' < n$, we find that also $a + m'g = a + mg - kng$ is divisible by n. Thus, we can replace m by m' to obtain that $0 \leq m < n$.

Denote by $\mathbf{H}'$ the relative divisible closure of $\mathbf{H}$ in $\mathbf{G}$. Suppose that in addition to the previous assumption, we have that $a \in H$. Since $g \in H$ and hence $a + mg \in H$, we obtain

$b \in H'$. This shows that $a + \mathbb{Z}g$ is divisible by n in $H'/\mathbb{Z}g$. We have thus proved that $H'/\mathbb{Z}g$ is divisible.

Now assume that $\mathbf{H} \subset \mathbf{G}$ is an extension of $\mathbb{Z}$-groups, both having the same least positive element g. Let $a \in G$ and $n \in \mathbb{N}$ such that $na \in H$. Choose $m \in \mathbb{N}$ such that $0 \leq m < n$ and $na + mg$ is divisible by n in H. Write $na + mg = nb$ with $b \in H$. Then $0 \leq b - a = \frac{m}{n}g < g$ shows that $a = b \in H$. We have thus proved that G/H is torsion free. □

When we consider $\wp$-adically closed fields, we will need:

Lemma 3.34 *Let $(H, <)$ be an ordered abelian group with least positive element g. Then there is an algebraic extension $(G, <)$ of $(H, <)$ such that $(G, <)$ is a $\mathbb{Z}$-group with least positive element g.*

Proof: Let $(G, <)$ be a subgroup of the divisible hull of $(H, <)$ containing $(H, <)$ and maximal with the property that g is the least positive element in $(G, <)$. (Such maximal subgroups exist by Zorn's Lemma). Suppose that $G/\mathbb{Z}\alpha$ is not divisible, and choose $a \in G$ and a prime p such that $a + \mathbb{Z}g$ is not divisible by p. Take $b \in \tilde{H} = \tilde{G}$ such that $pb = a$. Then $(G + \mathbb{Z}b : G) = p = ((G + \mathbb{Z}b)/\mathbb{Z}g : G/\mathbb{Z}g)$. This yields that $\mathbb{Z}g$ is still a convex subgroup of $G + \mathbb{Z}b$, contrary to the maximality of G. This proves that $G/\mathbb{Z}\alpha$ is divisible, i.e., $(G, <)$ is a $\mathbb{Z}$-group. □

Ordered abelian groups play an important role as value groups of valued fields. The relation between a valued field and its value group is a "one level higher analogue" to the relation between a valued group and its value set. Consequently, also initial segments of ordered abelian groups will play a role. For later use, let us state some auxiliary results.

Lemma 3.35 *Let Λ be an initial segment of the ordered abelian group Γ.*

a) For every $\Lambda' \subset \Gamma$, $\Lambda + \Lambda' = \{\alpha + \beta \mid \alpha \in \Lambda\,,\ \beta \in \Lambda'\}$ is again an initial segment of Γ.

b) For every natural number $n > 0$, the set $n \cdot \Lambda = \{n \cdot \alpha \mid \alpha \in \Lambda\}$ is cofinal in the n–fold sum $\Lambda + \Lambda + \ldots + \Lambda$ (which is an initial segment by part a)).

c) If Δ is a subgroup of Γ, then $\Lambda \cap \Delta$ is an initial segment of Δ.

Proof: a): Let $\alpha \in \Lambda$, $\beta \in \Lambda'$ and $\gamma \in \Gamma$ such that $\gamma < \alpha + \beta$. Then $\gamma - \beta < \alpha$, and since Λ is an initial segment of Γ, we have $\gamma - \beta \in \Lambda$. Hence, $\gamma = \gamma - \beta + \beta \in \Lambda + \Lambda'$.

b): Given $\alpha_i \in \Lambda$, $1 \leq i \leq n$, let $\alpha = \max_i \alpha_i$. Then $\alpha_1 + \ldots + \alpha_n \leq n\alpha$. Since the latter is again an element of the n–fold sum of Λ's, this shows our assertion.

c): An easy exercise. □

Exercise 3.7 *Let $\mathbf{P}$ be a positive cone of G. Show that the relation defined by $x \leq y \Leftrightarrow y - x \in \mathbf{P}$ is an ordering on G.*

Exercise 3.8 *Let v denote the natural valuation that we have defined above. We wish to show that v is indeed a valuation.*

a) Show that no nonzero element is archimedean equivalent to 0 and deduce that v satisfies $(V0)$.

b) Prove that $|a + b| \leq |a| + |b|$. Use this and (NV) to show that v satisfies (VT).

c) Deduce the following properties of the natural valuation:

(NV1) $\forall x, y \in G:\ v(x - y) > vx \Rightarrow \mathrm{sign}(x) = \mathrm{sign}(y)\,,$

(NV2) $\operatorname{sign}(\sum_{1\le i\le n} x_i) = \operatorname{sign}(x_m)$ *if* $vx_m < vx_i$ *for all* $i \neq m$,

(NV3) $x_m < x'_m \Rightarrow \sum_{1\le i\le n} x_i < \sum_{1\le i\le n} x'_i$ *if* $vx_m < vx_i$ *and* $vx'_m < vx'_i$ *for all* $i \neq m$.

Exercise 3.9 *Show that* $\mathbb{Z}\amalg\mathbb{Z}$ *does not admit proper immediate extensions. On the other hand, prove that every finite extension is again isomorphic to* $\mathbb{Z}\amalg\mathbb{Z}$ *and thus, their skeletons are isomorphic. Do the same with* $\mathbb{Q}\amalg\mathbb{Z}$ *and examples of different ranks.*

Exercise 3.10 *Show that the lexicographic product of two Hahn sums (resp. Hahn products) is a Hahn sum (resp. a Hahn product).*

3.6 A generalized Hensel's Lemma

In Part II of this book, we will meet the famous Hensel's Lemma for spherically complete valued fields. But the classical principle which is often used for its proof is actually a principle that applies already to valued abelian groups. In this section we will prove a theorem that may be viewed as the quintessence of this principle.

Let be given valued abelian groups (G, v) and (G', v'), a subset $\mathcal{S} \subset G$ and a map $f : \mathcal{S} \to G'$. We set $\mathcal{S}_0 = \mathcal{S} - \mathcal{S} := \{a - b \mid a, b \in \mathcal{S}\}$. A map $\Psi : \mathcal{S}_0 \to G'$ will be called a **valuation differential for** f **on** $\mathcal{S}$ if it satisfies

$$\forall a, b \in \mathcal{S} :\ f(a) \neq f(b) \ \Rightarrow\ v'(f(a) - f(b) - \Psi(a - b)) > v'(f(a) - f(b)) \ . \tag{3.9}$$

The latter inequality means that $f(a) - f(b)$ and $\Psi(a - b)$ have the same bones, and in particular, that $v'\Psi(a - b) = v'(f(a) - f(b))$.

The following theorem states the promised **generalization of Hensel's Lemma** for spherically continuous maps on spherically complete subsets of valued abelian groups. (For the notion of spherical continuity, cf. page 24.) We will see later that Hensel's Lemma for spherically complete valued fields is a consequence of this theorem. This shows that the only use of the field multiplication in Hensel's Lemma lies in the fact that the considered map on the field is given by a polynomial. The general principle itself works even in absence of multiplication. That is, we obtain a generalization for valued abelian groups on the one hand, and an assertion for arbitrary spherically continuous maps on the other hand. These maps are assumed to be given on a spherically complete subset of the valued group. Recall that all v-convex subsets of spherically complete valued abelian groups are spherically complete (Lemma 3.16).

Theorem 3.36 *Let* (G, v) *and* (G', v') *be valued abelian groups. Further, let* $\mathcal{S}$ *be a spherically complete subset of* (G, v) *and* $f : \mathcal{S} \to G'$ *a spherically continuous map. Then* $f(\mathcal{S})$ *is spherically complete and thus nice in* (G', v')*. If in addition, for a given* $c \in G'$ *there exists a valuation differential* Ψ *for* f *on* $\mathcal{S}$ *such that*

$$\forall b \in \mathcal{S} :\ f(b) \neq c \implies \exists b' \in \mathcal{S} :\ v'(f(b) - c - \Psi(b - b')) > v'(f(b) - c) \ , \tag{3.10}$$

then $c \in f(\mathcal{S})$*.*

Proof: Since $\mathcal{S}$ is assumed to be spherically complete, Lemma 2.8 shows that $f(\mathcal{S})$ is spherically complete and thus nice in (G', v'), according to Lemma 2.16.

Now assume in addition that $c \in G'$ and that Ψ is a valuation differential for f on $\mathcal{S}$ which satisfies (3.10). Since $f(\mathcal{S})$ is nice in (G', v'), there is some $b \in \mathcal{S}$ for which $v'(f(b)-c)$ is maximal. Assume that $f(b) \neq c$. Then (3.10) yields the existence of $b' \in \mathcal{S}$ such that

$$v'(f(b) - c - \Psi(b - b')) > v'(f(b) - c) \; .$$

Since Ψ is a valuation differential for f on $\mathcal{S}$, we have that $v'(f(b) - c - (f(b') - c) - \Psi(b - b')) = v'(f(b) - f(b') - \Psi(b - b')) > v'(f(b) - f(b')) = v'(f(b) - c - (f(b') - c))$. Now an exercise in valuation theoretical computation (cf. Exercise 3.11) shows that both inequalities together imply the inequality

$$v'(f(b') - c) > v'(f(b) - c) \; .$$

But this contradicts the maximality of b. This contradiction proves that $c = f(b) \in f(\mathcal{S})$. □

If we are looking for a zero of f, then we set $c = 0$. In this case, condition (3.10) reads as

$$\forall b \in \mathcal{S} : \; f(b) \neq 0 \implies \exists b' \in \mathcal{S} : \; v'(f(b) - \Psi(b - b')) > v'f(b) \; , \tag{3.11}$$

and $v'(f(b) - \Psi(b - b')) > v'f(b)$ implies that

$$v'f(b') > v'f(b)$$

which means that b' is a better approximation to a root of f than b. Observe that the case $c = 0$ actually implies the general case. Indeed, we just have to replace f by $f - c$. The reader may show that f is spherically continuous if and only if $f - c$ is. And trivially, Ψ is a valuation differential for f on $\mathcal{S}$ if it is for $f - c$.

If $\mathcal{S}$ is the coset of a subgroup H of (G, v), then $\mathcal{S} - \mathcal{S} = H$. In this case, condition (3.10) can be rewritten as

$$\forall b \in \mathcal{S} : \; f(b) \neq c \implies \exists a \in H : \; v'(f(b) - c - \Psi(a)) > v'(f(b) - c) \; , \tag{3.12}$$

This condition says that the bone of every element in the set $f(\mathcal{S}) - c$ appears as a bone of some element in the set $\Psi(H)$. This will in particular be the case if $f(\mathcal{S}) - c \subset \Psi(H)$. We thus obtain the assertion that $c \in f(\mathcal{S})$ if $f(\mathcal{S}) \subset c + \Psi(H)$.

We shall give a criterion for an arbitrary map f to be spherically continuous. The case of additive maps will then be treated in the next section. We have to introduce some definitions. The map $\Psi : \mathcal{S}_0 \to G'$ will be called **value-compatible on** $\mathcal{S}_0$ if it satisfies

$$va \geq vb \implies v'\Psi(a) \geq v'\Psi(b) \; , \tag{3.13}$$

that is, if Ψ induces a $\geq$-preserving map from $v\mathcal{S}_0 = \{va \mid 0 \neq a \in \mathcal{S}_0\}$ into $v'G'$. We will need an appropriate generalization of this definition. We define

$$\mathrm{Exb}^{\geq}(\Psi) := \{(vb, \mathrm{co}\, b) \mid b \in \mathcal{S}_0 \wedge \exists a \in \mathcal{S}_0 : \; va \geq vb \wedge v'\Psi(a) < v'\Psi(b)\} \; .$$

This set collects the **bad bones** of those elements b for which (3.13) does not hold. Hence,

$$va \geq vb \wedge (vb, \operatorname{co} b) \notin \operatorname{Exb}^{\geq}(\Psi) \implies v'\Psi(a) \geq v'\Psi(b)\,. \tag{3.14}$$

If the set $\operatorname{Exb}^{\geq}(\Psi)$ is finite, then Ψ will be called **almost value-compatible on $\mathcal{S}_0$**.

The existence of almost value-compatible valuation differentials is a criterion for spherical continuity:

Lemma 3.37 *Let (G, v) and (G', v') be two valued abelian groups, $\mathcal{S}$ a subset of G and $f : \mathcal{S} \to G'$ a map. Suppose that there exists an almost value-compatible valuation differential Ψ for f on $\mathcal{S}$. Then f is spherically continuous.*

Proof: Our proof is an adaptation of the proof of part a) of Lemma 2.9. Let $\mathbf{B}' \subset \mathbf{B}(f(\mathcal{S}))$ be a nest of balls in the ultrametric subspace $f(\mathcal{S})$ of (G', v'). If $\bigcap \mathbf{B}'$ contains an element $f(a)$, then we set $\mathbf{B} = \{B_\infty(a)\}$ and find that $f(B_\infty(a)) = \{f(a)\} \subset B'$ for every $B' \in \mathbf{B}'$. So let us now assume that $\bigcap \mathbf{B}' = \emptyset$. By the method described preceding to Lemma 2.52, we choose a coinitial sequence $(B'_\nu)_{\nu<\lambda}$ of balls $B'_\nu = B_{\alpha_\nu}(f(a_\nu))$ in $\mathbf{B}'$ such that $(f(a_\nu))_{\nu<\lambda}$ is a pseudo Cauchy sequence. So we have $v'(f(a_\nu) - f(a_{\nu+1})) < v'(f(a_{\nu+1}) - f(a_\mu))$ and $v'(f(a_\nu) - f(a_\mu)) = v'(f(a_\nu) - f(a_{\nu+1}))$ whenever $\nu < \mu < \lambda$.

Next, we clear the sequence $(a_\nu)_{\nu<\lambda}$ from the influence of bad bones. Among all differences $a_\mu - a_\nu$ whose bones are in $\operatorname{Exb}^{\geq}(\Psi)$, let $a_{\nu_0} - a_{\mu_0}$, $\nu_0 < \mu_0 < \lambda$, be one of smallest value $v(a_{\nu_0} - a_{\mu_0}) =: \alpha$. This exists since we have assumed $\operatorname{Exb}^{\geq}(\Psi)$ to be finite. We wish to extract a subsequence of $(a_\nu)_{\nu<\lambda}$ such that no difference of elements in this subsequence has value α. Observe that for every $\nu > \nu_0$ we must have that $v(a_{\nu_0+1} - a_\nu) \geq \alpha$ and that the bone of $a_{\nu_0+1} - a_\nu$ lies in $\operatorname{Exb}^{\geq}(\Psi)$ if equality holds. Indeed, $v'\Psi(a_{\nu_0+1} - a_\nu) = v'(f(a_{\nu_0+1}) - f(a_\nu)) > v'(f(a_{\nu_0}) - f(a_{\nu_0+1})) = v'(f(a_{\nu_0}) - f(a_{\mu_0})) = v'\Psi(a_{\nu_0} - a_{\mu_0})$. In view of (3.14), this shows that $v(a_{\nu_0+1} - a_\nu) \leq \alpha = v(a_{\nu_0} - a_{\mu_0})$ can only hold if the bone of $a_{\nu_0+1} - a_\nu$ lies in $\operatorname{Exb}^{\geq}(\Psi)$; but then, $v(a_{\nu_0+1} - a_\nu) = \alpha$ by the minimality of α. Since we have assumed $\operatorname{Exb}^{\geq}(\Psi)$ to be finite, there are only finitely many bones of value α in $\operatorname{Exb}^{\geq}(\Psi)$. So we can choose a maximal set of elements $a_{\nu_1}, \ldots, a_{\nu_n}$ with $\nu_i > \nu_0 + 1$ such that all $a_{\nu_0+1} - a_{\nu_i}$ have distinct bones of value α. We consider the sets

$$\{a_\nu \mid a_{\nu_0+1} - a_\nu \text{ has the same bone as } a_{\nu_0+1} - a_{\nu_i}\} = \{a_\nu \mid v(a_\nu - a_{\nu_i}) > \alpha\}\,,\ \ 1 \leq i \leq n\,,$$

together with the set $\{a_\nu \mid v(a_{\nu_0+1} - a_\nu) > \alpha\}$. By what we have shown before, the union of these sets contains every a_ν for $\nu \geq \nu_0$. Consequently, one of these sets must be a subsequence $\mathbf{S}$ of $(a_\nu)_{\nu<\lambda}$. If a_{ν_ρ} and a_{ν_σ} are two elements of $\mathbf{S}$, then either $a_{\nu_0+1} - a_{\nu_\rho}$ and $a_{\nu_0+1} - a_{\nu_\sigma}$ have the same bone, which yields that $v(a_{\nu_\rho} - a_{\nu_\sigma}) > \min\{a_{\nu_0+1} - a_{\nu_\rho}, a_{\nu_0+1} - a_{\nu_\sigma}\} = \alpha$, or $v(a_{\nu_0+1} - a_{\nu_\rho}) > \alpha$ and $v(a_{\nu_0+1} - a_{\nu_\sigma}) > \alpha$, which again yields $v(a_{\nu_\rho} - a_{\nu_\sigma}) > \alpha$. Hence, the value of all differences of elements of $\mathbf{S}$ is bigger than α. So $\mathbf{S}$ is the subsequence that we have looked for. Since we have assumed $\operatorname{Exb}^{\geq}(\Psi)$ to be finite, there are only finitely many values which appear as values of bones in this set. Iterating the above procedure, we can eliminate all of them, that is, we find a subsequence of $(a_\nu)_{\nu<\lambda}$ such that all differences of its elements have values that are not the value of any bad bone in $\operatorname{Exb}^{\geq}(\Psi)$. This sequence will again be denoted by $(a_\nu)_{\nu<\lambda}$. For this sequence, no difference $a_\nu - a_\mu$ has a bone that appears in $\operatorname{Exb}^{\geq}(\Psi)$. Hence, an analogue of the above computation using the properties of Ψ shows that $v(a_\nu - a_{\nu+1}) < v(a_{\nu+1} - a_\mu)$ if $\nu < \mu < \lambda$. That is, $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence.

We set $\beta_\nu := v(a_\nu - a_{\nu+1})$ and $B_\nu := B_{\beta_\nu}(\mathcal{S}, a_\nu) = B_{\beta_\nu}(a_\nu) \cap \mathcal{S}$. In particular, $a_\mu \in B_\nu$ for every $\mu > \nu$, showing that every two balls B_ν and B_μ have a nonempty intersection. This proves that $\mathbf{B} := \{B_\nu \mid \nu < \lambda\}$ is a nest of balls in $\mathcal{S}$. Further, $a \in B_\nu$ implies $v'(f(a_\nu) - f(a)) = v'\Psi(a_\nu - a) \geq v'\Psi(a_\nu - a_{\nu+1}) = v'(f(a_\nu) - f(a_{\nu+1}))$ (since the bone of $a_\nu - a_{\nu+1}$ is not in $\mathrm{Exb}^{\geq}(\Psi)$), and thus $f(a) \in B'_\nu$. That is, for every $B' \in \mathbf{B}'$ there is $\nu < \lambda$ such that $f(B_\nu) \subset B'_\nu \subset B'$. We have proved that f is spherically continuous. □

Note that only the first and the last paragraph of this proof are required if Ψ is value-compatible. From this lemma together with Theorem 3.36, we obtain

Corollary 3.38 *Let (G, v) and (G', v') be two valued abelian groups, $\mathcal{S}$ a spherically complete subset of G and $f : \mathcal{S} \to G'$ a map. Assume that there exists an almost value-compatible valuation differential Ψ for f on $\mathcal{S}$. Then $f(\mathcal{S})$ is spherically complete and thus nice in G'. If $c \in G'$ and Ψ satisfies (3.10), then $c \in f(\mathcal{S})$.*

To conclude this section, we wish to prove a multidimensional version of the foregoing corollary. Compared to Hensel's Lemma for valued fields, the generalized Hensel's Lemma for valued groups has the advantage that its multidimensional version is merely a corollary of the onedimensional case. This is because contrary to the case of field valuations, the product of finitely many valued groups admits a product of their valuations which is itself a group valuation. Let us give the precise definition. Let (G_i, v_i), $i \in I$, be valued abelian groups whose value sets $v_i G_i$ are all contained in a common ordered set, and assume that I is finite or $\bigcup_{i\in I} v_i G_i$ is wellordered. Then their **product** will be the product group $\prod_{i\in I} G_i$ equipped with the valuation $\prod_{i\in I} v_i : \prod_{i\in I} G_i \to (\bigcup_{i\in I} v_i G_i)\infty$ defined by

$$\prod_{i\in I} v_i \left((a_i)_{i\in I}\right) := \min_{i\in I} v_i a_i \ .$$

We leave it to the reader to verify that this map satisfies (V 0) and (VT). Note that every element of $\bigcup_{i\in I} v_i G_i$ appears as the value of a suitably chosen element of $\prod_{i\in I} G_i$. The reader may show that the underlying ultrametric space of the valued product group is just the product of the underlying ultrametric spaces of the groups (G_i, v_i). Consequently, if I is finite and all groups (G_i, v_i) are spherically complete, then Lemma 2.10 shows that also their product is spherically complete. More generally, if I is finite and $\mathcal{S}_i \subset G_i$ are spherically complete subsets, then their product is a spherically complete subset of the valued product group.

Let be given a valued group (G, v) and a natural number n. For the formulation of our corollary, we will work in the n-fold product G^n, endowed with the product valuation, which in this case will be denoted by $\underline{v}$. For $(a_1, \ldots, a_n) \in G^n$, it is given by

$$\underline{v}(a_1, \ldots, a_n) \ = \ \min_{1\leq i\leq n} v a_i \ .$$

Let $\mathcal{S}_i \subset G$, $1 \leq i \leq n$, be spherically complete subsets. Then $\underline{\mathcal{S}} := \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ is a spherically complete subset of G^n. Let $f_i : \mathcal{S}_i \to G$ be arbitrary maps. The map $(f_1, \ldots, f_n) : \underline{\mathcal{S}} \to G^n$ will be denoted by $\underline{f}$.

Corollary 3.39 *Let the situation be as described above. If there exists an almost value-compatible valuation differential Ψ for $\underline{f}$ on $\underline{\mathcal{S}}$, then $\underline{f}(\underline{\mathcal{S}})$ is spherically complete and thus nice in G^n. If for a given $\underline{c} \in G^n$, Ψ satisfies*

$$\forall \underline{b} \in \underline{\mathcal{S}} : \ \underline{f}(\underline{b}) \neq \underline{c} \implies \exists \underline{b}' \in \underline{\mathcal{S}} : \ v(\underline{f}(\underline{b}) - \underline{c} - \Psi(\underline{b} - \underline{b}')) > v(\underline{f}(\underline{b}) - \underline{c}) \ , \qquad (3.15)$$

then $\underline{c} \in \underline{f}(\underline{\mathcal{S}})$.

Exercise 3.11 *Show that the inequlities* $v(x-y) > vx$ *and* $v(x-x'-y) > v(x-x')$ *together imply the inequality* $vx' > vx$.

Exercise 3.12 *State and prove a multidimensional version of Theorem 3.36.*

3.7 Spherically continuous additive maps

An important specialization of the results of the last section can be obtained when we consider a map f which is **additive**, that is, $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathcal{S}$ (if $\mathcal{S} = G$, then this means that f is a homomorphism of the abelian group G). In this case, condition (3.9) holds trivially for $\Psi = f$, showing that f is its own valuation differential. Then we just set $\Psi = f$ to apply Theorem 3.36:

Corollary 3.40 *Let* (G, v) *and* (G', v') *be two valued abelian groups and* $f : G \to G'$ *a spherically continuous additive map. Assume that* (G, v) *is spherically complete. Then* $f(G)$ *is a spherically complete and thus a nice subgroup of* (G', v'). *If in addition,* $c \in G'$ *such that*

$$\forall b \in G : \ f(b) \neq c \implies \exists b' \in G : \ v'(f(b') - c) > v'(f(b) - c) \ , \tag{3.16}$$

then $c \in f(G)$. *If (3.16) holds for every* $c \in G'$, *that is, if* $(f(G), v') \subset (G', v')$ *is immediate, then* $f(G) = G'$.

In fact, this result can be derived directly from Lemma 2.8 and Lemma 2.16. It is particularly interesting for our theory of valued modules as developed in the next chapter, since multiplication by a scalar constitutes an additive map. (On the other hand, we may view every additive map $f : G \to G$ as multiplication by a scalar since G is a module over its endomorphism ring.)

Let us also note the following version of the foregoing corollary:

If in addition to the general assumptions of Corollary 3.40, H' *is a* v'*-convex subgroup of* G' *and condition (3.16) holds whenever* $v'(f(b) - c) < v'H'$, *then* $G' = f(G) + H'$.

Indeed, by Corollary 3.40, $f(G)$ is a nice subgroup of (G', v'). So for every $c \in G'$ there is some $b \in G$ such that $f(b) - c$ is of maximal value. Now $v'(f(b) - c) < v'H'$ is impossible since otherwise, by assumption there would exist some $b' \in G$ such that $v'(f(b') - c) > v'(f(b) - c)$. Hence, $f(b) - c \in H'$ because H' is assumed to be v'-convex. This proves that $c \in f(G) + H'$ for every $c \in G'$.

We shall now determine some criteria for additive maps to be spherically continuous. By an application of Lemma 3.37, again setting $\Psi = f$, we obtain:

Corollary 3.41 *Let* (G, v) *and* (G', v') *be two valued abelian groups and* $f : G \to G'$ *an additive map. If* f *is almost value-compatible, then* f *is spherically continuous.*

We wish to give a rather different criterion. In analogy to the set $\mathrm{Exb}^{\geq}(f)$, let us define the set

$$\mathrm{Exe}^{\geq}(f) := \{b \in G \mid \exists a \in G : \ va \geq vb \wedge v'f(a) < v'f(b)\}$$

of bad elements b for which (3.13) does not hold. Similarly, we define

$$\mathrm{Exe}^{>}(f) := \{b \in G \mid \exists a \in G:\ va > vb \wedge v'f(a) \leq v'f(b)\}\ .$$

We set $\mathrm{Exe}^{\geq,>}(f) := \mathrm{Exe}^{\geq}(f) \cup \mathrm{Exe}^{>}(f)$. Hence,

$$b \notin \mathrm{Exe}^{\geq,>}(f) \implies \begin{cases} va \geq vb \Rightarrow v'f(a) \geq v'f(b) \\ va > vb \Rightarrow v'f(a) > v'f(b) \end{cases} \tag{3.17}$$

Assume that f is additive. We leave it to the reader to show that $b \notin \mathrm{Exe}^{\geq}(f)$ if and only if $f(B_{vb}(b)) \subset B_{v'f(b)}(f(b))$, and that $b \notin \mathrm{Exe}^{>}(f)$ if and only if $f(B^{\circ}_{vb}(b)) \subset B^{\circ}_{v'f(b)}(f(b))$. If $b \notin \mathrm{Exe}^{\geq,>}(f)$, then $B^{\circ}_{vb}(b) \cap \mathrm{Exe}^{\geq,>}(f) = \emptyset$. This is seen as follows. For $c \in B^{\circ}_{vb}(b)$ we have that $v(b-c) > vb$ and $vb = vc$. The former implies that $v'(f(b)-f(c)) = v'f(b-c) > v'f(b)$ and consequently, $v'f(b) = v'f(c)$. Hence, $va \geq vc \Rightarrow va \geq vb \Rightarrow v'f(a) \geq v'f(b) \Rightarrow v'f(a) \geq v'f(c)$, and similarly for "$>$" in the place of "$\geq$".

Example 3.42 Let G_i, $1 \leq i \leq n$, be subgroups of the valued abelian group (G, v). Let $\underline{v} = \prod_{1 \leq i \leq n} v$ denote the product valuation on $\prod_{1 \leq i \leq n} G_i$. We let $f : \prod_i G_i \to G$ be the additive map which sends $(a_1, \ldots, a_n)$ to $a_1 + \ldots + a_n =: a$. Assume that $va_i \geq va$, $1 \leq i \leq n$. Then there is some j such that $va_j = va$, and we have $\underline{v}(a_1, \ldots, a_n) = va$. If $(b_1, \ldots, b_n) \in \prod_i G_i$ such that $\underline{v}(b_1, \ldots, b_n) \geq \underline{v}(a_1, \ldots, a_n)$, then $v(b_1 + \ldots + b_n) \geq \min_i vb_i = \underline{v}(b_1, \ldots, b_n) \geq \underline{v}(a_1, \ldots, a_n) = va$. If $\underline{v}(b_1, \ldots, b_n) > \underline{v}(a_1, \ldots, a_n)$, then we even obtain $v(b_1 + \ldots + b_n) > va$. This shows that $(a_1, \ldots, a_n) \notin \mathrm{Exe}^{\geq,>}(f)$.

For the converse, assume that $va_i < va$ for some i, say $i = 1$. The element $(a_1, 0, \ldots, 0)$ is sent to a_1. In view of $\underline{v}(a_1, 0, \ldots, 0) \geq \underline{v}(a_1, \ldots, a_n)$ and $va_1 < va$, we find that $(a_1, \ldots, a_n) \in \mathrm{Exe}^{\geq}(f)$. If in addition, $a = b_1 + \ldots + b_n$ with $\underline{v}(b_1, \ldots, b_n) > \underline{v}(a_1, \ldots, a_n)$ (which for instance is always the case if at least one G_i is equal to G), then $(a_1, \ldots, a_n) \in \mathrm{Exe}^{>}(f)$. $\diamondsuit$

Now we are able to state our second criterion for spherical continuity of additive maps.

Lemma 3.43 *Let (G, v) and (G', v') be two valued abelian groups and $f : G \to G'$ an additive map. Assume (G, v) to be spherically complete. Suppose that f satisfies*

$$\forall a \in G:\ f(a) \neq 0 \Rightarrow \exists b \in G \setminus \mathrm{Exe}^{\geq,>}(f):\ v'(f(a) - f(b)) > v'f(a)\ . \tag{3.18}$$

Then for every $a \in G$ there is some $\tilde{a} \in G \setminus \mathrm{Exe}^{\geq}(f)$ such that $f(a) = f(\tilde{a})$. Further, f is spherically continuous, and $f(G)$ is spherically complete and thus a nice subgroup of (G', v').

Proof: Let $a \in G$. We want to show the existence of $\tilde{a} \in G \setminus \mathrm{Exe}^{\geq}(f)$ such that $f(a) = f(\tilde{a})$. By induction (which stops if we happen to find $\tilde{a}$), we construct a strictly ascending sequence of values $\gamma_\nu \in v'G'$ and a descending chain of balls $B_\nu = B_{\beta_\nu}(b_\nu)$ in G such that

a) $f(B_\nu) \subset B_{\gamma_\nu}(f(a))$,
b) $v'(f(a) - f(b_\nu)) > \gamma_\nu$,
c) $\beta_\nu = vb'_\nu$ and $\gamma_\nu = v'f(b'_\nu)$ for some $b'_\nu \in G \setminus \mathrm{Exe}^{\geq,>}(f)$.

By hypothesis, there exists some $b \in G \setminus \mathrm{Exe}^{\geq,>}(f)$ such that $v'(f(a)-f(b)) > v'f(a)$. If $f(a) = f(b)$, then $\tilde{a} := b$ is the element that we have looked for, and we are done. Otherwise,

we set $\gamma_0 := v'(f(a)-f(b))$ and choose $b'_0 \in G\setminus \mathrm{Exe}^{\geq,>}(f)$ such that $v'(f(a)-f(b)-f(b'_0)) > v'(f(a)-f(b))$. This implies $v'f(b'_0) = v'(f(a)-f(b)) = \gamma_0$. We set $\beta_0 := vb'_0$ and $b_0 := b+b'_0$, and we define $B_0 := B_{\beta_0}(b_0)$. Observe that $b \in B_0$. By our choice of b'_0, we have that $v'(f(a)-f(b_0)) = v'(f(a)-f(b)-f(b'_0)) > v'(f(a)-f(b)) = \gamma_0$. Since $b'_0 \in G\setminus \mathrm{Exe}^{\geq,>}(f)$, we obtain that $v(b_0-c) \geq \beta_0$ implies $v'(f(b_0)-f(c)) = v'f(b_0-c) \geq v'f(b'_0) = \gamma_0$, that is, $v'(f(a)-f(c)) \geq \min\{v'(f(a)-f(b_0)), v'(f(b_0)-f(c))\} \geq \gamma_0$. Hence, $f(B_0) \subset B_{\gamma_0}(f(a))$. Further, let us note the following: $b'_0 \notin \mathrm{Exe}^{\geq}(f)$ and $v'f(b'_0) = v'(f(a)-f(b)) > v'f(a) = v'f(b)$ imply that $\beta_0 = vb'_0 > vb$. This yields that $B_0 \subset B^\circ_{vb}(b)$. So if we find some element $\tilde{a} \in B_0$ such that $f(a) = f(\tilde{a})$, then we are done because $B^\circ_{vb}(b) \cap \mathrm{Exe}^{\geq,>}(f) = \emptyset$.

Assume that we have already constructed γ_ν and B_ν. If γ_ν is the maximal element of $v'G'$, then b) implies that $f(a) = f(b_\nu)$, and we are done. So let us assume that $f(a) \neq f(b_\nu)$. Then γ_ν can not be the maximal element of $v'G'$. We apply the above procedure with b_ν in the place of b and $\gamma_{\nu+1} := v'(f(a)-f(b_\nu)) > \gamma_\nu$ in the place of γ_0. We choose $b'_{\nu+1} \in G\setminus \mathrm{Exe}^{\geq,>}(f)$ such that $v'(f(a)-f(b_\nu)-f(b'_{\nu+1})) > v'(f(a)-f(b_\nu))$ and set $\beta_{\nu+1} := vb'_{\nu+1}$ and $b_{\nu+1} := b_\nu + b'_{\nu+1}$. We obtain a ball $B_{\nu+1} = B_{\beta_{\nu+1}}(b_{\nu+1})$ which satisfies $f(B_{\nu+1}) \subset B_{\gamma_{\nu+1}}(f(a))$ and $v'(f(a)-f(b_{\nu+1})) > \gamma_{\nu+1}$. Further, $b'_{\nu+1} \notin \mathrm{Exe}^{\geq,>}(f)$ and $v'f(b'_{\nu+1}) = v'(f(a)-f(b_\nu)) > \gamma_\nu = v'f(b'_\nu)$ imply that $\beta_{\nu+1} = vb'_{\nu+1} > vb'_\nu = \beta_\nu$. In view of $b_\nu \in B_{\nu+1}$, this yields that $B_{\nu+1} \subset B_\nu$.

Now assume that μ is a limit ordinal and that we have constructed γ_ν and B_ν for all $\nu < \mu$. Since G is assumed to be spherically complete, there is some $\tilde{b}_\mu \in \bigcap_{\nu<\mu} B_\nu$. This element satisfies $\gamma_\mu := v'(f(a)-f(\tilde{b}_\mu)) \geq \gamma_\nu$ for all $\nu < \mu$. If the sequence $(\gamma_\nu)_{\nu<\mu}$ is cofinal in $v'G'$, then this implies $f(a) = f(\tilde{b}_\mu)$, and we are done. So let us assume that $f(a) \neq f(\tilde{b}_\mu)$ and consequently, that $(\gamma_\nu)_{\nu<\mu}$ is not cofinal in $v'G'$. To obtain the ball B_μ, we repeat the construction of B_0 with $\tilde{b}_\mu$ in the place of b. We choose $b'_\mu \in G\setminus \mathrm{Exe}^{\geq,>}(f)$ such that $v'(f(a)-f(\tilde{b}_\mu)-f(b'_\mu)) > v'(f(a)-f(\tilde{b}_\mu))$ and set $\beta_\mu := vb'_\mu$ and $b_\mu := \tilde{b}_\mu + b'_\mu$. We obtain a ball $B_\mu = B_{\beta_\mu}(b_\mu)$ which satisfies $f(B_\mu) \subset B_{\gamma_\mu}(f(a))$ and $v'(f(a)-f(b_\mu)) > \gamma_\mu$. Further, $b'_\mu \notin \mathrm{Exe}^{\geq,>}(f)$ and $v'f(b'_\mu) = v'(f(a)-f(\tilde{b}_\mu)) \geq \gamma_{\nu+1} > \gamma_\nu = v'f(b'_\nu)$ imply that $\beta_\mu = vb'_\mu > vb'_\nu = \beta_\nu$ for all $\nu < \mu$. In view of $\tilde{b}_\mu \in B_\mu$, this yields that $B_\mu \subset B_\nu$ for all $\nu < \mu$.

Unless we happen to find some element $\tilde{a} \in B_0$ such that $f(a) = f(\tilde{a})$, our construction will be repeated until the sequence $(\gamma_\nu)_\nu$ is cofinal in $v'G'$. Then either it has a maximal element which is also the maximal element in $v'G'$, or it has no maximal element. In both cases we are done, as we have already shown. We have now proved that for every $a \in G$ there is some $\tilde{a} \in G\setminus \mathrm{Exe}^{\geq,>}(f)$ such that $f(a) = f(\tilde{a})$.

We proceed to show that f is spherically continuous. Let $\mathbf{B}' \subset \mathbf{B}(f(G))$ be a nest of balls in the subgroup $f(G)$ of (G', v'). As in the proof of Lemma 3.37, we can assume that $\mathbf{B}'$ contains no minimal ball, and we can extract a coinitial sequence $(B'_\nu)_{\nu<\lambda}$ of balls $B'_\nu = B_{\alpha_\nu}(f(a_\nu))$ in $\mathbf{B}'$ in such a way that $(f(a_\nu))_{\nu<\lambda}$ is a pseudo Cauchy sequence. Again, we obtain $v'(f(a_\nu)-f(a_{\nu+1})) < v'(f(a_{\nu+1})-f(a_\mu))$ and $v'(f(a_\nu)-f(a_\mu)) = v'(f(a_\nu)-f(a_{\nu+1}))$ whenever $\nu < \mu < \lambda$. By induction on ν, we construct balls B_ν satisfying $f(B_\nu) \subset B'_\nu$.

We set $b_0 := a_0$. By what we have already shown, there is some $\tilde{a}_0 \in G\setminus \mathrm{Exe}^{\geq,>}(f)$ such that $f(\tilde{a}_0) = f(a_1) - f(a_0)$. We set $B_0 := B_{\beta_0}(b_0)$ with $\beta_0 := v\tilde{a}_0$. Let $b_1 := b_0 + \tilde{a}_0 \in B_0$. Note that $f(b_1) = f(b_0)+f(\tilde{a}_0) = f(a_0)+f(a_1)-f(a_0) = f(a_1)$. For every $c \in B_0$, we have that $v(b_0-c) \geq v\tilde{a}_0$, so $v'(f(a_0)-f(c)) = v'(f(b_0)-f(c)) \geq v'f(\tilde{a}_0) = v'(f(a_1)-f(a_0))$, showing that $f(c) \in B'_0$ because $f(a_0), f(a_1) \in B'_0$. Hence, $f(B_0) \subset B'_0$. Suppose that we have constructed $B_\nu = B_{\beta_\nu}(b_\nu)$ containing $b_{\nu+1}$ with $f(b_{\nu+1}) = f(a_{\nu+1})$ and such that

$f(B_\nu) \subset B'_\nu$. Then we repeat the above procedure with $\nu+1$ in the place of 0 and $\nu+2$ in the place of 1 to obtain a ball $B_{\nu+1} = B_{\beta_{\nu+1}}(b_{\nu+1})$ containing $b_{\nu+2}$ with $f(b_{\nu+2}) = f(a_{\nu+2})$ and such that $f(B_{\nu+1}) \subset B'_{\nu+1}$. Further, $\tilde{a}_{\nu+1} \in G \setminus \mathrm{Exe}^{\geq,>}(f)$ and $v'f(\tilde{a}_{\nu+1}) = v'(f(a_{\nu+1}) - f(a_{\nu+2})) > v'(f(a_\nu) - f(a_{\nu+1})) = v'f(\tilde{a}_\nu)$ imply that $\beta_{\nu+1} = v\tilde{a}_{\nu+1} > v\tilde{a}_\nu = \beta_\nu$. In view of $b_{\nu+1} \in B_\nu$, we find that $B_\nu \subset B_{\nu+1}$.

Now assume that $\mu < \lambda$ is a limit ordinal and that we have constructed such balls B_ν for all $\nu < \mu$. Since G is assumed to be spherically complete, there is some $a'_\mu \in \bigcap_\nu B_\nu$. This element satisfies $f(a'_\mu) \in B'_\nu$ for all $\nu < \mu$. We choose $\tilde{b}_\mu \in G \setminus \mathrm{Exe}^{\geq,>}(f)$ such that $f(\tilde{b}_\mu) = f(a_\mu) - f(a'_\mu)$. Then $b_\mu := a'_\mu + \tilde{b}_\mu \in \bigcap_\nu B_\nu$ for all $\nu < \mu$. Indeed, $v'f(\tilde{b}_\mu) = v'(f(a'_\mu) - f(a_\mu)) \geq \min\{v'(f(a'_\mu) - f(a_{\nu+1})), v'(f(a_{\nu+1}) - f(a_\mu))\} = \min\{v'(f(a'_\mu) - f(b_{\nu+1})), v'(f(b_{\nu+1}) - f(b_\mu))\} = v'(f(b_{\nu+1}) - f(b_{\nu+2})) > v'(f(b_\nu) - f(b_{\nu+1}))$, for all $\nu < \mu$. In view of $\tilde{b}_\mu \in G \setminus \mathrm{Exe}^{\geq,>}(f)$, this implies $v\tilde{b}_\mu > v(b_\nu - b_{\nu+1}) \geq \beta_\nu$. Hence, b_μ is contained in B_ν for every $\nu < \mu$, showing that $b_\mu \in \bigcap_{\nu<\mu} B_\nu$. Now we repeat the above procedure with b_μ in the place of b_0 and $b_{\mu+1}$ in the place of b_1. We choose $\tilde{a}_\mu \in G \setminus \mathrm{Exe}^{\geq,>}(f)$ such that $f(\tilde{a}_\mu) = f(a_{\mu+1}) - f(a_\mu)$. We set $B_\mu := B_{\beta_\mu}(b_\mu)$ with $\beta_\mu := v\tilde{a}_\mu$. Then B_μ contains $b_{\mu+1} := b_\mu + \tilde{a}_\mu$ which satisfies $f(b_{\mu+1}) = f(a_{\mu+1})$, and we have that $f(B_\mu) \subset B'_\mu$. Further, $\tilde{a}_\mu \in G \setminus \mathrm{Exe}^{\geq,>}(f)$ and $v'(f(\tilde{a}_\mu)) = v'(f(a_{\mu+1}) - f(a_\mu)) > v'(f(a_{\nu+1}) - f(a_\nu)) = v'(f(b_{\nu+1}) - f(b_\nu))$ imply that $\beta_\mu = v\tilde{a}_\mu > v(b_{\nu+1} - b_\nu) = \beta_\nu$ for all $\nu < \mu$. In view of $b_\mu \in B_\nu$, this shows that $B_\mu \subset B_\nu$ for all $\nu < \mu$. We have now proved that f is spherically continuous, which also gives the remaining assertions of our lemma. □

Let us illustrate our results by treating a special example. Suppose that $f : G \to G$ is an additive map which satisfies

$$\forall a \in G : \; vf(a) \geq va \; . \tag{3.19}$$

Natural examples for such maps are obtained if we view a valued abelian group as a $\mathbb{Z}$-module. Indeed, for every $n \in \mathbb{Z}$ we have that $vna \geq va$ by (VZ). More generally, we have the same situation for the scalar multiplication of component-compatibly valued modules, which we will introduce in the next chapter. There, we will also show that by virtue of (3.19), the additive map f induces an additive map f_α on the α-component of (G, v), for all $\alpha \in vG$.

Now assume that $b \in G$ such that $vf(b) = vb$. If $a \in G$ with $va \geq vb$, then $vf(a) \geq va \geq vb = vf(b)$, showing that $b \notin \mathrm{Exe}^{\geq}(f)$. Similarly, if $va > vb$, then $vf(a) \geq va > vb = vf(b)$, showing that $b \notin \mathrm{Exe}^{>}(f)$. So if we assume that

$$\forall a \in G : \; f(a) \neq 0 \; \Rightarrow \; \exists b \in G : \; vb = vf(b) \; \wedge \; v(f(a) - f(b)) > vf(a) \; ,$$

then condition (3.18) of the foregoing lemma is satisfied. Hence if (G, v) is spherically complete, then f will be spherically continuous and $f(G)$ will be spherically complete.

Now let us assume that f satisfies a little bit more:

$$\forall d \in G : \; d \neq 0 \; \Rightarrow \; \exists b \in G : \; vb = vf(b) \; \wedge \; v(d - f(b)) > vd \; .$$

Then we obtain that f also satisfies condition (3.16) of Lemma 3.40, for every $c \in G$. Indeed, if we set $d = f(b') - c$ and choose b such that $v(d - f(b)) > vd$, then $v(f(b') - c) = vd < v(d - f(b)) = v(f(b' - b) - c)$. We conclude from Lemma 3.40 that $f(G) = G$. Observe

that $vb = vf(b) \wedge v(d - f(b)) > vd$ just means that $vb = vd$ and $f_{vd}(\mathrm{co}_{vd} b) = \mathrm{co}_{vd} d$. That is, the above condition just says that every f_α is surjective. We have derived a result which is due to N. Alling [ALL2]:

Theorem 3.44 *Let (G, v) be a spherically complete valued abelian group and $f : G \to G$ an additive map which satisfies $vf(a) \geq va$ for all $a \in G$. If the maps induced by f on the components of (G, v) are all surjective, then f is surjective.*

If we choose $n \in \mathbb{Z}$ and take f to be the additive map $a \mapsto na$, then we obtain:

Corollary 3.45 *Let (G, v) be a spherically complete valued abelian group and $n \in \mathbb{N}$. If all components of (G, v) are divisible by n, then so is G.*

Consider an abelian p-group with its p-height valuation. If n is a natural number prime to p, then multiplication by n preserves the value and is thus a spherically continuous group endomorphism. On the other hand, if also multiplication by p is spherically continuous, then also multiplication by np^m is spherically continuous, for every natural number m. Hence in that case, multiplication by every natural number is spherically continuous.

Open Problem 3.1 Is multiplication by p a spherically continuous map on every abelian p-group (with its p-height valuation)? If not, give criteria for this to hold. What happens if we replace the p-group by some maximal immediate extension?

Exercise 3.13 *a) In the hypothesis of Lemma 3.43, replace the condition that (G, v) be spherically complete by the condition that $v'G'$ be countable. Further, replace "$v'(f(a) - f(b)) > v'f(a)$" by "$f(a) = f(b)$". Prove that again, f is spherically continuous. (Hint: the cofinality type of every initial segment of $v'G'$ is at most ω.)*

b) Show that multiplication by p is spherically continuous with respect to the p-height valuation if the p-group G has countable value set.

3.8 Sums of spherically complete subgroups

We are interested in a generalization of the last assertion of Corollary 3.40. What happens if for $1 \leq i \leq n$, the groups (G_i, v_i) are spherically complete and $f_i : G_i \to G'$ are spherically continuous additive maps such that the extension

$$(f_1(G_1) + \ldots + f_n(G_n), v') \subset (G', v')$$

is immediate? Indeed, if we would know that the subgroup $(f_1(G_1) + \ldots + f_n(G_n), v')$ is spherically complete, then we would know that the extension is trivial. So we are looking for a criterion which guarantees that the sum of finitely many spherically complete subgroups is again spherically complete. By use of the product valuation, we can derive a first, quite important criterion. It works equally well for subsets $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of a group G. We will call $\mathcal{S}_1, \ldots, \mathcal{S}_n$ **valuation independent** if for every sum $\sum_{i=1}^n a_i$ with $a_i \in \mathcal{S}_i$, we have that

$$v \sum_{i=1}^n a_i = \min_{1 \leq i \leq n} va_i \ .$$

In particular, this criterion implies that if $\sum a_i = 0$ for $a_i \in \mathcal{S}_i$, then $a_i = 0$ for all i. Indeed, if $a_j \neq 0$ for some j, then $v\sum_i a_i = \min_i va_i \leq va_j < \infty$, showing that $\sum a_i \neq 0$. Moreover, if $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are valuation independent, then the map

$$\mathcal{S}_1 \times \ldots \times \mathcal{S}_n \ni (a_1, \ldots, a_n) \ \mapsto \ a_1 + \ldots + a_n \in G$$

is a valuation preserving map from the product space $(\prod_i \mathcal{S}_i, \prod_i v)$ onto the sum $\mathcal{S}_1 + \ldots + \mathcal{S}_n$, because

$$(\prod_i v)(a_1, \ldots, a_n) = \min_i va_i = v(a_1 + \ldots + a_n) \ .$$

Since valuation preserving maps preserve spherical completeness, we can use Lemma 2.10 and Lemma 2.16 to conclude:

Lemma 3.46 *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be valuation independent subsets of the valued abelian group (G, v). If $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are spherically complete, then so is their sum $\mathcal{S}_1 + \ldots + \mathcal{S}_n$. If in addition, for every element $a \in G$ there are $a_i \in \mathcal{S}_i$ such that $v(a - \sum_{i=1}^n a_i) > va$, then $G = \mathcal{S}_1 + \ldots + \mathcal{S}_n$.*

This lemma gives no information about the sum of subgroups which are not valuation independent. But we will need such a criterion when we will consider the action of additive polynomials on valued fields of characteristic > 0. The following corollary to Lemma 3.43 gives a more general criterion for the sum of spherically complete subgroups to be spherically complete.

Corollary 3.47 *Let (G_i, v), $1 \leq i \leq n$, be spherically complete subgroups of the valued abelian group (G, v). Assume that for every element a in $G_1 + \ldots + G_n$ there are elements $a_i \in G_i$ of value $va_i \geq va$, $1 \leq i \leq n$, such that $v(a - \sum_{i=1}^n a_i) > va$. Then also the subgroup $(G_1 + \ldots + G_n, v)$ is spherically complete.*

Proof: By Lemma 2.10, the product $(\prod_{1\leq i\leq n} G_i, \prod_{1\leq i\leq n} v)$ is spherically complete. Let $f : \prod_i G_i \to G$ be the additive map $(a_1, \ldots, a_n) \mapsto a_1 + \ldots + a_n$. From the example preceding Lemma 3.43, we now see that the hypothesis of our corollary implies that f satisfies the hypothesis of Lemma 3.43. Hence, the assertion follows from that lemma. □

From Lemma 3.43 and the foregoing corollary, we now obtain the desired generalization of the last assertion of Corollary 3.40.

Corollary 3.48 *Let $G_1, \ldots, G_n$ be spherically complete subgroups of (G, v) and $f_i : G_i \to G'$ spherically continuous group homomorphisms. If for every $a \in f_1(G_1) + \ldots + f_n(G_n)$,*

$$\exists b_i \in G_i : \ vf_i(b_i) \geq va \ (1 \leq i \leq n) \quad \textit{and} \quad v(f_1(b_1) + \ldots + f_n(b_n) - a) > va \ , \qquad (3.20)$$

then $f_1(G_1) + \ldots + f_n(G_n)$ is spherically complete and a nice subgroup of (G', v'). If (3.20) holds for every $a \in G'$, then

$$G' = f_1(G_1) + \ldots + f_n(G_n) \ .$$

If H' is a v'-convex subgroup of (G', v') and (3.20) holds for every $a \in G'$ with $v'a < v'H'$, then

$$G' = f_1(G_1) + \ldots + f_n(G_n) + H' \ .$$

Open Problem 3.2 Is the sum of spherically complete subgroups always spherically complete? Under which conditions is a sum of nice subgroups again nice?

Chapter 4

Valued modules and vector spaces

4.1 Basic definitions

Let R be a ring, not necessarily commutative. **By an R-module, we will always mean a left R-module**. A **(general) valued R-module** (M, v) is a valued group (M, v) which also carries an R-module structure (with $(M, +)$ as its additive group). Every $r \in R$ can be viewed as an endomorphism of the group $(M, +)$; however, it may not respect the valuation or the coefficient map. On purpose, our definition does not say anything about the compatibility between valuation and module structure. So it enables us to view a valued abelian group (G, v) as a valued module of the endomorphism ring over the endomorphism ring of G, and it is general enough to permit a study of the different possible compatibility properties. In almost all cases that have been studied in the literature, valued modules satisfy the following compatibility axiom

(VM$\geq$) $\quad \forall a \in M \, \forall r \in R : \; vra \geq va \; .$

A valued module satisfying this axiom will be called **component-compatibly valued**. Every valued abelian group is a component-compatibly valued $\mathbb{Z}$-module. Indeed, as every abelian group is a $\mathbb{Z}$-module, every valued abelian group is a valued $\mathbb{Z}$-module. Indeed, for $R = \mathbb{Z}$, property (VM$\geq$) is the same as the law (VZ), which we have deduced from the ultrametric triangle law. As we will see later, this does not mean that (VM$\geq$) remains true for larger rings.

The following lemma shows the reason for the notion "component-compatibly valued"; its proof is straightforward:

Lemma 4.1 *Let $\mathbf{M} = (M, v)$ be a component-compatibly valued R-module. Then for every initial segment γ of vM,*

$$R\mathcal{O}_{\mathbf{M}}^{\gamma} \subset \mathcal{O}_{\mathbf{M}}^{\gamma} \quad \textit{and} \quad R\mathcal{M}_{\mathbf{M}}^{\gamma} \subset \mathcal{M}_{\mathbf{M}}^{\gamma} \; .$$

Hence, $\mathcal{O}_{\mathbf{M}}^{\gamma}$, $\mathcal{M}_{\mathbf{M}}^{\gamma}$ and $\mathrm{C}^{\gamma}\mathbf{M} = \mathcal{O}_{\mathbf{M}}^{\gamma}/\mathcal{M}_{\mathbf{M}}^{\gamma}$ are R-modules, and co_{γ} is R-linear, i.e.

$$\forall r \in R \, \forall a \in M : \; r \, \mathrm{co}_{\gamma} a = \mathrm{co}_{\gamma} ra \; .$$

In particular, for every $a \in M$ the annihilator of a is contained in the annihilator of $\mathrm{co}\, a$.

If we are given a skeleton $\mathrm{sk}_{\gamma\in\Gamma}C_\gamma$ where all C_γ are R-modules, then in a natural way, also Hahn sum and Hahn product over this skeleton are component-compatibly valued R-modules.

The above lemma has shown that every component of a component-compatibly valued R-module $\mathbf{M}$ is an R-module, which in particular means that every $r \in R$ induces an endomorphism of every component. Analogously, we will say that $\mathbf{M}$ is **value-compatible** if every $r \in R$ induces an $\leq$-preserving map of vM into itself, given by $va \mapsto vra$ for $0 \neq a \in M$. Using the terminology introduced in Section 3.6, $\mathbf{M}$ is value-compatible if every $r \in R$ induces a value-compatible map on $\mathbf{M}$. In a value-compatibly valued module, the value of vra only depends on va and not on $\mathrm{co}\, a$. Observe that for every invertible $r \in R$, the induced map $va \mapsto vra$ will then be an order preserving bijection. An important example for such valued modules is given by extensions $(L, v) \supset (K, v)$ of valued fields (see Part II). Here, (L, v) may be considered as a value-compatibly valued K-vector space. It is then of interest to study the valued K-subvector spaces of (L, v).

To classify a larger but still well-behaved class of valued modules, we invoke the skeleton. In short, we want to talk about those valued modules in which the scalar multiplication induces maps on the skeleton (or at least on almost all elements of the skeleton). Given a valued R-module $\mathbf{M}$ and any map f of M into itself, a bone $(\alpha, \zeta) \in \mathrm{sk}\,\mathbf{M}$ will be called an **exceptional bone for** f if there exist two elements a, b of M both having (α, ζ) as their bone, such that $f(a)$ and $f(b)$ have distinct bones. Hence f induces a map on all bones that are not exceptional for f. If $(va, \mathrm{co}\, a)$ is an exceptional bone for f, then a is called an **exceptional element for** f, and va is called an **exceptional value for** f. Further, $\mathrm{Exb}_f\mathbf{M}$ will denote the set of all exceptional bones for f, $\mathrm{Exe}_f\mathbf{M}$ will denote the set of all exceptional elements for f, and $\mathrm{Exv}_f\mathbf{M}$ will denote the set of all exceptional values for f. Every element of the ring R can be identified with the endomorphism that it induces on M. This leads to the following definitions. A bone will be called **exceptional bone of** $\mathbf{M}$ if it is exceptional for some $r \in R$. The set of all exceptional bones of $\mathbf{M}$ will be denoted by $\mathrm{Exb}\mathbf{M}$. Further, an element of $\mathbf{M}$ is called **exceptional element of** $\mathbf{M}$ if it is exceptional for some $r \in R$, and the set of all exceptional elements of $\mathbf{M}$ is denoted by $\mathrm{Exe}\mathbf{M}$. Similarly, we define the **exceptional values of** $\mathbf{M}$ and the set $\mathrm{Exv}\mathbf{M}$.

Example 4.2 The exceptional elements in the valued $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}\amalg\mathbb{Z}/3\mathbb{Z}$ are precisely $(1,0)$, $(1,1)$ and $(1,2)$. All have the same bone, but we have $2(1,0) = (0,0)$, whereas $2(1,1) \neq (0,0)$ and $2(1,2) \neq (0,0)$. On the other hand, the trivially valued $\mathbb{Z}$-module $\mathbb{Z}/3\mathbb{Z}$ has no exceptional elements and correspondingly, the elements $(0,0)$, $(0,1)$ and $(0,2)$ of $\mathbb{Z}/2\mathbb{Z}\amalg\mathbb{Z}/3\mathbb{Z}$ are not exceptional. If Γ is any ordered set and p a prime, then $\coprod_\Gamma \mathbb{Z}/p\mathbb{Z}$ is a valued $\mathbb{Z}$-module and a valued $\mathbb{Z}/p\mathbb{Z}$-vector space without exceptional elements. $\diamondsuit$

Bones, elements or values which are not exceptional will be called **ordinary**. A valued module without exceptional elements will be called an **ordinary valued module**. For example, an ordered abelian group equipped with its natural valuation is an ordinary valued $\mathbb{Z}$-module. Also, the valued field extensions mentioned above are ordinary valued vector spaces. The following is an important characterization of ordinary valued modules:

Lemma 4.3 *Let* $\mathbf{M}$ *be a valued* R*-module. Then* $a \in M$ *is ordinary for* $r \in R$ *if and only if*

$$va < va' \implies vra < vra' \vee ra = ra' = 0 \tag{4.1}$$

for every $a' \in M$. Hence, $\mathbf{M}$ *is ordinary if and only if (4.1) holds for all $a, a' \in M$ and every $r \in R$. Further,* $\mathrm{Exe}_r\mathbf{M}$ *is the set of all $a \in M$ for which (4.1) does not hold for all $a' \in M$.*

Proof: Suppose that $a \in M$ is ordinary for $r \in R$ and let $a' \in M$ such that $ra' \neq 0$ and $va < va'$. Then by (3.4), a and $b := a' + a$ have the same bone. By assumption, it follows that ra and rb have the same bone. Since $ra' \neq 0$, this implies that $ra \neq 0$ and again by (3.4), $vra < v(ra - rb) = v(-ra') = vra'$.

For the converse, let $a \neq 0$ and b have the same bone. Then by (3.4), $va < va'$ for $a' := a - b$. If this implies that $vra < vra'$, then $vra < v(ra - rb)$, and (3.4) shows that ra and rb have the same bone. □

A map f on the valued R-module $\mathbf{M}$ will be called a **finitely exceptional map** if there are only finitely many exceptional bones for f on $\mathbf{M}$. So $\mathbf{M}$ will be called a **finitely exceptional valued module** if every $r \in R$ has only finitely many exceptional bones on $\mathbf{M}$. In the same spirit, $\mathbf{M}$ will be called an **almost value-compatible valued module** if every $r \in R$ is an almost value-compatible map on M (the latter notion has been introduced in Section 3.6). At present, these definitions may seem a bit weird. But for the study of spherical completeness and spherical continuity, the finitely exceptional and the almost value-compatible modules provide an appropriate generalization of the notions of ordinary and value-compatible modules. Moreover, from the model theoretical point of view, finitely many "exceptions" are not too bad since they can be pinned down by axioms. We will meet our most important example of finitely exceptional almost value-compatible modules in Section 11.4 when we study valued fields of characteristic $p > 0$ as valued modules over rings of additive polynomials. For the time being, it may suffice to present an infinite but finitely exceptional $\mathbb{Z}$-module by just replacing "$\mathbb{Z}/3\mathbb{Z}$" by "$\mathbb{Z}$" in our last example.

Every element $r \in R$ operates on the set $\mathrm{sk}\,\mathbf{M} \setminus \mathrm{Exb}_r\mathbf{M}$ by $(va, \mathrm{co}\, a) \mapsto (vra, \mathrm{co}\, ra)$. Recall that two bones (α_1, ζ_1) and (α_2, ζ_2) can be added if and only if they lie in the same component (i.e., $\alpha_1 = \alpha_2$). So we will call a finite formal sum $\sum_i r_i(\alpha_i, \zeta_i)$ with $r_i \in R$ an **admissible linear combination** if for every i, the bone (α_i, ζ_i) is not exceptional for r_i and if all bones $r_i(\alpha_i, \zeta_i)$ lie in the same component. If $\mathcal{B} \subset \mathbf{M}$, then $\underline{\mathcal{B}} := \{(vb, \mathrm{co}\, b) \mid b \in \mathcal{B}\}$ will be the set of the bones of all elements of $\mathcal{B}$. Sets of bones will be denoted by $\underline{\mathcal{B}}$ since it is always possible to find $\mathcal{B} \subset \mathbf{M}$ such that $\underline{\mathcal{B}}$ is precisely the set of bones of all elements in $\mathcal{B}$.

A valued group or module is called **spherically complete** resp. **complete** if it has these properties as an ultrametric space.

Remark 4.4 A special type of valued vector spaces is studied in non-archimedean analysis. Here, the value set is a subset of the reals, the field K carries a valuation whose value group is a subgroup of the reals, and on the valued K-vector space (V, v), the scalar multiplication acts as follows: $vrx = vr + vx$ for all $r \in K$ and all $x \in V$. (In a similar way, the scalar multiplication bevaves for the valued vector spaces that we will derive from valued field extensions.) For details on these **real-valued vector spaces**, see [GRU1], [MON], [VDP] and their references. In particular, A. W. Ingleton [ING] shows that real-valued vector spaces satisfy the Hahn-Banach Theorem if and only if they are spherically complete.

An **embedding** (resp. **isomorphism**) **of valued modules** is an embedding (resp. isomorphism) of valued abelian groups which also preserves the module structure. An **extension of valued modules** is an extension of valued groups which at the same time is an extension of modules; in this case, we will write $\mathbf{M} \subset \mathbf{N}$ and call $\mathbf{M}$ a **valued**

submodule of N. We will say that $\mathbf{M} \subset \mathbf{N}$ is a **finite extension** (or that **N** is **finitely generated over M**) if N/M is finitely generated as an R-module. An extension of valued modules is called **immediate** if it is immediate as an extension of valued groups. Thus, the same cardinality bound as for valued groups also holds for valued modules, showing that

Lemma 4.5 *Every valued module admits some maximal immediate extension.*

Such maximal immediate extension are **maximal modules** in the sense that they do not admit proper immediate module extensions. But note that for $R \neq \mathbb{Z}$ it is not a priori clear whether every maximal R-module is also a maximal group. In the next section, we will show that Hahn products are always maximal.

Exercise 4.1 *Let Γ be an ordered set, n_γ a natural number for every $\gamma \in \Gamma$ and $C_\gamma = \mathbb{Z}/n_\gamma\mathbb{Z}$. Give conditions on the numbers n_γ and the set Γ which yield that $\coprod_{\gamma\in\Gamma} C_\gamma$ is an ordinary $\mathbb{Z}$-module. More generally, give conditions on the components and the value set of a valued R-module which yield that it is ordinary. (Hint: consider annihilators.)*

4.2 Nice submodules

We will say that **M** is a **nice submodule of N** or just **nice in N** if as an ultrametric space, it is nice in **N**. This is equivalent to the condition that for every $b \in N \setminus M$, the subset $v(b+M) = \{v(b+c) \mid c \in M\} = \{v(b-c) \mid c \in M\}$ of vN admits a maximum. If vb is this maximum, then b is called a **proper representative for the coset** $b+M$. By equivalence (3.6) on page 57 we see that b is a proper representative for the coset $b+M$ if and only if $(vb, \operatorname{co} b) \notin \operatorname{sk} \mathbf{M}$.

Remark 4.6 We would have preferred to use the name "nice representative" (in particular in view of our use of the word "proper" as explained in section 1.3). But the notion "proper representative" is the one to be found in the literature.

Note that **M** is a nice submodule of **N** if and only if for every $b \in N$, **M** is a nice submodule of $\mathbf{M} + Rb$. If **M** is a nice submodule of **N**, then it is also a nice submodule of every $\mathbf{N}'$ such that $\mathbf{M} \subset \mathbf{N}' \subset \mathbf{N}$. From Lemma 2.15 we know that both properties "nice in" and "immediate extension" are transitive. From Lemma 2.16 and Corollary 3.10, we obtain:

Theorem 4.7 *A spherically complete valued module is nice in every valued module extension and in particular, it is maximal.*

Together with Lemma 3.14, this gives:

Theorem 4.8 *Every Hahn product is maximal.*

Remark 4.9 It is an interesting question to ask whether the converse of Theorem 4.7 holds: is every maximal module spherically complete? Or at least: does every valued module admit some immediate extension which is a spherically complete module? For certain classes of valued modules, this is known to be true. If a valued abelian group is embedded inbetween the Hahn sum and the Hahn product over its skeleton, then the Hahn product is an immediate spherically complete extension. We will show that component-compatibly valued vector spaces can be embedded in such a way (this is the Hahn Embedding

Theorem for valued vector spaces, cf. Theorem 4.52). Moreover, by a simple model theoretic construction, it can be shown that every maximal ordinary valued vector space is spherically complete.

According to I. Fleischer [FL1], every maximal component-compatibly valued R-module is spherically complete if R is a principal ideal ring. (We will prove this result in Section 4.8 below.) In the same article, an example is given showing that for valued modules, the maximal immediate extensions are not necessarily unique if torsion interferes. It is essentially the example that we have given in 3.15. The groups that we have constructed there are valued $\mathbb{Z}$-modules. The group H of that example has two nonisomorphic maximal immediate extensions, one of which is even a valued $\mathbb{F}_2$-vectorspace like **H** while the other is not.

For valued modules with value preserving scalar multiplication (cf. section 4.4), N. Sankaran and R. Venkataraman [SAN–VE] have shown that there is always a maximal immediate extension which is spherically complete. For component-compatibly valued modules in general, they are still able to show the existence of what they call an **almost ultracomplete** immediate extension.

For valued fields, the above question is answered to the affirmative by Ostrowski's and Kaplansky's theory of immediate extensions of valued fields; see section 7.10.

The quotient by a nice submodule can be endowed with a canonical valuation, which we will call the **quotient valuation**:

Lemma 4.10 *Suppose that* $\mathbf{M}$ *is a nice submodule of* $\mathbf{N}$*, and set*

$$w(b+M) := \max_{c\in M} v(b-c) \quad \text{for every } b \in N\,.$$

Then w is well-defined, and it is a valuation of N/M. Further, $w(N/M) \subset vN$, and $w(b+M) = vb$ holds if and only if b is a proper representative. If v is component-compatible, then so is w.

Proof: Since $\max_{c\in M} v(b-c)$ does not depend on the representative b of the coset $b+M$, it follows that w is well-defined. We have $w(0+M) = \max_{c\in M} v(0-c) = \infty$, but if $b \notin M$, then $0 \notin b+M$ and $w(b+M) = \max_{c\in M} v(b-c) \neq \infty$; the maximum exists by our assumption that $\mathbf{M}$ be nice in $\mathbf{N}$. By definition, b is a proper representative if and only if this maximum is equal to vb. To prove that $w(b+b'+M) \geq \min\{w(b+M), w(b'+M)\}$, we can assume without loss of generality that b, b' are proper representatives. Then

$$\begin{aligned} w(b+b'+M) &= \max_{c\in M} v(b+b'-c) \;\geq\; v(b+b') \\ &\geq \min\{vb, vb'\} \;=\; \min\{w(b+M), w(b'+M)\}\,. \end{aligned}$$

(Note that in general, $b+b'$ is not a proper representative of $b+b'+M = (b+M)+(b'+M)$.) If we assume in addition that v is component-compatible, then we obtain for every $r \in R$:

$$w(rb+M) \geq vrb \geq vb = w(b+M)\,,$$

which shows that w is component-compatible. □

Lemma 4.11 *Let* $\mathbf{M} \subset \mathbf{N}' \subset \mathbf{N}$ *be valued modules with* $\mathbf{M}$ *nice in* $\mathbf{N}$.
a) $\mathbf{N}'$ *is nice in* $\mathbf{N}$ *if and only if* $(N'/M, w)$ *is nice in* $(N/M, w)$.
b) $\mathbf{N}' \subset \mathbf{N}$ *is immediate if and only if* $(N'/M, w) \subset (N/M, w)$ *is.*

Proof:
a): $\Rightarrow$: Suppose that $\mathbf{N}'$ is nice in $\mathbf{N}$. Then for every $b \in N$ there is $b' \in N'$ such that $v(b-b') = \max_{c'\in N'} v(b-c')$. Hence, $w(b-b'+M) = \max_{c\in M} v(b-b'-c) = v(b-b') =$

$\max_{c'\in N'} \max_{c\in M} v(b-c'-c) = \max_{c'\in N'} w(b-c'+M)$, showing that $(N'/M, w)$ is nice in $(N/M, w)$.
$\Leftarrow$: Suppose that $(N'/M, w)$ is nice in $(N/M, w)$. Then for every every $b \in N$ there is $b' \in N'$ (which we can choose to be a proper representative of $b'+M$) such that $v(b-b') = w(b-b'+M) = \max_{c'\in N'} w(b-c'+M) = \max_{c'\in N'} \max_{c\in M} v(b-c'-c) = \max_{c'\in N'} v(b-c')$, showing that $\mathbf{N}'$ is nice in $\mathbf{N}$.

b): $\Rightarrow$: Suppose that $\mathbf{N}' \subset \mathbf{N}$ is immediate. We show that for every $b \in N$ there exists some $b' \in N'$ such that $w(b-b'+M) > w(b+M)$. We assume that b is a proper representative of $b+M$. By hypothesis, there exists some $b' \in N'$ such that $v(b-b') > vb$. This implies that $w(b-b'+M) = \max_{c\in M} v(b-b'-c) \geq v(b-b') > vb = w(b+M)$.
$\Leftarrow$: Suppose that $(N'/M, w) \subset (N/M, w)$ is immediate, and let $b \in N$. There exists some $c' \in N'$ such that $w(b-c'+M) > w(b+M)$, i.e. $\max_{c\in M} v(b-c'-c) > \max_{c\in M} v(b-c) \geq vb$. Hence there exists $c \in M$ such that $v(b-c'-c) > vb$, so $b' = c'+c \in N'$ is an element which satisfies $v(b-b') > vb$. □

Exercise 4.2 *Assume that* $\mathbf{M}$ *is nice in* $\mathbf{N}$.
a) Show that the converse of the last assertion of Lemma 4.10 does not hold in general, that is, w *may be component-compatible while* v *is not.*
b) Assume that for every $r \in R$*, if* b *is a proper representative of* $b+M$ *then* rb *is a proper representative of* $rb+M$*. Use Lemma 4.3 to show that* $(N/M, w)$ *is ordinary if* $\mathbf{N}$ *is ordinary. Does this also hold without our additional assumption? Show that the assumption holds if* R *is a (skew) field.*

4.3 Valuation independence

In this section, we will consider the following situation:

R	a ring
$\mathbf{N} = (N, v)$	a valued R-module
$\mathbf{M}$	an R-submodule of $\mathbf{N}$, equipped with the restriction of v (which we will again denote by v).

If $\mathcal{B}$ is a subset of N, then we let $M + R\mathcal{B}$ denote the submodule of N and $\mathbf{M} + R\mathcal{B}$ the valued submodule of $\mathbf{N}$ generated over M by the elements of $\mathcal{B}$. Similarly, if $\underline{\mathcal{B}}$ is a subset of $\operatorname{sk}\mathbf{N}$, then $\operatorname{sk}\mathbf{M} + R\underline{\mathcal{B}}$ denotes the subskeleton of $\operatorname{sk}\mathbf{N}$ generated over $\operatorname{sk}\mathbf{M}$ by the elements of $\underline{\mathcal{B}}$, that is, it is the set of all admissible linear combinations of bones in $\operatorname{sk}\mathbf{M} \cup \underline{\mathcal{B}}$. If $\mathbf{N}$ is component-compatibly valued, then for every $\alpha \in vN$, the α-component of $\operatorname{sk}\mathbf{M} + R\underline{\mathcal{B}}$ is just the R-module generated over the α-component of $\operatorname{sk}\mathbf{M}$ by $\{\zeta \mid (\alpha, \zeta) \in \underline{\mathcal{B}}\}$.

A subset $\mathcal{B} \subset N$ is called **R-independent over $\mathbf{M}$** if $0 \notin \mathcal{B}$ and for every choice of finitely many $r_i \in R$, $b_i \in \mathcal{B}$, where we let i range over some finite index I, and for every $a \in M$,

$$\sum_{i\in I} r_i b_i + a = 0 \implies \forall i \in I:\ r_i b_i = 0 \ \text{ and } \ a = 0\ .$$

Note: whenever we write "$\sum r_i b_i$ with $b_i \in \mathcal{B}$", then we implicitly mean that all b_i are distinct.

We will say that $\mathcal{B} \subset N$ is **R-valuation independent over $\mathbf{M}$** if $0 \notin \mathcal{B}$ and for every choice of finitely many $r_i \in R$, $b_i \in \mathcal{B}$, $i \in I$, and every $a \in M$,

$$v\left(\sum_{i\in I} r_i b_i + a\right) = \min_{i\in I}\{vr_i b_i, va\}\ .$$

Further, $\mathcal{B} \subset N$ is called **R-valuation independent** if it is R-valuation independent over the zero module. Recall that we have already defined the notion of valuation independent subsets of a valued abelian group. Using this notion, our above definition reads as follows: $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$ if and only if $0 \notin \mathcal{B}$ and for every n, every choice of $b_1, \ldots, b_n \in \mathcal{B}$ and every $a \in M$, the submodules $Rb_1, \ldots, Rb_n, M$ are valuation independent in $\mathbf{N}$. The reader may prove the following characterization:

Lemma 4.12 *A subset $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if every finite subset of $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$. A finite subset $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if the ultrametric space $(M + R\mathcal{B}, v)$ is the product of the ultrametric spaces (M, v) and (Rb, v), $b \in \mathcal{B}$.*

If $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$, then it is R-independent over $\mathbf{M}$ since otherwise, there would exist finitely many $r_i \in R$, $b_i \in \mathcal{B}$, $i \in I$, and $a \in M$ such that some $r_i b_i \neq 0$ and

$$v\left(\sum_{i\in I} r_i b_i + a\right) = \infty \neq \min_{i\in I}\{vr_i b_i, va\} \ .$$

This motivates the next definition. If a set $\mathcal{B}$ of generators of the extension $\mathbf{M} \subset \mathbf{N}$ is R-valuation independent over $\mathbf{M}$, then it will be called an **R-valuation basis of $\mathbf{N}$ over $\mathbf{M}$** or an **R-valuation basis of $\mathbf{M} \subset \mathbf{N}$**. Further, $\mathbf{M} \subset \mathbf{N}$ will be called a **defectless extension of valued modules** if every finite subextension admits an R-valuation basis over $\mathbf{M}$. Note that this does not imply that $\mathbf{N}$ itself admits an R-valuation basis over $\mathbf{M}$. For instance, the extension $0 \amalg \mathbb{Z} \subset \mathbb{Q} \amalg \mathbb{Z}$ is a defectless extension of valued $\mathbb{Z}$-modules, but $\mathbb{Q} \amalg \mathbb{Z}$ does not even admit a $\mathbb{Z}$-basis over $0 \amalg \mathbb{Z}$. Later, we will meet examples of another type (cf. Examples 4.63). Note that it is not really necessary to exclude 0 from R-independent and R-valuation independent sets. We have done that in view of the above definition of an R-valuation basis and our later definition of the valuation basis of an extension of valued fields, since a "basis" is usually not supposed to contain 0. However, with the above definition, it may still contain elements $b \in N$ such that $Rb = \{0\}$, but this is only possible if R does not contain 1.

Lemma 4.13 *The set $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if for every finite sum $b = \sum_{i\in I} r_i b_i + a$ of elements $b_i \in \mathcal{B}$ with $r_i \in R$ and $a \in M$,*

$$\min_{i\in I} vr_i b_i \ = \ \max_{c\in M} v(b-c) \ . \tag{4.2}$$

If this equation holds, then $v(b-a) = \max_{c\in M} v(b-c)$.

Proof: Suppose first that $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$. Then for every $c \in M$, $v(b-a) = \min_{i\in I} vr_i b_i \geq \min_{i\in I}\{vr_i b_i, v(a-c)\} = v(b-c)$, which yields equation (4.2). For the converse, suppose that (4.2) holds for all b. Let $\gamma = \min_{i\in I} vr_i b_i$. If $v(a-c) < \gamma$, then

$$v\left(\sum_{i\in I} r_i b_i + a - c\right) = v(a-c) = \min_{i\in I}\{vr_i b_i, v(a-c)\} \ . \tag{4.3}$$

Now let $v(a-c) \geq \gamma$. Then by (4.2), $v(b-c) \leq \min_{i\in I} vr_i b_i = \min_{i\in I}\{vr_i b_i, v(a-c)\} \leq v(b-c)$, hence equality holds everywhere.

Now suppose that (4.2) holds for b. Then $v(b-a) \geq \min_{i\in I} vr_i b_i = \max_{c\in M} v(b-c) \geq v(b-a)$, hence equality holds everywhere. This completes our proof. □

Corollary 4.14 *If* $\mathbf{N}$ *admits an* R*-valuation basis* $\mathcal{B}$ *over* $\mathbf{M}$*, then* $\mathbf{M}$ *is nice in* $\mathbf{N}$*. The same holds if* $\mathbf{M} \subset \mathbf{N}$ *is a defectless extension.*

Proof: First, let $\mathcal{B}$ be an R-valuation basis of $\mathbf{N}$ over $\mathbf{M}$. If $b \in N \setminus M$ then write $b = \sum_{i\in I} r_i b_i + a$ with elements $b_i \in \mathcal{B}$, $r_i \in R$, $a \in M$. Then by the previous lemma, $b - a$ is a proper representative for $b + M$.

Now assume that $\mathbf{M} \subset \mathbf{N}$ is a defectless extension. Then for every $b \in N$, the module $\mathbf{M} + Rb$ admits an R-valuation basis over $\mathbf{M}$. By what we have shown, it follows that $\mathbf{M}$ is nice in $\mathbf{M} + Rb$. Hence, $\mathbf{M}$ is nice in $\mathbf{N}$. □

Lemma 4.15 *Assume that* $\mathbf{M}$ *is nice in* $\mathbf{N}$*, and let* $\mathcal{B} \subset N$*. Then* $\mathcal{B}$ *is* R*-valuation-independent with respect to* v *over* $\mathbf{M}$ *if and only if* $\{b + M\,;\, b \in \mathcal{B}\}$ *is* R*-valuation-independent with respect to* w *and for all* $r \in R$ *and* $b \in \mathcal{B}$*, the elements* rb *are proper representatives of* $rb + M$*. In this case, every finite sum* $\sum_{i\in I} r_i b_i$ *with* $b_i \in \mathcal{B}$ *is a proper representative of* $\sum_{i\in I} r_i(b_i + M)$*.*

Proof: If b is R-valuation-independent with respect to v over $\mathbf{M}$, then it is a proper representative of $b + M$ by virtue of Lemma 4.13. Since

$$\max_{c\in M} v\left(\sum_{i\in I} r_i b_i + a - c\right) = w\left(\sum_{i\in I} r_i b_i + M\right) = w\left(\sum_{i\in I} r_i(b_i + M)\right)$$

and

$$\min_{i\in I} v r_i b_i = \min_{i\in I} w(r_i b_i + M)$$

for the proper representatives $r_i b_i$, our assertion follows from Lemma 4.13. □

From this lemma and the foregoing corollary, we obtain:

Corollary 4.16 *If* $\mathbf{N}$ *admits an* R*-valuation basis over* $\mathbf{M}$*, then* $\mathbf{M}$ *is nice in* $\mathbf{N}$ *and* $(N/M, w)$ *admits an* R*-valuation basis. If* $\mathbf{M} \subset \mathbf{N}$ *is defectless, then* $\mathbf{M}$ *is nice in* $\mathbf{N}$ *and* $(N/M, w)$ *is a defectless extension of* 0*.*

For ordinary valued vector spaces, also the converses of these assertions hold; cf. Corollary 4.24 below.

The proof of the following lemma is left to the reader.

Lemma 4.17 *Let* $\mathcal{B}$*,* $\mathcal{B}'$ *be subsets of* N*. Then* $\mathcal{B} \cup \mathcal{B}'$ *is* R*-valuation independent over* $\mathbf{M}$ *if and only if* $\mathcal{B}$ *is* R*-valuation independent over* $\mathbf{M}$ *and* $\mathcal{B}'$ *is* R*-valuation independent over* $\mathbf{M} + R\mathcal{B}$*. Further, the union of an ascending chain of* R*-valuation independent sets over* $\mathbf{M}$ *is again* R*-valuation independent over* $\mathbf{M}$*.*

A subset $\underline{\mathcal{B}} \subset \operatorname{sk}\mathbf{N}$ will be called R**-independent over** $\operatorname{sk}\mathbf{M}$ if for every admissible R-linear combination $\sum_i r_i(\alpha_i, \zeta_i)$ of bones in $\underline{\mathcal{B}} \cup \operatorname{sk}\mathbf{M}$,

$$\sum_i r_i(\alpha_i, \zeta_i) = (\alpha, 0) \implies \forall i:\ r_i(\alpha_i, \zeta_i) = (\infty, 0)\ .$$

Lemma 4.18 *Let* $\mathcal{B} \subset N \setminus \operatorname{Exe}\mathbf{N}$*. Then* $\mathcal{B}$ *is* R*-valuation independent over* $\mathbf{M}$ *if and only if* $\underline{\mathcal{B}}$ *is* R*-independent over* $\operatorname{sk}\mathbf{M}$*.*

Proof: $\Rightarrow$: Suppose there are $b_1, \ldots, b_n \in \mathcal{B}$ and $r_1, \ldots, r_n \in R$, not all equal to zero, and $a \in M$ such that $\sum r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$ is an admissible linear combination which equates to zero. Then $vr_1b_1 = \ldots = vr_nb_n = va$ and $\sum r_i \operatorname{co}_{va} b_i + \operatorname{co}_{va} a = 0$. We obtain

$$v(r_1b_1 + \ldots + r_nb_n + a) > va = \min_{i\in I}\{vr_ib_i, va\} \ ,$$

which shows that $\mathcal{B}$ is not R-valuation independent over $\mathbf{M}$.

$\Leftarrow$: Let $\sum_{i\in I} r_ib_i$ be a finite sum of elements $b_i \in \mathcal{B}$ with $r_i \in R$ and let $a \in M$. Let $\gamma = \min_{i\in I} vr_ib_i$. If $va < \gamma$, then (4.3) holds. Assume now that $va \geq \gamma$. Let $J \subset I$ contain precisely the indices for which $r_ib_i \neq 0$ and $vr_ib_i = \gamma$. By assumption, the bones of b_j, $j \in J$, are R-independent over $\operatorname{sk}\mathbf{M}$. Since all b_i are ordinary, this implies

$$\operatorname{co}_\gamma\left(\sum_{i\in J} r_ib_i + a\right) = \sum_{i\in J} \operatorname{co}_\gamma r_ib_i + \operatorname{co}_\gamma a \neq 0 \ ,$$

whence

$$v\left(\sum_{i\in I} r_ib_i + a\right) = \gamma = \min_{i\in I} vb_i = \min_{i\in I}\{vb_i, va\} \ .$$

□

Corollary 4.19 *Let $\mathcal{B} \subset N \setminus \operatorname{Exe}\mathbf{N}$ be R-valuation independent over $\mathbf{M}$. Then $\underline{\mathcal{B}}$ is an R-independent set of generators of $\operatorname{sk}(\mathbf{M} + R\mathcal{B})$ over $\operatorname{sk}\mathbf{M}$.*

Proof: In view of the last lemma, we only have to show that $\mathcal{B}$ generates $\operatorname{sk}(\mathbf{M} + R\mathcal{B})$ over $\operatorname{sk}\mathbf{M}$. For $b \in M + R\mathcal{B}$, let us write

$$b = \sum_{i=1}^{n} r_ib_i + a \quad \text{with} \quad r_i \in R,\ b_i \in \mathcal{B},\ a \in M \ . \tag{4.4}$$

We assume (after a suitable renumeration) that precisely $r_1b_1, \ldots, r_mb_m, a$ are the summands of least value γ in (4.4). If va is smaller than the value of the other summands, then omit the elements r_ib_i and their bones in the following; if on the other hand, va is greater than the value of some summand r_ib_i, then omit a and its bone. Since all b_i are ordinary, the linear combination $\sum_{i=1}^m r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$ is admissible. In view of our assumption and the foregoing lemma, $\sum_{i=1}^m \operatorname{co}_\gamma r_ib_i + \operatorname{co}_\gamma a \neq 0$. Hence the bone of b is equal to $\sum_{i=1}^m r_i(vb_i, \operatorname{co} b_i) + (va, \operatorname{co} a)$. □

For the conclusion of this section, we will consider ordinary valued K-vector spaces, where K is a field or a skew field. We need the following characterization of ordinary valued vector spaces:

Lemma 4.20 *Let (V, v) be a valued K-vector space. Then (V, v) is ordinary if and only if*

$$va < va' \iff vra < vra' \tag{4.5}$$

holds for all $a, a' \in V$ and every $r \in K \setminus \{0\}$.

This is a direct consequence of Lemma 4.3 since every $r \in K \setminus \{0\}$ is invertible. For the same reason, for an extension $(W, v) \subset (V, v)$ of ordinary valued K-vector spaces every bone in $\operatorname{sk}(V, v) \setminus \operatorname{sk}(W, v)$ is already K-independent over $\operatorname{sk}(W, v)$. This shows:

Lemma 4.21 *Let $(W,v) \subset (V,v)$ be an extension of ordinary valued K-vector spaces and $b \in V$. If $(vb, \operatorname{co} b) \notin \operatorname{sk}(W,v)$, that is, if $vb = \max\{v(b-c) \mid c \in W\}$, then $\{b\}$ is a valuation basis of $(W + Kb, v)$ over W. Hence if (W,v) is nice in (V,v), then every K-subvector space (V',v) of (V,v) of dimension 1 over W admits a valuation basis over W.*

In view of Lemma 4.18 and Corollary 4.19 we deduce that if $\mathcal{B} \subset V$ is a maximal subset which is K-valuation independent over W then $\operatorname{sk}(W + K\mathcal{B}) = \operatorname{sk} W + K\underline{\mathcal{B}} = \operatorname{sk} V$. The existence of such maximal $\mathcal{B}$ is shown by Zorn's Lemma together with Lemma 4.17. We have proved:

Lemma 4.22 *For every extension $(W,v) \subset (V,v)$ of ordinary valued K-vector spaces there exists a subextension $(W,v) \subset (V',v)$ such that (V',v) admits a K-valuation basis over W and $(V',v) \subset (V,v)$ is immediate. Every subset of V which is maximal K-valuation independent over W, generates a subextension V' with these properties.*

For ordinary valued vector spaces, we can improve our results on quotients:

Lemma 4.23 *Let $(W,v) \subset (V,v)$ be ordinary valued K-vector spaces. If $b \in V$ and $0 \neq r \in K$, then b is a proper representative of $b + W$ if and only if rb is a proper representative of $rb + W$.*

Proof: From Lemma 4.20, we obtain that $vb = \max_{c \in W} v(b-c)$ if and only if $vrb = \max_{c \in W} v(rb - rc)$. Since $rW = W$, we have $\max_{c \in W} v(rb - rc) = \max_{c \in W} v(rb - c)$, which proves our assertion. □

From this lemma together with Lemma 4.15, we deduce:

Corollary 4.24 *Assume $(W,v) \subset (V,v)$ to be ordinary valued K-vector spaces with (W,v) nice in (V,v). Let $\mathcal{B} \subset V$. If $\{b + W\,;\, b \in \mathcal{B}\}$ is K-valuation-independent in $(V/W, w)$ and every $b \in \mathcal{B}$ is a proper representative of $b + W$, then $\mathcal{B}$ is K-valuation-independent over W. Consequently, if $(V/W, w)$ admits an K-valuation basis, then (V,v) admits a K-valuation basis over W, and if $(V/W, w)$ is a defectless extension of 0, then $(W,v) \subset (V,v)$ is defectless.*

See Section 4.7 below for further details on defectless extensions of ordinary valued vector spaces.

4.4 Value preserving scalar multiplication

A valued R-module $\mathbf{M}$ will be said to have **value preserving scalar multiplication** if it satisfies

(VPSM) $\quad \forall x \in M\, \forall r \in R \setminus \{0\} :\ vrx = vx\,.$

In particular, every valued R-module with value preserving scalar multiplication is component-compatibly valued. Conversely, if a component-compatibly module is a vector space, then it has value preserving scalar multiplication. Indeed, for every nonzero invertible $r \in R$, we must have $vx = vr^{-1}rx \geq vrx \geq vx$, hence equality. We can generalizes this observation as follows. Let $\mathbf{M}$ be a component-compatibly valued R-module, $a \in M$ and $0 \neq r \in R$. Then $r \operatorname{co} a = 0$ if and only if $vra > va$. This proves:

Lemma 4.25 *A valued R-module has value preserving scalar multiplication if and only if it is component-compatible and every component is a torsion free R-module. In particular, a valued abelian group has value preserving scalar multiplication if and only if every component is torsion free.*

Examples 4.26 Every ordered group with its natural valuation is a valued $\mathbb{Z}$-module with value preserving scalar multiplication since it satisfies (NVZ). If in addition, this group is divisible, then it is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication.

The **socle of a p-group** is the set of all elements of order p. It is consequently an $\mathbb{F}_p$-vector space. Restricting the height function to the socle, we obtain a p-valuation on this $\mathbb{F}_p$-vector space with value preserving scalar multiplication.

The set $K^\times/K^{\times 2}$ of square classes of a field K is an $\mathbb{F}_2$-vector space. If the field has a valuation, then under certain conditions, it induces a valuation on $K^\times/K^{\times 2}$ ♢

As a direct consequence of Lemma 4.3, we get:

Lemma 4.27 *Every valued module with value preserving scalar multiplication is ordinary.*

In the case of value preserving scalar multiplication, a linear combination of bones is admissible if and only if all bones show the same value in their first component. Then we have $\sum_i r_i(\alpha,\zeta_i) = (\alpha, \sum_i r_i\zeta_i)$. Consequently, the condition that a set $\underline{\mathcal{B}}$ of bones be R-independent over $\operatorname{sk}\mathbf{M}$ is equivalent to the condition that for all $\alpha \in vN$, the set $\underline{\mathcal{B}}_\alpha := \{\zeta \mid (\alpha,\zeta) \in \underline{\mathcal{B}}\}$ is R-independent over $\mathrm{C}^\alpha\mathbf{M}$. Hence, the following is a corollary to Lemma 4.18:

Corollary 4.28 *Suppose that $\mathbf{N}$ has value preserving scalar multiplication. Then $\mathcal{B} \subset N$ is R-valuation independent over $\mathbf{M}$ if and only if for every $b_1,\dots,b_n \in \mathcal{B}$ of equal value γ, the elements $\operatorname{co}_\gamma b_1,\dots,\operatorname{co}_\gamma b_n \in \mathrm{C}^\gamma\mathbf{N}$ are R-independent over $\mathrm{C}^\gamma\mathbf{M}$.*

Observe that in view of Lemma 4.25 which tells us that the components are torsion free R-modules, R-independence is equivalent to R-linear independence: The elements $\zeta_1,\dots,\zeta_n$ are said to be **R-linear independent** if $\sum r_i\zeta_i = 0$ implies $\forall i:\ r_i = 0$.

The above corollary yields that $\mathcal{B} \subset N$ is maximal with the property of being R-valuation independent over $\mathbf{M}$ if and only if for every $\alpha \in vN$, $\underline{\mathcal{B}}_\alpha$ is maximal with the property of being R-independent over $\mathrm{C}^\alpha\mathbf{M}$. In the case of vector spaces, where R is a field (or a skew field), a maximal R-independent set is just a basis. So we obtain the following corollary to Lemma 4.18:

Corollary 4.29 *Let K be a field or a skew field and $\mathbf{W} \subset \mathbf{V}$ an extension of valued K-vector spaces with value preserving scalar multiplication.*

a) A subset $\mathcal{B} \subset V$ is maximal with the property of being K-valuation independent over W if and only if for every $\alpha \in vV$, the set $\underline{\mathcal{B}}_\alpha = \{\operatorname{co}_\alpha b \mid b \in \mathcal{B} \text{ and } vb = \alpha\}$ forms a basis of $\mathrm{C}^\alpha\mathbf{V}$ over $\mathrm{C}^\alpha\mathbf{W}$.

b) Let $V' \subset V$ be as in Lemma 4.22. Reading the dimensions as (finite or infinite) cardinals,

$$\dim_K V/W \geq \dim_K V'/W = \sum_{\alpha\in vV} \dim_K \mathrm{C}^\alpha\mathbf{V}\,/\mathrm{C}^\alpha\mathbf{W} \geq |vV \setminus vW|\,. \tag{4.6}$$

The last inequality of (4.6) holds since for every $\alpha \in vV \setminus vW$, we have $\mathrm{C}^\alpha\mathbf{W} = \{0\}$ whereas $\mathrm{C}^\alpha\mathbf{V}$ contains at least one copy of K, showing that $\dim_K \mathrm{C}^\alpha\mathbf{V}\,/\mathrm{C}^\alpha\mathbf{W} \geq 1$.

We apply this corollary to the divisible hulls of ordered abelian groups.

Corollary 4.30 *Let $(H,<) \subset (G,<)$ be an extension of ordered abelian groups and let v be the natural valuation of G. Then*

$$\mathrm{rr}\, G/H \geq \sum_{\alpha\in vG} \mathrm{rr}\ (\mathrm{C}^\alpha\mathbf{G}\,/\mathrm{C}^\alpha\mathbf{H}) \geq |vG \setminus vH|\,. \tag{4.7}$$

In particular,

$$\mathrm{rr}\, G \geq \sum_{\alpha\in vG} \mathrm{rr}\, \mathrm{C}^\alpha\mathbf{G} \geq |vG| \tag{4.8}$$

and if G has finite rank, then $\mathrm{rr}\, G \geq \mathrm{rk}\, G$.

Proof: The reader may verify that for every extension $H \subset G$ of abelian groups, $\mathbb{Q} \otimes_{\mathbb{Z}} G/H \cong (\mathbb{Q} \otimes_{\mathbb{Z}} G)/(\mathbb{Q} \otimes_{\mathbb{Z}} H)$, whence $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} G/H = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G)/(\mathbb{Q} \otimes_{\mathbb{Z}} H)$. Applying this to the given extension as well as to the extensions of components, inequality (4.7) follows from part b) of the previous corollary.

Inequalities (4.8) follow from (4.7) with $H = \{0\}$. For groups of finite rank, the rank is equal to the principal rank $|vG|$, so the inequality $\mathrm{rr}\, G \geq \mathrm{rk}\, G$ follows from (4.8). □

It can be shown that also for infinite rank, the cardinality of the rank is equal to the cardinality of the principal rank. So replacing the rank by its cardinality, the last inequality remains true for arbitrary ordered groups.

Using the upper bound for $vV \setminus vW$ given in part b) of Corollary 4.29, we can show the following important theorem. It was exploited by R. Brown [BRW1] to classify countably-dimensional valued vector spaces with value preserving scalar multiplication by their skeletons (see Theorem 4.47 below).

Theorem 4.31 *Every countably-dimensional valued vector space with value preserving scalar multiplication admits a valuation basis.*

Proof: Let (V,v) be such a valued K-vector space. Then it is the union over finite-dimensional valued vector space (V_i,v), $i \in \mathbb{N}$, such that $V_1 = 0$ and $\dim_K V_{i+1}/V_i = 1$ for every i. By part b) of Corollary 4.29, every value set vV_i is finite and thus, every (V_i,v) is spherically complete. Hence by Lemma 4.7, every (V_i,v) is maximal and consequently, the extension $(V_i,v) \subset (V_{i+1},v)$ is not immediate. In view of Lemma 4.22 and $\dim_K V_{i+1}/V_i = 1$, it follows that every (V_{i+1},v) admits a K-valuation basis $\{b_i\}$ over V_i. Now it follows from Lemma 4.17 that $\{b_i \mid i \in \mathbb{N}\}$ is a K-valuation basis of (V,v) over $V_1 = 0$. □

This theorem was generalized by S. Kuhlmann [KUS1] in the following way: *If $(W,v) \subset (V,v)$ is a countably generated extension of valued K-vector spaces with value preserving scalar multiplication and if (W,v) admits a K-valuation basis, then also (V,v) admits a K-valuation basis.* Note that this does not mean that (V,v) would admit a K-valuation basis over W: this is not true in general. However, it is true if (W,v) is nice in (V,v):

Corollary 4.32 *Let $(W,v) \subset (V,v)$ be valued vector spaces with value preserving scalar multiplication. If (W,v) is nice in (V,v) and V/W is of countable dimension, then (V,v) admits a valuation basis over W.*

Proof: We endow V/W with the quotient valuation w. By Theorem 4.31, $(V/W, w)$ admits a valuation basis. From Lemma 4.27 we know that (V, v) is ordinary valued. Now our assertion follows from Corollary 4.24. □

From this corollary together with Corollary 4.14, we obtain the following result that was proved by S. Kuhlmann in [KUS1]:

Theorem 4.33 *Let $(W, v) \subset (V, v)$ be valued vector spaces with value preserving scalar multiplication. Then (W, v) is nice in (V, v) if and only if the extension $(W, v) \subset (V, v)$ is defectless.*

Note that Theorem 4.31 does not remain true if the assumption of value preserving scalar multiplication is omitted. The simplest counterexample is that of a proper immediate extension $(K, v) \subset (L, v)$ of valued fields (see below): if we choose $a \in L \setminus K$ then the 2-dimensional valued K-vector space $(K + Ka, v)$ does not admit a K-valuation basis.

We should mention that in a category of valued vector spaces with value preserving scalar multiplication with suitable morphisms, those admitting a valuation basis are precisely the **free** ones. See L. Fuchs [FU2], p. 29.

If all R-modules C_γ are torsion free, then the Hahn sum $\coprod_{\gamma\in\Gamma} C_\gamma$ and the Hahn product $\mathbf{H}_{\gamma\in\Gamma} C_\gamma$ are valued R-modules with value preserving scalar multiplication. If R is a field, then the Hahn sum admits a natural R-valuation basis. Indeed, for every γ we take an R-basis $\underline{\mathcal{B}}_\gamma$ of C_γ and for every element ζ of this basis, we take e_ζ to be the tuple which has ζ at the γ-th entry and zeros everywhere else. Then the elements $\{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\gamma \wedge \gamma \in \Gamma\}$ generate $\coprod_{\gamma\in\Gamma} C_\gamma$. On the other hand, they are R-valuation independent since their bones are R-independent. This shows that $\{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\gamma \wedge \gamma \in \Gamma\}$ is an R-valuation basis of $\coprod_{\gamma\in\Gamma} C_\gamma$.

Remark 4.34 For a more detailed investigation of valued vector spaces with value preserving scalar multiplication, see [BRW1], [FU2], [FU3], [GRA1], [HI1], [HI2], [HI–WH], [KUS1] and their references. Much of the material on valued vector spaces as exposed in this chapter, is taken over from S. Kuhlmann's thesis [KUS1].

Exercise 4.3 *Let $(W, v) \subset (V', v) \subset (V, v)$ be valued vector spaces with value preserving scalar multiplication. Suppose that (W, v) is nice in (V, v) and that V'/W is of countable dimension. Prove that (V', v) is nice in (V, v).*

Exercise 4.4 *Prove that every finitely generated valued R-module $\mathbf{M}$ with value preserving scalar multiplication has a finite value set. More precisely, assume that M is generated by n elements, and show that $|vM| \leq n$. Assume in addition that all components are free R-modules, and show that $\mathbf{M}$ admits a valuation basis with at most n elements.*

4.5 p-bases and straight bases of p-groups

If we consider an abelian group with a p-valuation as a valued $\mathbb{Z}$-module, then we do not have value preserving scalar multiplication. Nevertheless, there is an analogue of Brown's Theorem (4.31), which we shall discuss now. See Section 3.4 for the definition of the p-height function, the p-height valuation and the order valuation.

Throughout this section, let G be an abelian p-group. We will consider G with its p-height valuation v. If ν is the p-height of G, i.e. $\nu\infty$ is the set of heights in G, then $vG = \nu$

or $vG = \nu + 1$ depending on whether G is reduced or not. If $va < \nu$, then $vpa > va$, that is, $p \operatorname{co}_{va} a = 0$. This shows that for every $\mu < \nu$, the μ-component $\mathrm{C}_\mu(G, v)$ is an $\mathbb{F}_p$-vector space. This is not true for $\mathrm{C}_\nu(G, v)$ if it is nontrivial.

In the following, we will prove some facts for arbitrary valuations v on the p-group G. For this, we need some preparation.

Lemma 4.35 *Let (G, v) be an arbitrary valued group, $a \in G$ and p a prime.*
a) If $\mathrm{C}_{va}(G, v)$ is an $\mathbb{F}_p$-vector space, then $vpa > va$ and $vna = va$ for every integer n which is prime to p.
b) If G is a p-group, then $vna = va$ for every integer n which is prime to p.

Proof: a): Let $\mathrm{C}_{va}(G, v)$ be an $\mathbb{F}_p$-vector space. Then $p \operatorname{co}_{va} a = 0$ and thus $vpa > va$. If n prime to p, then there exist integers r, s such that $1 = rn + sp$. By (VZ) and what we have already shown, $vspa \geq vpa > va$. Hence $va = v(rn + sp)a = v(rna + spa) = vrna$. Again by (VZ), $vrna \geq vna \geq va$. Altogether, we obtain that $va = vna = nrna$.
b): Since in a p-group every element has order a power of p, the assertion follows from Lemma 3.1. □
Note that the assertion of part b) in particular holds for the p-height valuation v, even if $\mathrm{C}_\nu(G, v)$ is nontrivial, that is, G is not reduced.

Lemma 4.36 *Let G be an abelian p-group and v an arbitrary valuation on G. Let $\mathcal{B} \subset G \setminus \{0\}$ and assume that for every $\mu \in \{vb \mid b \in \mathcal{B}\}$, the component $\mathrm{C}_\mu(G, v)$ is an $\mathbb{F}_p$-vector space. Then the following conditions are equivalent:*
1) For every $\mathbb{Z}$-linear combination $\sum_{i=1}^m n_i b_i$ with $b_i \in \mathcal{B}$ and n_i integers prime to p,

$$v\left(\sum_{i=1}^m n_i b_i\right) = \min vn_i b_i \ .$$

(Since n_i is prime to p, we could write "vb_i" in the place of "$vn_i b_i$".)
2) Condition 1) holds whenever $0 < n_i < p$.
3) For every $\mu \in \{vb \mid b \in \mathcal{B}\}$, the coefficients $\operatorname{co}_\mu b$, $b \in \mathcal{B}$ of value $vb = \mu$, are $\mathbb{F}_p$-independent (we could say: the bones of the elements $b \in \mathcal{B}$ are $\mathbb{F}_p$-independent).
Recall that whenever we write "$\sum n_i b_i$ with $b_i \in \mathcal{B}$", then we implicitly mean that all b_i are distinct (unless stated otherwise).

Proof: 1)⇒2) is trivial. We prove 2)⇒3). Let be given a linear combination $\sum \overline{n}_i \operatorname{co}_\mu b_i$, where b_i are elements of value μ of $\mathcal{B}$. We choose natural numbers $0 < n_i < p$ whose residues modulo p are $\overline{n}_i$. Condition 2) tells us that $v(\sum n_i b_i) = \mu$. Hence, $\sum \overline{n}_i \operatorname{co}_\mu b_i = \operatorname{co}_\mu \sum n_i b_i \neq 0$.

Now we prove 3)⇒1). Let be given a linear combination $\sum n_i b_i$, where b_i are elements of $\mathcal{B}$ and n_i are integers, prime to p. Let $\mu = \min_i vn_i b_i = \min_i vb_i$. We have $\operatorname{co}_\mu \sum n_i b_i = \sum \overline{n}_i \operatorname{co}_\mu b_i$, where $\overline{n}_i$ denotes the residue of n_i modulo p. Since n_i is assumed to be prime to p, we have $\overline{n}_i \neq 0$ for all i. By our choice of μ, there is at least one i such that $\operatorname{co}_\mu b_i \neq 0$. By condition 3), $\sum \overline{n}_i \operatorname{co}_\mu b_i \neq 0$, showing that $v \sum n_i b_i = \mu = \min_i vn_i b_i$. □

A subset $\mathcal{B} \subset G$ will be called **p'-valuation independent in** (G, v) (read: "p-prime valuation independent") if it satisfies condition 1) of this lemma. This looks like valuation independence, but it is restricted to scalar multiplication with scalars which are prime to p (that is, preserve the value). Further, $\mathcal{B} \subset G \setminus \{0\}$ will be called a **v-p-basis of** G if

(vpB1) the elements of $\mathcal{B}$ are p'-valuation independent in (G, v),

(vpB2) $\mathcal{B}$ generates G,

(vpB3) $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$.

A subset $\mathcal{B} \subset G \setminus \{0\}$ of a p-group G is called a **p-basis of** G if it is a v-p-basis of G with respect to the p-height valuation v of G. A p-group admitting a p-basis is called **simply presented**.

We have used condition 1) of the above lemma for the definition of p'-valuation independence and thus also in the definition of a p-basis. In the literature, condition 2) in the place of condition 1) is commonly used for the definition of a p-basis. Nevertheless, this is equivalent to our definition. To show this, we establish the equivalence of conditions 1) and 2) for every subset $\mathcal{B}$ of a p-group, provided that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$. Even without this provision, the equivalence holds for all reduced p-groups by means of the foregoing lemma since for them every component is an $\mathbb{F}_p$-vector space. For the general case, we now give a proof that does not use Lemma 4.36.

Lemma 4.37 *Let G be a p-group and $\mathcal{B} \subset G \setminus \{0\}$ such that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$. Then every finite sum $a = \sum_{i=1}^{m} n_i b_i$, with b_i (in this case not necessarily distinct) elements from $\mathcal{B}$ and n_i arbitrary integers, can be written as a (possibly trivial) sum $a = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ with $0 < \tilde{n}_i < p$ and $\tilde{b}_i$ distinct elements from the finite set*

$$\{p^j b_i \mid 1 \le i \le m \text{ and } j \ge 0 \text{ with } p^j b_i \ne 0\} \subset \mathcal{B} .$$

If all b_i are distinct and all n_i are prime to p, then the latter sum is nontrivial (i.e., $\tilde{m} \ge 1$); in particular, every element of maximal order among the b_i will appear as some $\tilde{b}_i$.

As a consequence, conditions 1) and 2) of Lemma 4.36 are equivalent for every valued p-group (G, v) and $\mathcal{B} \subset G \setminus \{0\}$, provided that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$.

Proof: We shall prove the first assertion by induction on the maximal order of the elements b_i appearing in the sum. Beforehand, if the elements b_i are not all distinct, we rewrite the sum such that all appearing b_i are distinct. Further, we can assume without loss of generality that all n_i are positive. Indeed, if $n_i < 0$, then we choose ν_i such that $p^{\nu_i} b_i = 0$ and add a multiple of p^{ν_i} to n_i in order to obtain a positive coefficient (which is still prime to p if n_i was). Now we use the p-adic expansion of every n_i: We represent n_i as a finite sum $\sum n_{i,j} p^j$ in which $0 \le n_{i,j} < p$ for $j \ge 0$. Then $a = \sum_{i,j} n_{i,j} p^j b_i$. If all b_i have order p, then this sum is equal to $\sum_{i,0} n_{i,0} b_i$. If all b_i were distinct from the start and if all n_i were prime to p, then all $n_{i,0}$ are nonzero. This proves our assertion in the case where all b_i have order p.

Let us assume that we have proved our assertion for all sums in which the order of all elements b_i is smaller than p^k. Consider a sum $a = \sum_{i=1}^{m} n_i b_i$ where the elements $b_1, \ldots, b_{m'}$ have order p^k and the elements $b_{m'+1}, \ldots, b_m$ have smaller order, with $m' \le m$. Proceeding as before, we find that

$$a = \sum_{i,j} n_{i,j} p^j b_i = \sum_{i=1}^{m'} n_{i,0} b_i + \sum_{i > m' \vee j > 0} n_{i,j} p^j b_i .$$

Since $\mathrm{ord}(pb) < \mathrm{ord}(b)$ for every element b in a p-group, all $p^j b_i$ appearing in the latter sum have order smaller than p^k. So we can apply the induction hypothesis to find that the

latter sum can be written as a sum $\sum_{i=m'+1}^{m''} \tilde{n}_i \tilde{b}_i$ with $0 < \tilde{n}_i < p$ and $\tilde{b}_i$ distinct elements from $\{p^j b_i \mid 1 \leq i \leq m \text{ and } j \geq 0 \text{ with } p^j b_i \neq 0 \text{ and } i \leq m' \Rightarrow j > 0\} \subset \mathcal{B}$. This latter set contains only elements of order smaller than p^k, so it does not contain $b_1, \ldots, b_{m'}$. Putting $\tilde{b}_i := b_i$ and $\tilde{n}_i := n_{i,0}$ for $1 \leq i \leq m'$, then omitting $\tilde{n}_i \tilde{b}_i$ if $\tilde{n}_i = 0$ and renumbering suitably, we arrive at a sum $a = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ which is of the required form. Again, if all b_i were distinct from the start and if all n_i were prime to p, then all $n_{i,0}$ are nonzero. This completes the proof of our first assertion.

Now we turn to the equivalence of conditions 1) and 2) of Lemma 4.36. 1)⇒2) is trivial. For the proof of 2)⇒1), let be given a finite sum $\sum_{i=1}^{m} n_i b_i$, where $b_i \in \mathcal{B}$ and n_i are integers prime to p. Let $\mu = \min_i vn_i b_i = \min_i vb_i$. If we are able to prove that $v \sum_{vb_i = \mu} n_i b_i = \mu$, then also $v \sum n_i b_i = \mu = \min_i vn_i b_i$. So we can assume from the start that $\mu = vb_i$ for all i. We write $\sum_{i=1}^{m} n_i b_i = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ according to our first assertion. Then all $\tilde{b}_i$ have a value $\geq \mu$ (this follows from (VZ) since every $\tilde{b}_i$ is a multiple of some b_j). On the other hand, at least one b_i (of maximal order) appears among the $\tilde{b}_i$, so condition 1) yields that $v \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i = \mu$, as required. □

We will also need the following corollary:

Corollary 4.38 *Assume that $\mathcal{B}$ is a v-p-basis of the valued p-group G. Suppose that*

$$v(nb - p \sum_{i=1}^{m} n_i b_i) > vnb$$

with b, b_i (not necessarily distinct) elements in $\mathcal{B}$, n_i arbitrary integers and n prime to p. Then there is some $j \geq 1$ and some i such that $b = p^j b_i$.

Proof: According to the foregoing lemma, we write $p \sum_{i=1}^{m} n_i b_i = \sum_{i=1}^{m} n_i p b_i = \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i$ with $0 < \tilde{n}_i < p$ and $\tilde{b}_i$ distinct elements from the set $\{p^j p b_i \mid 1 \leq i \leq m \text{ and } j \geq 0 \text{ with } p^j p b_i \neq 0\} \subset \mathcal{B}$. But $b \in \mathcal{B}$ and $v(nb - \sum_{i=1}^{\tilde{m}} \tilde{n}_i \tilde{b}_i) > vnb$ then show that some $\tilde{b}_i$ must be equal to b, by property (vpB1) of the v-p-basis $\mathcal{B}$. On the other hand, $\tilde{b}_i = p^j b_i$ with $j \geq 1$. □

In the common definitions of p-bases, a stronger version of condition (vpB2) is used. Namely, it is required that every element $a \in G$ *has a unique representation* $a = \sum n_i b_i$ with $b_i \in \mathcal{B}$ and $0 < n_i < p$. However, in the presence of (vpB1) and (vpB3), this condition is equivalent to condition (vpB2). Indeed, (vpB2) and (vpB3) imply the existence of such a representation, as follows from the first assertion of the foregoing lemma. The uniqueness of this representation follows from condition (vpB1): Assume that $a = \sum n_i b_i = \sum m_i b_i$ with $0 \leq n_i < p$ and $0 \leq m_i < p$ (we admit zero coefficients in order to let the sums run over the same set of elements b_i in $\mathcal{B}$). Then $0 = \sum (n_i - m_i) b_i$ with $n_i - m_i$ either zero or prime to p. If $n_i \neq m_i$ would hold for some i, then by p'-valuation independence, the value of this sum would be equal to the value of some b_i and thus $< \infty$, a contradiction. This proves the uniqueness of the representation.

For later use, we give a procedure for extending the p-basis of a subgroup of a p-group:

Lemma 4.39 *Let (G, v) be a valued p-group and H a subgroup of G with v-p-basis $\mathcal{B}_H$. If $a \in G$ is a proper representative of $a + H$ and such that $pa \in \mathcal{B}_H \cup \{0\}$, then $\mathcal{B}_H \cup \{a\}$ is a v-p-basis of $H + \mathbb{Z}a$.*

Proof: Since $\mathcal{B}_H$ generates H, we see that $\mathcal{B}_H \cup \{a\}$ generates $H + \mathbb{Z}a$. Since $p\mathcal{B}_H \subset \mathcal{B}_H \cup \{0\}$ and $pa \in \mathcal{B}_H \cup \{0\}$, it follows that $p(\mathcal{B}_H \cup \{a\}) \subset \mathcal{B}_H \cup \{0\}$. The p'-valuation independence is seen as follows. We only have to show that va is the value of every linear combination $na + \sum n_i b_i$ with $b_i \in \mathcal{B}_H$ of value $vb_i = va$ and n, n_i prime to p. To see this, we multiply by an integer n' which satisfies $n'n \equiv 1$ modulo the order p^ν of a. Then $n'na = a$, and we obtain that $va \le v(na + \sum n_i b_i) \le v(a + \sum n' n_i b_i) \le va$, where the latter inequality holds since a was assumed to be a proper representative. This proves that the value of the sum is indeed equal to va. □

The elements of a p-basis $\mathcal{B}$ will in general not be $\mathbb{Z}$-valuation independent unless all of them have order p, since if $b \in \mathcal{B}$ with $pb \neq 0$, then the linear combination $1 \cdot (pb) + (-p) \cdot b$ is zero. But throwing away the multiples, one can select subsets of $\mathcal{B}$ which are almost $\mathbb{Z}$-valuation independent: A subset $\mathcal{B} \subset G$ will be called **weakly $\mathbb{Z}$-valuation independent** if every $\mathbb{Z}$-independent subset of $\mathcal{B}$ is $\mathbb{Z}$-valuation independent. If in addition $\mathcal{B}$ generates G, then it will be called a **weak $\mathbb{Z}$-valuation basis of** G.

Lemma 4.40 *If $\mathcal{B}$ is a p-basis of the p-group G, then the subset $\mathcal{B}_0 \subset \mathcal{B}$ of elements of p-height 0 is weakly $\mathbb{Z}$-valuation independent and generates all elements in $\mathcal{B}$ of p-height $< \infty$. In particular, if G is reduced, then a weak $\mathbb{Z}$-valuation basis can be selected from every p-basis. Conversely, if $\mathcal{B}'$ is a weak $\mathbb{Z}$-valuation basis of the reduced p-group G, then $\mathcal{B} = \bigcup_{i=0}^{\infty} p^i \mathcal{B}'$ is a p-basis of G.*

Proof: Let $\mathcal{B}$ be a p-basis of the p-group G. Let $\mathcal{B}_0 \subset \mathcal{B}$ be the set of all elements of p-height 0 in $\mathcal{B}$. Since $p^j \mathcal{B}_0 \subset \mathcal{B}$ for all $j \in \mathbb{N}$, every $\mathbb{Z}$-linear combination of elements in $\mathcal{B}_0$ can be written as a $\mathbb{Z}$-linear combination of elements in $\mathcal{B}$ with coefficients prime to p; since $p^j b = p^j b'$ is possible for $b, b' \in \mathcal{B}_0$ even if $b \neq b'$, we have to admit that an element of $\mathcal{B}$ appears more than once in the latter linear combination. If no element of $\mathcal{B}$ appears more than once, then the value of this linear combination is equal to the minimal value of its summands, because $\mathcal{B}$ is a p-basis. But if some element of $\mathcal{B}$ appears at least twice, then this means that the elements of $\mathcal{B}_0$ appearing in our original $\mathbb{Z}$-linear combination are not $\mathbb{Z}$-independent. This proves that the elements of $\mathcal{B}_0$ are weakly $\mathbb{Z}$-valuation independent. Further, $\bigcup_{i=0}^{\infty} p^i \mathcal{B}_0$ contains all elements in $\mathcal{B}$ of p-height $< \infty$. Indeed, let $b \in \mathcal{B}$ be of p-height $< \infty$, and assume that our assertion is already proved for all elements of $\mathcal{B}$ of value $< vb$. By (pH1′) there is some a such that $pa = b$ and $va < vb$. We write $a = \sum_{i=1}^{m} n_i b_i$ with $b_i \in \mathcal{B}$ and arbitrary integers n_i. Consequently, $b = p \sum_{i=1}^{m} n_i b_i$. According to Corollary 4.38, $b = p^j b_i$ for some i and $j \ge 1$. Since b is of p-height $< \infty$, the latter implies $vb_i < vb$. By assumption, $b_i \in p^k \mathcal{B}_0$ for some integer $k \ge 0$. Consequently, $b \in p^{k+j} \mathcal{B}_0$.

The proof of the remaining assertions is left to the reader. □

This procedure of selecting a weak valuation basis from a p-basis is made possible by the finite descent from an element of arbitrary height $< \infty$ to an element of height 0. The same principle can be found in very different settings; see e.g. the selection of a $K[\varphi]$-basis from a Frobenius-closed K-basis in section 11.2.

If a p-group G admits a $\mathbb{Z}$-valuation basis $\mathcal{B}'$ (with respect to its p-height valuation), then the elements of $\mathcal{B}'$ are $\mathbb{Z}$-independent and thus, G is the direct sum $\bigoplus_{b \in \mathcal{B}'} \mathbb{Z}b$. Such a group is reduced, and all elements have finite p-height. Consequently, a p-group with elements of infinite height can not admit a $\mathbb{Z}$-valuation basis.

The following theorem is the promised analogue of Brown's Theorem. Its first assertion is a result of R. Hunter and E. Walker ([HU–WA], Corollary 4.11).

Theorem 4.41 *If G is a countable p-group, then it admits a p-basis. If in addition, G is reduced and v is the p-height valuation, then (G, v) admits a weak $\mathbb{Z}$-valuation basis. If all elements of the reduced p-group G have height $< \omega$, then (G, v) admits a $\mathbb{Z}$-valuation basis.*

Proof: Let v be the p-height valuation of G. Let a_j , $j \in \mathbb{N}$, be an enumeration of the elements of G. It may be chosen in such a way that for every $j \in \mathbb{N}$, the element pa_j is contained in the group generated by $a_1, \ldots, a_{j-1}$. We set $G_1 := \{0\}$ and $\mathcal{B}_1 = \emptyset$. Suppose that we have already constructed a finite subgroup G_j of G containing $a_1, \ldots, a_{j-1}$ and admitting a v-p-basis $\mathcal{B}_j$ with the following property:

$$\forall b \in \mathcal{B}_j : \; vb = \mu + 1 < \nu \; \Rightarrow \; \exists b' \in \mathcal{B}_j : \; pb' = b \wedge vb' = \mu \; , \tag{4.9}$$

where ν is the p-height of G. We show how to construct a finite subgroup G_{j+1} of G containing G_j and a_j and admitting a v-p-basis $\mathcal{B}_{j+1}$ containing $\mathcal{B}_j$ and having again property (4.9).

Since G_j is finite, its value set vG_j is also finite. Consequently, G_j is nice in (G, v). Hence, there is a proper representative a'_j for $a_j + G_j$. Since also $pa'_j \in G_j$ and a_j and a'_j generate the same abelian group extension of G_j , we can assume from the start that a_j is itself a proper representative. If $pa_j \in \mathcal{B}_j$, then $\mathcal{B}''_j := \mathcal{B}_j \cup \{a_j\}$ is a p-basis of $G''_j := G'_j + \mathbb{Z}a_j$, by virtue of Lemma 4.39.

If $pa_j \notin \mathcal{B}_j$, then we carry through the following construction. Let $pa_j = \sum n_i b_i$ with $b_i \in \mathcal{B}_j$ and n_i prime to p. We distinguish two cases.

Case 1: $va_j < \nu$. Then $vb_i \geq vpa_j > va_j$ for all i. By property (4.9), for all i with $vb_i < \nu$ a successor ordinal, there are elements $b'_i \in \mathcal{B}_j$ such that $pb'_i = b_i$ and $vb'_i \geq va_j$. For b_i with vb_i a limit ordinal, (pH1′) allows us to choose an element b'_i of value $vb'_i \geq va_j$ such that $pb'_i = b_i$. If $\mathcal{B}_j$ does not contain such an element, then we just choose it in G. Since vb_i is a limit ordinal and vG_j is finite, there are values in vG which lie properly between vb_i and all values of vG_j that are smaller than vb_i . By virtue of (pH1′), we can choose b'_i with such a value. Then $vb'_i \notin vG_j$ and $vb'_i > va_j$. From the former property, it follows that $\mathcal{B}_j \cup \{b'_i\}$ is a v-p-basis of $G'_j = G_j + \mathbb{Z}b'_i$.

Finally, for b_i with $vb_i = \nu$, we choose b'_i such that $pb'_i = b_i$ and $vb'_i = vb_i = \nu$ (there is such an element in G since $p^\nu G$ is divisible). If possible, we choose $b'_i \in \mathcal{B}_j$. If $\mathcal{B}_j$ does not contain such an element, then we just choose it in G, and we have to show that $\mathcal{B}_j \cup \{b'_i\}$ is a v-p-basis of $G'_j = G_j + \mathbb{Z}b'_i$. If this were not the case, then there would exist a $\mathbb{Z}$-linear combination $nb'_i + \sum_k n''_k b''_k$ with $b''_k \in \mathcal{B}_j$ of value ν and n''_k, n prime to p, such that $v(nb'_i + \sum_k n''_k b''_k > vnb'_i = \nu$. But then, the value of $nb_i + p\sum_k n''_k b''_k = p(nb'_i + \sum_k n''_k b''_k)$ is bigger than $\nu = vnb_i$. Now Corollary 4.38 shows that $b_i = p^\ell b''_k$ for some k and $\ell \geq 1$, hence $b_i = pp^{\ell-1}b''_k$ with $p^{\ell-1}b''_k \in \mathcal{B}_j$ of value ν, which contradicts our assumption on b_i .

Since $vb'_i > va_j$ in all cases where we have to adjoin b'_i to $\mathcal{B}_j$, we find that a_j will still be a proper representative of $a_j + G'_j$. If necessary, we repeat this procedure for other elements b_i . After a finite number of repetitions, we arrive at a group which we will again call G'_j , with a v-p-basis $\mathcal{B}'_j$ such that for every i there is $b'_i \in \mathcal{B}'_j$ satisfying $vb'_i \geq va_j$ and $pb'_i = b_i$. Moreover, a_j will still be a proper representative of $a_j + G_j$. We set $\tilde{a}_j = a_j - \sum n_i b'_i$. Then $v\tilde{a}_j \geq va_j$, which yields that $v\tilde{a}_j = va_j$ and that $\tilde{a}_j$ is still

a proper representative of $a_j + G_j$. Since $p\tilde{a}_j = 0$ by construction, it now follows from Lemma 4.39 that $\mathcal{B}_j'' := \mathcal{B}_j' \cup \{\tilde{a}_j\}$ is a v-p-basis of $G_j'' := G_j' + \mathbb{Z}\tilde{a}_j$.

Case 2: $va_j = \nu$. As in case 1, we choose b_i' such that $pb_i' = b_i$ and $vb_i' = vb_i = \nu$. Again, $\mathcal{B}_j \cup \{b_i'\}$ will be a v-p-basis of $G_j' = G_j + \mathbb{Z}b_i'$, but we can not guarantee that a_j is still a proper representative of $a_j + G_j'$. After a finite number of repetitions, we arrive at a group which we will again call G_j', with a p-basis $\mathcal{B}_j'$ such that for every i there is $b_i' \in \mathcal{B}_j'$ satisfying $vb_i' = va_j = \nu$ and $pb_i' = b_i$. We set $\tilde{a}_j = a_j - \sum n_i b_i'$. Then $v\tilde{a}_j \geq va_j$. If "=" holds, then we proceed as in case 1. If ">" holds, then that means that $a_j = \sum n_i b_i'$. In this case, we just set $\mathcal{B}_j'' := \mathcal{B}_j'$ and $G_j'' := G_j'$.

Now we have to enlarge $\mathcal{B}_j''$ and G_j'' to obtain a v-p-basis that has property (4.9). Assume that $b \in \mathcal{B}_j''$ has value $\mu + 1 < \nu$ and that there is no $b' \in \mathcal{B}_j''$ of value μ such that $pb' = b$. (Note that by our construction of $\mathcal{B}_j''$, there are only finitely many such b.) We pick an element $b' \in G$ with the above properties. If there is no element in $\mathcal{B}_j''$ of value μ, then we can just adjoin b' to $\mathcal{B}_j''$ to obtain a p-basis of the enlarged group $G_j'' + \mathbb{Z}b'$. Now assume that there are elements of value μ in $\mathcal{B}_j''$. Suppose that $\mathcal{B}_j'' \cup \{b'\}$ is not p'-valuation independent. Then there is a linear combination $nb' + \sum n_i b_i$ with elements $b_i \in \mathcal{B}_j''$ of value μ and n, n_i prime to p, whose value is $> \mu$. But then, the value of $nb + p\sum n_i b_i = p(nb' + \sum n_i b_i)$ is $> \mu+1 = vb$ (using the assumption that $\mu+1 = vb < \nu$). Hence by virtue of Corollary 4.38, $b = p^j b_i$ for some i and $j \geq 1$. Then $j = 1$ since otherwise, $vb = vp^j b_i \geq \mu + 2 > vb$, a contradiction. But $b = pb_i$ contradicts our assumption on b. This shows that $\mathcal{B}_j'' \cup \{b'\}$ is p'-valuation independent. So again, we can adjoin b' to $\mathcal{B}_j''$. The descent from a successor ordinal to the next lower limit ordinal (or 0) is finite. Hence, after a finite repetition of this procedure we will arrive at a finite subgroup G_{j+1} of G with a v-p-basis $\mathcal{B}_{j+1}$ that has property (4.9). This establishes the induction step. The union $\mathcal{B} := \bigcup_{j\in\mathbb{N}} \mathcal{B}_j$ is the desired p-basis of G.

The second assertion of our theorem follows from what we just proved, together with the previous lemma. The third assertion is seen as follows. If G is reduced and only ordinals $< \omega$ appear as values of elements of G, then for every b being put into the p-basis by the above procedure, we only need to add once an element b' such that $pb' = b$, namely, that one satisfying $vb = vb' + 1$. That is, in the constructed p-basis $\mathcal{B}$ there will be no two elements $b_1 \neq b_2$ with $pb_1 = pb_2$. This implies that the elements of $\mathcal{B}_0 = \{b \in \mathcal{B} \mid vb = 0\}$ are $\mathbb{Z}$-independent and thus, they are $\mathbb{Z}$-valuation independent. If G is reduced, hence generated by $\mathcal{B}_0$, this implies that $\mathcal{B}_0$ is a $\mathbb{Z}$-valuation basis of (G, v). □

From now on, let v be the order valuation of the abelian p-group G. With this valuation, all components of (G, v) are $\mathbb{F}_p$-vector spaces. A p'-valuation independent set $\mathcal{B}$ of generators of (G, v) is called a **straight basis of** G (we do not require that $p\mathcal{B} \subset \mathcal{B} \cup \{0\}$). For details on straight bases, see K. Benabdallah and K. Honda [BE–HO]. Here, we prove:

Theorem 4.42 *Every p-group G has a straight basis.*

Proof: For every $m \in vG$, we choose elements $b_{m,i} \in G$, $i \in I_m$, such that $vb_{m,i} = m$ and the elements $\mathrm{co}_m b_{m,i}$, $i \in I_m$, form an $\mathbb{F}_p$-basis of $\mathrm{C}_m(G, v)$. Then by Lemma 4.36, the elements $b_{m,i}$ are p'-valuation independent. Let $a \in G$ and $m = va$. Then there are finitely many elements $b_{m,i}$ and integers $0 < n_{m,i} < p$ such that $a - \sum n_{m,i} b_{m,i}$ has value $\geq m + 1$. By a finite repetition of this procedure, we find finitely many elements $b_{j,i}$ and integers

$0 < n_{j,i} < p$, $m \leq j < 0$, such that $a = \sum_{j,i} n_{j,i} b_{j,i}$. Hence, $\mathcal{B} = \{b_{m,i} \mid m \in vG \wedge i \in I_m\}$ generates G. By what we have shown before, $\mathcal{B}$ is thus a straight basis of G. □

Finally, let us say some words about linear independence of the elements of p-bases and straight bases. Given elements $b_1, \ldots, b_m \in G$, we will call them **p'-independent** if no nontrivial $\mathbb{Z}$-linear combination $\sum_{i=1}^{m} n_i b_i$, with n_i integers prime to p, equates to zero. p'-valuation independence implies p'-independence. Thus, the elements of p-bases and straight bases are p'-independent. But contrary to the case of a p-basis, for fixed $m \in vG$, the elements of value m of a straight basis $\mathcal{B}$ are even $\mathbb{Z}$-independent. This is seen as follows. Let $\sum_{i=1}^{k} n_i b_i$ be any $\mathbb{Z}$-linear combination with $b_i \in \mathcal{B}$ of value m and $n_i b_i \neq 0$. Let p^n be the highest power of p which divides all n_i, and write $n_i = n_i' p^n$. Note that $m + n = vp^n b_i \leq v n_i b_i < \infty$. Without loss of generality, we may assume that there is j with $1 < j \leq k$ such that $n_1', \ldots, n_j'$ are prime to p and $n_{j+1}', \ldots, n_m'$ are divisible by p. Now $v \sum_{i=1}^{j} n_i' b_i = m$ and $v \sum_{i=j+1}^{k} n_i' b_i \geq m + 1$, which shows that $v \sum_{i=1}^{k} n_i' b_i = m$. Consequently, $v \sum_{i=1}^{k} n_i b_i = m + n < \infty$, which proves that $\sum_{i=1}^{k} n_i b_i \neq 0$.

Exercise 4.5 *Show that p'-independence of the elements of $\mathcal{B}$ does not imply the uniqueness of a representation $a = \sum n_i b_i$ with $b_i \in \mathcal{B}$ and n_i prime to p.*

Exercise 4.6 *Let G be a divisible p-group. Construct a p-basis $\mathcal{B}$ of G (hint: start from an $\mathbb{F}_p$-basis of the socle of G). Show that in the case of divisible p-groups, condition 1) for a p-basis is equivalent to 1′) the elements of $\mathcal{B}$ are p'-independent.*

4.6 Embedding lemmas

Valuation independent sets can serve to obtain isomorphisms between valued modules, as the following lemma shows.

Lemma 4.43 *Let $\mathbf{N}$ and $\mathbf{N}'$ be valued R-modules containing $\mathbf{M}$ as a common valued submodule. Let $\mathcal{B} \subset N \setminus \mathrm{Exe}\mathbf{N}$ and $\mathcal{B}' \subset N' \setminus \mathrm{Exe}\mathbf{N}$ be R-valuation independent over $\mathbf{M}$. Suppose that there exists an R-linear embedding*

$$\sigma : \mathrm{sk}\,\mathbf{N} \longrightarrow \mathrm{sk}\,\mathbf{N}'$$

over $\mathrm{sk}\,\mathbf{M}$ and a bijection

$$\psi : \mathcal{B} \longrightarrow \mathcal{B}'$$

preserving σ, i.e.

$$\forall b \in \mathcal{B} : (v\psi b, \mathrm{co}\,\psi b) = \sigma(vb, \mathrm{co}\, b) .$$

Then ψ extends R-linearly to an isomorphism

$$\iota : M + R\mathcal{B} \longrightarrow M + R\mathcal{B}'$$

of valued R-modules which also preserves σ.

If in addition, $\mathbf{N}$ and $\mathbf{N}'$ are ordered and v is their natural valuation, and if σ is an embedding of ordered skeletons (i.e. it preserves the ordering on the components), then ι also preserves the ordering.

Proof: Since $\mathcal{B}$ and $\mathcal{B}'$ are R-independent over M, ψ extends R-linearly to an isomorphism ι between $M + R\mathcal{B}$ and $M + R\mathcal{B}'$. We have to show that it is an isomorphism of valued R-modules which preserves σ. Given $b \in M + R\mathcal{B}$, let us write b in the form (4.4). We assume (after a suitable renumeration) that precisely $r_1b_1, \ldots, r_mb_m, a$ are the summands of least value γ in (4.4). If va is smaller than the value of the other summands, then omit the elements r_ib_i in the following; if on the other hand, va is greater than the value of some summand r_ib_i, then omit a everywhere. By our hypothesis on ψ it follows that precisely $r_1\psi b_1, \ldots, r_m\psi b_m, a$ are the summands of least value in $\iota b = \sum r_i\psi b_i + a$. Since $\mathcal{B}$ is R-valuation independent over $\mathbf{M}$, we have $(vb, \operatorname{co} b) = r_1(vb_1, \operatorname{co} b_1) + \ldots + r_m(vb_m, \operatorname{co} b_m) + (va, \operatorname{co} b)$. Since also $\mathcal{B}'$ is R-valuation independent over $\mathbf{M}$,

$$\begin{aligned}(v\iota b, \operatorname{co} \iota b) &= r_1(v\psi b_1, \operatorname{co} \psi b_1) + \ldots + r_m(v\psi b_m, \operatorname{co} \psi b_m) + (va, \operatorname{co} b)\\ &= r_1\sigma(vb_1, \operatorname{co} b_1) + \ldots + r_m\sigma(vb_m, \operatorname{co} b_m) + (va, \operatorname{co} b)\\ &= \sigma(r_1(vb_1, \operatorname{co} b_1) + \ldots + r_m(vb_m, \operatorname{co} b_m) + (va, \operatorname{co} b)) = \sigma(vb, \operatorname{co} b)\ .\end{aligned}$$

This shows that ι preserves σ, which in turn yields that ι is an isomorphism of valued R-modules.

The last assertion follows directly from Corollary 3.26. □

Let us apply this lemma to ordered abelian groups. As we have mentioned already, with their natural valuation they are ordinary valued $\mathbb{Z}$-modules.

Corollary 4.44 *Let $(G, <)$ and $(G', <)$ be ordered abelian groups, equipped with their natural valuation v and containing $(H, <)$ as a common ordered subgroup. Let $\mathcal{B} \subset G$ and $\mathcal{B}' \subset G'$ be $\mathbb{Z}$-valuation independent over $\mathbf{H}$. Suppose that there exists a $\mathbb{Z}$-linear embedding $\sigma : \operatorname{sk}\mathbf{G} \to \operatorname{sk}\mathbf{G}'$ of ordered skeletons over $\operatorname{sk}\mathbf{H}$ and let ψ be as in the lemma. Then ψ extends linearly to an isomorphism $\iota : H + \mathbb{Z}\mathcal{B} \to H + \mathbb{Z}\mathcal{B}'$ of ordered abelian groups which also preserves σ.*

By Lemma 3.31, there is a unique extension of the orders from $M + \mathbb{Z}\mathcal{B}$ to $M + \mathbb{Q}\mathcal{B}$ and from $M + \mathbb{Z}\mathcal{B}'$ to $M + \mathbb{Q}\mathcal{B}'$. Consequently, ι also extends to an isomorphism $\iota : M + \mathbb{Q}\mathcal{B} \to M + \mathbb{Q}\mathcal{B}'$ of ordered abelian groups, and if they are contained in $\mathbf{N}$ resp. $\mathbf{N}'$ then ι preserves σ on their skeletons.

The following is an interesting special case of the foregoing corollary; for an application, see [KU4].

Corollary 4.45 *Let $(G, <)$, $(G', <)$, $(H, <)$ and v be as in the foregoing corollary. Suppose that b_i and b'_i, $i \in I$, are elements of G and G' respectively, such that*

1) all vb_i, $i \in I$, are distinct and not contained in vH,

2) $\operatorname{sign}(b_i) = \operatorname{sign}(b'_i)$ for all $i \in I$, and the assignment $vb_i \mapsto vb'_i$ establishes an order isomorphism

$$\rho :\ vH \cup \{vb_i \mid i \in I\} \to vH \cup \{vb'_i \mid i \in I\}\ .$$

Then the assignment $b_i \mapsto b'_i$ defines order and valuation preserving isomorphisms

$$\begin{aligned}H + \sum_{i\in I} \mathbb{Z}b_i &\longrightarrow H + \sum_{i\in I} \mathbb{Z}b'_i\\ H + \sum_{i\in I} \mathbb{Q}b_i &\longrightarrow H + \sum_{i\in I} \mathbb{Q}b'_i\end{aligned}$$

both preserving ρ. (Here, the groups are endowed with the restrictions of the orders of the divisible hulls of G and G' which are uniquely determined by those of G and G'.) Note that $vH \cup \{vb_i \mid i \in I\}$ is the value set of both groups on the left hand side, and $vH \cup \{vb'_i \mid i \in I\}$ is the value set of both groups on the right hand side.

Alternatively, assume 1) and the following condition:

2') the assignment $b_i \mapsto b'_i$ establishes an isomorphism

$$\iota : H \cup \{b_i \mid i \in I\} \to H \cup \{b'_i \mid i \in I\}$$

as ordered sets.

Then ι extends linearly to order and valuation preserving isomorphisms of the above groups.

The equivalence of the conditions 2) and 2') is seen as follows. Let be given an $i \in I$. Condition 1) says that $vb_i \notin vH \cup \{vb_j \mid i \neq j \in I\}$. Hence, (3.7) gives $|b_i| > |a| \Leftrightarrow vb_i < va$ for all $a \in H \cup \{b_j \mid i \neq j \in I\}$. Note that by virtue of $H \cap \{b_i \mid i \in I\} = \emptyset$, the order on $H \cup \{b_i \mid i \in I\}$ is already determined by the cuts induced in H by the elements b_i and by the order on the set $\{b_i \mid i \in I\}$. Further, observe that every element of G of the same sign and value as b_i will induce the same cut as b_i in H. Cuts induced in H by elements of value not in vH, have special properties; such cuts are studied in the preliminaries of [KU4] under the name **v-cuts**.

Another corollary to Lemma 4.43 is the following characterization of those valued vector spaces which are isomorphic to the Hahn sum over their skeleton:

Corollary 4.46 *Let (V, v) be a valued K-vector space with value preserving scalar multiplication which admits a K-valuation basis. Then there is an isomorphism from (V, v) onto the valued Hahn sum over its skeleton; this isomorphism can be chosen such that it preserves the coefficient map.*

If in addition, V is an ordered K-vector space and v is its natural valuation, then the isomorphism can be chosen as to preserve the ordering.

Conversely, every valued Hahn sum whose components are K-vector spaces is a valued K-vector space with value preserving scalar multiplication which admits a K-valuation basis.

Proof: Assume that (V, v) admits a K-valuation basis $\mathcal{B}$. Then by Corollary 4.29 where we take $W = 0$, the set $\underline{\mathcal{B}}_\alpha = \{\mathrm{co}_\alpha b \mid b \in \mathcal{B} \text{ and } vb = \alpha\}$ forms a basis of $\mathrm{C}^\alpha\mathbf{V}$ for every $\alpha \in vV$. Using these bases of the components, we form a K-valuation basis $\mathcal{B}' = \{e_\zeta \mid \zeta \in \underline{\mathcal{B}}_\alpha\mathbf{V} \wedge \alpha \in vV\}$ of the Hahn sum $\coprod_{\alpha \in vV} \mathrm{C}^\alpha\mathbf{V}$. Now our first and second assertion follow from Lemma 4.43, where we take σ to be the identity.

The converse was already proved in the last section. □

According to Theorem 4.31, every countably-dimensional valued vector space with value preserving scalar multiplication admits a valuation basis. Hence, the foregoing corollary yields the following theorem of R. Brown [BRW1]:

Theorem 4.47 *Two countably-dimensional valued vector spaces with value preserving scalar multiplication are isomorphic if and only if their skeletons are isomorphic. Similarly, two countably-dimensional ordered vector spaces are isomorphic if and only if their natural skeletons are isomorphic.*

The following example is meant to prevent a common error:

Example 4.48 Given an element a in a valued module, let us call the representation $a = a_1 + \ldots + a_n$ a **convex decomposition of a of length** n if all values $va_1, \ldots, va_n$ are distinct. It is *not* true that every element in every Hahn sum has a convex decomposition of maximal length. As soon as the value set admits an infinite ascending chain β_i, $i \in \mathbb{N}$ of values starting from $\beta_1 = va$, we can carry out the following construction. We choose elements b_i such that $vb_i = \beta_i$ and $b_1 = a$. To obtain a decomposition of a of length n for arbitrary $n \in \mathbb{N}$, we just have to set $a_i = b_i - b_{i+1}$ for $i < n$ and $a_n = b_n$. The sum of the a_i's is a, and their values $va_i = vb_i$ are all distinct.

For instance, this construction works for every nonzero element in the Hahn sum $\coprod_{\mathbb{N}} \mathbb{Q}$, which is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication. $\diamondsuit$

Remark 4.49 This example shows a theorem of Banaschewski ([BAN], Satz, p. 435) to be false. This theorem characterizes the Hahn sums to be those valued vector spaces with value preserving scalar multiplication in which every element admits a convex decomposition of maximal length. (Note that Banaschewski uses the names "schwache Hahnsche Summe" for the Hahn sum and "Hahnsche Summe" for the Hahn product.) The gap in Banaschewski's proof is that he assumes as obvious that a tuple of the Hahn product with only one nonzero entry admits only convex decompositions of length 1. (His verification reads "wie man sofort sieht"; we have carefully avoided to use corresponding english phrases in this book, in order not to mark our errors.)

Most remarkably, Fleischer [FL2] cites the wrong theorem of Banaschewski in order to disprove a theorem of Hill and Mott ([HI–MO], Theorem 5.1). This theorem states that a countable ordered abelian group G whose components are all isomorphic to $\mathbb{Z}$, can be embedded in the ordered Hahn sum $\coprod_{vG} \mathbb{Z}$ (without claiming that the resulting extension is immediate). Fleischer gives an interesting example which, because of Banaschewski's error, does not show what he wants but rather lends credibility to the theorem of Hill and Mott. Consider the Hahn product $\mathbf{H}_{\mathbb{N}}\mathbb{Z}$ and its subgroup of all cofinitely constant tuples. The latter is generated over the Hahn sum $\coprod_{\mathbb{N}} \mathbb{Z}$ by just one element: the tuple $(1, 1, 1, \ldots)$ which has a 1 at every entry. According to Banaschewski's theorem, this group can not be isomorphic to the Hahn sum over its skeleton and hence, it cannot admit a $\mathbb{Z}$-valuation basis. But it does: the elements $(0, \ldots, 0, 1, 1, 1, \ldots)$ which are constantly 1 from the i-th entry on and 0 before, for $i \in \mathbb{N}$. (I am endebted to S. Kuhlmann for bringing my attention to the cited articles, for detecting the mentioned errors and for providing the valuation basis.)

Let us consider once more the assertion of Lemma 4.22. It says that under the assumptions of that lemma, an extension $(W, v) \subset (V, v)$ can be broken up into one which is generated by a system of valuation independent elements and one which is immediate. This principle will play an important role throughout this book. Having treated the first type of extensions in the last lemma and corollary, the question arises what we can say about immediate extensions. By the characterization of immediate extensions of ultrametric spaces given in Lemma 2.21 we know that the approximation type of every element in such an extension is immediate. On the other hand, we have already Lemma 2.38 at hand. If $\mathbf{M}$ is a valued module, $\mathbf{N}$ an immediate and $\mathbf{N}'$ an arbitrary extension of $\mathbf{M}$, and if $b \in N$ and $b' \in N'$ have the same approximation type over $\mathbf{M}$, then this lemma tells us that $v(b-a) = v(b'-a)$ for all $a \in M$. The question is, whether also $v(rb-a) = v(rb'-a)$ for all $a \in M$ and all $r \in R$ because then, $b \mapsto b'$ will define an isomorphism of the valued modules $\mathbf{M} + Rb$ and $\mathbf{M} + Rb'$ over $\mathbf{M}$. If both $\mathbf{N}$ and $\mathbf{N}'$ are valued vector spaces with value preserving scalar multiplication, then we are done: if $r = 0$, there is nothing to show, and if $r \neq 0$, then $v(rb - a) = v(b - r^{-1}a) = v(b' - r^{-1}a) = v(rb' - a)$. But we want to prove a little bit more.

Lemma 4.50 *Let* $\mathbf{N}$, $\mathbf{N}'$ *be torsion free ordinary valued* R*-modules, containing a common valued submodule* $\mathbf{M}$ *such that* $\mathbf{M} \subset \mathbf{N}$ *is an immediate extension. Assume that for every* $r \in R$*, the* R*-module* $r\mathbf{M}$ *is nice in* $\mathbf{M}$ *(note that this condition is void if* R *is a field). Given elements* $b \in N \setminus M$ *and* $b' \in N'$ *such that* $\operatorname{appr}(b, M) = \operatorname{appr}(b', M)$*, then the assignment* $b \mapsto b'$ *induces an embedding of* $(M + Rb, v)$ *in* $\mathbf{N}'$ *over* $\mathbf{M}$*.*

If in addition, $\mathbf{N}$ *and* $\mathbf{N}'$ *are ordered and* v *is their natural valuation, then the embedding also preserves the ordering.*

Proof: From Lemma 2.38 we know that $v(b-a) = v(b'-a)$ for all $a \in M$. We show that $v(rb-a) = v(rb'-a)$ for all $r \in R \setminus \{0\}$ and $a \in M$. Since $r\mathbf{M}$ is assumed to be nice in $\mathbf{M}$, there is some $c \in M$ such that $v(a-rc) = \max\{v(a-rd) \mid d \in M\}$. Since $\mathbf{M} \subset \mathbf{N}$ is an immediate extension, Lemma 3.9 shows that there is some $d \in M$ such that $v(b-d) > v(b-c)$ (note that $b-c \neq 0$ since $b \notin M$). By assumption, the R-module N is torsion free and $r \neq 0$, so we have $r(b-c) \neq 0$. Hence by Lemma 4.3, $v(b-d) > v(b-c)$ yields that $v(rb-rd) > v(rb-rc)$. Now we compute: $v(rb-rd) > v(rb-rc) \geq \min\{v(rb-a), v(a-rc)\} \geq \min\{v(rb-a), v(a-rd)\}$, showing that $v(rb-a) = v(a-rd)$ by virtue of (V=). Since we know already that $v(b-c) = v(b'-c)$ and $v(b-d) = v(b'-d)$, we can replace b by b' everywhere in the above computation. This gives $v(rb-a) = v(a-rd) = v(rb'-a)$, as desired. Furthermore, we observe that $v(rb-a) = v(rb'-a) \leq v(rb-rc) < \infty$ for all $r \in R \setminus \{0\}$ and $a \in M$. This proves $rb \notin M$ and $rb' \notin M$. Thus, $b \mapsto b'$ induces an embedding of $M + Rb$ in N' over M, and our computation of the values shows that this embedding preserves the valuation.

Since $\mathbf{M} \subset \mathbf{M} + Rb$ is immediate as a subextension of the immediate extension $\mathbf{M} \subset \mathbf{N}$, the same is true for the extension $\mathbf{M} \subset \mathbf{M} + Rb'$. Hence trivially, the embedding preserves the identity on their skeletons, which are equal to the skeleton of $\mathbf{M}$. If $\mathbf{N}$ and $\mathbf{N}'$ are ordered and v is their natural valuation, then by Corollary 3.26 the embedding will also preserve the ordering. □

The significance of this lemma is that it deduces information about the valued modules $M + Rb$ and $M + Rb'$ from information that is encoded in the ultrametric spaces $M \cup \{b\}$ and $M \cup \{b'\}$. This principle of reduction to simpler structures is the basic idea in the applications of approximation types and will be met in different forms at several instances in this book (cf. e.g. the Ax–Kochen–Ershov Principle introduced in Section 14.2 and Lemma 14.8).

Let us consider for a moment the assumptions of the lemma. If R is a (skew) field, then for all $r \in R \setminus \{0\}$ we have $r\mathbf{M} = \mathbf{M}$, which is nice in $\mathbf{M}$. Having in mind the ordered abelian groups, let us also consider (left) valued R-modules $\mathbf{M}$ with value preserving scalar multiplication, and assume in addition that the ring R admits a field K of (left) fractions (cf. [COHN2], chapter 0, section 0.8). Then we can consider the K-module $K \otimes_R M$. Because of value preserving scalar multiplication, M is R-torsion free and thus a submodule of $K \otimes_R M$ (cf. [COHN2], chapter 0, Proposition 9.1). There is a unique extension of the valuation from M to $K \otimes_R M$ such that the scalar multiplication remains value preserving. Indeed, for every element $b \in K \otimes_R M$ there is some $r \in R$ such that $rb \in M$, and we have $vb = vrb$. Equipped with this extension of the valuation, the K-module $K \otimes_R M$ will be denoted by $K \otimes_R \mathbf{M}$. (Without the assumption of value preserving scalar multiplication, existence and uniqueness of such an extension are not a priori guaranteed.) Now if $\mathbf{M}$ is nice in $K \otimes_R \mathbf{M}$, then for every $r \in R \setminus \{0\}$ and every $a \in M$ there is some $c \in M$

such that $v(r^{-1}a - c) = \max\{v(r^{-1}a - d) \mid d \in M\}$. Since the scalar multiplication is value preserving, we find $v(a - rc) = \max\{v(a - rd) \mid d \in M\}$. Hence: *if* $\mathbf{M}$ *is nice in* $K \otimes_R \mathbf{M}$*, then* $r\mathbf{M}$ *is nice in* $\mathbf{M}$ *for every* $r \in R \setminus \{0\}$. With the assumptions modified correspondingly, the lemma can now be applied to ordered abelian groups, with v the natural valuation, $R = \mathbb{Z}$ and $K = \mathbb{Q}$. In this case, $K \otimes_R \mathbf{M}$ is just the divisible hull of M, equipped with the natural valuation with respect to the unique extension of the ordering (cf. Lemma 3.31). For divisible ordered abelian groups, the niceness condition is trivially satisfied. Another important class of ordered abelian groups satisfying this condition is the class of $\mathbb{Z}$-groups:

Lemma 4.51 *Every $\mathbb{Z}$-group, equipped with its natural valuation, is nice in its divisible hull.*

Proof: Let $(G, <)$ be a $\mathbb{Z}$-group with smallest positive element g. Let $a \in G$ and $n \in \mathbb{N}$; we wish to show that $\{v(\frac{a}{n} - d) \mid d \in G\}$ admits a maximum. According to Lemma 3.33 we write $a = nb - mg$ with $b \in G$ and $m \in \mathbb{N}$. If n divides m, then $\frac{a}{n} \in G$ and the assertion is trivial. If n does not divide m, then $\frac{a}{n} = b - \frac{m}{n}g \notin G$. In this case, $\infty \notin \{v(\frac{a}{n} - d) \mid d \in G\}$, and hence, the maximum is $v\frac{m}{n}g = vg = v(\frac{a}{n} - b)$ (which is the maximum of vG). □

Now we are able to prove the Hahn Embedding Theorem for valued and ordered vector spaces. In particular, it shows that every maximal component-compatibly valued vector space is isomorphic to the Hahn product over its skeleton, and thus, it is spherically complete. (Recall that every component-compatibly valued vector space has value preserving scalar multiplication).

Theorem 4.52 *Let K be a field or a skew field and (V, v) an ordinary valued K-vector space with value preserving scalar multiplication. Then there is an isomorphism ι of the valued Hahn sum $\coprod_{\alpha \in vV} \mathrm{C}^\alpha \mathbf{V}$ onto a valued subvector space (V', v) of (V, v) which also preserves the coefficient map. On the other hand, ι^{-1} can be extended to an embedding of (V, v) in the valued Hahn product $\mathbf{H}_{\alpha \in vV} \mathrm{C}^\alpha \mathbf{V}$ which also preserves the coefficient map. This embedding is onto if and only if (V, v) is maximal. Hence, for component-compatibly valued vector spaces the properties "maximal", "spherically complete" and "being (isomorphic to) a Hahn product" coincide.*

If in addition, V is an ordered K-vector space and v is its natural valuation, then the embeddings can be chosen as to preserve the ordering.

Proof: By Lemma 4.22 where we take $W = 0$, we obtain the existence of some valued subvector space (V', v) of (V, v) which admits a K-valuation basis $\mathcal{B}$ and such that $(V', v) \subset (V, v)$ is an immediate extension. Hence, (V', v) has the same skeleton as (V, v). By Corollary 4.46, (V', v) is isomorphic to the Hahn sum over this skeleton, that is, to $\coprod_{\alpha \in vV} \mathrm{C}^\alpha \mathbf{V}$. The isomorphism can be chosen as to preserve the coefficient map and, in the ordered case, to preserve the ordering.

We identify the Hahn sum with its isomorphic image (V', v). We wish to show that (V, v) is embeddable in $\mathbf{H}_{\alpha \in vV} \mathrm{C}^\alpha \mathbf{V}$ over V'. Consider the set of all such embeddings of subspaces of (V, v). A union over an ascending chain of such embeddings is again an embedding of a subspace of (V, v). By Zorn's Lemma, there is a maximal subspace (V'', v) which is embeddable in this way. We have to show $V'' = V$. Otherwise, there is some $b \in V \setminus V''$. Let us identify (V'', v) with its image in $\mathbf{H}_{\alpha \in vV} \mathrm{C}^\alpha \mathbf{V}$. Since the extension $(V'', v) \subset (V, v)$ is immediate, $\mathrm{appr}\,(b, V'')$ is immediate by Lemma 2.21. Since

$\mathbf{H}_{\alpha\in vV}\,\mathrm{C}^{\alpha}\mathbf{V}$ is spherically complete (Lemma 3.14), Lemma 2.42 shows that $\mathrm{appr}\,(b,V'')$ is realized by some element b' in the Hahn product, i.e., $\mathrm{appr}\,(b,V'')=\mathrm{appr}\,(b',V'')$. Now Lemma 4.50 shows that also $(V''+Kb,v)$ can be embedded, contrary to our maximality assumption. This shows that (V,v) can be embedded over V' in the Hahn product. Since $(V',v)\subset(V,v)$ is immediate, for every $b\in V$ there is some $a\in V'$ such that $v(b-a)>vb$. If b' is the image of b under the embedding, then also $v(b'-a)>vb=vb'$. It follows that $\mathrm{co}_{vb}b=\mathrm{co}_{vb}a=\mathrm{co}_{vb'}a=\mathrm{co}_{vb'}b'$, showing that the embedding also preserves the coefficient map. In the ordered case, all embeddings are taken to be order preserving, and we use the corresponding additional assertion of Lemma 4.50.

The Hahn product $\mathbf{H}_{\alpha\in vV}\,\mathrm{C}^{\alpha}\mathbf{V}$ is an immediate extension of the embedded image of (V,v). If (V,v) is maximal, then this extension must be trivial. Conversely, if the embedding is an isomorphism onto the Hahn product, then in view of Theorem 4.8 and Lemma 3.14, (V,v) must be maximal and spherically complete. □

As a corollary, we obtain the following counterpart to Brown's Theorem (Theorem 4.47):

Theorem 4.53 *Two maximal valued vector spaces with value preserving scalar multiplication are isomorphic if and only if their skeletons are isomorphic. Similarly, two maximal ordered vector spaces are isomorphic if and only if their natural skeletons are isomorphic.*

Since a divisible ordered abelian group equipped with its natural valuation is a valued $\mathbb{Q}$-vector space with value preserving scalar multiplication, we obtain:

Corollary 4.54 *Every divisible ordered abelian group is embeddable inbetween the ordered Hahn sum and the ordered Hahn product over its natural skeleton; the embedding is an isomorphism onto the Hahn product if and only if the group is maximal with respect to its natural valuation. Hence, for divisible ordered abelian groups the properties "maximal", "spherically complete" and "being (isomorphic to) an ordered Hahn product" coincide.*

Given an arbitrary ordered abelian group $(G,<)$, the ordering can be extended to the divisible hull of $\tilde{G}$, according to Lemma 3.31. From the preceding Corollary, we thus obtain

Corollary 4.55 *Every ordered abelian group is embeddable in the ordered Hahn product over the natural skeleton of its divisible hull.*

From Lemma 3.31 we also know the natural skeleton of $\tilde{G}$. In fact, we have $v\tilde{G}=vG$ and for every $\alpha\in vG$, the α-component of $\tilde{G}$ is the divisible hull of the α-component of G. So the corollary shows that $(G,<)$ is embeddable in the ordered Hahn product $\coprod_{\alpha\in vG}\mathrm{C}^{\alpha}\tilde{\mathbf{G}}$. If we wish to replace these components by a uniform one, we can employ the following theorem which is due to Hlder [HL]:

Theorem 4.56 *Every archimedean ordered abelian group can be embedded in the ordered additive group of the reals.*

For the proof and additional remarks, see [PC1], Chapter I, §3. Cf. also Exercise 4.7. This lemma shows that every component $\mathrm{C}^{\alpha}\tilde{\mathbf{G}}$ can be viewed as an ordered subgroup of $\mathbb{R}$. Thus, the ordered Hahn product $\coprod_{\alpha\in vG}\mathrm{C}^{\alpha}\tilde{\mathbf{G}}$ is an ordered subgroup of the ordered Hahn product $\coprod_{\alpha\in vG}\mathbb{R}$. (At this point, we should note that all embeddings of ordered abelian groups that have ocurred so far can be chosen as to preserve also the natural valuation.) We have now deduced the original Hahn Embedding Theorem:

Corollary 4.57 *Every ordered abelian group $(G,<)$ with natural valuation v is embeddable in the ordered Hahn product $\mathbf{H}_{\alpha\in vG}\,\mathbb{R}$.*

Remark 4.58 Hahn proved this theorem in 1907, cf. [HAHN]. For a brief history of generalizations and different proofs and for a proof following Banaschewski's approach, see Prie-Crampe [PC1]. Corollary 4.54 is due to Banaschewski [BAN]. Our own valuation theoretical approach follows the spririt of Gravett's papers [GRA1], [GRA2].

It is not true that an arbitrary ordered abelian group is embeddable in the ordered Hahn product over its own natural skeleton. Apparently, the first counterexample was given by B. Gordon, as cited by A. H. Clifford in [CLI]. Contrary to the statements in P. Ribenboim's papers [RIB5] and [RIB6], it is not even true if the group is regular in the sense of these papers. Counterexamples were given by P. Hill and J. L. Mott in [HI–MO]. For instance, there are $\mathbb{Z}$-groups of rank 2 and rational rank 2 with archimedean components $\mathbb{Z}$ and $\mathbb{Q}$ such that the convex subgroup $\mathbb{Z}$ is not a direct summand (cf. Theorem 3.1 and Proposition 3.2 of [HI–MO]). It should be noted that if the group is represented as a subgroup of $\mathbb{Q}\times\mathbb{Q}$, the projections on the second component will not coincide with the archimedean component $\mathbb{Z}$. Otherwise, $\mathbb{Z}$ would be a direct summand, as was pointed out by G. Sabbagh. In this book, we use the word "component" in the sense of α-components and archimedean components, and these may differ significantly from the **components** which are understood as images under the coordinate projections of tuples.

For ordered abelian groups, the above lemmas are only of interest in the case where the natural valuation is nontrivial, i.e. where the ordering on the group is non-archimedean. The results are void for the archimedean case, which in particular is of interest for the embedding of components. So let us give an auxiliary embedding lemma for this case, without aiming at the best possible result.

Lemma 4.59 *Let $(G,<)\subset(H,<)$ and $(G,<)\subset(H',<)$ be two extensions of ordered abelian groups and $b\in H\setminus G$, $b'\in H'\setminus G$. Assume further that G is divisible. If b and b' induce the same cut in G, then the assignment $b\mapsto b'$ defines order preserving isomorphisms from $G+\mathbb{Z}b$ onto $G+\mathbb{Z}b'$ and from $G+\mathbb{Q}b$ onto $G+\mathbb{Q}b'$.*

Proof: Since $b,b'\notin G$, the assignment $b\mapsto b'$ defines isomorphisms from $G+\mathbb{Z}b$ onto $G+\mathbb{Z}b'$ and from $G+\mathbb{Q}b$ onto $G+\mathbb{Q}b'$. Let (Λ_1,Λ_2) be the cut induced by b and b' in G. Given $m,n\in\mathbb{N}$ and $a\in G$, then $\frac{m}{n}a\in G$ and $b>\frac{m}{n}a \;\Leftrightarrow\; -\frac{m}{n}a\in\Lambda_1 \;\Leftrightarrow\; b'>\frac{m}{n}a$. Hence, $\frac{n}{m}b>a \;\Leftrightarrow\; \frac{n}{m}b'>a$. This also yields that $-\frac{n}{m}b>a \;\Leftrightarrow\; -\frac{n}{m}b'>a$. We have thus shown that the isomorphisms are order preserving. □

Exercise 4.7 *a) Try to prove Lemma 4.59 without the hypothesis that G be divisible. What is the difficulty? Compare with our remark succeeding to Lemma 4.50. Prove Lemma 4.59 under the assumption that for every $n\in\mathbb{N}$, between every two elements of G there is one which is divisible by n. Show that G has this property already if it is p-divisible for an arbitrary prime p.*

b) Prove Theorem 4.56, using Lemma 4.59, Lemma 3.31 and the fact that $\mathbb{R}$ is **Dedekind complete** *(that is, every cut in $\mathbb{R}$ is already realized in $\mathbb{R}$, provided that both sets in the cut are nonempty).*

4.7 Defectless extensions of valued vector spaces

We have defined defectless extensions of valued modules in section 4.3 to be those extensions for which every finite subextension admits a valuation basis. It is a direct consequence of this definition that every subextension of a defectless extension is again defectless. The most important questions arising in this connection are: Is an extension defectless if it admits a valuation basis? If an extension admits a valuation basis, does this also hold for

every subextension? We will consider these questions in the case of ordinary valued vector spaces. In what follows, we let K be an arbitrary field (or a skew field).

Vector spaces have the following **basis exchange property**: If V is a K-vector space with basis $\mathcal{B}$ over W and if V' is a subvector space of V with basis $\mathcal{B}'$ over W, then there is a subset $\mathcal{B}'' \subset \mathcal{B}$ which is a basis of V over V'. This implies that $\mathcal{B}' \cap \mathcal{B}'' = \emptyset$ and that $\mathcal{B}' \cup \mathcal{B}''$ is a basis of V over W. We will now show the same for valuation bases for the case where V' is finite-dimensional over W. In difference to the cited situation, now we also have to show the existence of the K-valuation basis $\mathcal{B}'$.

Let (V, v) be an ordinary valued K-vector space and (W, v) a valued subspace. Assume that $\mathcal{B}$ is a K-valuation basis of (V, v) over W. Let $b \in V \setminus W$. We write $b = a + \sum_i r_i b_i$ with $r_i \in K$, $b_i \in \mathcal{B}$ and $a \in W$. Set $b' := b - a$. Since the elements b_i are K-valuation independent over W, the same holds for $b' = \sum_i r_i b_i$. Hence, $\mathcal{B}' := \{b'\}$ is a K-valuation basis of $(W + Kb, v)$ over W. Let j be some index among the i for which the summand $r_j b_j$ is of minimal value. Set $\mathcal{B}'' := \mathcal{B} \setminus \{b_j\}$. We wish to show that $\mathcal{B}''$ is a K-valuation basis of (V, v) over $W + Kb$. This is seen as follows. Given a finite linear combination $a + rb' + \sum_{i \neq j} r_i' b_i$ with $r, r_i \in K$, $b_i \in \mathcal{B}''$ and $a \in W$, the only critical case appears if $vrb' = \min_{i \neq j}\{va, vr_i' b_i\}$. In this case, we have to verify that the value of the linear combination is equal to vrb'. By Lemma 4.3 we know that $rr_j b_j$ is a summand of minimal value in $rb' = \sum_i r r_i b_i$ since $r_j b_j$ was of minimal value in $b' = \sum_i r_i b_i$. Hence, $vrb' = vrr_j b_j$. Rewriting the given linear combination as $a + rr_j b_j + \sum_{i \neq j}(rr_i + r_i')b_i$, we now see that its value must be vrb' by the valuation independence of the b_i.

Repeating this argument and using Lemma 4.17, the following can be shown by induction on $\dim_K V'/W$:

Lemma 4.60 *Let (V, v) be an ordinary valued K-vector space admitting a K-valuation basis $\mathcal{B}$ over the subspace W. If $W \subset V'$ is a finite subextension of $W \subset V$ (i.e., $\dim_K V'/W < \infty$), then (V', v) admits a K-valuation basis $\mathcal{B}'$ over W and there is $\mathcal{B}'' \subset \mathcal{B}$ such that $\mathcal{B}''$ is a K-valuation basis of (V, v) over V'.*

The lemma shows that the first question stated at the beginning can be answered to the affirmative:

Corollary 4.61 *Let (V, v) be an ordinary valued K-vector space with subspace W. If (V, v) admits a K-valuation basis over W, then $(W, v) \subset (V, v)$ is a defectless extension of valued vector spaces.*

If in the situation of Lemma 4.60, $W \subset V'$ is a subextension of countably infinite dimension, then it admits a basis indexed by the natural numbers, and by the same induction procedure, it can be deduced that V' admits a K-valuation basis over W:

Corollary 4.62 *Let (V, v) be an ordinary valued K-vector space admitting a K-valuation basis $\mathcal{B}$ over the subspace W. Then every subspace (V', v) of (V, v) of countable dimension over W admits a K-valuation basis over W.*

On the other hand, if in the same situation, V' is not of finite dimension over W, then the other part of the assertion of Lemma 4.60 may not remain true:

Examples 4.63 Let K be an arbitrary field and consider the Hahn sum $V := \coprod_{\mathbb{N}} K$, which is an ordinary valued K-vector space with value preserving scalar multiplication.

Let e_i be the tuple which has a 1 at the i-th entry and zeros everywhere else. Then e_i, $i \in \mathbb{N}$, is a K-valuation basis of (V, v) (over the zero subspace). Consider the countably dimensional subspace V' which is spanned by all elements $e_i - e_{i+1}$, $i \in \mathbb{N}$. These elements even form a K-valuation basis of V'. Then $V = V' \oplus Ke_1$. On the other hand, the skeleton of both V and V' is $\mathrm{sk}_{\mathbb{N}} K$ and thus, the extension $(V', v) \subset (V, v)$ is immediate. This shows that (V, v) does not admit a K-valuation basis over V'. Even more, V' is dense in (V, v). This follows from the fact that for every $i \in \mathbb{N}$, the element $e_1 - e_i$ lies in V' and we have $v(e_1 - (e_1 - e_i)) = ve_i = i$.

By a slight modification of this example, given by L. Fuchs in [FU3], one can show that Corollary 4.62 does not remain true if the subextension is not countably generated. Let us now take $V := \coprod_\lambda K$ where λ is a limit ordinal of cofinality type $> \omega$, and let us take V' to be the subspace generated by all $e_i - e_j$ for $i, j \in \lambda$. Assume to the absurd that V' admits a K-valuation basis; it is of the form b_i, $i \in \lambda$ with $vb_i = i$. Consider the elements $e_0 - e_j$ and their unique representations as linear combinations $\sum r_i b_i$, $r_i \in K$. If $j \leq k$, then $v((e_0 - e_j) - (e_0 - e_k)) = v(e_k - e_j) = j$ showing that for all $i < j$, the i-th coefficient must be the same in the representations of both $e_0 - e_j$ and $e_0 - e_k$. We form an increasing sequence j_ν, $\nu \in \mathbb{N}$, of elements in λ as follows. Let $j_0 > 0$. Observe that for all $j > 0$ we have $v(e_0 - e_j) = 0$ and thus, b_0 must appear in the representation of $e_0 - e_j$. If $e_0 - e_{j_\nu} = \sum r_i b_i$, then $e_0 \neq \sum_{i<j_\nu} r_i b_i$ since $e_0 \notin V'$. Hence, we can choose $j_{\nu+1} > v(e_0 - \sum_{i<j_\nu} r_i b_i) =: k$. For $j \geq j_{\nu+1}$, we have $v(e_0 - (e_0 - e_j)) = ve_j = j > k$ showing that b_k must appear in the representation of $e_0 - e_j$. By construction, for every $\nu \in \mathbb{N}$, the representation of $e_0 - e_{j_\nu}$ requires at least ν basis elements b_i of value $< j_\nu$, and these are also required in the representations of all $e_0 - e_j$ for $j \geq j_\nu$. Since the cofinality type of λ is not ω, there must be some $j \in \lambda$ which is larger than all j_ν. But then, the representation of $e_0 - e_j \in V'$ requires infinitely many basis elements b_i, a contradiction. We have proved that V' does not admit a K-valuation basis. Observe that as before, V' is dense in (V, v).

In this last example we have constructed a valued vector space not admitting a valuation basis. There is another method of constructing such vector spaces, using a cardinality argument. For example, consider $\coprod_{\mathbb{N}} \mathbb{Q}$ and $\mathbf{H}_{\mathbb{N}} \mathbb{Q}$. The first is countable since it is a sum of countably many countable groups. Similarly, the skeleton is countable. The second $\mathbb{Q}$-vector space has cardinality $2^{\aleph_0}$ since this is the cardinality of all maps from $\mathbb{N}$ into $\mathbb{Q}$ (note that every subset of $\mathbb{N}$ is wellordered, so every map from $\mathbb{N}$ into $\mathbb{Q}$ is an element of the Hahn product). Since the Hahn product has the same countable skeleton as the Hahn sum, every $\mathbb{Q}$-valuation independent set contained in it must be countable; hence, it can not be a basis of the Hahn product. On the other hand, Theorem 4.31 shows that the Hahn product is a defectless extension of the trivial vector space 0. Even more, Theorem 4.31 in combination with Lemma 4.60 shows that the Hahn product is a defectless extension of every of its finite-dimensional subspaces. ♢

Our example has shown that there exist defectless extensions $(W, v) \subset (V, v)$ of ordinary valued K-vector spaces which admit a subextension $(W, v) \subset (V', v)$ such that $(V', v) \subset (V, v)$ is a proper immediate extension and thus not defectless. On the other hand, we can prove the following **transitivity of defectless extensions**:

Lemma 4.64 *Let $(W, v) \subset (V, v)$ be an extension of ordinary valued K-vector spaces and $(W, v) \subset (V', v)$ a subextension. If $(W, v) \subset (V', v)$ and $(V', v) \subset (V, v)$ are defectless extensions, then so is $(W, v) \subset (V, v)$.*

Proof: Let $(W,v) \subset (V_0,v)$ be a finite subextension of $(W,v) \subset (V,v)$. We have to show that it is defectless. Choose any K-basis $b_1,\ldots,b_n$ of V_0 over W and let V_1 be the K-subvector space of V generated by $b_1,\ldots,b_n$ over V'. Since we have assumed $(V',v) \subset (V,v)$ to be a defectless extension, we know that the finite extension $(V',v) \subset (V_1,v)$ admits a K-valuation basis $\mathcal{B}_1$. To write down the elements $b_1,\ldots,b_n$ as K-linear combinations of the basis elements from $\mathcal{B}_1$ and elements from V', we only need n elements $b'_1,\ldots,b'_n$ from V'. Let V_2 be the K-subvector space of V' generated by $b'_1,\ldots,b'_n$ over W. Since we have assumed $(W,v) \subset (V',v)$ to be a defectless extension, it follows that the finite extension $(W,v) \subset (V_2,v)$ admits a K-valuation basis $\mathcal{B}_2$. Since $\mathcal{B}_1$ is K-valuation independent over (V',v), it is also K-valuation independent over (V_2,v). Hence $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$ is K-valuation independent over (W,v). By construction, the K-subvector space V_3 of V generated by $\mathcal{B}$ over W contains V_0. Since $\mathcal{B}$ is a K-valuation basis of the extension $(W,v) \subset (V_3,v)$, this extension is defectless by Corollary 4.61. Hence, also its subextension $(W,v) \subset (V_0,v)$ is defectless. □

For the conclusion of this section, we wish to discuss the relation of "nice" and "defectless" for ordinary valued vector spaces (which is not quite as nice as for vector spaces with value preserving scalar multiplication).

Lemma 4.65 *Let $(W,v) \subset (V,v)$ be an extension of ordinary valued K-vector spaces. If every K-subvector space (V',v) of (V,v) of arbitrary dimension over W is nice in (V,v), then (V,v) admits a K-valuation basis over W.*

Proof: Take a well-ordered K-basis $\{b_\nu \mid \nu < \nu_0\}$ of V over W, where ν_0 is some ordinal. We construct a K-valuation basis of $(W,v) \subset (V,v)$ by induction on ν. Let V_μ be the K-vector space generated by $\{b_\nu \mid \nu \leq \mu\}$ over W. Assume that we have already constructed valuation bases $\mathcal{B}_\nu$ for all $\nu < \mu$ such that $\mathcal{B}_\nu \subset \mathcal{B}_{\nu'}$ if $\nu \leq \nu'$. If μ is a limit ordinal, then we set $\mathcal{B}_\mu := \bigcup_{\nu<\mu} \mathcal{B}_\nu$. This is a valuation basis over W since every finite set of elements in $\mathcal{B}_\mu$ is already included in some $\mathcal{B}_\nu$ with $\nu < \mu$. Now let $\mu = \mu' + 1$ be a successor ordinal. By hypothesis, $(V_{\mu'},v)$ is nice in (V,v) and hence also in (V_μ,v). Since V_μ is of dimension 1 over $V_{\mu'}$, Lemma 4.21 shows that (V_μ,v) admits a K-valuation basis $\mathcal{B}'_\mu$ over $V_{\mu'}$. It follows that $\mathcal{B}_\mu := \mathcal{B}_{\mu'} \cup \mathcal{B}'_\mu$ is a K-valuation basis of (V_μ,v) over W, cf. Lemma 4.17. (For $\mu = 0$, just replace $(V_{\mu'},v)$ by (W,v) and $\mathcal{B}_{\mu'}$ by $\emptyset$ in this argument.) Now $\mathcal{B}_{\nu_0}$ is the desired K-valuation basis of $(W,v) \subset (V,v)$. □

The following result was proved by F. Delon for the case of valued vector spaces which are induced by valued fields, cf. [DEL5], Théorème, equivalence of a) and b).

Corollary 4.66 *The extension $(W,v) \subset (V,v)$ of ordinary valued K-vector spaces is defectless if and only if every K-subvector space (V',v) of (V,v) of finite dimension over W is nice in (V,v).*

Proof: Let $(W,v) \subset (V,v)$ be defectless and (V',v) a K-subvector space of (V,v) of finite dimension over W. We have to show that (V',v) is nice in every K-subvector space (V'',v) of (V,v) of dimension 1 over V'. But V'' is also of finite dimension over W, and thus, it admits a K-valuation basis over W by hypothesis. Hence, it admits a K-valuation basis over V' by virtue of Lemma 4.60. Corollary 4.14 now shows that (V',v) is nice in (V'',v).

For the converse, assume that every K-subvector space (V',v) of (V,v) of finite dimension over W is nice in (V,v). Given a K-subvector space (V'',v) of (V,v) of finite dimension

over W, we have to show that it admits a K-valuation basis over W. By hypothesis, every K-subvector space (V', v) of (V'', v) of finite dimension over W is nice in (V, v) and thus also in (V'', v). Thus, the existence of the desired valuation basis follows from the foregoing lemma. □

Exercise 4.8 *What happens to the above examples if $\mathbb{N}$ is replaced by $\omega + 1$ and λ is replaced by $\lambda + 1$?*

Exercise 4.9 *Let (V, v) be a valued K-vector space with value preserving scalar multiplication. Prove that every finite-dimensional subspace (W, v) is nice in (V, v) (and consequently, $(W, v) \subset (V, v)$ is defectless). Show that this is not true for all ordinary valued vector spaces.*

4.8 Immediate extensions of valued modules

Let $\mathbf{M} = (M, v)$ be a component-compatibly valued left R-module. Assume $\mathbf{A}$ to be a nontrivial immediate approximation type over $\mathbf{M}$.

For $r \in R$ and $c \in M$, define

$$\Upsilon(\mathbf{A}, r, c) \ :\Leftrightarrow\ \forall \alpha \in \Lambda(\mathbf{A})\, \exists a \in \mathbf{A}_\alpha :\ v(c - ra) \geq \alpha$$

and

$$I_{\mathbf{A}} := \{r \in R \mid \Upsilon(\mathbf{A}, r, c) \text{ for some } c \in M\}\ .$$

Note that "$\exists a \in \mathbf{A}_\alpha :\ v(c - ra) \geq \alpha$" is equivalent to "$\forall a \in \mathbf{A}_\alpha :\ v(c - ra) \geq \alpha$". Indeed, if $a, a' \in \mathbf{A}_\alpha$, then $v(ra - ra') = vr(a - a') \geq v(a - a') \geq \alpha$ and if $v(c - ra) \geq \alpha$, then $v(c - ra') \geq \min\{v(c - ra), v(ra - ra')\} \geq \alpha$. Further, note that $I_{\mathbf{A}}$ always contains 0 since for $r = 0$, the condition is satisfied by $c = 0$.

In the case of value preserving scalar multiplication, the reader may show that the sets $r\mathbf{A}_\alpha$, $\alpha \in \Lambda(\mathbf{A})$, form a nest of balls in rM, and that its intersection is empty since $\mathbf{A}$ is assumed to be a nontrivial immediate approximation type. Then $I_{\mathbf{A}}$ is the collection of all $r \in R$ for which this nest of balls is realized in M. Hence, $r \notin I_{\mathbf{A}}$ if rM is spherically closed in M. Recall that by Lemma 2.16 this holds if and only if rM is nice in M. Recall further that this is always the case if rM is spherically complete.

For arbitrary rings R, we have:

Lemma 4.67 *For every component-compatibly valued left R-module $\mathbf{M}$ and every nontrivial immediate approximation type $\mathbf{A}$ over $\mathbf{M}$, the set $I_{\mathbf{A}}$ is a proper left ideal of R.*

Proof: For every element $r \in I_{\mathbf{A}}$ we choose an element $c_r \in M$ such that $\Upsilon(\mathbf{A}, r, c_r)$. Let $r \in I_{\mathbf{A}}$ and $s \in R$. Then $v(sc_r - sry) \geq v(c_r - ry) \geq \alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, showing that $sr \in I_{\mathbf{A}}$. Now let also $s \in I_{\mathbf{A}}$. Then $v(c_r + c_s - (r + s)y) \geq \min\{v(c_r - ry), v(c_s - sy)\} \geq \alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, showing that $r + s \in I_{\mathbf{A}}$. Finally, the intersection of all $\mathbf{A}_\alpha$ for $\alpha \in \Lambda(\mathbf{A})$ is empty by virtue of Lemma 2.30, because $\mathbf{A}$ is a nontrivial immediate approximation type. This shows that there is no $c_1 \in M$ such that $\Upsilon(\mathbf{A}, 1, c_1)$ (recall that $\exists a \in \mathbf{A}_\alpha : v(c_1 - a) \geq \alpha$ implies that $c_1 \in \mathbf{A}_\alpha$). Consequently, $1 \notin I_{\mathbf{A}}$ and $I_{\mathbf{A}}$ is a proper left ideal of R. □

Lemma 4.68 *Assume that $I_{\mathbf{A}} = \{0\}$. Then there is a component-compatibly valued left R-module $(M \oplus Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M \oplus Rx, v)$ is immediate and x realizes $\mathbf{A}$ in $(M \oplus Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$, then there is an isomorphism $(M \oplus Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y. In particular, y is R-independent and torsion free over M.*

Proof: We let Rx be the free R-module on the generator x and $M \oplus Rx$ the direct sum of M with Rx. We have to extend the valuation v from M to $M \oplus Rx$. So let $c \in M$ and $r \in R$; we have to define $v(c-rx)$. By our assumption on $I_{\mathbf{A}}$, there is some $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c-ra) < \alpha$. Then we set $v(c-rx) = v(c-ra)$. This does not depend on the choice of α and a. Indeed, let $\alpha' \in \Lambda(\mathbf{A})$ and $a' \in \mathbf{A}_{\alpha'}$ such that $v(c-ra') < \alpha'$. Without loss of generality, we may assume that $\alpha \leq \alpha'$. Then $v(ra-ra') = vr(a-a') \geq v(a-a') \geq \alpha$, and $v(c-ra') = v(c-ra+ra-ra') = \min\{v(c-ra), v(ra-ra')\} = v(c-ra)$. In particular, this shows that $v(c-ra') < \alpha$ for all $a' \in \mathbf{A}_\alpha$.

Let us show that our definition yields a group valuation on $M \oplus Rx$. So let $c, c' \in M$ and $r, r' \in R$; we have to show that $v(c-rx-c'+r'x) \geq \min\{v(c-rx), v(c'-r'x)\}$. By what we have just shown, we may choose $\alpha \in \Lambda(\mathbf{A})$ large enough and $a \in \mathbf{A}_\alpha$ such that $v(c-rx) = v(c-ra)$, $v(c'-r'x) = v(c-ra)$ and $v(c-c'-(r-r')x) = v(c-c'-(r-r')a)$. Then we have that $v(c-rx-c'+r'x) = v(c-c'-(r-r')a) \geq \min\{v(c-ra), v(c'-r'a)\} = \min\{v(c-rx), v(c'-r'x)\}$, as required. In the same way, it is shown that $vr'(c-rx) \geq v(c-rx)$, that is, the valuation on $M \oplus Rx$ is component-compatible.

We show that x realizes $\mathbf{A}$. Let $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$. Then we choose $\alpha' > \alpha$ and $a' \in \mathbf{A}_{\alpha'}$ such that $v(a-x) = v(a-a')$. But $v(a-a') \geq \alpha$, hence $v(a-x) \geq \alpha$, as required. Now we can also show that the extension is immediate. Given $c-rx \in M \oplus Rx$, we choose $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c-rx) = v(c-ra) < \alpha$ and $v(a-x) \geq \alpha$. Then $v(c-rx-c+ra) = vr(a-x) \geq v(a-x) \geq \alpha > v(c-rx)$. This proves that for every $b \in M \oplus Rx$ there is some $b_0 \in M$ such that $v(b-b_0) > vb$. By Lemma 3.9 it follows that the extension is immediate.

Now assume that y realizes $\mathbf{A}$. Given $c \in M$ and $r \in R$, we choose again $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c-ra) < \alpha$. On the other hand, $v(a-y) \geq \alpha$ by our assumption, which yields that $v(ra-ry) \geq \alpha$. Hence, we have $v(c-ry) = \min\{v(c-ra), v(ra-ry)\} = v(c-ra) = v(c-rx)$. That is, the map $c-rx \mapsto c-ry$ is value preserving. Consequently, $c-ry \neq 0$ if $c \neq 0$ or $r \neq 0$, showing that this map is an isomorphism of R-modules over M and that y is R-independent over M. □

Lemma 4.69 *Assume that $I_{\mathbf{A}} \neq \{0\}$ is principal. Let r_0 be a generator of $I_{\mathbf{A}}$ and assume that r_0 is not a zero divisor. Then there is a component-compatibly valued left R-module $(M+Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M+Rx, v)$ is immediate, $r_0 x \in M$ and x realizes $\mathbf{A}$ in $(M+Rx, v)$. Moreover, if $(M, v) \subset (M+Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$ and $r_0 y = r_0 x \in M$, then there is an isomorphism $(M+Rx, v) \cong (M+Ry, v)$ of valued R-modules over M which sends x to y.*

Proof: We choose $c_0 \in M$ such that $\Upsilon(\mathbf{A}, r_0, c_0)$. We let Rz be the free R-module on the generator z, and we define N to be the module $(M \oplus Rz)/R(c_0 - r_0 z)$. Observe that $r(c_0 - r_0 z) \in M$ will imply that $rr_0 = 0$. Since R is assumed to be a domain, this in turn will imply that $r = 0$. That is, N contains M as a submodule. Choosing x to be the image of z under the canonical epimorphism, we find that x is a generator of N over M, and we can write $N = M + Rx$. By definition, $r_0 x = c_0 \in M$. We extend the valuation v from M to N as follows. Given the element $c - rx \in N$, we can assume that $r \notin I_{\mathbf{A}}$; otherwise, $c - rx \in M$ and there is nothing to define. But if $r \notin I_{\mathbf{A}}$ then we may define $v(c-rx)$ just as in the proof of the last lemma. The independence of the choice of α and a is shown as it was done there. To prove that $v(c-rx)$ is well-defined, it remains to show that our definition assigns the same value to

$c - rx$ and $c' - r'x$ if $c - rz$ and $c' - r'z$ are equivalent modulo $R(c_0 - r_0 z)$ and if $r \neq 0$, $r' \neq 0$ and $r - r' \neq 0$. But the latter means that there is $r'' \in R$ such that $r - r' = r''r_0$ and $c - c' = r''c_0$. We choose $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c - ra) < \alpha$, $v(c' - r'a) < \alpha$ and $v(c - c' - ra + r'a) = vr''(c_0 - r_0 a) \geq v(c_0 - r_0 a) \geq \alpha$. We find that in view of our definition, $v(c - rx) = v(c - ra) = \min\{v(c' - r'a), v(c - c' - ra + r'a)\} = v(c' - r'a) = v(c' - r'x)$, as required. Having proved this, the triangle inequality follows as in the proof of the last lemma, with one additional provision: We have to take the representation of the elements $c - rx$ and $c' - r'x$ in such a way that $r = 0$ if $r \in I_{\mathbf{A}}$ and $r' = 0$ if $r' \in I_{\mathbf{A}}$, and if $r - r' \in I_{\mathbf{A}}$, then we take $r' = r$. Again with the provision that $r = 0$ if $r \in I_{\mathbf{A}}$, it is shown as in the proof of the last lemma that the extension is immediate, and the proof for the fact that x realizes $\mathbf{A}$ can be taken over literally.

To show that v is component-compatible on $M + Rx$, let $c \in M$ and $r, r' \in R$, $r \notin I_{\mathbf{A}}$. We choose $\alpha \in \Lambda(\mathbf{A})$ and $a \in \mathbf{A}_\alpha$ such that $v(c - ra) < \alpha$. Then $v(c - rx) = v(c - ra) \leq vr'(c - ra) = v(r'c - r'ra)$. If $r'r \notin I_{\mathbf{A}}$, then we may choose α as large as to guarantee that $v(r'c - r'ra) < \alpha$, and we obtain $v(r'c - r'ra) = v(r'c - r'rx)$. Now assume that $r'r \in I_{\mathbf{A}}$ and write $r'r = r''r_0$. Then $v(r''c_0 - r'ra) = vr''(c_0 - r_0 a) \geq \alpha$, and in view of $v(c - ra) < \alpha$ we obtain that $v(c - rx) \leq \min\{v(r'c - r'ra), v(r''c_0 - r'ra)\} = v(r'c - r''c_0)$. But since $r''c_0 = r''r_0 x = r'rx$, we obtain that $v(c - rx) \leq v(r'c - r''c_0) = vr'(c - rx)$. We have thus proved that v is component-compatible on $M + Rx$.

Now assume that y realizes $\mathbf{A}$. The map $c - rx \mapsto c - ry$ is well-defined by virtue of our assumption that $r_0 y = c_0$. We leave it to the reader to show that it is value preserving. Consequently, $c - ry \neq 0$ if $c - rx \neq 0$, showing that this map is an isomorphism of R-modules over M. □

As a consequence of the last two lemmata, and in view of Theorem 4.7 and Lemma 2.40, we obtain the following result of I. Fleischer (cf. [FL1]):

Theorem 4.70 *Let R be a left principal ideal domain. Then a component-compatibly valued left R-module $\mathbf{M}$ is maximal if and only if every immediate approximation type over $\mathbf{M}$ is trivial, that is, if and only if $\mathbf{M}$ is spherically complete.*

As a consequence, every component-compatibly valued left R-module admits an immediate extension which is spherically complete, provided that R is a left principal ideal ring.

With $R = \mathbb{Z}$, we obtain:

Corollary 4.71 *A valued abelian group is maximal if and only if it is spherically complete.*

Remark 4.72 E. Schrner [SCH] has obtained the assertion of this corollary, using an induction method to show that a valued abelian group admits an immediate extension if it admits a nontrivial immediate approximation type.

For the second half of this section, **let $\mathbf{M} = (M, v)$ be a finitely exceptional valued left R-module.** In fact, all what we prove in the following, will already work if for every $r \in R$, the set of exceptional values $\mathrm{Exv}_r\mathbf{M}$ is finite. This is always true in the case of finitely exceptional modules. By Lemma 4.3, we can write $\mathrm{Exv}_r\mathbf{M} = \{va \mid a \in M \wedge \exists a' \in M : \text{(4.1) does not hold}\}$. In particular,

$$va < va' \wedge va \notin \mathrm{Exv}_r\mathbf{M} \implies vra < vra' \vee ra = ra' = 0 .$$

It turns out that in the case of finitely exceptional modules, it is convenient to work with pseudo Cauchy sequences, in view of the following fact. Let $\mathbf{S} := (a_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in some valued module $\mathbf{M}$. If $r \in R$ is finitely exceptional on $\mathbf{M}$, then $\mathrm{Exv}_r\mathbf{M}$ is finite and hence, there is some $\nu_0 < \lambda$ such that $\alpha_\nu := v(a_\nu - a_{\nu+1}) \notin \mathrm{Exv}_r\mathbf{M}$ for $\nu_0 \leq \nu < \lambda$. From this, it follows that $v(ra_\nu - ra_{\nu+1}) < v(ra_{\nu+1} - ra_\mu)$ or $ra_\nu = ra_{\nu+1} = ra_\mu$ whenever $\nu_0 \leq \nu \leq \mu < \lambda$. Hence $(ra_\nu)_{\nu<\lambda}$ is also a pseudo Cauchy sequence, and we will denote it by $\mathbf{S}_r$. Let ν_0 be as in the definition of pseudo Cauchy sequences (Section 2.6).

Assume that $c \in M$ is not a limit of $\mathbf{S}$. Then in view of Lemma 2.50, there is some μ_0 such that $\nu_0 \leq \mu_0 < \lambda$ and $v(c - a_\nu) < v(a_\nu - a_{\nu+1})$ whenever $\mu_0 < \nu < \lambda$. Suppose that $v(c - a_\nu) \notin \mathrm{Exv}_r\mathbf{M}$ for arbitrarily high $\nu \geq \nu_0$. From $v(c - a_\nu) < v(a_\nu - a_{\nu+1})$ with such a $\nu > \mu_0$, we then find that $v(rc - ra_\nu) < v(ra_\nu - ra_{\nu+1})$ or $rc = ra_\nu = ra_{\nu+1}$. In the latter case, it follows that r is not injective on M and that $\mathbf{S}_r$ is eventually constant. Consequently, if $\mathbf{S}_r$ is not eventually constant, then rc will not be a limit of $\mathbf{S}_r$. Hence if c is not a limit of $\mathbf{S}$, but rc is a limit of $\mathbf{S}_r$ and this is not eventually constant, then there is some $\nu_1 \geq \nu_0$ such that $v(a_\nu - a_{\nu+1}) > v(c - a_\nu) = v(c - a_{\nu_1}) \in \mathrm{Exv}_r\mathbf{M}$ whenever $\nu_1 \leq \nu < \lambda$. Note that the same will then hold for every $c' \in M$ with $rc' = rc$. In particular, our consideration shows the following:
If r is ordinary and injective on $\mathbf{M}$ and $\mathbf{S}$ has no limit in M, then $\mathbf{S}_r$ has no limit in rM.

Now assume again that $\mathbf{A}$ is a nontrivial immediate approximation type over $\mathbf{M}$. We choose a pseudo Cauchy sequence $\mathbf{S} = \mathbf{S_A}$ without a limit in $\mathbf{M}$, according to Lemma 2.53. For $r \in R$, we define

$$J_\mathbf{A} := \{r \in R \mid \mathbf{S}_r \text{ has a limit in } \mathbf{M}\} .$$

We leave it to the reader to show that $J_\mathbf{A}$ does not depend on the choice of the pseudo Cauchy sequence $\mathbf{S}$. Since $\mathbf{S}_0$ has 0 as a limit, $J_\mathbf{A}$ always contains 0. But $J_\mathbf{A}$ does not contain 1, because $\mathbf{S}$ has no limit in $\mathbf{M}$. Suppose that c is a limit of $\mathbf{S}_r$, and let $s \in R$. We wish to show that sc is a limit of $\mathbf{S}_{sr}$. By the definition of a limit, we have that $v(c - ra_\nu) < v(c - ra_\mu)$ for all large enough μ, ν such that $\nu < \mu < \lambda$. But for large enough ν we also have that $v(c - ra_\nu) \notin \mathrm{Exv}_s\mathbf{M}$. This yields that $v(sc - sra_\nu) < v(sc - sra_\mu)$ or $sc = ra_\nu = ra_\mu$ for all large enough μ, ν such that $\nu < \mu < \lambda$. In both cases, we obtain that sc is a limit of $\mathbf{S}_{sr}$. We have proved:

Lemma 4.73 *For every finitely exceptional valued left R-module $\mathbf{M}$ and every nontrivial approximation type $\mathbf{A}$ over $\mathbf{M}$, $J_\mathbf{A}$ is a proper subset of R containing 0 but not 1, and satisfying $RJ_\mathbf{A} \subset J_\mathbf{A}$. If $r \notin J_\mathbf{A}$ and the bone of $c - ra_\nu$ is ordinary in $\mathbf{M}$ for $s \in R$, then $sr \notin J_\mathbf{A}$ or sc is a limit of $\mathbf{S}_{sr}$ and $sc = sra_\nu$ for all large enough ν.*

Now we prove the following analogue of Lemma 4.68:

Lemma 4.74 *Assume that $J_\mathbf{A} = \{0\}$. Then there is a finitely exceptional valued left R-module $(M \oplus Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M \oplus Rx, v)$ is immediate and x realizes $\mathbf{A}$ in $(M \oplus Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$, then there is an isomorphism $(M \oplus Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y. In particular, y is R-independent and torsion free over M.*

Proof: We let Rx be the free R-module on the generator x and $M \oplus Rx$ the direct sum of M with Rx. We have to extend the valuation v from M to $M \oplus Rx$. So let $c \in M$

and $0 \neq r \in R$; we have to define $v(c - rx)$. By our assumption on $J_{\mathbf{A}}$, we know that c is not a limit of $\mathbf{S}_r$. So by Lemma 2.50, there exists some μ_0 such that $\nu_0 \leq \mu_0 < \lambda$ and $v(c - ra_\nu) = v(c - ra_{\mu_0})$ whenever $\mu_0 \leq \nu < \lambda$. We set $v(c - rx) = v(c - ra_{\mu_0})$.

Let us show that our definition gives a group valuation on $M \oplus Rx$. If $r \neq 0$, then $ra_\nu \neq c$ since $\mathbf{S}_r$ has no limit in $\mathbf{M}$. Hence $v(c - rx) = v(c - ra_\nu) \neq \infty$, showing that (V0) holds. Now let $c, c' \in M$ and $r, r' \in R$; we have to show that $v(c - rx - c' + r'x) \geq \min\{v(c - rx), v(c' - r'x)\}$. By what we have just shown, we may choose $\nu < \lambda$ large enough such that $v(c - rx) = v(c - ra_\nu)$, $v(c' - r'x) = v(c - ra_\nu)$ and $v(c + c' - (r + r')x) = v(c + c' - (r + r')a_\nu)$. Then we have $v(c - rx - c' + r'x) = v(c + c' - (r + r')a_\nu) \geq \min\{v(c - ra_\nu), v(c' - r'a_\nu)\} = \min\{v(c - rx), v(c' - r'x)\}$, as required.

We show that x is a limit of $\mathbf{S}$ and thus realizes $\mathbf{A}$. For $\nu < \mu < \lambda$, we have that $v(a_\nu - a_\mu) = v(a_\nu - a_{\nu+1})$. Hence our definition yields that $v(a_\nu - x) = v(a_\nu - a_{\nu+1})$, as required. Similarly, it is shown that rx is a limit of $\mathbf{S}_r$ for every $r \in R$. Now we can also prove that the extension is immediate. Given $c - rx \in M \oplus Rx$ with $r \neq 0$, we choose $\nu < \lambda$ such that $v(c - rx) = v(c - ra_\nu) < v(ra_\nu - ra_{\nu+1})$ and $v(ra_\nu - rx) = v(ra_\nu - ra_{\nu+1})$. Then $v(c - rx - c + ra_\nu) = v(ra_\nu - rx) = v(ra_\nu - ra_{\nu+1}) > v(c - rx)$. This proves that for every $b \in M \oplus Rx$ there is some $b_0 \in M$ such that $v(b - b_0) > vb$. By Lemma 3.9 it follows that the extension is immediate. Further, we note that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ for large enough ν. Trivially, this also holds if $r = 0$. Hence if in addition $s \in R$, then the bone of $sc - srx$ is equal to the bone of $sc - sra_\nu$, which in turn only depends on s and the bone of $c - ra_\nu \in M$, if the latter is ordinary. This shows that the bone of $c - rx$ is ordinary in $(M \oplus Rx, v)$ for s if the same holds for the bone of $c - ra_\nu$. We have thus proved that also $(M \oplus Rx, v)$ is finitely exceptional, with the same exceptional bones as $\mathbf{M}$.

Now assume that also y realizes $\mathbf{A}$. It follows that y is a limit of $\mathbf{S}$, like x. Given $c \in M$ and $r \in R$, we choose $\nu < \lambda$ such that $v(c - ra_\nu) < v(ra_\nu - ra_{\nu+1})$. Since ry is a limit of $\mathbf{S}_r$ (which is shown as for rx), we have that $v(ra_\nu - ry) = v(ra_\nu - ra_{\nu+1}) > v(c - ra_\nu)$. Hence, $v(c - ry) = \min\{v(c - ra_\nu), v(ra_\nu - ry)\} = v(c - ra_\nu) = v(c - rx)$. That is, the map $c - rx \mapsto c - ry$ is value preserving. Consequently, $c - ry \neq 0$ if $c \neq 0$ or $r \neq 0$, showing that this map is an isomorphism of R-modules over M and that y is R-independent and torsion free over M. □

The following is the analogue of Lemma 4.69:

Lemma 4.75 *Assume that R is a left euclidean domain with respect to a degree function* deg, *and assume that $J_{\mathbf{A}} \neq \{0\}$. Let $r_0 \neq 0$ be an element of minimal degree* $\deg r_0$ *in $J_{\mathbf{A}}$. Then there is a finitely exceptional valued left R-module $(M + Rx, v)$ extending $\mathbf{M}$ such that $(M, v) \subset (M + Rx, v)$ is immediate, $r_0 x \in M$ and x realizes $\mathbf{A}$ in $(M + Rx, v)$. Moreover, if $(M, v) \subset (M + Ry, v)$ is another extension of valued left R-modules such that y realizes $\mathbf{A}$ and $r_0 y = r_0 x \in M$, then there is an isomorphism $(M + Rx, v) \cong (M + Ry, v)$ of valued R-modules over M which sends x to y.*

Proof: We choose $c_0 \in M$ to be a limit of $\mathbf{S}_{r_0}$. We let Rz be the free R-module on the generator z, and we define N to be the module $(M \oplus Rz)/R(c_0 - r_0 z)$. As in the proof of Lemma 4.69, N is shown to contain M as a submodule. Choosing x to be the image of z under the canonical epimorphism, we find that x is a generator of N over M, and we can write $N = M + Rx$. By construction, $r_0 x = c_0 \in M$. We extend the valuation v from M to N as follows. Let be given the element $c - rx \in N$. We can assume that

$\deg r < \deg r_0$: Since R is a left euclidean domain by hypothesis, for every $r \in R \setminus \{0\}$ there is $r' \in R$ with $\deg r' < \deg r$ such that $r - r' \in Rr_0$. But then, $c - rz$ is equivalent modulo $R(c_0 - r_0 z)$ to $c' - r'z$ for some c'. So every element of N can be given in the form $c - rx$ with $\deg r < \deg r_0$. This yields that $r = 0$ or $r \notin J_{\mathbf{A}}$. In the first case, $c - rx = c \in M$ and there is nothing to define. In the second case, we can define $v(c - rx)$ just as in the proof of the last lemma. Again it follows that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ for large enough ν (which again proves that the extension is immediate). To prove that $v(c - rx)$ is well-defined, it remains to show that our definition assigns the same value to $c - rx$ and $c' - r'x$ if $c - rz$ and $c' - r'z$ are equivalent modulo $R(c_0 - r_0 z)$ and if $\deg r < \deg r_0$ and $\deg r' < \deg r_0$.

We shall show a little bit more. We just assume that $c - rz$ and $c' - r'z$ are equivalent modulo $R(c_0 - r_0 z)$. Then there is $r'' \in R$ such that $r - r' = r''r_0$ and $c - c' = r''c_0$. Hence,

$$c - ra_\nu - (c' - r'a_\nu) = c - c' - (r - r')a_\nu = r''c_0 - r''r_0 a_\nu \; .$$

Since c_0 is a limit of $\mathbf{S}_{r_0}$, we know that $r''c_0$ is a limit of $\mathbf{S}_{r''r_0}$. Hence by the definition of a limit, $v(r''c_0 - r''r_0 a_\nu)$ increases with ν, or $r''c_0 = r''r_0 a_\nu$ for all large enough ν. In the second case, $c - ra_\nu = c' - r'a_\nu$ for all large enough ν. Now assume the first case, and assume in addition that $v(c - ra_\nu)$ and $v(c' - r'a_\nu)$ are constant for all large enough ν. But this can only be the case if $v(r''c_0 - r''r_0 a_\nu) > v(c - ra_\nu)$ for large enough ν. Since this means that $v(c - ra_\nu - (c' - r'a_\nu)) > v(c - ra_\nu)$, it follows that the bones of $c - ra_\nu$ and $c' - r'a_\nu$ are equal for all large enough ν, as in the second case. Note that if $c' - r'a_\nu$ is constantly 0 for all large enough ν, then we must have the first case and also $c - ra_\nu$ is constantly 0 for all large enough ν.

Now if $r, r' \notin J_{\mathbf{A}}$, then c, c' are not limits of $\mathbf{S}_r$ resp. $\mathbf{S}_{r'}$. Hence by Lemma 2.50, $v(c - ra_\nu)$ and $v(c' - r'a_\nu)$ are constant for all large enough ν, and we obtain the equality of the bones of $c - ra_\nu$ and $c' - r'a_\nu$ are equal for large enough ν. If in addition $\deg r < \deg r_0$, then we know already that the bone of $c - rx$ is equal to the bone of $c - ra_\nu$ and thus also to the bone of $c' - r'a_\nu$ for large enough ν. If we assume further that $\deg r' < \deg r_0$, then we obtain that $v(c - rx) = v(c - ra_\nu) = v(c' - r'a_\nu) = v(c' - r'x)$ for large enough ν, which proves that $v(c - rx)$ is well-defined.

Having proved this, the triangle inequality follows as in the proof of the last lemma, with one additional provision: We have to take the representation of the elements $c - rx$ and $c' - r'x$ in such a way that $r = 0$ if $r \in Rr_0$ and $r' = 0$ if $r' \in Rr_0$, and if $r - r' \in Rr_0$, then we take $r' = r$. The proof for the fact that x realizes $\mathbf{A}$ can be taken over literally.

To prove that (N, v) is finitely exceptional, we need an additional argument. Assume that $c - rx \in N$ with $\deg r < \deg r_0$. Let $s \in R$ and assume that the bone of $c - ra_\nu$ is ordinary in $\mathbf{M}$ for s. Then by Lemma 4.73, $sr \notin J_{\mathbf{A}}$, or sc is a limit of $\mathbf{S}_{sr}$ and $sc = sra_\nu$ for all large enough ν. In the first case, by what we have shown above, the bone of $sc - srx$ is equal to the bone of $sc - sra_\nu$ for large enough ν. In the second case, let $\tilde{r} \in R$ and $\tilde{c} \in M$ be such that $\deg \tilde{r} < \deg r_0$ and that $sc - srz$ and $\tilde{c} - \tilde{r}z$ are equivalent modulo $R(c_0 - r_0 z)$. Since $\tilde{r} \notin J_{\mathbf{A}}$, we know that $v(\tilde{c} - \tilde{r}a_\nu)$ is constant for all large enough ν. In view of $sc - sra_\nu = 0$, we can infer from our above computation that $sc - srx = \tilde{c} - \tilde{r}a_\nu = c - ra_\nu = 0$ for all large enough ν. So also in this case, the bones of $sc - sra_\nu$ and $c - rx$ are equal. But the bone of $sc - sra_\nu$ only depends on s and the bone of $c - ra_\nu$ since this is assumed to be ordinary in $\mathbf{M}$ for s. Hence $(M \oplus Rx, v)$ is finitely exceptional, with the same exceptional bones as $\mathbf{M}$.

Now assume that y realizes $\mathbf{A}$. The map $c - rx \mapsto c - ry$ is well-defined by virtue of our assumption that $r_0 y = c_0$. We leave it to the reader to show that it is value preserving. Consequently, $c - ry \neq 0$ if $c - rx \neq 0$, showing that this map is an isomorphism of R-modules over M. □

As a consequence of Lemma 4.74 and Lemma 4.75, and in view of Theorem 4.7 and Lemma 2.40, we obtain:

Theorem 4.76 *Let R be a left euclidean domain. Then a finitely exceptional valued left R-module $\mathbf{M}$ is maximal if and only if every immediate approximation type over $\mathbf{M}$ is trivial, that is, if and only if $\mathbf{M}$ is spherically complete.*

As a consequence, every finitely exceptional valued left R-module admits an immediate extension which is spherically complete, provided that R is a left euclidean domain.

Let $M \subset N$ be an extension of left R-modules and $b \in R$. We will call b **R-algebraic over** M if there is some $r \in R \setminus \{0\}$ such that $rb \in M$, and we will call $M \subset N$ an **R-algebraic extension** if N is generated over M by a set of elements which are R-algebraic over M. If R is commutative, then this implies that every element of N is R-algebraic over M. If $R = \mathbb{Z}$, then our definition coincides with our notion of "algebraic extension" for abelian groups. We say that a valued R-module $\mathbf{M}$ is **algebraically maximal** if it does not admit a proper immediate R-algebraic extension. Now let $\mathbf{M}$ be a component-compatible resp. a finitely exceptional valued left R-module and $\mathbf{A}$ a nontrivial immediate approximation type over $\mathbf{M}$. Let us call $\mathbf{A}$ a **transcendental immediate approximation type** if $I_{\mathbf{A}} = \{0\}$ resp. $J_{\mathbf{A}} = \{0\}$; otherwise, we call it an **algebraic immediate approximation type**. Lemma 4.69 and Lemma 4.69 have shown that if $\mathbf{M}$ admits a nontrivial algebraic approximation type, then it admits an immediate R-algebraic extension. Let us show the converse. Let $(M, v) \subset (M + Rx, v)$ be immediate and $0 \neq r_0 \in R$ such $r_0 x = c_0 \in M$. Then by Lemma 2.21, $\mathbf{A} := \operatorname{appr}(x, M)$ is immediate. On the other hand, $v(c_0 - r_0 x) = \infty \geq \alpha$ for all $\alpha \in \Lambda(x, M)$, showing that $r_0 \in I_{\mathbf{A}}$. We have proved:

Theorem 4.77
a) Let R be a left principal ideal domain. Then a component-compatibly valued left R-module $\mathbf{M}$ is algebraically maximal if and only if every algebraic immediate approximation type over $\mathbf{M}$ is trivial.
b) Let R be a left euclidean domain. Then a finitely exceptional valued left R-module $\mathbf{M}$ is algebraically maximal if and only if every algebraic immediate approximation type over $\mathbf{M}$ is trivial.

Example 4.78 Let G be a $\mathbb{Z}$-group and v its natural valuation. Then (G, v) is an ordinary valued $\mathbb{Z}$-module. If $0 \neq n \in \mathbb{Z}$, then nG is again a $\mathbb{Z}$-group. According to Lemma 4.51, nG is nice in its divisible hull and thus also in G. In fact, this was verified directly in the argument preceding that lemma. Now it follows from the above theorem that (G, v) is algebraically maximal. ♢

If R is a field or a skew field, then $I_{\mathbf{A}} = \{0\}$ resp. $J_{\mathbf{A}} = \{0\}$ for every nontrivial approximation type $\mathbf{A}$ over $\mathbf{M}$, because $I_{\mathbf{A}}$ is an ideal of R, resp. $1 \notin RJ_{\mathbf{A}}$. In particular, we obtain:

Corollary 4.79 *Every ordinary valued vector space (V, v) is algebraically maximal. Its maximal immediate extensions are spherically complete and unique up to valuation preserving isomorphism over V.*

Proof: Only the uniqueness remains to be proved. Let (V_1, v_1) and (V_2, v_2) be two ordinary valued K-vector spaces which are maximal immediate extensions of the ordinary valued K-vector space (V, v), and let (V', v) be a maximal valued subvector space of (V_1, v_1) which admits a valuation preserving embedding in (V_2, v_2) over V. We identify (V', v) with its image in (V_2, v_2). Suppose that $V' \neq V_1$. Then we choose $x \in V_1 \setminus V'$. Since $(V', v) \subset (V_1, v_1)$ is immediate like $(V, v) \subset (V_1, v_1)$, we know by Lemma 2.21 that $\mathbf{A} := \operatorname{appr}(x, V')$ is immediate. Since (V', v_1) is algebraically maximal, this approximation type is transcendental. Since (V_2, v_2) is maximal, it must contain an element y which realizes $\mathbf{A}$. Now Lemma 4.74 shows that there is a valuation preserving isomorphism of $(V' + Kx, v_1)$ and $(V' + Ky, v_2)$ over V'. This contradicts the maximality of V' and thus shows that $V' = V_1$. Let V_2' denote the isomorphic image of V_1 in V_2. Then (V_2', v_2) is maximal (cf. Exercise 4.10). On the other hand, the extension $(V_2', v_2) \subset (V_2, v_2)$ is immediate. This proves that $V_2' = V_2$ and that (V_1, v_1) and (V_2, v_2) admit a valuation preserving isomorphism over V. □

Exercise 4.10 *Show that maximality (of valued abelian groups, valued modules etc.) is preserved under valuation preserving isomorphisms.*

Exercise 4.11 *Show that every component-compatibly resp. finitely exceptional valued module has an immediate extension which does not admit any transcendental immediate approximation type.*

Exercise 4.12 *Let $\mathbf{G}$ be a valued group such that only finitely many of its components are not divisible. Assume that $vna = va$ for all $n \in \mathbb{Z} \setminus \{0\}$. Prove that $\mathbf{G}$ is algebraically maximal. Try to do the same without the latter hypothesis.*

Chapter 5

Valued fields

5.1 Valuations, valuation rings and places

Let K be a field and $v: K \ni x \mapsto vx \in \Gamma\infty$ a valuation of its additive group $(K, +, 0)$. Suppose that Γ also admits an addition $+$ such that $(\Gamma, +, 0, <)$ is an ordered abelian group. We extend the connective $+$ to $\Gamma\infty$ by setting $\infty + \alpha = \alpha + \infty = \infty + \infty = \infty$, and the ordering by $\alpha < \infty$, for every $\alpha \in \Gamma$. Then the group valuation v is called a **valuation of the field** K if it is a homomorphism from the multiplicative group of K onto the additive group Γ. Hence, a map $v: K \ni x \mapsto vx \in \Gamma\infty$ is a valuation of K if it satisfies

(V0) $\quad vx = \infty \Longleftrightarrow x = 0\,,$

(VT) $\quad v(x - y) \geq \min\{vx, vy\}\,,$

(VH) $\quad v(xy) = vx + vy\,.$

for all $x, y \in K$. In this case, (K, v) is called a **valued field**, and $vK := \Gamma$ is called the **value group of** K. Henceforth, we will write $vK\infty$ in the place of $\Gamma\infty$. We will also use $\mathbf{K}$ and $\mathbf{L}$ to denote valued fields (K, v) and (L, v) respectively. From (VH), we obtain $v1 = v(1 \cdot 1) = v1 + v1$, which implies $v1 = 0$. With this, (VH) gives $vx + vx^{-1} = 0$ as a rule for inverses. So every field valuation satisfies

(V1) $\quad v1 = 0\,,$

(VI) $\quad vx^{-1} = -vx\,.$

Recall also the properties (VS), (V$\neq$), (V=) and (VM) that v has since it is a group valuation. Equivalence of valuations is defined as in the case of group valuations, but here the isomorphism between the value sets is required to be an isomorphism of ordered abelian groups. A valuation is said to be **trivial on** K if $vK = \{0\}$. Further, we define the **rank of** (K, v) to be the rank of the value group vK. So (K, v) is of rank 1 if and only if vK is archimedean.

The subgroup

$$\mathcal{O}_{\mathbf{K}} := \mathcal{O}^0_{\mathbf{K}} = \{x \in K \mid vx \geq 0\}$$

of the additive group of K is in fact a subring of K, called the **valuation ring of K** (or: the **valuation ring of** v and then denoted by $\mathcal{O}_v$). The subgroup

$$\mathcal{M}_{\mathbf{K}} := \mathcal{M}^0_{\mathbf{K}} = \{x \in K \mid vx > 0\}$$

is an ideal of $\mathcal{O}_\mathbf{K}$, called the **valuation ideal of K** (or: **of** v). Now $\mathcal{O}_\mathbf{K} \setminus \mathcal{M}_\mathbf{K}$ is the set of elements of value 0, and in view of (VI), these are precisely the elements which are invertible in $\mathcal{O}_\mathbf{K}$. So we find that $\mathcal{M}_\mathbf{K}$ is the unique maximal ideal of $\mathcal{O}_\mathbf{K}$, i.e., $\mathcal{O}_\mathbf{K}$ is a **local ring**. Since $\mathcal{M}_\mathbf{K}$ is maximal, $\mathcal{O}_\mathbf{K}/\mathcal{M}_\mathbf{K}$ is a field, called the **residue field of K** and denoted by $\overline{K}$, or by Kv if the valuation has to be specified. The canonical epimorphism

$$\mathcal{O}_\mathbf{K} \longrightarrow \mathcal{O}_\mathbf{K}/\mathcal{M}_\mathbf{K} = \overline{K}\,, \qquad a \mapsto \overline{a}$$

is called the **residue map of K**. The element $\overline{a}$ is called the **residue of** a. We will also use the notation av if it is necessary to indicate the valuation. The characteristic of the field $\overline{K}$ is called the **residue characteristic of K**. If $\operatorname{char} K = p > 0$, then the residue characteristic is also equal to p. Indeed, if $p \cdot 1$ equals 0 in K, then the same is true in the ring $\mathcal{O}_\mathbf{K}$ and thus also in the residue field $\mathcal{O}_\mathbf{K}/\mathcal{M}_\mathbf{K}$. In other words: if $\operatorname{char} K > 0$ then $\operatorname{char} \overline{K} = \operatorname{char} K$. But there is also the **mixed characteristic case** where $\operatorname{char} K = 0$ and $\operatorname{char} \overline{K} > 0$, as we will see in the next section.

An element of $\mathcal{O}_\mathbf{K}$ is invertible in $\mathcal{O}_\mathbf{K}$ if and only if its residue is nonzero. Such an element is called a **unit**, and $\mathcal{O}_\mathbf{K}^\times$ resp. $\mathcal{O}_v^\times$ denotes the multiplicative group of all units. Since the group homomorphism $v : K^\times \to \Gamma$ has kernel $\mathcal{O}_\mathbf{K}^\times$, we have an isomorphism

$$K^\times/\mathcal{O}_\mathbf{K}^\times \cong \Gamma\,.$$

An element with residue 1 is called a **1-unit**, and we observe that the inverse of a 1-unit is again a 1-unit. Note that $1 + \mathcal{M}_\mathbf{K}$ is the multiplicative group of all 1-units. The elements $a, a' \in \mathcal{O}_\mathbf{K}^\times$ have the same residue if and only if a/a' is a 1-unit. Consequently, the residue map, being a homomorphism from the multiplicative group $\mathcal{O}_\mathbf{K}^\times$ onto the multiplicative group $\overline{K}^\times$ with kernel $1 + \mathcal{M}_\mathbf{K}$, induces an isomorphism

$$\mathcal{O}_\mathbf{K}^\times/1 + \mathcal{M}_\mathbf{K} \cong \overline{K}^\times\,.$$

From (VI), we see that the elements of negative value are precisely the inverses of the elements in $\mathcal{M}_\mathbf{K}$. Since the residue map sends every element in $\mathcal{M}_\mathbf{K}$ to 0, an extension of it to the whole of K should send every element of negative value to "$\frac{1}{0}$", an element not contained in $\overline{K}$ and that we will denote by ∞ (there is not too much danger of a confusion with the maximal element ∞ which we adjoin to value sets). Now this extension is a map from K onto $\overline{K} \cup \{\infty\}$. It is called the **place associated with** v and denoted by P, $P_\mathbf{K}$ or P_v. It is customary to write the application of a place from the right, that is, for $a \in K$, its image under P is denoted by aP. In this spirit, the residue field $\overline{K}$ can also be written as KP. Note that as we did with value sets, we exclude ∞ from KP.

Two places P and P' of K are called **equivalent** if there is a field isomorphism $\sigma : KP \to KP'$ such that $P' = \sigma \circ P$. As for valuations, we will identify equivalent places. A place is said to be **trivial on** K if it is an isomorphism on K, that is, if it is equivalent to the identity on K.

We axiomatize places as follows. A map $P : K \to KP \cup \{\infty\}$ is a **place of** K if KP is a field and there exists a subring $\mathcal{O}$ of K such that P satisfies:

(PH) P is a homomorphism from $\mathcal{O}$ onto KP, and $(K \setminus \mathcal{O})P = \{\infty\}$,

(PI) $\forall x \neq 0 :\ xP = 0 \iff \frac{1}{x}P = \infty$.

The ring $\mathcal{O}$ is called the **valuation ring of** P and will also be written as $\mathcal{O}_P$. Note that axiom (PH) yields that there is some element $a \in K$ such that $aP \neq 0, \infty$. Axioms (PH) and (PI) together imply that $\mathcal{O} = \{a \in K \mid aP \neq \infty\}$; moreover, we deduce

$$aP \neq 0, \infty \implies a, \tfrac{1}{a} \in \mathcal{O} \wedge \tfrac{1}{a}P \neq 0, \infty \,. \tag{5.1}$$

If (K, v) is a valued field and P is the place associated with v, then with $\mathcal{O} = \mathcal{O}_{\mathbf{K}}$, it satisfies axioms (PH) and (PI). Given a place P on a field K, how do we associate a valuation? The most direct way is to associate a relation first. Define

$$a \,v\, b \;:\Longleftrightarrow\; a \neq 0 \wedge \tfrac{b}{a}P \neq \infty \,. \tag{5.2}$$

This relation is called **valuation divisibility relation**. Instead of $a \,v\, b$ we will prefer to write $va \leq vb$, even if we formally work with the relation. We derive an equivalence relation by setting $a \sim_P b :\Leftrightarrow a \,v\, b \wedge b \,v\, a$. The set of equivalence classes can be ordered by setting $a/\!\sim_P \leq b/\!\sim_P$ if $a \,v\, b$. Observe that $\infty := 0/\!\sim_P$ is the maximal element of the so obtained ordered set. We introduce addition on the set of all equivalence classes $\neq 0/\!\sim_P$ by $a/\!\sim_P + b/\!\sim_P := ab/\!\sim_P$. We leave it to the reader to show that we have obtained an ordered abelian group Γ and a valuation $v : K \ni a \mapsto a/\!\sim_P \in \Gamma\infty$. We see that for fields, the concepts of "valuation" and of "place" are interchangeable.

The third concept which is interchangeable with both valuation and place, is that of a valuation ring which we have already met in connection with them. So we will axiomatize it too. A ring $\mathcal{O}$ is called a **valuation ring of** K if it is a nontrivial subring of K and satisfies

(VR) $\qquad \forall x \neq 0 : \; x \in \mathcal{O} \vee \tfrac{1}{x} \in \mathcal{O} \,.$

Note that (VR) implies that $1 \in \mathcal{O}$ and that K is the quotient field of $\mathcal{O}$. Let $\mathcal{M}$ be the subset of all non-units in $\mathcal{O}$. We show that $\mathcal{M}$ is a maximal ideal of $\mathcal{O}$. Let a and $b \neq 0$ be non-units in $\mathcal{O}$. By (VR), we can assume w.l.o.g. that $\frac{a}{b} \in \mathcal{O}$, hence also $\frac{a+b}{b} = 1 + \frac{a}{b} \in \mathcal{O}$. If $a + b$ were a unit in $\mathcal{O}$ then it would follow that $\frac{1}{b} \in \mathcal{O}$, contradicting the assumption that b is not a unit. Further, for every $c \in \mathcal{O}$, also ca is a non-unit because if ca were invertible, the same would hold for a. We have shown that $\mathcal{M}$ is an ideal of $\mathcal{O}$. Since $\mathcal{O} \setminus \mathcal{M}$ contains only units, every proper ideal of $\mathcal{O}$ must be contained in $\mathcal{M}$, showing that $\mathcal{M}$ is the unique maximal ideal of $\mathcal{O}$. That is, $\mathcal{O}$ is a local ring. By definition of $\mathcal{M}$,

$$\forall x \neq 0 : \; x \in K \setminus \mathcal{O} \iff \frac{1}{x} \in \mathcal{M} \,. \tag{5.3}$$

A valuation ring associated with a valuation v satisfies axiom (VR) since $vx \geq 0 \vee v\frac{1}{x} \geq 0$. For the valuation ring of a place, this is seen from (PI) and (5.1). Given a valuation ring $\mathcal{O}$ of the field K, we now wish to show how to obtain a valuation and a place to which it is the associated valuation ring. Since $\mathcal{M}$ is a maximal ideal, $\mathcal{O}/\mathcal{M}$ is a field. The canonical epimorphism $P : \mathcal{O} \to \mathcal{O}/\mathcal{M} =: KP$, extended to K by setting $(K \setminus \mathcal{O})P = \{\infty\}$, satisfies (PH). By definition of $\mathcal{M}$ and (VR) we have $x \in \mathcal{M} \Leftrightarrow \frac{1}{x} \in \mathcal{O} \setminus \mathcal{M}$, showing that P satisfies (PI).

In order to obtain a valuation from $\mathcal{O}$, we consider the map v which associates with every element of $0 \neq a \in K$ the principal ideal $a\mathcal{O}$ generated by a, which is a fractional ideal if and only if $a \in K \setminus \mathcal{O}$. Under ideal multiplication (which is just given by multiplication of their

generators), these ideals form an abelian group. Let us show that it is totally ordered by inclusion. Indeed, if a, b are nonzero elements of K, then by (VR), $\frac{a}{b} \in \mathcal{O}$ or $\frac{b}{a} \in \mathcal{O}$, that is, $a\mathcal{O} \subset b\mathcal{O}$ or $b\mathcal{O} \subset a\mathcal{O}$. Hence, also the reversed ordering defined by $a\mathcal{O} \geq b\mathcal{O} :\Leftrightarrow a\mathcal{O} \subset b\mathcal{O}$ is total. It is straightforward to show that this ordering is compatible with the ideal multiplication. Further, if $b\mathcal{O} \subset a\mathcal{O}$ then $(a-b)\mathcal{O} \subset a\mathcal{O} - b\mathcal{O} \subset a\mathcal{O}$. That is, $b\mathcal{O} \geq a\mathcal{O}$ implies that $(a-b)\mathcal{O} \geq a\mathcal{O}$, proving that v satisfies (VT).

We have now shown that the three concepts "valuation", "place" and "valuation ring" are interchangeable. The reader may show that two valuations (resp. two places) are equivalent if and only if their valuation rings are equal. So every single valuation ring corresponds to precisely one equivalence class of valuations and precisely one equivalence class of places. In addition, every valuation ring $\mathcal{O}$ is already uniquely determined by its maximal ideal $\mathcal{M}$ since $\mathcal{O} = K \setminus \{x^{-1} \mid 0 \neq x \in \mathcal{M}\}$.

A valuation ring is a proper subring of K if and only if the associated valuation is nontrivial on K, which is the case if and only if the associated place is nontrivial on K. Let us state the following property of valuation rings:

Lemma 5.1 *Every valuation ring of a field K is integrally closed in K.*

Proof: Let $\mathcal{O}$ be a valuation ring of K and P the associated place. Further, let $x \in K$ be an element which is **integrally dependent on** $\mathcal{O}$, that is, there are elements $a_0, \ldots, a_{n-1} \in \mathcal{O}$ such that $x^n + \ldots + a_0 = 0$. Dividing by x^n, we obtain

$$1 = -a_{n-1}\left(\frac{1}{x}\right) - a_{n-2}\left(\frac{1}{x}\right)^2 - \ldots - a_0\left(\frac{1}{x}\right)^n .$$

If $x \notin \mathcal{O}$, then $\frac{1}{x} \in \mathcal{M}$ and thus $\frac{1}{x}P = 0$. Applying P to the above equation, we find $1 = 1P = 0$, a contradiction. This shows $x \in \mathcal{O}$ and proves that $\mathcal{O}$ is integrally closed. □

$(L, w)|(K, v)$ is called a **valued field extension** or an **extension of valued fields** if $L|K$ is a field extension and the **restriction of** w **to** K, denoted by $w|_K$, coincides with v. In that case, we will call w an **extension of** v **from** K **to** L. Later, when we are dealing with only one extension, we will denote the valuation on L also by v. Then, the extension will also be denoted as $(L|K, v)$ or as $\mathbf{L|K}$.

Since the axioms for a valuation are universal, the restriction of a valuation v of L to a subfield K of L is again a valuation, and $(L|K, v)$ is a valued field extension (cf. section 13.1). The same is true for the restriction of places and valuation rings. Here, some further remarks are necessary. Given places Q on L and P on K, then we will say that (L, Q) **is an extension of** (K, P) and write $(L, Q) \supset (K, P)$ if the restriction of Q to K coincides with P. The reader may show that this is the case already when the restriction of Q to $\mathcal{O}_P$ coincides with P. Using the interchangeability of places and valuations, it can also be shown that $(L, w)|(K, v)$ *is a valued field extension if and only if* $(L, P_w) \supset (K, P_v)$.

As to the valuation rings, it is not true that $(L, w)|(K, v)$ is a valued field extension if $\mathcal{O}_w \supset \mathcal{O}_v$. Analyzing our construction of places from valuation rings, we see that it is a crucial point to set $xP = \infty$ for *all* $x \notin \mathcal{O}$. So the required condition is $\mathcal{O}_w \cap K = \mathcal{O}_v$. In view of (5.3), this condition is equivalent to $\mathcal{M}_w \cap \mathcal{O}_v = \mathcal{M}_v$. This gives rise to the following definition. Let $\mathcal{O}, \mathcal{O}'$ be local rings with maximal ideals $\mathcal{M}, \mathcal{M}'$ respectively. Then we will say that $\mathcal{O}$ **lies above** $\mathcal{O}'$ if $\mathcal{O} \supset \mathcal{O}'$ and $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$. The latter condition can as well be replaced by the condition $\mathcal{M} \supset \mathcal{M}'$. Indeed, $\mathcal{O}' \setminus \mathcal{M}'$ consists only of units,

and these can not be elements of $\mathcal{M}$. This shows that $\mathcal{O} \supset \mathcal{O}'$ together with $\mathcal{M} \supset \mathcal{M}'$ will imply $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$. We have: $(L, w)|(K, v)$ *is a valued field extension if and only if* $\mathcal{O}_w$ *lies above* $\mathcal{O}_v$.

5.2 Examples of valued fields

In the following, we give two basic examples of field valuations:

1) **Valuations on a rational function field** $k(x)$**.**

Let k be a field and $k(x)$ the rational function field in one variable over k. Many mathematicians will remember that in their schooldays they were (more or less) tortured with the discussion of the properties of suitably chosen rational functions (with real coefficients). One task was to determine the zeros, poles and their orders. Now for every $a \in k$ we define a map $v_{x-a} : k(x) \to \mathbb{Z}$ by taking $v_{x-a}r(x)$ to be the zero order of $r(x)$ at a if a is not a pole of $r(x)$, or otherwise to be the *negative* pole order of $r(x)$ at a (take a representation of $r(x)$ with relatively prime numerator and denominator so that removable singularities do not interfere). For example, $v_{x-a}(x-a) = 1$, $v_{x-a}(\frac{1}{(x-a)^2}) = -2$, $v_{x-b}(x-a) = 0$ if $b \neq a$. Hence, the valuation ring is $\mathcal{O} = \{r(x) \mid r(x) \text{ has no pole at } a\}$ and its maximal ideal is $\mathcal{M} = \{r(x) \in \mathcal{O} \mid r(a) = 0\}$, which is just the ideal generated by the polynomial $x - a$. Consequently, the residue map sends x to the same residue class as a. Further, k is contained in $\mathcal{O}$, which yields that $k^\times \subset \mathcal{O}^\times$ and $v_{x-a}(k^\times) = \{0\}$ and that the residue map acts on k as an isomorphism (which is induced on k by the natural epimorphism $\mathcal{O} \to \mathcal{O}/\mathcal{M}$). Composition of the residue map with the inverse of this isomorphism shows that modulo equivalence, the associated place P_{x-a} can be chosen to act as identity on k. Hence, P_{x-a} can be understood to be the **evaluation map** which sends $r(x)$ to $r(a)$, and we have $k(x)P_{x-a} = k$. Note that P_{x-a} sends $x - a$ to 0. If $r(x)$ has a pole at a, the place P_{x-a} will send it to ∞.

Let $p(x)$ be any irreducible polynomial in $k[x]$. Since $k[x]$ admits unique factorization, we can write every polynomial $f(x) \neq 0$ in the form $p^n(x)g(x)$ with $g(x)$ prime to $p(x)$. Define $v_{p(x)}f(x) = n$ and $v0 = \infty$. In view of (VH), this determines the value of every rational function. Then $v_{p(x)}$ is a valuation: (V 0) holds by our definition, and (VH) is an easy consequence. To show (VT), let $r(x), s(x) \in k(x)$. After multiplication by a suitable polynomial, it suffices to show (VT) under the assumption that r, s are polynomials. Now we only have to observe that if r, s are both divisible by $p^n(x)$, then so is $r - s$. We have shown that $v_{p(x)}$ is a valuation. It is called the $p(x)$**-adic valuation**. The elements of value ≥ 0 are those rational functions which can be written as a quotient of two polynomials, the one in the denominator being prime to $p(x)$. Consequently, the valuation ring of $v_{p(x)}$ is the localization $k[x]_{(p(x))} = \{\frac{f(x)}{g(x)} \mid f(x), g(x) \in k[x] \text{ and } g(x) \text{ is prime to } p(x)\}$. The valuation ideal is the ideal generated by $p(x)$. The associated place $P_{p(x)}$ sends $p(x)$ to 0. The residue field is $k(x)P_{p(x)} = k[x]_{(p(x))}/(p(x)) = k[x]/(p(x))$ (for the last equality, see the corresponding argument in the next example).

Besides these valuations $v_{p(x)}$ there exists v_∞ with associated place P_∞ which sends $\frac{1}{x}$ to 0 (do you remember from your schooldays how to compute the asymptotic behaviour of $r(x)$ for $x \to \infty$?). P_∞ is called the **place at infinity**. To compute $v_\infty r(x)$, write $r(x) = \frac{1}{x}^n \frac{f_0}{g_0}$ where f_0 and g_0 are polynomials in $\frac{1}{x}$ with non-vanishing constant terms (hence P_∞ sends them neither to 0 nor to ∞). Then $v_\infty r(x) = n$. If $r(x) = \frac{f(x)}{g(x)}$ with

suitable polynomials f, g, then you will find that $n = \deg g - \deg f$.

Let us determine all valuations v of $k(x)$ which are trivial on k, or equivalently, all places P of $k(x)$ which are isomorphisms on k. Assume that P is nontrivial and $xP \neq \infty$. Then $k[x] \subset \mathcal{O}$, and $\mathcal{M} \cap k[x]$ will be a nontrivial prime ideal of $k[x]$. Since $k[x]$ is a principle ideal domain, this ideal is generated by one element $p(x) \in k[x]$, and since the ideal is prime, $p(x)$ is irreducible. By virtue of $p(x) \in \mathcal{M}$, we have $p(x)P = 0$. Since a place is (up to equivalence) determined by the valuation ideal, it follows that $P = P_{p(x)}$ and thus, $v_P = v_{p(x)}$.

Now assume that let $xP = \infty$. Then by (VI), $\frac{1}{x}P = 0$. As we have already seen above, this determines the valuation and thus also the place on the field $K(x) = K(\frac{1}{x})$ completely (up to equivalence, as always), and we have $P = P_\infty$. We have shown:
If a valuation v of $k(x)$ is nontrivial, but trivial on k, then $v = v_\infty$ or $v = v_{p(x)}$ for some irreducible $p(x) \in k[x]$. If a place P of $k(x)$ is the identity on k but not on $k(x)$, then $P = P_\infty$ $P = P_{p(x)}$ for some irreducible $p(x) \in k[x]$.

If k is algebraically closed, then every irreducible polynomial is linear, i.e., of the form $x - a$ with $a \in k$. Hence, all places are of the form P_{x-a} or P_∞. Their residue field is always k.

If k is not algebraically closed, then there will exist irreducible polynomials $p(x)$ of degree $\deg p(x) > 1$. For every such $p(x)$, the residue field will be a proper algebraic extension of k of degree $\deg p(x)$ since xP must be a zero of $p(x)$ (because P is a homomorphism on $\mathcal{O}$).

In all cases, the value group is $\mathbb{Z}$. Valuations with value group (isomorphic to) $\mathbb{Z}$ are called **discrete valuations**.

2) **Valuations on $\mathbb{Q}$.**

In the preceding example, for every prime element (i.e. irreducible polynomial) $p(x)$ in the ring $k[x]$ there was a valuation $v_{p(x)}$ of the quotient field $k(x)$ with $vp(x) = 1$ and residue field $k[x]/(p(x))$. The same is true for the ring $\mathbb{Z}$ and its quotient field $\mathbb{Q}$. For every prime number p there exists a valuation v_p that counts how often the factor p appears in an integer m, i.e. $v_p m = e$ if $m = p^e m'$ where $(p, m') = 1$. For a rational $r = \frac{m}{n}$ we put $v_p r = v_p m - v_p n$ according to (VH). v_p is called the **p-adic valuation**. Note that on $\mathbb{Z}$ it coincides with the p-valuation given by the height function; but on $\mathbb{Q}$, the height of every element is ∞, so we have to distinguish well between p-valuation and p-adic valuation. Valuation ring and valuation ideal of v_p are $\mathcal{O}_p = \{\frac{m}{n} \in \mathbb{Q} \mid (p, n) = 1\}$ and $\mathcal{M}_p = \{\frac{m}{n} \mid (p, n) = 1 \text{ and } (p, m) = p\} = p\mathcal{O}_p$. The residue field is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ (for every $\frac{m}{n} \in \mathbb{Q}$ with $(p, n) = 1$ the residue $\overline{n}$ of n modulo p is invertible in $\mathbb{Z}/p\mathbb{Z}$, hence the residue of $\frac{m}{n}$ is just $\overline{m} \cdot \overline{n}^{-1} \in \mathbb{Z}/p\mathbb{Z}$), and the value group is $\mathbb{Z}$. Every v_p yields a metric $d_p(x, y) = p^{-v_p(x-y)}$ on $\mathbb{Q}$ and thus a completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to this (ultra)metric, called the **field of p–adic numbers**. Contrary to the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the metric induced by the absolute value $|x|$, in $\mathbb{Q}_p$ two integers are near to each other if their difference is divisible by a high power of p. This reminds us of the ultrametric associated to a valuation. In fact, we will define a completion for every valued field using this ultrametric (cf. section 5.3). In the case of $\mathbb{Q}$ with the p-adic valuation, this definition just gives $\mathbb{Q}_p$.

Observe that in our first example, the characteristic of the residue field $k(x)P$ was the same as that of the valued field $k(x)$, since it contained k. It can be 0 or a prime number

p. This is the **equal characteristic case**. In the second example, the valued field $\mathbb{Q}$ has characteristic 0 whereas its residue field $\mathbb{F}_p$ with respect to the p-adic valuation has characteristic p. This is the **mixed characteristic case**. These are all possible cases because $\operatorname{char} K = p > 0$ implies $\operatorname{char} \overline{K} = p$, as we have already seen.

All examples that we gave so far were discrete valuations. Later, we will use the theory of extensions and compositions of valuations to give examples of non-discrete valuations. Another way of giving examples is to equip Hahn products with a multiplication, cf. section 5.3. From the model theoretic point of view, being discrete is not quite the property of a valuation that we want to investigate (cf. section 13.1). Rather, it is the property of admitting a prime element (which is also shared by certain non-discrete valuations). An element π in a valued field (K, v) is called a **prime element** (also: **local parameter** or **uniformizing parameter**) if its value $v\pi$ is the least positive element in the value group vK. For instance, $(k(x), v_x)$ admits x as a prime element, and $(\mathbb{Q}, v_p)$ admits p as a prime element.

If (K, v) is a valued field, then for any field extension L of K, v can be extended to a valuation of L (cf. Corollary 7.37 below). In particular, on $k(x)$ there may exist valuations which are extensions of a nontrivial valuation of k. On the other hand, we see that any of the valuations described above in example 1) can be extended to any (algebraic or transcendental) extension K of $k(x)$. This fact is used for the study of algebraic function fields (for their definition, see Section 6.5).

For an arbitrary field extension $K|k$, a place P of K is called a **place of** $K|k$ if it is an isomorphism on k. In this case, we will always assume that it is the identity on k since otherwise, we could find an isomorphism σ of KP such that the equivalent place $\sigma \circ P$ is the identity on k. In particular, our assumption yields that $k \subset KP$. If $k = KP$, then P is called a **rational place of** $K|k$. We can characterize places of $K|k$ as follows (the proof is left to the reader):

Lemma 5.2 *A place P of K is a place of $K|k$ if and only if k is contained in its valuation ring.*

Exercise 5.1 *Prove: if P is a place of $K|k$ then it is an isomorphism on the relative algebraic closure of k in K (which is called the* **(exact) constant field of** *$K|k$). Cf. Corollary 7.38.*

5.3 The valued field as an upgrading of its valued additive group

By our definition of the valuation v on a field K, the additive group of K together with v is a valued group. So we have to consider how the concepts developed for valued groups carry over to valued fields. We have seen that the value set becomes an ordered abelian group. What can be said about the components of the skeleton? In the same way as the introduction of addition has led to the identification of certain polygons in an ultrametric space, the introduction of multiplication leads to the identification of the components in a valued group. To see this, we have to consider the groups $\mathcal{O}_{\mathbf{K}}^{\alpha}$ and $\mathcal{M}_{\mathbf{K}}^{\alpha}$ for $\alpha \in vK$. These are $\mathcal{O}_{\mathbf{K}}$-modules: by virtue of (VT), they are closed under addition; they are closed under multiplication with elements of $\mathcal{O}_{\mathbf{K}}$ since this can only increase the value, in view of (VH). These modules are in fact ideals of $\mathcal{O}_{\mathbf{K}}$ (fractional, if $\alpha < 0$). Choosing an element $t^{\alpha} \in K$

with $vt^\alpha = \alpha$, we find that $v(a/t^\alpha) \geq 0$ for every element $a \in \mathcal{O}_{\mathbf{K}}^\alpha$, and $v(b/t^\alpha) > 0$ for every element $b \in \mathcal{M}_{\mathbf{K}}^\alpha$. Consequently, $\mathcal{O}_{\mathbf{K}}^\alpha = t^\alpha \mathcal{O}_{\mathbf{K}}$ and $\mathcal{M}_{\mathbf{K}}^\alpha = t^\alpha \mathcal{M}_{\mathbf{K}}$. This shows that every component $\mathrm{C}^\alpha\mathbf{K}$ is isomorphic to $\mathrm{C}^0\mathbf{K} = \mathcal{O}_{\mathbf{K}}/\mathcal{M}_{\mathbf{K}}$, which is just the additive group of the residue field $\overline{K}$. The isomorphism can be explicitly given by $x + \mathcal{M}_{\mathbf{K}}^\alpha \mapsto xt^{-\alpha} + \mathcal{M}_{\mathbf{K}}$ for $x \in \mathcal{O}_{\mathbf{K}}^\alpha$. We compose the coefficient map co_α with this isomorphism. Doing this for every $\alpha \in vK$, and setting $\mathrm{co}\,0 := 0 \in \overline{K}$, we obtain a coefficient map $\mathrm{co}\colon K \to \overline{K}$ which sends a to $\overline{at^{-va}}$ and satisfies (CO<), (CO+) and (3.3). But we have to note that it depends on the choice of the elements t^α. In the presence of multiplication, it would be desirable that it is moreover a homomorphism from the multiplicative group of K onto the multiplicative group of $\overline{K}$. Hence, we define: a map $\mathrm{co} : K \to \overline{K}$ will be called a **multiplicative coefficient map of the valued field K** if it satisfies

(CO 0) $\quad \mathrm{co}\,0 = 0\,,$

(COR) $\quad \forall x \in \mathcal{O}_{\mathbf{K}}^\times :\ \mathrm{co}\,x = \overline{x}\,,$

(COM) $\quad \forall x, y \in K :\ \mathrm{co}\,xy = \mathrm{co}\,x\,\mathrm{co}\,y\,.$

The reader may show that (CO<), (CO+) and (3.3) are consequences of these axioms. In general, a valued field may not admit a coefficient map satisfying (COM). A valued field without a multiplicative coefficient map was constructed by J. Pas [PAS].

Consider the map $\mathrm{co} : K \to \overline{K}$ that we have constructed above. For $a, b \in K$ we compute: $\mathrm{co}\,ab = \overline{abt^{-vab}} = \overline{at^{-va}} \cdot \overline{bt^{-vb}} \cdot \overline{t^{-vab}/t^{-va}t^{-vb}} = \mathrm{co}\,a \cdot \mathrm{co}\,b \cdot \overline{t^{-va-vb}/t^{-va}t^{-vb}}$. So we have

$$\mathrm{co}\,ab = \mathrm{co}\,a \cdot \mathrm{co}\,b \cdot \zeta_{va,vb} \quad \text{where} \quad \zeta_{\alpha,\beta} := \overline{t^{-\alpha-\beta}/t^{-\alpha}t^{-\beta}} \ \text{ for } \alpha, \beta \in vK\,. \tag{5.4}$$

Hence, co is multiplicative if and only if all $\zeta_{\alpha,\beta} = 1$, which holds if and only if $\overline{t^\alpha t^\beta} = \overline{t^{\alpha+\beta}}$ for all $\alpha, \beta \in vK$. In particular, this will be true if $t^\alpha t^\beta = t^{\alpha+\beta}$ for all $\alpha, \beta \in vK$. In this case, $t : vK \to K$, $\alpha \mapsto t^\alpha$, is a homomorphism from vK into the multiplicative group of K such that $v \circ t$ is the identity on vK. Such a homomorphism is called a **cross-section**. Later, we will give conditions which imply the existence of a cross-section (cf. Section 14.10).

We have seen that for valued fields, the concept of a skeleton is replaced by the concept of value group and residue field. So we have to say a word about the definition of the notions **embedding** resp. **isomorphism of valued fields**. As for valued groups, we may define them to be field embeddings resp. isomorphisms which preserve the valuation divisibility relation $vx < vy$. That is, $\iota : (K, v) \to (K', v')$ is an embedding of valued fields if it is an embedding of K in K' and satisfies

$$vx < vy \ \Leftrightarrow\ v'\iota x < v'\iota y\,.$$

In analogy to the case of abelian groups, the reader may show that such an embedding induces an embedding $\rho : vK \to vK'$ and an embedding $\sigma : \overline{K} \to \overline{K'}$ satisfying

$$v\iota a = \rho v a \ \text{ and } \ \overline{\iota a} = \sigma \overline{a} \quad \text{for all } a \in K\,.$$

If this holds, then we say that ι **preserves** ρ and σ (or **induces** ρ and σ). If ι is an isomorphism, then so are ρ and σ. This consideration shows that we could as well define embeddings of valued fields to be embeddings of structures of the form (K, vK, v) or

$(K, vK, \overline{K}, v, ^-)$. In any case, an embedding of valued fields is a field embedding which at the same time is an embedding of the valued additive groups.

If ι is a field embedding of K into K' and v is a valuation of K', then $v\iota : K \to vK'\infty$, $x \mapsto v(\iota x)$, is a valuation of K. Its valuation ring is $\iota^{-1}\mathcal{O}_v$ and its valuation ideal is $\iota^{-1}\mathcal{M}_v$. Value group and residue field of v and $v\iota$ coincide. Moreover, $\iota : (K, v\iota) \to (K', v)$ is an embedding of valued fields. If P is a place of K', then the map $P^\iota : K \to K'P\infty$, $x \mapsto (\iota x)P$, is a place of K'. Its associated valuation is $v_{P^\iota} = (v_P)\iota$. Note that $P^{\iota\iota'} = (P^\iota)^{\iota'}$. Let k be a common subfield of K and K' and ι an embedding of K in K' over k. If v is an extension of a valuation v' from k to K', then $v\iota$ is an extension of v' from k to K. If P is a place of $K|k$ and ι is an automorphism of $K|k$, then P^ι is again a place of $K|k$. The proof of these facts is left to the reader. See also Exercise 5.2 below.

The question arises whether we can explicitly upgrade certain valued abelian groups by introducing a multiplication on them. This is indeed possible. Let us fix an arbitrary ordered abelian group Γ and a field k. Viewing k as an abelian group and Γ as an ordered set, we have thus given a skeleton with $B_\gamma = k$ for all $\gamma \in \Gamma$, and we can form the Hahn product $\mathbf{H}_{\gamma\in\Gamma}\, k$. Its elements are called **formal power series**, or just **power series**. The product of two power series $a = (a_\gamma)_{\gamma\in\Gamma}$ and $b = (b_\gamma)_{\gamma\in\Gamma}$ is given as follows: for every $\gamma \in \Gamma$, we set

$$(a \cdot b)_\gamma := \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \ . \tag{5.5}$$

Since the supports of a and b are well-ordered by the definition of the Hahn product, these sums are finite for all γ. To generalize the construction, one can introduce a **factor set** $\{\zeta_{\alpha,\beta} \mid \alpha, \beta \in \Gamma\}$ and define $(a \cdot b)_\gamma := \sum_{\alpha+\beta=\gamma} \zeta_{\alpha,\beta} a_\alpha b_\beta$; these factor sets have to satisfy certain compatability conditions. Since we will not need these constructions using factor sets, we omit the details and refer the reader to [KOC]. Recall that the valuation of the Hahn product assigns to every element the minimum of its support. So we observe that by (5.5), we automatically obtain $v(a \cdot b) = va + vb$, which is axiom (VH). We leave it to the reader to show that our multiplication turns the Hahn product into an integral domain. It will be denoted by $k((\Gamma))$, and v will be called its **canonical valuation**. Here, we take over the job to show the existence of inverses, following [PC2]. Beforehand, let us note that $k((\Gamma))$ admits a cross-section (which is due to the fact that definition (5.5) uses the trivial factor set $\zeta_{\alpha,\beta} = 1$). Indeed, for t^α we only have to take the power series given by $t^\alpha_\alpha = 1$ and $t^\alpha_\beta = 0$ for $\beta \neq \alpha$. Then $t^\alpha \cdot t^\beta = t^{\alpha+\beta}$ is a direct consequence of (5.5). In particular, note that $1/t^\alpha = t^{-\alpha}$. In view of the notation "t^α" some authors write $k((t^\Gamma))$ for $k((\Gamma))$. Instead of $k((\mathbb{Z}))$ we will write $k((t))$ and it is understood that t is an element of $k((t))$ which has value $1 \in \mathbb{Z}$ for the canonical valuation v. That is, v is equal to the **t-adic valuation** v_t in the sense of Section 5.2 1), t being understood as a prime polynomial in $k[t]$. The place associated to this valuation sends t to 0. Further, t is a prime element of both $(k(t), v_t)$ and $(k((t)), v_t)$. Observe that in a valued field $\mathbf{K}$, an element t is a prime element if and only if $\mathcal{M}_\mathbf{K} = t \cdot \mathcal{O}_\mathbf{K}$.

Here is the promised existence of inverses:

Lemma 5.3 *$k((\Gamma))$ is a field.*

Proof: Let $a = (a_\gamma)_{\gamma\in\Gamma} \in k((\Gamma))$ and set $\alpha := va$. Consequently, $a_\alpha \neq 0$. Then $a = a_\alpha \cdot t^\alpha + a'$ with $va' > \alpha$. This shows that for $b := a \cdot a_\alpha^{-1} t^{-\alpha} - 1$ we have $vb > 0$.

Now we consider the map $\Xi : k((\Gamma)) \to k((\Gamma))$ given by $x \mapsto a_\alpha^{-1}t^{-\alpha} - xb$. This map is contractive since $v(\Xi x - \Xi y) = v(-xb + yb) = v(x-y) + vb > v(x-y)$. By Lemma 3.14, the valued group $k((\Gamma))$ is spherically complete. By the Ultrametric Fixed Point Theorem (2.11), there is a unique fixed point $x \in k((\Gamma))$ for Ξ. It satisfies $x = a_\alpha^{-1}t^{-\alpha} - xb$, that is, $x(1+b) = a_\alpha^{-1}t^{-\alpha}$, whence $x \cdot a \cdot a_\alpha^{-1}t^{-\alpha} = a_\alpha^{-1}t^{-\alpha}$, which gives $x \cdot a = 1$. □

A field of the form $k((\Gamma))$ is called a **power series field**. Together with its canonical valuation it is a valued field, and a power series contained in it lies in the valuation ring if and only if its support contains only positive elements of Γ. This valuation ring will be denoted by $k[[\Gamma]]$. A ring of this form is called a **power series ring**. In the case of the power series field $k((t))$, we call its elements **power series in** t (in the literature, also the name **(formal) Laurent series** is in use and in that case, "power series" is often meant to denote only the series with positive support). The valuation ring is denoted by $k[[t]]$, and the valuation ideal can be written as $t \cdot k[[t]]$.

For every valued field extension $(L|K, v)$, the value group vK is a subgroup of vL, and the residue field $\overline{K} = KP_v$ is a subfield of $\overline{L} = LP_v$. Recall that we are identifying equivalent valuations and places; otherwise, we would have to speak of embeddings of vK in vL and of $\overline{K}$ in $\overline{L}$. If both extensions $vL|vK$ and $\overline{L}|\overline{K}$ are trivial, then $(L|K, v)$ is called **immediate**. A valued field is called **maximal** if it does not admit any proper immediate extensions. It will be called a **spherically complete field** if its underlying ultrametric space is spherically complete. At this point, let us mention:

Lemma 5.4 *A valued field (K, v) is spherically complete if and only if $\mathcal{O}_\mathbf{K}$ is spherically complete (as a valued abelian group), and this is the case if and only if $\mathcal{M}_\mathbf{K}$ is spherically complete.*

Proof: If a nest of balls lies entirely in $\mathcal{O}_\mathbf{K}$ then so does its intersection. Hence if (K, v) is spherically complete then so is $\mathcal{O}_\mathbf{K}$. For the converse, consider an arbitrary nest $\mathbf{B}$ of balls in (K, v). Pick some ball B in $\mathbf{B}$ and let $\mathbf{B}'$ be the subset of all balls in $\mathbf{B}$ which are contained in B. Then the intersection of $\mathbf{B}'$ as the same as that of $\mathbf{B}$. Now let a be any center of B and let α be its radius. Then all elements of B have value $\geq \min\{va, \alpha\} =: \beta$. Choose $b \in K$ with $vb > -\beta$. Then $bB = \{bc \mid c \in B\}$ is a subset of $\mathcal{O}_\mathbf{K}$. So $b\mathbf{B}' = \{bB' \mid B' \in \mathbf{B}'\}$ is a nest of balls in $\mathcal{O}_\mathbf{K}$. If it admits an element c in its intersection, then $b^{-1}c$ lies in the intersection of $\mathbf{B}'$ and thus also of $\mathbf{B}$. This shows that (K, v) is spherically complete if $\mathcal{O}_\mathbf{K}$ is. For the second equivalence, just note that $\mathcal{O}_\mathbf{K}$ can be replaced by $\mathcal{M}_\mathbf{K}$ everywhere in this proof. □

In fact, the assertion of this lemma remains true if we replace $\mathcal{O}_\mathbf{K}$ resp. $\mathcal{M}_\mathbf{K}$ by $\mathcal{O}_\mathbf{K}^\alpha$ resp. $\mathcal{M}_\mathbf{K}^\alpha$ for any $\alpha \in vK$. Cf. also Lemma 3.17.

We have already considered above the skeleton of the valued additive group of a valued field (K, v). Its value set is vK, and we have shown that the components are all isomorphic to the additive group of the residue field. This isomorphism is induced by the multiplication with an element t^α, from which it follows that it extends canonically to every valued field extension. This proves: *an extension of valued fields is immediate if and only if the underlying extension of their valued additive groups is immediate.* As a corollary to Lemma 3.9, Corollary 3.10 and Lemma 3.14, we thus obtain:

Corollary 5.5 *The extension $(L|K, v)$ is immediate if and only if for every $b \in L \setminus \{0\}$ there is some $a \in K$ such that $v(b-a) > vb$. This is the case if and only if the underlying*

extension of ultrametric spaces is immediate. A spherically complete field is maximal. Every power series field $k((\Gamma))$ is spherically complete and thus maximal.

Our construction of $k((\Gamma))$ gives an answer also to the question whether for a given ordered abelian group Γ and a given field k there exists a valued field with value group Γ and residue field k. In view of its maximality, $k((\Gamma))$ is a "large field" satisfying our conditions. Inspired by the observation that the Hahn sums appear to be the smallest valued groups with a given skeleton, we may ask for "small" valued fields with given value group and residue field. It turns out that this is indeed an important question, as we will see when we deal with the defect of valued fields. The word "smallest" will be made precise by means of the notion "valuation transcendence basis", cf. section 8.7.

Note that $k((\Gamma))$ is a valued field which has the same residue characteristic as k. If the characteristic of a given residue field is $p > 0$, then the question arises whether we can also construct a valued field of characteristic 0 having this residue field. The analogue of our above power series field construction is the famous theory of **Witt vectors**. We refer the reader to [HAS] or [JAC]. It is of special importance for the structure theory of maximal fields of mixed characteristic. If one is only interested in the problem of constructing arbitrary fields with given value group and residue field, then one can again follow the approach using valuation transcendence bases. It also works in the mixed characteristic case, starting from the prime field $\mathbb{Q}$ with its p-adic valuation. The pure existence of maximal fields with the same value group and residue field is then guaranteed by Theorem 5.7 below which states the existence of maximal immediate extensions.

In particular, as a corollary to Lemma 3.12 we obtain:

Lemma 5.6 *For every valued field (K, v),*

$$|K| \leq |\overline{K}|^{|vK|} \quad .$$

As in the case of valued abelian groups, we derive:

Theorem 5.7 *Every valued field admits a maximal immediate extension.*

For the construction of immediate extensions and a characterization of maximal fields, see section 7.10. In that section, we will treat Ostrowski's and Kaplansky's theory of immediate extensions of valued fields. One main result will be that maximal fields are spherically complete (cf. our discussion in connection with maximal valued modules in Remark 4.9).

Remark 5.8 From the beginning of valuation theory until now, the maximal fields and thus also the power series fields have been objects of particular interest. This is mainly due to the fact that they have the important properties of being spherically complete and thus henselian and defectless (cf. Theorem 7.124, Theorem 8.26 and Section 9.1). The beginning of the theory of valued fields may be seen in the works of Hensel [HE1-10]. In accordance to the number theoretical problems that valuation theory was born from, the first maximal fields considered were complete discretely valued fields (in particular $\mathbb{Q}_p$). We will introduce completions of valued fields below. With the appearance of other valued fields, non-discretely valued or even of higher rank, the completions turned out to be unfit for guaranteeing the validity of Hensel's Lemma. Switching to maximal fields, which are always henselian, the question arose whether every valued field admits an extension which is a maximal field and in particular, a maximal immediate extension.

W. Krull [KRU7] was the first to use the estimation for the cardinality of a valued field to show the existence of maximal immediate extensions. His proof of Lemma 5.6 appears to be correct but rather circuitous. It is a good example of a phenomenon that is to be observed till the present day: the unnecessary inclination to work inside power series fields. Bursting these chains, K. A. H. Gravett [GRA3] gave a beautiful short proof. (I would like to thank N. Alling for bringing it to my attention.) Gravett's proof has inspired our generalization to ultrametric spaces, cf. Lemma 2.13.

A different approach was to show that a given valued field can be embedded in a power series field with same value group and residue field, which will then be a maximal immediate extension of the embedded field. Having proved the existence of maximal immediate extensions, the question is unavoidable whether they are unique (up to isomorphism). I. Kaplansky [KAP1] approached this question by characterizing the maximal fields as those which are spherically complete (in his language: in which every pseudo Cauchy sequence admits a limit). Under a certain additional condition (Kaplansky's hypothesis A), Kaplansky was then able to prove uniqueness (see also [KU–PA–ROQ]).

While the structural analysis of maximal fields, the power series fields and the Witt vector constructions are of significant importance for many applications, the concept of maximal fields has turned out to be too coarse for contemporary valuation theoretical problems. Replacing it by the notion of henselian fields and henselizations allows us to work with algebraic extensions of given valued fields in the place of maximal immediate extensions, which in general are of high transcendence degree. In contrast to maximality, the property of being henselian is elementary (cf. section 13.1), which is of fundamental importance for the model theory of valued fields and for the application of model theoretic tools to problems in valuation theory.

There is one further construction that we can upgrade from the case of valued groups, namely the **completion of a valued field $\mathbf{K} = (K, v)$**, denoted by $\mathbf{K}^c$, $(K, v)^c$ or (K^c, v^c). We define $\mathbf{K}^c$ to be the completion of its underlying valued additive group. For extending the multiplication from K to K^c, we will use the following auxiliary result.

Lemma 5.9 *If $B_\alpha(x_\alpha)$ and $B_\alpha(x'_\alpha)$ are balls in $\mathbf{K}$, then $B_\alpha(x_\alpha) \cdot B_\alpha(x'_\alpha) = B_\beta(x_\alpha \cdot x'_\alpha)$ where $\beta = \alpha + \min\{vx_\alpha, vx'_\alpha, \alpha\}$.*

Proof: We compute $B_\alpha(x_\alpha) \cdot B_\alpha(x'_\alpha) = (x_\alpha + \mathcal{O}_\mathbf{K}^\alpha) \cdot (x'_\alpha + \mathcal{O}_\mathbf{K}^\alpha) = x_\alpha x'_\alpha + x_\alpha \mathcal{O}_\mathbf{K}^\alpha + x'_\alpha \mathcal{O}_\mathbf{K}^\alpha + \mathcal{O}_\mathbf{K}^\alpha \cdot \mathcal{O}_\mathbf{K}^\alpha$. In the $\mathcal{O}_\mathbf{K}^\alpha$-module $M := x_\alpha \mathcal{O}_\mathbf{K}^\alpha + x'_\alpha \mathcal{O}_\mathbf{K}^\alpha + \mathcal{O}_\mathbf{K}^\alpha \cdot \mathcal{O}_\mathbf{K}^\alpha$, every element has value at least $\beta = \alpha + \min\{vx_\alpha, vx'_\alpha, \alpha\}$; hence, $M \subset \mathcal{O}_\mathbf{K}^\beta$. On the other hand, M contains an element b of value β, whence $\mathcal{O}_\mathbf{K}^\beta = b\mathcal{O}_\mathbf{K} \subset M$. Consequently, $M = \mathcal{O}_\mathbf{K}^\beta$. We thus find $B_\alpha(x_\alpha) \cdot B_\alpha(x'_\alpha) = B_\beta(x_\alpha x'_\alpha)$. □

Let $\mathbf{A}$ and $\mathbf{A}'$ be two completion types in $\mathbf{K}^c$. By virtue of the preceding lemma, $\mathbf{B} := \{\mathbf{A}_\alpha \cdot \mathbf{A}'_\alpha \mid \alpha \in vK\}$ is a nest of balls. We will now show that the values appearing as radii of balls in $\mathbf{B}$ form a cofinal subset of vK; then $\mathbf{B}$ admits a unique associated completion type, which we will take to be $\mathbf{A} \cdot \mathbf{A}'$ (see the construction preceding to Lemma 2.40). Assume that $\mathbf{A}_\alpha = B_\alpha(x_\alpha)$ and $\mathbf{A}'_\alpha = B_\alpha(x'_\alpha)$ for all $\alpha \in vK$. Now let $0 < \alpha \in vK$. If $vx_\alpha \geq 0$ and $vx'_\alpha \geq 0$, then by the last lemma, $B_\alpha(x_\alpha) \cdot B_\alpha(x'_\alpha)$ is a ball of radius $\alpha + \min\{vx_\alpha, vx'_\alpha, \alpha\} \geq \alpha$ in $\mathbf{B}$. If $vx_\alpha < 0$ or $vx'_\alpha < 0$ then let $\gamma := \alpha - \min\{vx_\alpha, vx'_\alpha, \alpha\} > \alpha$. Then $\min\{vx_\gamma, vx'_\gamma, \gamma\} \geq \min\{vx_\alpha, vx'_\alpha, \alpha\}$ since otherwise, $vx_\gamma < vx_\alpha < \alpha$ or $vx'_\gamma < vx'_\alpha < \alpha$. But the first would contradict the fact that $x_\gamma \in B_\alpha(x_\alpha)$, and the second would contradict $x'_\gamma \in B_\alpha(x'_\alpha)$. Consequently, $B_\gamma(x_\gamma) \cdot B_\gamma(x'_\gamma)$ is a ball of radius $\gamma + \min\{vx_\gamma, vx'_\gamma, \gamma\} \geq \gamma + \min\{vx_\alpha, vx'_\alpha, \alpha\} = \alpha$ in $\mathbf{B}$. This proves the cofinality of the radii.

By the definition, it is obvious that the multiplication on K^c coincides with the usual multiplication on the embedded field K. With this multiplication, K^c is an integral domain; the proof is straightforward and therefore left to the reader. Here, we wish to show the existence of inverses. So let $\mathbf{A}$ be a completion type in $\mathbf{K}$ and let $\mathbf{A}_\alpha = \{B_\alpha(x_\alpha)$ for all

$\alpha \in vK\}$. Assume that $\mathbf{A}$ does not represent the element 0. That is, there is some γ such that $0 \notin B_\gamma(x_\gamma)$, or in other words, $vx_\gamma < \gamma$. Then for all $\beta > \gamma$, we have $vx_\beta = vx_\alpha$ by (VT), since $x_\beta \in B_\gamma(x_\gamma)$. Since it suffices to invert the element $x_\gamma^{-1}\mathbf{A}$, which is the completion type associated with the nest of balls $\{x_\gamma^{-1}\mathbf{A}_\alpha = B_\alpha(x_\gamma^{-1}x_\alpha) \mid \alpha \in vK\}$, we can assume from the start that $vx_\alpha = 0$ for all $\alpha \geq \gamma$. Now we set $\mathbf{B} = \{B_\alpha(x_\alpha^{-1}) \mid \gamma \leq \alpha \in vK\}$. This is a nest of balls: for every $\beta \geq \alpha \geq \gamma$, we have $x_\alpha - x_\beta \in \mathcal{O}_\mathbf{K}^\alpha$, hence $x_\beta^{-1} - x_\alpha^{-1} \in x_\alpha x_\beta \mathcal{O}_\mathbf{K}^\alpha = \mathcal{O}_\mathbf{K}^\alpha$, that is, $B_\beta(x_\beta^{-1}) \subset B_\alpha(x_\alpha^{-1})$. By the same argument as before, $\mathbf{B}$ admits a unique associated completion type, which we will call $\mathbf{A}^{-1}$. It is evident from the definition that $\mathbf{A} \cdot \mathbf{A}^{-1} = 1$.

From the underlying valued groups, we obtain

Lemma 5.10 $\mathbf{K}$ *is dense in* $\mathbf{K}^c$ *and* $\mathbf{K}^c|\mathbf{K}$ *is an immediate extension.*

The proof of the following uniqueness statement and its analogues for ultrametric spaces and valued groups is left to the reader:

Lemma 5.11 *If* $\mathbf{L}|\mathbf{K}$ *is an extension of valued fields such that* $\mathbf{K}$ *is dense in* $\mathbf{L}$ *and* $\mathbf{L}$ *is complete, then* $\mathbf{L}$ *is isomorphic to* $\mathbf{K}^c$ *over* $\mathbf{K}$*.*

Remark 5.12 The author has to confess that he is cheating when he pretends that the proof of this lemma is straightforward; it may not be so for the unexperienced reader. The topologically trained reader may use the continuity of $+$, $\cdot$ and $^{-1}$, together with Lemma 2.38 or the results of Section 4.8. We will give an alternative proof in Section 7.10. Actually, we do not essentially need Lemma 5.11 before that section, although we cite it sometimes. Historically, the completion has played an important role in the development of valuation theory. Nevertheless, the author has an ambivalent relation to completions. It is not any more adequate to employ them in the extension theory of valuations of arbitrary rank, and for the study of the fundamental inequality, the henselization has taken over their place. As modern valuation theory does not confine itself to the study of discrete or rank 1 valuations and generalizes most historical results to valuations of arbitrary valuations, completions have lost their original importance. On the other hand, there was discovered a quite different and rather unexpected role that completions can play even for valuations of arbitrary rank, which is connected with the notion "completion defect". If it were not for this application, we would probably have avoided to mention completions in this book.

As an application of this Lemma 5.11, we obtain that $k((t))$ with its t-adic valuation v_t is *the* completion of its subfield $(k(t), v_t)$ (since we frequently identify isomorphic objects). Indeed, from Corollary 5.5 we know that $(k((t)), v_t)$ is maximal and thus complete. On the other hand, $(k(t), v_t)$ is dense in $(k((t)), v_t)$: Every power series in $k((t))$ is of the form

$$\sum_{i=-n}^{\infty} c_i t^i \quad \text{with } n \in \mathbb{N}\,,\ c_i \in k\,.$$

Given an arbitrary $m \in \mathbb{Z} = v_t k((t))\,$, we have

$$v_t\left(\sum_{i=-n}^{\infty} c_i t^i \ - \ \sum_{i=-n}^{m} c_i t^i\right) = v_t \sum_{i=m+1}^{\infty} c_i t^i \geq m+1$$

with $\sum_{i=-n}^{m} c_i t^i \in k(t)$.

In fact, $(k((t)), v_t)$ is even a maximal immediate extension of $(k(t), v_t)$, unique up to isomorphism over $(k(t), v_t)$. The power series ring $k[[t]]$ can be viewed as the completion of the polynomial ring $k[t]$ (cf. Exercise 5.3).

Now assume that (K,v) is a valued field with $vK = \mathbb{Z}$ and $\overline{K} = k \subset K$ (we assume that the residue map induces the identity on k). Let t be a prime element of (K,v). Given an arbitrary element $a \in K$, we wish to represent a as a power series in $k((t))$. This is done as follows. For $i \in \mathbb{Z}$, we choose $c_i \in k$, beginning with $i = va$. Since $v(at^{-va}) = 0$, there is some $c_i \in k$ such that $\overline{at^{-va}} = \overline{c_i} = c_i$, so that $v(at^{-va} - c_i) > 0$ and $v(a - c_i t^{va}) > vt^{va} = va$. Having constructed $\sum_{i=va}^{m} c_i t^i$ such that $v\left(a - \sum_{i=va}^{m} c_i t^i\right) \geq m+1$, we set $c_{m+1} = 0$ if "$>$" holds, and otherwise we apply the same procedure to $a - \sum_{i=va}^{m} c_i t^i$ in the place of a to find c_{m+1} such that $v\left(a - \sum_{i=va}^{m+1} c_i t^i\right) \geq m+2$. In this way we find that $(k(t), v_t)$ is dense in (K,v). Since density is a transitive property (as the reader may show), we conclude that $(k(t), v_t)$ is dense in the completion of (K,v). By the foregoing lemma, this completion of (K,v) is at the same time the completion of $(k(t), v_t)$, showing that (K,v) is embeddable in the completion of $(k(t), v_t)$. In fact, our above construction has produced the power series which represents the element a under the unique embedding; we can write

$$a = \sum_{i=va}^{\infty} c_i t^i \quad \text{with } c_i \in k \; .$$

This power series is called the **t-adic expansion of** a (also **v-adic expansion** or **P-adic expansion** if $P = P_v$ is the place associated with v).

A main application of this expansion procedure is given in the case where $K|k$ is an algebraic function field and $P = P_v$ is a rational place of $K|k$. We will see later (Corollary 7.32) that an algebraic function field in one variable endowed with a rational place always has value group (isomorphic to) $\mathbb{Z}$; so the expansion procedure can be applied to it.

The expansion procedure can be generalized to the case where the residue field $\overline{K}$ is not embedded in K (in particular, if they have different characteristic). In this case, we replace k by a suitable set of representatives for $\overline{K}$, but our expansions will in general not anymore define an embedding in a power series field.

If (K,v) is a valued field whose value group vK is a subgroup of $\mathbb{R}$, then we can define a metric by choosing some real number $r > 1$ and setting $d(a,b) = r^{-v(a-b)}$ for all $a, b \in K$. The reader may show that the completion of K with respect to this metric is the same as the completion with respect to v that we have defined above. In particular, in the case of $K = \mathbb{Q}$ and $v = v_p$ the p-adic valuation, one usually considers the metric $d_p(a,b) = p^{-v_p(a-b)}$, and the so obtained completion $\mathbb{Q}_p$ equals our completion of $\mathbb{Q}$ with respect to v_p (the field of p-adic numbers). In particular, $(\mathbb{Q}_p, v_p)$ is an immediate extension of $(\mathbb{Q}, v_p)$, showing that p is a prime element for both fields, and both have residue field $\mathbb{F}_p$. Taking $0, 1, \ldots, p-1$ as a set of representatives for this residue field, the expansion with respect to the prime element p gives the **p-adic expansion** for every element $a \in \mathbb{Q}_p$:

$$a = \sum_{i=va}^{\infty} c_i p^i \quad \text{with } c_i \in \{0, 1, \ldots, p-1\} \; .$$

Exercise 5.2 *Show that $\iota : (K,v) \to (K',v')$ is an embedding of valued fields if and only if it is an embedding of K in L with $\iota(\mathcal{O}_{(K,v)}) \subset \mathcal{O}_{(K',v')}$ and $\iota(\mathcal{M}_{(K,v)}) \subset \mathcal{M}_{(K',v')}$. Show that neither the condition "$\iota(\mathcal{O}_{(K,v)}) \subset \mathcal{O}_{(K',v')}$" nor the condition "$\iota(\mathcal{M}_{(K,v)}) \subset \mathcal{M}_{(K',v')}$" can be omitted. Further, show that ι is an isomorphism of valued fields if and only if it is an isomorphism of K onto L and $\iota(\mathcal{O}_{(K,v)}) = \mathcal{O}_{(K',v')}$. Prove that in this case, the latter condition is equivalent to the condition $\iota(\mathcal{M}_{(K,v)}) = \mathcal{M}_{(K',v')}$.*

Exercise 5.3 *Adapt the construction of the completion in order to obtain the* **completion of a valued ring**. *Which assumptions on the ring are necessary? Apply to the case of valued polynomial rings in several variables. Any difficulties?*

5.4 Ordered fields

One of the main examples for group valuations was the natural valuation of an ordered abelian group. Let us upgrade ordered groups. A field K together with a relation $<$ is called an **ordered field** and $<$ is called an **ordering of** K if its additive group together with $<$ is an ordered abelian group and the ordering is compatible with the multiplication:

(OM) $\qquad 0 < x \wedge 0 < y \implies 0 < xy\,.$

For the **positive cone of an ordered field**, the corresponding additional axiom is:

(PC$\cdot$) $\qquad \mathbf{P} \cdot \mathbf{P} \subset \mathbf{P}\,.$

Since $-\mathbf{P} \cdot -\mathbf{P} = \mathbf{P} \cdot \mathbf{P} \subset \mathbf{P}$, all squares of K and thus also all sums of squares are contained in $\mathbf{P}$. Since $-1 \in -\mathbf{P}$ and $\mathbf{P} \cap -\mathbf{P} = \{0\}$, it follows that $-1 \notin \mathbf{P}$ and in particular, -1 is not a sum of squares. From this, we see that the characteristic of an ordered field must be zero (if it would be $p > 0$, then -1 would be the sum of $p - 1$ 1's and hence a sum of squares).

Since the correspondence between orderings and positive cones is bijective, we may identify the ordering with its positive cone. In this sense, $\mathbf{X}_K$ will denote the set of all orderings resp. positive cones of K.

Let us consider the natural valuation of the additive ordered group of the ordered field $(K, <)$. Through the definition $va + vb := vab$, its value set vK becomes an ordered abelian group and v becomes a homomorphism from the multiplicative group of K onto vK. We have obtained the **natural valuation of the ordered field** $(K, <)$. The place associated with the natural valuation will be called **natural place of the ordered field** $(K, <)$. Since the natural valuation of an ordered field is in fact the natural valuation of its ordered additive group, it inherits the properties of the latter. In particular, the natural valuation of an ordered field is trivial if and only if the ordering is archimedean. We have seen in the last section that if we consider a field valuation as a group valuation, then the components in its associated skeleton are all isomorphic to the additive group of the residue field. Since the components of natural group valuation are archimedean, it follows that the residue field of the natural valuation is an archimedean ordered group. We leave it to the reader to show that the so induced ordering is compatible with the multiplication (we will prove this fact later in a more general setting), and that the residue field is thus an archimedean ordered field. By a theorem of D. Hilbert, which is an analogue of the Theorem of Hölder (Theorem 4.56), every archimedean ordered field can be embedded into the reals $\mathbb{R}$, as an ordered field (cf. [HILB]; see also S. Prieß-Crampe [PC1], Chapter II, §3, or L. Fuchs [FU1], Chapter VIII, §1, for the proof in the more general case of archimedean rings). So if P is the natural place of $(K, <)$, then there is an embedding σ of KP in $\mathbb{R}$. Since $\sigma \circ P$ and P are equivalent places (which we usually identify), we can always assume without loss of generality that

The residue field of the natural valuation resp. the natural place) is a subfield of $\mathbb{R}$.

A field K is called **formally real** if it can be ordered. The ordering may not be unique, so there may be a variety of natural valuations on K. An arbitrary place P of K is called a **real place** if its residue field KP is formally real. Every subfield of $\mathbb{R}$ is formally real, so every natural place is a real place. We are going to state some basic facts about these valuations and places.

Remark 5.13 The natural valuation was introduced by R. Baer under the name "Ordnungsbewertung" (cf. [BAER]). For W. Krull, the main reason for the introduction of general valuations was the investigation of the formally real fields, as introduced by E. Artin and O. Schreier [AR–SCHR]. In his far-sighted article "Allgemeine Bewertungstheorie" ([KRU7]), Krull says:

> Nachdem nun durch Artin und Schreier der Begriff der allgemeinen reellen Krper rein algebraisch eingefhrt wurde, liegt die Frage nahe, ob es nicht mglich ist, den Bewertungsbegriff so zu verallgemeinern, da er zu einer — ihrer Natur nach halb algebraischen, halb analytischen — Behandlung der reellen Krper brauchbar wird. ... Im Rahmen dieses Programms erscheint die Weiterentwicklung der Theorie der geordneten Krper als ein — sogar verhltnismig einfacher — Spezialfall der Theorie der allgemein bewerteten Krper. Im brigen knnen auch die "transzendentesten" geordneten Krper mit ... unsrer allgemeinen Bewertungstheorie behandelt werden. Das erscheint mir, wie bereits zu Beginn der Einleitung hervorgehoben, als das wichtigste Ergebnis und die eigentliche Rechtfertigung der Einfhrung des verallgemeinerten Bewertungsbegriffes.

For the further development of the valuation theoretical approach to formally real fields, see S. Lang [LANG2] (1953), R. Brown [BRW2] (1971), M. Knebusch [KN1] (1972) and [KN2] (1973), A. Prestel [PR1] (1975), M. Knebusch and M. Wright [KN–WR] (1976), T. Y. Lam [LAM1] (1980) and [LAM2] (1983), S. Prieß-Crampe [PC1] (1983).

The algebraic theory of formally real fields is known as the **Artin-Schreier-Theory**. One of its basic results is the following: *A field is formally real if and only if* -1 *is not a sum of squares in that field, and this holds if and only if nontrivial sums of squares in* K *never equate to zero* (cf. [PR1] Theorem 1.8 or [JAC], Chapter VI). Using this characterization, the following is easy to prove:

Lemma 5.14 *If the field* K *admits a real place, then it is formally real.*

Proof: Let P be real place of K and $\sum_i a_i^2$ a nontrivial finite sum of squares in K, that is, there is at least one nonzero summand. We may assume that all a_i lie in $\mathcal{O}_P$ and that at least one of them lies in $\mathcal{O}_P^\times$; otherwise, we pick some a_j with $v_P a_j$ minimal, and we divide the whole sum by a_j^2, thereby replacing the a_i by the elements a_i/a_j which satisfy our assumption. Now we apply P and obtain the finite sum $(\sum_i a_i^2)P = \sum_i (a_i P)^2$ of squares in KP. By our assumption on the a_i , this sum has at least one nonzero summand. Since KP is formally real by assumption, the sum is nonzero. Hence, the sum $\sum_i a_i^2$ is nonzero (and it lies in $\mathcal{O}_P^\times$). This proves that K is formally real. □

The question arises whether an ordering of K can be found such that the real place P is the natural place with respect to this ordering. But this can not be true if the formally real residue field KP has no archimedean ordering (e.g., if its cardinality is bigger than $|\mathbb{R}|$, then it can not be embedded in $\mathbb{R}$ and thus can not admit an archimedean ordering). Instead of "natural place", we have to use the notions **compatible with the ordering**, **convex valuation** and v**-compatible ordering** which we define as in the group case (cf. page 65). We note:

Lemma 5.15 *Let $(K,<)$ be an ordered field and v a valuation of K. Then v is compatible with the ordering if and only if it satisfies*

(COMP) $\forall x,y \in K:\ 0<x\le y \Rightarrow vx \ge vy$.

This in turn is equivalent to each of the following assertions:
1) the valuation ring $\mathcal{O}_v$ is a convex subset of $(K,<)$,
2) the valuation ideal $\mathcal{M}_v$ (or equivalently, $1+\mathcal{M}_v$) is a convex subset of $(K,<)$,
3) the positive cone $\mathbf{P}$ of $(K,<)$ contains $1+\mathcal{M}_v$,
4) $\mathcal{M}_v < 1$.

Proof: If v is compatible with the ordering then by definition, it satisfies (COMP). Now assume (COMP) and let $0\le a\le b$ with $b\in\mathcal{O}_v$. Then $va\ge vb$ and hence, $a\in\mathcal{O}_v$. Since $\mathcal{O}_v$ is closed under $x\mapsto -x$, this yields that $\mathcal{O}_v$ is convex.

Assume that $\mathcal{O}_v$ is convex and let $0\le a\le b$ with $b\in\mathcal{M}_v$. Then $0\le b^{-1}\le a^{-1}$ and $b^{-1}\notin\mathcal{O}_v$. By the convexity of $\mathcal{O}_v$, we find $a^{-1}\notin\mathcal{O}_v$, that is, $a\in\mathcal{M}_v$. Since $\mathcal{M}_v$ is closed under $x\mapsto -x$, this yields that $\mathcal{M}_v$ is convex.

Assume that $\mathcal{M}_v$ is convex. Recall that in an ordered abelian group $(G,<)$ for every $g\in G$, a subset $M\subset G$ is convex if and only if $g+M$ is. Hence, $\mathcal{M}_v$ is convex if and only if $1+\mathcal{M}_v$ is. If the latter is the case, then $1+\mathcal{M}_v\subset\mathbf{P}$ since otherwise, it would contain an element <0 and in view of $1\in 1+\mathcal{M}_v$ and convexity, it would follow that $0\in 1+\mathcal{M}_v$, which is impossible.

Assume that $1+\mathcal{M}_v\subset\mathbf{P}$. Since $1\notin\mathcal{M}_v$ and $\mathcal{M}_v$ is closed under $x\mapsto -x$, this yields that $1-a>0$ for all $a\in\mathcal{M}_v$, that is, $\mathcal{M}_v<1$.

Assume that $\mathcal{M}_v<1$, and let $a,b\in K$ such that $va<vb$. Then $b/a\in\mathcal{M}_v$, hence also $n|b|/|a|\in\mathcal{M}_v$ for every $n\in\mathbb{N}$. Our assumption then yields that $n|b|/|a|<1$, i.e. $n|b|<|a|$. We have thus proved that $|a|\le n|b|\Rightarrow va\ge vb$, showing that v is compatible with the order, by our original definition given for the group case. □

We let $\mathbf{X}_K^v$ denote the set of all v-compatible orderings (resp. positive cones) of K.

Recall from the group case that a valuation is compatible with the order if and only if it is a coarsening of the natural valuation. If v is the natural valuation of $(K,<)$, then by definition, $\mathcal{M}_v$ is precisely the set of all $a\in K$ which satisfy $|a|<1/n$ for all $n\in\mathbb{N}$; such an element a is called **infinitesimal**. We see that precisely the infinitesimals are sent to 0 by the natural place. The existence of nonzero infinitesimals characterizes the non-archimedean ordered fields (in particular, every nonstandard model of the reals has a nontrivial natural valuation and nonzero infinitesimals). Observe that every convex valuation v of $(K,<)$ will satisfy $\mathcal{M}_v<1/n$ for all $n\in\mathbb{N}$; this follows from $\mathcal{M}_v<1$. We thus find that a valuation is convex if and only if its valuation ideal is contained in the valuation ideal of the natural valuation. Hence, it is a coarsening of the natural valuation (cf. Section 10).

Let v be a convex valuation of $(K,<)$ with associated place $P=P_v$. Then by the foregoing lemma, $\mathcal{M}_v$ is a convex subset of $(K,<)$. Hence, the ordered additive group $\mathcal{M}_v$ is a convex subgroup of the ordered additive group $\mathcal{O}_v$. By Lemma 3.21, the ordering of $\mathcal{O}_v$ thus induces an ordering of the additive group of $\overline{K}=\mathcal{O}_v/\mathcal{M}_v$. If $\mathbf{P}$ denotes the positive cone of $(K,<)$, then $\mathbf{P}\cap\mathcal{O}_v$ is the positive cone of $(\mathcal{O}_v,<)$, and $\overline{\mathbf{P}}:=(\mathbf{P}\cap\mathcal{O}_v)/\mathcal{M}_v=(\mathbf{P}\cap\mathcal{O}_v)P$ is the positive cone of the induced ordering of $\overline{K}$. Observe that for $a\in\mathcal{O}_v$,

$a \in \mathbf{P}$ if and only if $\overline{a} \in \overline{\mathbf{P}}$. Since $(ab)P = aP \cdot bP$, we have $\overline{\mathbf{P}} \cdot \overline{\mathbf{P}} \subset \overline{\mathbf{P}}$, which shows that the induced ordering is in fact an ordering of the field $\overline{K}$. In particular, we conclude:

Lemma 5.16 *The place associated with a convex valuation is a real place.*

The ordering (resp. the positive cone $\mathbf{P}$) of $(K, <)$ induces also some structure on the value group of the convex valuation v, which we will describe now. The ordering induces the **character** (that is, a multiplicative group homomorphism) $\mathrm{sign}_{\mathbf{P}} : K^\times \to \{1, -1\}$ sending a to 1 if and only if $a > 0$. Since all squares are positive, $K^{\times 2}$ lies in the kernel of this homomorphism, and we obtain a character $\chi_{\mathbf{P}}$: $K^\times/K^{\times 2} \to \{1, -1\}$ (observe that $\mathbf{P}$ is uniquely determined by $\chi_{\mathbf{P}}$). On the other hand, composing the valuation v with the canonical epimorphism $vK \to vK/2vK$, we obtain an epimorphism $K^\times \to vK/2vK$. Again, the squares are in the kernel, so we obtain an epimorphism h_v: $K^\times/K^{\times 2} \to vK/2vK$. Both groups $K^\times/K^{\times 2}$ and $vK/2vK$ are $\mathbb{F}_2$-vector spaces. Hence there is an embedding $s : vK/2vK \to K^\times/K^{\times 2}$ such that $h_v \circ s = \mathrm{id}$. Now $\sigma_{\mathbf{P}} := \chi_{\mathbf{P}} \circ s$: $vK/2vK \to \{1, -1\}$ is a character induced by $\mathbf{P}$. Note that it depends on our choice of s. With a fixed s, the following holds:

Theorem 5.17 *Let v be a valuation of the field K. The map*

$$\mathbf{X}_K^v \ni \mathbf{P} \mapsto (\overline{\mathbf{P}}, \sigma_{\mathbf{P}}) \in \mathbf{X}_{\overline{K}} \times \mathrm{Hom}(vK/2vK, \{1, -1\}) \tag{5.6}$$

is a bijection.

Proof: We choose $\mathcal{B} \subset K^\times$ such that the elements $bK^{\times 2}$, $b \in \mathcal{B}$, form an $\mathbb{F}_2$-basis of $s(vK/2vK)$. It follows that the elements $h_v(bK^{\times 2}) = vb + 2vK$, $b \in \mathcal{B}$, form an $\mathbb{F}_2$-basis of $vK/2vK$. Hence, for every element $a \in K \setminus \{0\}$, there are elements $b_1, \ldots, b_n \in \mathcal{B}$ and $c \in K \setminus \{0\}$ such that $va = vb_1 + \ldots + vb_n + 2vc$, with $b_1, \ldots, b_n$ uniquely determined. Consequently, $a = b_1 \cdot \ldots \cdot b_n u c^2$ with a unit $u \in \mathcal{O}_v^\times$. Since c^2 is positive for every ordering, the sign of a only depends on the signs of $b_1, \ldots, b_n$ and u. By our definition of $\sigma_{\mathbf{P}}$, we have $\sigma_{\mathbf{P}}(vb + 2vK) = 1 \Leftrightarrow \chi_{\mathbf{P}}(bK^{\times 2}) = \chi_{\mathbf{P}} \circ s(vb + 2vK) = 1 \Leftrightarrow b \in \mathbf{P}$. For a positive cone $\mathbf{P}$ compatible with v, the sign of u with respect to $\mathbf{P}$ is the same as the sign of $\overline{u}$ with respect to $\overline{\mathbf{P}}$. We conclude that $\mathbf{P}$ is uniquely determined by $(\overline{\mathbf{P}}, \sigma_{\mathbf{P}})$.

It remains to show the surjectivity of (5.6). Given a character $\sigma : vK/2vK \to \{1, -1\}$ and a positive cone $\mathbf{P}_{\overline{K}}$ of $\overline{K}$, we define

$$\mathrm{sign}(b_1 \cdot \ldots \cdot b_n u c^2) = \sigma(b_1) \cdot \ldots \cdot \sigma(b_n) \cdot \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) \; .$$

This is well-defined since it does not depend on the choice of u and c. To show that $\mathbf{P} = \{a \in K \mid \mathrm{sign}(a) = 1\} \cup \{0\}$ is a positive cone of K, it suffices to show $\mathbf{P} + \mathbf{P} \subset \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{P} \subset \mathbf{P}$. The latter is immediate since the above defined sign function is multiplicative. To show the former, let $a, a' \in \mathbf{P} \setminus \{0\}$ and let a be represented as above. If $va < va'$, then $a + a' = a(1 + a'/a) = b_1 \cdot \ldots \cdot b_n u(1 + a'/a)c^2$ where $(1 + a'/a)$ is a 1-unit; since u and $u(1 + a'/a)$ have the same residue, it follows that $\mathrm{sign}(a + a') = \mathrm{sign}(a)$, showing that $a + a' \in \mathbf{P}$. Now assume that $va = va'$. Then there is a representation $a' = b_1 \cdot \ldots \cdot b_n u' c^2$, and $\overline{u}$ and $\overline{u'}$ have the same sign with respect to $\mathbf{P}_{\overline{K}}$. Since $\mathbf{P}_{\overline{K}}$ is a positive cone, $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u} + \overline{u'})$ is equal $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u'})$. Consequently, $\mathrm{sign}(u + u') = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u + u'}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u} + \overline{u'}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = \mathrm{sign}(u)$. Hence again, $\mathrm{sign}(a + a') = \mathrm{sign}(a)$, and $a + a' \in \mathbf{P}$.

Further, $\mathbf{P}$ contains all 1-units since $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = 1$ for every 1-unit u. This proves that $\mathbf{P}$ is compatible with v. By our definition, $\mathrm{sign}(u) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u})$, showing that $\overline{\mathbf{P}} = \mathbf{P}_{\overline{K}}$. Finally, $\sigma_{\mathbf{P}} = \sigma$ since $\sigma_{\mathbf{P}}$ and σ are uniquely determined by their action on the basis elements $vb + 2vK$, $b \in \mathcal{B}$. □

There are the following special cases of this theorem. If $\overline{K}$ admits a unique ordering (e.g. if $\overline{K} = \mathbb{R}$), then (5.6) is in fact a bijection between $\mathbf{X}_K^v$ and $\mathrm{Hom}(vK/2vK, \{1, -1\})$. If on the other hand, vK is 2-divisible, then $\mathrm{Hom}(vK/2vK, \{1, -1\})$ contains only the trivial character, and (5.6) is a bijection between $\mathbf{X}_K^v$ and $\mathbf{X}_{\overline{K}}$. If $\overline{K}$ admits a unique ordering and vK is 2-divisible, then K admits precisely one v-compatible ordering. In any case, we have the following answer to our initial question:

Corollary 5.18 *If v is a valuation of the field K such that $\overline{K}$ is formally real, then there exists a v-compatible ordering of K. In other words, if P is a real place of K, then v_P is compatible with some ordering of K.*

A further consequence of our theorem is the following. Assume that (L, v) is an immediate extension of (K, v), that is, value group and residue field remain unchanged. If $\mathcal{B}$ and s are as above, we observe that the values vb, $b \in \mathcal{B}$, remain $\mathbb{F}_2$-independent over $2vK = 2vL$. Hence, the elements $b \in \mathcal{B}$ will also be $\mathbb{F}_2$-independent over $L^{\times 2}$. Consequently, s remains to be an embedding of $vK/2vK$ in $L^\times/L^{\times 2}$. The same construction as in the proof of the theorem then gives a bijection between $\mathbf{X}_L^v$ and $\mathbf{X}_{\overline{K}} \times \mathrm{Hom}(vK/2vK, \{1, -1\})$. This proves:

Corollary 5.19 *Let $(L|K, v)$ be an immediate extension of (K, v). Then every v-compatible ordering of K admits a unique extension to a v-compatible ordering of L.*

The extension can easily be defined: By assumption, for every $a \in L$ there is $a' \in K$ such that $v(a - a') > va$. Then we (have to) set $\mathrm{sign}(a) = \mathrm{sign}(a')$. Indeed, if the order is compatible with v on L, then its positive cone contains $1 + \mathcal{M}_{\mathbf{L}}$ and thus, $\mathrm{sign}(a) = \mathrm{sign}(a(1 + (a' - a)/a)) = \mathrm{sign}(a')$ since $1 + (a' - a)/a$ is a 1-unit.

Example 5.20 Let us see in how many ways a power series field $K = k((\Gamma))$ over a formally real field k can be ordered. Let v be the canonical valuation of K. First, we note that every element of $1 + \mathcal{M}_v$ is a square in K. The reader may prove this by computing explicitly the expansion of the square roots of a given 1-unit. Later, we will see that this is a consequence of the fact that v is henselian and the residue field k has characteristic 0. (In fact, the same arguments show that the group of 1-units is even divisible.) It follows from Lemma 5.15 that v is compatible with every ordering on K, that is, $\mathbf{X}_K = \mathbf{X}_K^v$. It also follows that the sign of a power series $a = \sum_\alpha a_\alpha t^\alpha$ must be the same as that of $a_{va} t^{va}$ (since $a/a_{va}t^{va}$ is a 1-unit). If $\alpha \in 2\Gamma$, then $t^\alpha = (t^{\alpha/2})^2$ is a square and thus positive under every ordering of K. If we choose a maximal set of values $\beta \in \Gamma$, $\mathbb{F}_2$-independent over 2Γ, then the signs of all t^α are already determined by the signs of the elements t^β. On the other hand, we have freedom to choose the signs of all t^β and to determine the sign of the coefficients $a_\alpha \in k$ by choosing an ordering of k. We obtain $\mathbf{X}_K = \mathbf{X}_K^v \leftrightarrow \mathbf{X}_k \times \mathrm{Hom}(vK/2vK, \{1, -1\})$ in accordance with Theorem 5.17. ♢

Loosely speaking, we have shown above how to transfer the property "formally real" upwards (from the residue field to the field) and downwards (from the field to the residue field) by a place. There are many other properties that admit such a treatment, e.g. "real closed", "Rolle field", "half-ordered", "algebraically closed", "perfect". Since we will need certain properties of places and valuations, like "henselian" or "algebraically maximal", we will discuss these cases later. Here, we will give some examples which all use the idea of the proof of Lemma 5.14.

A field is called **euclidean** if it is formally real and every element or its negative is a square. Consequently, *every valuation of a euclidean field has a 2-divisible value group.* Suppose that $(K,<)$ is an ordered field with convex valuation v. Every element of $\overline{K}$ may be written as $\overline{a}$ with $a \in \mathcal{O}_{\mathbf{K}}^{\times}$. Then a or $-a$ and thus also $\overline{a}$ or $-\overline{a}$ is a square. This proves: *The residue field of a euclidean field with respect to a convex valuation is euclidean.*

The **Pythagoras number** of a field K is the smallest number $n \leq \infty$ such that every sum of squares in K equals a sum of at most n squares. If $n = 1$, that is, if every sum of squares is a square, then K is called **pythagorean**. Assume K to be formally real and the Pythagoras number of K to be n. Let v be any convex valuation of K. Given a sum $\sum_{i=1}^{m} \overline{a}_i^2$ of squares in $\overline{K}$ with $m \geq n$, we can find $b_1, \ldots, b_n \in K$ such that $\sum_{i=1}^{m} a_i^2 = \sum_{i=1}^{n} b_i^2$. Then all b_i are in $\mathcal{O}_v$. Indeed, if this would not be true, we could choose some b_j of minimal value and divide the second sum by b_j to obtain a sum of squares which are all in $\mathcal{O}_v$ but not all in $\mathcal{M}_v$ such that the sum is in $\mathcal{M}_v$. Passing to the residue field, we would obtain a nontrivial sum of squares which sums up to zero. This contradicts the fact that $\overline{K}$ is formally real, being the residue field of a convex valuation. Hence, all b_i are in $\mathcal{O}_v$, and passing to the residue field, we find that $\sum_{i=1}^{m} \overline{a}_i^2 = \sum_{i=1}^{n} \overline{b}_i^2$. We have proved: *If v is a convex valuation of the formally real field K, then the Pythagoras number of $\overline{K}$ is smaller or equal to the Pythagoras number of K. In particular, if K is pythagorean, then so is $\overline{K}$.*

If the field K is not formally real, then -1 is a sum of squares in K. The **level** of K is the smallest number n such that -1 is a sum of n squares. If K is formally real, then we define the level of K to be ∞. Let $-1 = \sum_{i=1}^{n} a_i^2$ in K, and let v be any valuation of K. Since $-1 \notin \mathcal{M}_v$, not all a_i can lie in $\mathcal{M}_v$. If all a_i lie in $\mathcal{O}_v$, then $-1 = \sum_{i=1}^{n} \overline{a}_i^2$. If there are some a_i of value < 0, then by the same procedure as in our last argument, we find a nontrivial sum of at most n squares in $\overline{K}$ which sums up to zero. Dividing by an arbitrary nonzero square in this sum, we find that -1 is a sum of less than n squares in $\overline{K}$. This proves: *If v is an arbitrary valuation of the field K, then the level of $\overline{K}$ is smaller or equal to the level of K.*

Remark 5.21 There is a version of Theorem 5.17 for valuations such that Kv is a subfield of $\mathbb{R}$ (for every v-compatible ordering, v will thus be the natural valuation); see [BRW2] and [LAM1] for details. Further, Theorem 5.17 can be generalized in order to treat simultaneously orderings and semiorderings; see [PR1], Chapter 7. For an exemplary application of real places in the theory of quadratic forms (namely, the characterization of fields which satisfy the Weak Hasse Principle), see Theorem 9.1 of [PR1].

Exercise 5.4 *Let $(K,<)$ be an ordered field. Then its prime field is $\mathbb{Q}$. Let v be a valuation of K and show that v is compatible with $<$ if and only if $\mathcal{M}_v < r$ for every positive $r \in \mathbb{Q}$, and that this is the case if and only if $\mathcal{M}_v < r$ for some positive $r \in \mathbb{Q}$.*

Exercise 5.5 *Let $(K,<)$ be an ordered field with natural valuation v and $a \in K$. Show that $va = 0$ (that is, a is archimedean equivalent to 1) if and only if there exists some $n \in \mathbb{N}$ such that $1/n < |a| < n$. What happens if we replace v by an arbitrary convex valuation?*

Exercise 5.6 *Compare Theorem 5.17 and Corollary 5.19 with Corollary 3.25. Explain why in Corollary 3.25, an ordering is uniquely determined by the skeleton, whereas in Theorem 5.17, the valuation v and the ordering of $\overline{K}$ do in general not suffice to fix a unique ordering of the field. Prove a generalization of Corollary 3.25 for arbitrary valuations, analogous to our above approach.*

5.5 Non-archimedean exponential fields

Remark 5.22 A special class of formally real fields are those which are equipped with an exponential. The basic example is $(\mathbb{R}, \exp)$, the reals with the usual exponential function. In the last years, model theoretic algebra has witnessed the solution of one of its most renowned problems: A. Wilkie [WIL2] showed that the theory of $(\mathbb{R}, \exp)$ is model complete. However, this solution left open a lot of questions, among them the request for a general structure theory comparable to that of formally real fields. While the existence of non-archimedean ordered exponential fields is guaranteed by model theory (cf. section 13.3), the explicit construction of such fields is of interest since it gives a deeper insight in the structure theory and the possible axiomatizations of exponentials. We devote this section to the structure theory of non-archimedean exponential fields since to our opinion, it shows a very convincing application of the theory of valued vector spaces to field theory. This application was worked out in detail by S. Kuhlmann in her thesis [KUS1], and the first part of this section will follow closely her paper [KUS2]. Her approach has extended and refined the methods and results of N. Alling [ALL1] who studied already in 1962 the structure which an exponential induces on the value group and residue field of the natural valuation. Apparently, Alling gave the first (to some extent explicit) construction of non-archimedean exponential fields.

The second part of this section will sketch some ideas from the paper [KU-KUS1]. This paper gives valuation theoretical interpretations of growth and Taylor axioms (as satisfied by the usual exponential function). One of them will be discussed here since it plays a crucial role for the construction of non-archimedean exponential fields which have the same elementary properties as $(\mathbb{R}, \exp)$. Finally, we will discuss a map which is induced on the natural value group by an exponential.

For the background and the recent progress in the theory of exponential fields and exponential functions, see N. Alling [ALL1], L. van den Dries [VDD 2,3,7], L. van den Dries and Levitz [VDD–LEV], C. W. Henson and L. A. Rubel [HEN–RUB], A. Macintyre [MAC], D. Richardson [RICS], B. I. Dahn and H. Wolter [DAH–WOL1,2] H. Wolter [WOL1,2,3,4,5], B. I. Dahn [DAH], B. I. Dahn and P. Gring [DAH–GÖ], A. Wilkie [WIL1,2], J.-P. Ressayre [RES], M. H. Mourgues [MOU], L. van den Dries, A. Macintyre and D. Marker [VDD–MAC–MAR1,2], C. Miller [MIL1,2], L. van den Dries and C. Miller [VDD-MIL]. For a study of the theory of all analytic functions of $\mathbb{R}$, restricted to the interval $[0,1]$, see L. van den Dries [VDD8], J. Denef and L. van den Dries [DEN–VDD]. Some of these papers make explicit use of the natural valuation for the study of non-archimedean exponential fields or ordered power series fields.

Throughout this section, let $(K, <)$ be an ordered field with natural valuation v, and define $K^{>0} := \{a \in K \mid a > 0\}$. The additive group $(K, +, 0, <)$ and the multiplicative group $(K^{>0}, \cdot, 1, <)$ are ordered groups. We shall ask for the order isomorphisms between these two groups, in the case where v is nontrivial, i.e., $(K, <)$ is a non-archimedean ordered field.

Let us discuss the structure of these groups. The natural valuation v of K is at the same time the natural valuation of $(K, +, 0, <)$. The valuation ring $\mathcal{O}_v$ is a convex subgroup of $(K, +, 0, <)$, and the valuation ideal $\mathcal{M}_v$ is a convex subgroup of $\mathcal{O}_v$.

The **set of positive units** $\mathcal{U}_v^{>0} := \{a \in \mathcal{O}_v^\times \mid a > 0\}$ is a subgroup of $(K^{>0}, \cdot, 1, <)$. Using (COMP), we see that the map

$$\begin{array}{rcl} (K^{>0}, \cdot, 1, <) & \to & (vK, +, 0, <) \\ a & \mapsto & -va = va^{-1} \end{array}$$

is a surjective group homomorphism preserving $\leq$, with kernel $\mathcal{U}_v^{>0}$. It follows from Lemma 3.20 that $\mathcal{U}_v^{>0}$ is a convex subgroup of $(K^{>0}, \cdot, 1, <)$, and from Corollary 3.22

we obtain

$$(K^{>0},\cdot,1,<)/\mathcal{U}_v^{>0} \;\cong\; (vK,+,0,<)\;. \tag{5.7}$$

Further, the map

$$\begin{array}{rcl}(\mathcal{U}_v^{>0},\cdot,1,<) & \to & (\overline{K}^{>0},\cdot,1,<)\\ a & \mapsto & \overline{a}\end{array}$$

is a surjective group homomorphism preserving $\leq$, with kernel $1+\mathcal{M}_v$, the set of 1-units in (K,v) (recall that all 1-units are positive, by virtue of Lemma 5.15). It follows that $(1+\mathcal{M}_v,\cdot,1,<)$ is a convex subgroup of $(\mathcal{U}_v^{>0},\cdot,1,<)$, and

$$(\mathcal{U}_v^{>0},\cdot,1,<)/1+\mathcal{M}_v \;\cong\; (\overline{K}^{>0},\cdot,1,<)\;. \tag{5.8}$$

Since v is the natural valuation of K, we know that $(\overline{K},<)$ is archimedean. Consequently, the same is true for the multiplicative ordered group $(\overline{K}^{>0},\cdot,1,<)$ (if $b>a>1$ then choose $n\in\mathbb{N}$ such that $n(a-1)\geq b$ to obtain that $a^n=(1+a-1)^n\geq 1+n(a-1)>b$). As the element 2 lies in $\mathcal{U}_v^{>0}$ but not in $1+\mathcal{M}_v$, we see that:

a) $\mathcal{U}_v^{>0}$ is the smallest convex subgroup of $(K^{>0},\cdot,1,<)$ containing 2

b) $1+\mathcal{M}_v$ is the biggest convex subgroup of $(K^{>0},\cdot,1,<)$ not containing 2

c) $(\overline{K}^{>0},\cdot,1,<)$ is the component of 2 in $(K^{>0},\cdot,1,<)$.

Let us use the name **exponential** for an isomorphism

$$f:\;(K,+,0,<)\longrightarrow(K^{>0},\cdot,1,<) \tag{5.9}$$

which is normed by one of the equivalent conditions given in the following lemma:

Lemma 5.23 *The following conditions for an isomorphism (5.9) are equivalent:*
1) $f(\mathcal{O}_v)=\mathcal{U}_v^{>0}$ *and* $f(\mathcal{M}_v)=1+\mathcal{M}_v$,
2) $v(f(1)-1)=0$,
3) there is $n\in\mathbb{N}$ *such that* $1+1/n<f(1)<n$,
4) $f(1)$ *is archimedean equivalent to* 2 *in* $(K^{>0},\cdot,1,<)$.

Proof: Assume that 1) holds. Then for every $a\in\mathcal{O}_v$ we have $vf(a)=0$ and thus, $v(f(a)-1)\geq 0$. By virtue of $f(\mathcal{M}_v)=1+\mathcal{M}_v$, we obtain that $v(f(a)-1)=0$ for all $a\in\mathcal{O}_v\setminus\mathcal{M}_v$. Since $1\in\mathcal{O}_v\setminus\mathcal{M}_v$, we now see that 2) holds.

Condition 2) says that $f(1)-1$ is archimedean equivalent to 1 in $(K,+,0,<)$. Now we leave it to the reader to show that this is equivalent to condition 3) (cf. Exercise 5.5).

Assume condition 2), that is, $v(f(1)-1)=0=v2$. Choose $n\in\mathbb{N}$ such that $n|f(1)-1|\geq 2$ and $n\cdot 2\geq|f(1)-1|$; w.l.o.g., we assume $n>2$. Observe that $f(1)>f(0)=1$. Hence, we have $f(1)^n=(1+f(1)-1)^n\geq n(f(1)-1)\geq 2$ and $2^n\geq 2n+1\geq f(1)$, showing that 4) holds.

Assume that 4) holds. Then the order isomorphism f must send the smallest convex subgroup of $(K,+,0,<)$ containing 1 onto the smallest convex subgroup of $(K^{>0},\cdot,1,<)$ containing $f(1)$. The former is just $\mathcal{O}_v$, the latter is $(\mathcal{U}_v^{>0},\cdot,1,<)$ since $f(1)$ is archimedean equivalent to 2 by assumption. Further, f must send the biggest convex subgroup of $(K,+,0,<)$ not containing 1 onto the biggest convex subgroup of $(K^{>0},\cdot,1,<)$ not containing $f(1)$ (or 2). Hence, $f(\mathcal{M}_v)=1+\mathcal{M}_v$. We have proved that 1) holds. □

Note that every isomorphism (5.9) gives rise to an exponential. Indeed, if f is any such isomorphism, then we may take $a \in K^{>0}$ such that $f(a) = 2$; setting $e(x) := f(ax)$, we thus obtain an isomorphism e which satisfies $e(1) = 2$. So it makes sense to call an ordered field an **exponential field** as soon as it admits an isomorphism (5.9). Instead of "exponential", one may also speak of "weak exponential" since the usual exponential exp on $\mathbb{R}$ has many more properties that do not follow from just being such a (normed) isomorphism.

In the last section, we have studied the structure induced by orderings on the value group and residue field of a compatible valuation. We will do the same here for an exponential f and the natural valuation v of $(K, <)$.

First of all, we see that in view of condition 1) of the foregoing lemma, f induces an isomorphism $(\mathcal{M}_v, +, 0, <) \to (1+\mathcal{M}_v, \cdot, 1, <)$, which we will denote by f_R. Again in view of that condition, f induces an isomorphism $(\mathcal{O}_v, +, 0, <)/\mathcal{M}_v \to (\mathcal{U}_v^{>0}, \cdot, 1, <)/1 + \mathcal{M}_v$. Composing it with the isomorphism (5.8), we obtain an induced isomorphism

$$\begin{array}{rcl} \overline{f} : (\overline{K}, +, 0, <) & \to & (\overline{K}^{>0}, \cdot, 1, <) \\ \overline{a} & \mapsto & \overline{f(a)} \end{array}$$

which is an exponential on $\overline{K}$.

Using once more our condition that $f(\mathcal{O}_v) = \mathcal{U}_v^{>0}$, we find that f induces an order isomorphism $(K, +, 0, <)/\mathcal{O}_v \to (\overline{K}^{>0}, \cdot, 1, <)/\mathcal{U}_v^{>0}$. By (5.7), the latter group is order isomorphic to the value group $(vK, +, 0, <)$. Let us write $\mathbf{G} = (G, <)$ for $(vK, +, 0, <)$, and let $v_{\mathbf{G}}$ denote the natural valuation of this ordered group. So the exponential f induces an order isomorphism $(K, +, 0, <)/\mathcal{O}_v \to \mathbf{G}$, which we denote by f_L. This isomorphism in turn induces an isomorphism between the ordered skeletons of these groups. The value set of $(K, +, 0, <)$ is G, and its components are all isomorphic to $(\overline{K}, +, 0, <)$. From Corollary 3.23 we infer that the value set of $(K, +, 0, <)/\mathcal{O}_v$ is $G^{<0} = \{\alpha \in G \mid \alpha < 0\}$, and its components are all isomorphic to $(\overline{K}, +, 0, <)$. So f induces an isomorphism

$$\begin{array}{rcl} \varphi_f : (G^{<0}, <) & \to & (v_{\mathbf{G}}G, <) \\ va & \mapsto & v_{\mathbf{G}}(-vf(a)) = v_{\mathbf{G}}(vf(a)) \end{array}$$

of ordered sets, and isomorphisms from $(\overline{K}, +, 0, <)$ onto every component of $\mathbf{G}$. We have proved the following result due to N. Alling (cf. [ALL1], Section 1):

Theorem 5.24 *Let $(K, <)$ be an ordered field with natural valuation v and value group $\mathbf{G}$. Let $v_{\mathbf{G}}$ be the natural valuation of $\mathbf{G}$. If K admits an exponential f, then*
a) there is an isomorphism $f_L : (K, +, 0, <)/\mathcal{O}_v \to \mathbf{G}$,
b) there is an isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$ of ordered sets,
c) all components of $\mathbf{G}$ are isomorphic to $(\overline{K}, +, 0, <)$,
d) there is an exponential $\overline{f}$ on $\overline{K}$,
e) there is an isomorphism $f_R : (\mathcal{M}_v, +, 0, <) \to (1 + \mathcal{M}_v, \cdot, 1, <)$.

Now the question arises whether there exists a converse of this theorem. This would mean that we have to build an exponential from three different parts: an order isomorphism f_R, an exponential $\overline{f}$ on the residue field, and an order isomorphism f_L. We have to lift $\overline{f}$ and f_L up into the additive and positive multiplicative group of K. Since the

additive group of a field of characteristic 0 is always divisible, there is a group complement $\mathbf{A}' \cong (\overline{K}, +, 0, <)$ for $\mathcal{M}_v$ in $\mathcal{O}_v$, and there is a group complement $\mathbf{A} \cong (K, +, 0, <)/\mathcal{O}_v$ for $\mathcal{O}_v$ in $(K, +, 0, <)$. The reader may show that we have a lexicographic decomposition

$$(K, +, 0, <) \cong \mathbf{A} \amalg \mathbf{A}' \amalg \mathcal{M}_v \,. \tag{5.10}$$

To do the same for the group $(K^{>0}, \cdot, 1, <)$, we have to assume that it is divisible, that is, the field is **root closed for positive elements**. Note that this condition is necessary for the existence of an exponential since an isomorphism between additive and positive multiplicative group yields that the latter is divisible. Under this condition, there is a group complement $\mathbf{B}' \cong (\overline{K}^{>0}, \cdot, 1, <)$ for $1 + \mathcal{M}_v$ in $\mathcal{U}_v^{>0}$, and there is a group complement $\mathbf{B} \cong (vK, +, 0, <)$ for $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$. This gives a lexicographic decomposition

$$(K^{>0}, \cdot, 1, <) \cong \mathbf{B} \amalg \mathbf{B}' \amalg 1 + \mathcal{M}_v \,. \tag{5.11}$$

Now f_R may be viewed as an exponential of the right sides of these decompositions, and we thus call it a **right exponential**. Similarly, f_L may be viewed as an exponential of the left sides of these decompositions, and we call it a **left exponential**. Putting the exponentials f_L, $\overline{f}$ and f_R together, we obtain an exponential of K which induces these partial exponentials. The problem is now to find criteria for the existence of right and left exponentials.

In [ALL1], Theorem 3.1, Alling gives a criterion for the existence of f_R. He shows that on every power series field, a right exponential from the valuation ideal onto the set of 1-units can be defined by the usual Taylor expansion for exp. Thus, an arbitrary ordered field admits a right exponential if it is embeddable into a power series field in such a way that the image of its natural valuation ideal is closed under that right power series exponential and its inverse.

We have deduced information about the skeletons of the groups from the existence of an exponential. To some extent, this procedure can be reversed. Recall that divisible ordered abelian groups are ordered $\mathbb{Q}$-vector spaces. The idea introduced by S. Kuhlmann in [KUS1] and [KUS2] is to employ the fact that in two special cases, these are determined up to isomorphism by their skeletons, namely if they are countable (cf. Theorem 4.47) or if they are maximal (cf. Theorem 4.53). (Recall that by Theorem 4.52, for divisible ordered abelian groups the properties "maximal", "spherically complete" and "being (isomorphic to) a Hahn product" coincide.) In order to apply this idea, S. Kuhlmann computes the skeleton of $(1 + \mathcal{M}_v, \cdot, 1, <)$. Let w denote its natural valuation. Implicitly, we have already above determined the relation between v and w. Explicitly, it reads as follows (up to equivalence, as usual):

Theorem 5.25 *Let v be the natural valuation of $(K, <)$. Then*

$$w(1 + a) = va \quad \textit{for all } a \in \mathcal{M}_v \tag{5.12}$$

is the natural valuation on $(1 + \mathcal{M}_v, \cdot, 1, <)$. Hence, its value set is $w(1 + \mathcal{M}_v) = v\mathcal{M}_v = (vK)^{>0}$. The components are all isomorphic to $(\overline{K}, +, 0, <)$. Consequently, $(\mathcal{M}_v, +, 0, <)$ and $(1 + \mathcal{M}_v, \cdot, 1, <)$ have the same skeleton.

Proof: Suppose that $a, b \in \mathcal{M}_v$ with $a, b > 0$. Assume that $va \geq vb$. Then there is $n \in \mathbb{N}$ such that $nb \geq a$. This yields that $(1 + b)^n \geq 1 + nb \geq 1 + a$, hence $w(1 + a) \geq w(1 + b)$.

Conversely, assume that $w(1+a) \geq w(1+b)$ and choose $n \in \mathbb{N}$ such that $(1+b)^n \geq 1+a$. Hence $(1+b)^n - 1 \geq a$, showing that $va \geq v((1+b)^n - 1)$. Since $vb > 0$, we have $v((1+b)^n - 1) = v(nb + \ldots + b^n) = vnb = vb$, which proves that $va \geq vb$. We have shown that $va \geq vb \Leftrightarrow w(1+a) \geq w(1+b)$; consequently, $va = w(1+a)$ for all positive $a \in \mathcal{M}_v$ (up to equivalence of valuations). To deduce that this also holds for negative $a \in \mathcal{M}_v$, it suffices to show that $w(1+b) = w(1-b)$ for every positive $b \in \mathcal{M}_v$, or in other words, that $1+b$ is archimedean equivalent to $1-b$ in the multiplicative group $1+\mathcal{M}_v$. Since b is positive, we have that $(1-b)^{-1} > 1+b$. On the other hand, $(1+b)^2(1-b) = 1+b-b^2-b^3$ which is > 1, since $vb < 2vb = v(b^2+b^3)$ yields that $b > b^2+b^3$. This proves that $(1+b)^2 > (1-b)^{-1}$. Hence indeed, $1+b$ is archimedean equivalent to $1-b$. Now it also follows that $w(1+\mathcal{M}_v) = v\mathcal{M}_v = (vK)^{>0}$.

Let $\alpha \in w(1+\mathcal{M}_v) = (vK)^{>0}$. From what we have proved, we see that $\{c \in 1+\mathcal{M}_v \mid wc \geq \alpha\} = 1+\mathcal{O}_v^\alpha$ and $\{c \in 1+\mathcal{M}_v \mid wc > \alpha\} = 1+\mathcal{M}_v^\alpha$. Hence, the α-component of $(1+\mathcal{M}_v, \cdot, 1, <)$ is $(1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha)$. Define a map

$$\mathcal{O}_v^\alpha \ni a \mapsto (1+a)(1+\mathcal{M}_v^\alpha) \in (1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha) \ .$$

Let $va = vb = \alpha$. Since $\alpha > 0$, we have that $vab > \alpha$ and consequently, $(1+a+b+ab)/(1+a+b) \in 1+\mathcal{M}_v^\alpha$. This yields that $(1+a+b)(1+\mathcal{M}_v^\alpha) = (1+a+b+ab)(1+\mathcal{M}_v^\alpha) = (1+a)(1+b)(1+\mathcal{M}_v^\alpha)$. Hence, our map is a group homomorphism. Since $a \mapsto a+1$ preserves the order and since the order on $(1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha)$ is the order induced by $(1+\mathcal{O}_v^\alpha, \cdot, 1, <)$, our map preserves $\leq$. Its kernel is $\mathcal{M}_v^\alpha$. Now Corollary 3.22 shows that it induces an order isomorphism from the component $(\mathcal{O}_v^\alpha, +, 0, <)/\mathcal{M}_v^\alpha$ (which is isomorphic to $(\overline{K}, +, 0, <)$) onto the component $(1+\mathcal{O}_v^\alpha, \cdot, 1, <)/(1+\mathcal{M}_v^\alpha)$. □

Since every exponential f (and every right exponential f_R) induces an isomorphism of the skeletons of $\mathcal{M}_v$ and $1+\mathcal{M}_v$, the equality of the value sets shows that it induces an automorphism ψ_f of $(vK)^{>0}$.

From (5.12), we deduce

$$w\left(\frac{1+a}{1+b}\right) = v\left(\frac{1+a}{1+b} - 1\right) = v\left(\frac{a-b}{1+b}\right) = v(a-b)$$

for all $a, b \in \mathcal{M}_v$. This shows that

$$\mathcal{M}_v \ni \ a \ \mapsto \ 1+a \ \in 1+\mathcal{M}_v$$

is an isomorphism of ultrametric spaces. In view of Lemma 5.4 and Theorem 4.7, we obtain:

Corollary 5.26 *If K is spherically complete with respect to its natural valuation, then the groups $(\mathcal{M}_v, +, 0, <)$ and $(1+\mathcal{M}_v, \cdot, 1, <)$ are spherically complete (and hence maximal) with respect to their natural valuations.*

We have mentioned in the last section that the group $1+\mathcal{M}_v$ of 1-units (with respect to the natural valuation) is divisible. Hence, it is a $\mathbb{Q}$-vector space like the additive group $\mathcal{M}_v$. Hence, we may use Theorem 4.47 and Theorem 4.53 to obtain:

Corollary 5.27 *Suppose that $(K, <)$ is countable or spherically complete with respect to its natural valuation. Then $(K, <)$ admits a right exponential.*

The conditions for the skeletons of $\mathbf{A} \cong (\overline{K}, +, 0, <)/\mathcal{O}_v$ and $\mathbf{B} \cong \mathbf{G}$ to be isomorphic are given by b) and c) of Theorem 5.24. Hence, we have: *If $(K, <)$ is countable or spherically complete with respect to its natural valuation, then conditions b), c) and d) of Theorem 5.24 imply the existence of an exponential on K.* We shall improve this result by determining all possible exponentials (in the spirit of Theorem 5.17). But this is quite hopeless if we allow the exponentials to be too messy on the components. So we will confine ourselves to **$\overline{K}$-linear exponentials**. To explain what we mean by this notion, we show that we are actually working with $\overline{K}$-vector spaces. Observe that the components of $(\mathcal{M}_v, +, 0, <)$, $(1 + \mathcal{M}_v, \cdot, 1, <)$, and $(\overline{K}, +, 0, <)/\mathcal{O}_v$ are all $\overline{K}$, and the same is true for $\mathbf{G}$ if it satisfies condition c). We shall use the following fact:

Lemma 5.28 *Let k be a field and $\mathbf{G}$ a countable or spherically complete divisible ordered (resp. valued) group whose components are k-vector spaces. Then $\mathbf{G}$ is a k-vector space (with value preserving scalar multiplication).*

Proof: If $\mathbf{G}$ is countable, then by Theorem 4.47 it is a Hahn sum over its skeleton. If $\mathbf{G}$ is spherically complete, then by Theorem 4.52 it is a Hahn product over its skeleton. In both cases, k-scalar multiplication can be defined componentwise since the components are k-vector spaces. Multiplication by a nonzero scalar does not change the minimum of the support and therefore, it preserves the value. □

It now remains to consider the component $(\mathcal{U}_v^{>0}, \cdot, 1, <)/1 + \mathcal{M}_v \cong (\overline{K}^{>0}, \cdot, 1, <)$. If $\overline{f}$ is an exponential on $\overline{K}$, then it is isomorphic to $(\overline{K}, +, 0, <)$ and through this isomorphism, it inherits the structure of a $\overline{K}$-vector space, with respect to which $\overline{f}$ is $\overline{K}$-linear. With this in mind, we will consider f to be $\overline{K}$-linear if and only if f_L and f_R are $\overline{K}$-linear. If $\mathbf{G}$ satisfies condition c), then every $\overline{K}$-linear f_L is uniquely determined by the order isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$. Similarly, every $\overline{K}$-linear f_R is uniquely determined by the order automorphism ψ_f of $G^{>0}$. We leave it to the reader to prove the following:

Lemma 5.29 *Let k be a field and $(V, <)$ and $(V', <)$ two ordered Hahn sums or Hahn products whose components are all equal to $(k, +, 0, <)$ (hence, we view them as k-vector spaces). Then for every isomorphism φ from the value set of V onto the value set of V' there is a unique k-linear isomorphism form $(V, <)$ onto $(V', <)$ which induces φ.*

Fixing the isomorphism of the considered vector spaces onto the respective Hahn sums (or Hahn products), we now obtain the desired description of $\overline{K}$-linear exponentials:

Theorem 5.30 *Suppose that $(K, <)$ is root closed for positive elements and countable. Assume further that all components of its value group $\mathbf{G}$ are isomorphic to $(\overline{K}, +, 0, <)$. Then there is a bijection*

$$f \mapsto (\varphi_f, \overline{f}, \psi_f)$$

from the set of all $\overline{K}$-linear exponentials of $(K, <)$ onto the product of the set of all order isomorphisms $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$, the set of all exponentials on $\overline{K}$ and the set of all order automorphisms ψ_f of $G^{>0}$.

The same holds if $(K, <)$ and $\mathbf{G}$ are spherically complete with respect to their natural valuations.

Observe that our theorem does not assert the existence of the isomorphism φ_f (while ψ_f can always be taken to be the identity). And in fact, not every value group admits such

an isomorphism. In the countable case, the value group of an exponential field K is of the form

$$\coprod_{\mathbb{Q}} (\overline{K}, +, 0, <) \ .$$

Indeed, the value group $\mathbf{G}$ of a root closed field must be divisible. If it is countable, it is thus isomorphic to a Hahn sum, by virtue of Theorem 4.47. From divisibility, it also follows that its ordering is dense and without endpoints and that the same is true for $G^{<0}$. If G and hence also $G^{<0}$ are countable, then $G^{<0}$ is a countable dense linearly ordered set without endpoints and as such, it is isomorphic to the ordered set $\mathbb{Q}$ (cf. [ROS]). Hence, a nontrivial countable divisible ordered abelian group $\mathbf{G}$ admits an isomorphism $G^{<0} \cong v_{\mathbf{G}} G$ if and only if it is (isomorphic to) a Hahn sum with value set $\mathbb{Q}$. We see that there exist countable groups satisfying conditions b) and c), so our theorem indeed yields the existence of countable exponential fields as soon as we are able to construct countable valued fields with given value group and residue field. This will be done in section 7.6. As to the problem of the existence of exponential power series fields, see [KU–KUS2].

Let us now consider the valuation theoretical interpretation of growth axioms for an exponential of a nonarchimedean ordered field. As an example, we choose an axiom scheme which plays a distinguished role in the model theory of exponential fields, according to a theorem of Ressayre ([RES]; cf. also Theorem of [VDD–MAC–MAR1]). It describes the growth of the exponential f in comparison with polynomials:

(GRE) $\qquad x > n^2 \implies f(x) > x^n \qquad\qquad (n \in \mathbb{N})\,.$

Because of the condition "$x > n^2$", this axiom scheme is void for infinitesimals. That is, it gives information only in the case of $vx \leq 0$. It holds in the case $vx = 0$ if f coincides with exp on a maximal archimedean subfield of K (viewed as a subfield of $\mathbb{R}$). (Recall that exp itself satisfies (GRE) and even $x \geq n^2 \ \Rightarrow \ f(x) > x^n$.) Indeed, assume that $vx = 0$; then $x = r + \varepsilon$ with $r \in \mathbb{R}$ and $v\varepsilon > 0$, and if $x > n^2$, then $r \geq n^2$. So we have $f(r) = \exp(r) > r^n$. But $f(r)$ is the residue of $f(x)$ and r^n is the residue of x^n modulo the natural valuation v since ε is infinitesimal. Hence, $f(r) > r^n$ implies that $f(x) > x^n$.

Now we have to consider the case of $vx < 0$. In this case, "$x > n^2$" holds for all $n \in \mathbb{N}$ if only x is positive. Restricted to $K \setminus \mathcal{O}_{\mathbf{K}}$, axiom scheme (GRE) is thus equivalent to the assertion

$$vx < 0 \ \wedge \ x > 0 \implies \forall n \in \mathbb{N} : \ f(x) > x^n \ . \tag{5.13}$$

But "$\forall n \in \mathbb{N} : \ f(x) > x^n$" means that $wf(x) < wx$. Observe that the condition "$vx < 0 \wedge x > 0$" implies that $x, f(x) \in K^{>0} \setminus \mathcal{U}_v^{>0}$. Through the isomorphism (5.7) which is induced by the map $a \mapsto -va$, we see that the natural valuation w is given on $K^{>0} \setminus \mathcal{U}_v^{>0}$ by

$$wa = v_{\mathbf{G}}(va) \quad \text{for all } a \in K^{>0} \setminus \mathcal{U}_v^{>0}$$

(note that $v_{\mathbf{G}}(-va) = v_{\mathbf{G}}(va)$). Hence, $wf(x) < wx$ is equivalent to $v_{\mathbf{G}}(vf(x)) < v_{\mathbf{G}}(vx)$. In view of $vx < 0$ and the definition of φ_f, this in turn is equivalent to $\varphi_f(vx) < v_{\mathbf{G}}(vx)$. Therefore, assertion (5.13) is equivalent to

$$\varphi_f \, g < v_{\mathbf{G}} \, g \quad \text{for all } g \in G^{<0} \ . \tag{5.14}$$

For the conclusion of this section, let us sketch how to obtain a map induced on the natural value group by the exponential f. One way to define this map is to compose the

natural valuation $v_{\mathbf{G}}$ with the inverse of the isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$. This gives a map

$$\varphi_f^{-1} \circ v_{\mathbf{G}} : \; G \setminus \{0\} \; \to \; G^{<0} \; . \tag{5.15}$$

Using the definition of φ_f, we find that this map sends $vf(a)$ to va, provided that $va < 0$. Setting $b = f^{-1}(a)$, we see that the above map is in fact induced by the **logarithm** f^{-1}:

$$\begin{array}{rcl} G \setminus \{0\} & \to & G^{<0} \\ vb & \mapsto & vf^{-1}(b) \end{array}$$

Since the natural valuation $v_{\mathbf{G}}$ sends archimedean equivalent elements to the same value, it is seen immediately from (5.15) that the above map does the same. In other words, the map is constant on every archimedean class (which consists of two convex sets, one in $G^{<0}$ and one in $G^{>0}$). Nevertheless, its image is all of $G^{<0}$ since both $v_{\mathbf{G}}$ and φ_f^{-1} are surjective. This shows that ordered groups admitting such maps resp. an isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$ must be "quite big". For the study of their properties it turned out to be convenient to change these maps a little bit. We define

$$\chi_f \, g := \varphi_f^{-1}(v_{\mathbf{G}} \, g) \quad \text{for all } g \in G^{<0}$$

and extend χ_f to all of G by symmetry, setting $\chi_f \, 0 = 0$. in this way, we obtain a map $\chi_f : G \to G$ which satisfies the following axioms:

(C0) $\chi_f \, x = 0 \; \Leftrightarrow \; x = 0 \, ,$

(C−) $\chi_f(-x) = -\chi_f \, x \, ,$

(CA) $v_{\mathbf{G}} \, x = v_{\mathbf{G}} \, y \; \wedge \; \text{sign}(x) = \text{sign}(y) \; \Rightarrow \; \chi_f \, x = \chi_f \, y \, ,$

(C≤) χ_f preserves $\leq$,

(CS) χ_f is surjective.

The axioms (C0) and (C−) are direct consequences of our definition. Axiom (CA) follows from the fact that (5.15) is constant on archimedean classes, and the extension by symmetry. Axiom (C≤) is seen as follows. On $G^{<0}$, the natural valuation $v_{\mathbf{G}}$ preserves $\leq$ (cf. (3.7)), so its composition with the order preserving map φ_f^{-1} also preserves $\leq$. On $G^{>0}$, $v_{\mathbf{G}}$ reverses $\leq$, but χ_f again preserves $\leq$ since there it is equal to $-\varphi_f^{-1} \circ v_{\mathbf{G}}$ by virtue of our symmetric extension. Finally, axiom (CS) is a consequence of symmetry and the fact that the image of $\varphi_f^{-1} \circ v_{\mathbf{G}}$ is all of $G^{<0}$. The advantage of the map χ_f in comparison with $\varphi_f^{-1} \circ v_{\mathbf{G}}$ is that it is a map from G to G and that it preserves $\leq$ on all of G. Observe that all axioms except (CS) are universal (cf. Section 13.1). Since axiom (CA) says that the map contracts archimedean classes to just two points, we call a map χ_f a **precontraction** if it satisfies the universal axioms (C0), (C−), (CA) and (C≤), and we call it a **contraction** if it satisfies in addition axiom (CS).

Furthermore, assertion (5.14) is equivalent to

$$g < \chi_f \, g \quad \text{for all } g \in G^{<0} \; .$$

In view of the symmetry axiom (C−), this in turn is equivalent to the axiom

(CP) $x \neq 0 \; \Rightarrow \; |\chi_f \, x| < |x| \, .$

This axiom expresses the assertion that χ_f sends the elements towards the middle point 0 of the group. Therefore, we call a precontraction **centripetal** if it satisfies (CP). For a precontraction, (CP) is equivalent to the seemingly stronger assertion

$$v_{\mathbf{G}}(\chi_f\, g) > v_{\mathbf{G}}\, g \quad \text{for all } g \in G \setminus \{0\}\ . \tag{5.16}$$

Indeed, (CP) implies $v_{\mathbf{G}}\chi_f\, g \geq v_{\mathbf{G}}\, g$. But equality can not hold for $g \neq 0$ since otherwise we would obtain $\chi_f\chi_f\, g = \chi_f\, g$ by virtue of (CA), in contradiction to (CP). We have proved:

Lemma 5.31 *Every exponential f on $(K,<)$ induces a contraction χ_f on its natural value group. Further, f satisfies (5.13) if and only if χ_f is centripetal.*

Ordered abelian groups with contractions and precontractions are studied in detail in the papers [KU4] and [KU5], making intense use of the natural valuation of such groups.

Remark 5.32 In Ressayre's proof of his main theorem in [RES], a crucial tool is the **integer part** (also called **integral part**) of an ordered field $(K,<)$, that is, a subring R of K which satisfies

(IP1) $(R,<)$ is discretely ordered (i.e., $\forall x \in R : x \leq 0 \vee x \geq 1$),

(IP2) $\forall x \in K\ \exists y \in R :\ |x-y| < 1\,.$

M. H. Mourgues and J.-P. Ressayre [MOU–RES] showed that every real closed field (cf. Section 9.6) has an integer part; see also S. Boughattas [BOUG]. If f is an exponential on $(K,<)$, then an integer part will be called **exponential integer part** if it is closed under f. Ressayre has shown that every real closed field has an exponential integer part, provided that the exponential satisfies $f(1) = 2$ (such an exponential is commonly denoted by 2^x). The reader may show the following:

Let $(K,<)$ be an ordered field with natural valuation v. If the group complement $\mathbf{A}$ of $\mathcal{O}_v$ in $(K,+,0)$ is closed under multiplication, then the subring R of K generated by $\mathbf{A}$ and $\mathbb{Z}$ is an integer part of $(K,<)$. If in addition, f is an exponential on $(K,<)$ satisfying $f(1) = 2$ and if $\mathbf{A}$ is closed under f, then R is an exponential integer part of $(K,<,f)$.

Exercise 5.7 *For an exponential f satisfying (GRE), deduce (5.16) directly from the definition $\chi_f(vb) = vf^{-1}(b)$.*

Part II

The structure of valued fields and their extensions

Chapter 6

Preliminaries from algebra

6.1 Algebraic field extensions

Let $L|K$ be an arbitrary field extension. If $\mathcal{P}$ is any property of field extensions, then we will say that $L|K$ is a **tower of extensions with property** $\mathcal{P}$ if there are intermediate fields $K = K_0 \subset K_1 \subset \ldots \subset K_n = L$ such that $K_i|K_{i-1}$ are extensions with property $\mathcal{P}$, for $i = 1, \ldots, n$. A homomorphism $L \to F$ will be called a **homomorphism over** K if it fixes K elementwise. An automorphism of L will be called **automorphism of** $L|K$ if it is a homomorphism over K. An element $b \in L$ is **algebraic over** K, if it is the root of some nonzero polynomial f with coefficients from K. For every homomorphism $\sigma : L \to F$ over K, the element σb is again a root of f, hence it is algebraic over K. The **relative algebraic closure of** K **in** L is the subfield of L consisting of all elements which are algebraic over K; in particular, it contains K. We see that every automorphism of $L|K$ sends the relative algebraic closure into itself.

A field extension $L|K$ is called **algebraic** if every element $b \in L$ is algebraic over K. Every **finite field extension** $L|K$ (i.e., $[L : K] < \infty$) is algebraic. A field extension of the form $K(b)|K$ is called **simple**. It is algebraic if and only if b is algebraic over K, and this is the case if and only if the field $K(b)$ is equal to the ring $K[b]$. If $L|K$ and $L'|L$ are algebraic field extensions, then so is $L'|K$. There exist maximal algebraic extensions, called **algebraic closures of** K. An algebraic closure L of K is itself **algebraically closed**, that is, every algebraic extension $L'|L$ is **trivial**: $L' = L$. Over an algebraically closed field, every polynomial splits into linear factors. All algebraic closures of a field K are isomorphic over K, so we shall speak of *the* algebraic closure of K and denote it by $\tilde{K}$. In this sense, the algebraic closure of K coincides with the relative algebraic closure of K in every algebraically closed extension field. We will often use such an identification of isomorphic objects without further mentioning. If $\sigma : L \to F$ is a homomorphism over K, then it is injective, and we will call it an **embedding of** L **in** F **over** K. If in addition F is algebraically closed, then σ can be extended to an embedding of $\tilde{L}$ in F over K. In particular, if $L|K$ is algebraic, then every embedding of L in $\tilde{L} = \tilde{K}$ over K can be extended to an automorphism of $\tilde{K}|K$.

Let b be algebraic over K. Then $\{f \in K[X] \mid f(b) = 0\}$ is a nontrivial proper ideal of the ring $K[X]$. Since $K[X]$ is a euclidean ring with respect to the usual degree function for polynomials, this ideal is principal. It has precisely one monic generator f, which is called the **minimal polynomial of** b **over** K**.** (Recall that a polynomial

$f = c_n X^n + \ldots + c_1 x + c_0$ is called **monic** if its **leading coefficient** c_n is equal to 1.) Thus, a polynomial $g \in K[X]$ admits b as a zero if and only if f divides g in the ring $K[X]$. In particular, f is irreducible over K. Every embedding σ of $K(b)$ in $\tilde{K}$ over K sends b to a root σb of f. Conversely, if a is a second root of f, then there is an isomorphism σ of the fields $K(b) = K[b]$ and $K(a) = K[a]$ over K which sends b to a. This isomorphism extends to an automorphism of $\tilde{K}|K$. Hence, the finite set of roots of f in $\tilde{K}$ coincides with the set $\{\sigma b \mid \sigma \text{ is an automorphism of } \tilde{K}|K\}$; its elements are called the **conjugates of** b. Two elements $a, b \in \tilde{K}$ are called **conjugate over** K if there exists an automorphism σ of $\tilde{K}|K$ such that $a = \sigma b$.

Lemma 6.1 *a) Let $g, h \in K[X]$. Then g is prime to h in the ring $K[X]$ if and only if g and h admit no common root in $\tilde{K}$.*

b) Assume that $gh \in K[X]$ with $g, h \in \tilde{K}[X]$ such that g is monic, g and h admit no common root in $\tilde{K}$ and the set of roots of g in $\tilde{K}$ is closed under conjugates. Then $g, h \in K[X]$.

Proof: a): If f is a nonconstant common divisor of g and h, then it admits a root in $\tilde{K}$, which is thus a common root of g and h. Conversely, if b is a common root of g and h in $\tilde{K}$, then the minimal polynomial of b over K divides both g and h, showing that g is not prime to h in $K[X]$.

b): Suppose that we have verified our assertion for all products gh with $\deg g < n$ (for $g = 1$, the assertion is trivial). Now let g be of degree n and gh satisfy the assumptions of part b). Let b be a root of g in $\tilde{K}$. Since it is also a root of $gh \in K[X]$, the minimal polynomial f of b over K must divide gh. Every root of f is a conjugate of b over K and by assumption, it is also a root of g. If it appears m times in f, then it also appears m times in g, because g and h have no roots in common. Hence, f divides g in $\tilde{K}[X]$. So we can write $gh = f\tilde{g}h$ with $\tilde{g} \in \tilde{K}[X]$ of degree $< n$. Since f and gh are elements of $K[X]$, we can apply the euclidean algorithm (or in other words, the polynomial division with remainder) to f and gh in $K[X]$. The result can not be different from the result in the ring $\tilde{K}[X]$, which shows that $\tilde{g}h \in K[X]$. Now we can apply the induction hypothesis to conclude that $\tilde{g}, h \in K[X]$ and thus also $g = f\tilde{g} \in K[X]$. □

In the following, let $L|K$ be an algebraic extension, not necessarily finite. It is called **normal** if every automorphism of $\tilde{K}|K$ sends L into itself. In particular, $\tilde{K}|K$ is normal. If $K'|K$ is a subextension of $L|K$ and $L|K$ is normal, then also $L|K'$ is normal since every automorphism of $\tilde{K}|K'$ is an automorphism of $\tilde{K}|K$. On every simple extension $K(b)$ of K, the restriction of a given automorphism σ of $\tilde{K}|K$ is uniquely determined by the root σb of the minimal polynomial of b over K. Consequently, $K(b)|K$ is normal if and only if all conjugates of b lie in $K(b)$. Moreover, it shows that the embeddings of $K(b)$ in $\tilde{K}$ over K correspond bijectively to the roots of the minimal polynomial f of b over K. Since $\deg f$ is equal to the degree $[K(b) : K]$ of the extension $K(b)|K$, and since f has at most $\deg f$ many distinct roots in $\tilde{K}$, it follows that there are at most $[K(b) : K]$ distinct embeddings of $K(b)$ in $\tilde{K}$ over K.

Every element $b \in L$ is already contained in the finite algebraic subextension $K(b)|K$. This shows that L is the union over all finite subextensions. So we can write $L = \bigcup_{i \in I} L_i$ where L_i runs through all finite extensions of K which are contained in L (the index set may be taken e.g. to be a subset of the cardinal $2^{|L|}$). But it also suffices to take the fields L_i to be simple extensions of K. On the other hand, every finite extension $L|K$ is a tower

of simple extensions. The degree of field extensions is multiplicative, that is, if $L|K'$ and $K'|K$ are finite extensions, then $[L : K] = [L : K'] \cdot [K' : K]$. Since also the number of embeddings is multiplicative in a similar way, one deduces by induction on the extensions in the tower that every finite extension $L|K$ admits at most $[L : K]$ distinct embeddings in $\tilde{K}$ over K. The number of distinct embeddings is called the **separable degree of** $L|K$, denoted by $[L : K]_{\text{sep}}$. If $[L : K] = [L : K]_{\text{sep}}$, then $L|K$ is called **separable**. Hence $K(b)|K$ is separable if and only if the minimal polynomial of b has no multiple roots; in this case, the element b and its minimal polynomial are called **separable over** K. At the other extreme, an arbitrary algebraic extension $L|K$ is called **purely inseparable** if it only admits one embedding in $\tilde{K}$ over K (which is the identity if we assume L to lie in $\tilde{K}$, as we will usually do). Hence $K(b)|K$ is purely inseparable if and only if its separable degree is 1, i.e. if the minimal polynomial of b admits only b as a root; in this case, the element b and its minimal polynomial are called **separable over** K. We note that

The extension $L|K$ is purely inseparable if and only if every $b \in L$ is purely inseparable over K.

Indeed, if $K(b)$ would admit two distinct embeddings in $\tilde{K}$ over K, then these could be extended to distinct embeddings of L in $\tilde{K}$ over K. If on the other hand $L|K$ admits two distinct embeddings, then there must be some $b \in L$ on which these act differently.

An element a is a multiple root of a polynomial f if and only if it is also a root of the derivative f' of f. If a is a simple root of some polynomial over K, then it is also a simple root of its minimal polynomial over K. Hence, an element a in an arbitrary extension of K is separable algebraic over K if (and only if) there is some $f \in K[X]$ such that $f(a) = 0$ and $f'(a) \neq 0$. See Lemma 9.16 for a "multidimensional" version of this criterion, working simultaneously for n elements by use of n polynomials.

If $L|K$ is finite and separable, then it is simple (cf. [LANG3], Chapter VII, §6, Theorem 14). An element $b \in L$ satisfying $L = K(b)$ is called a **primitive element**.

Like the degree $[L : K]$, also the separable degree $[L : K]_{\text{sep}}$ is multiplicative, that is, if $L|K'$ and $K'|K$ are finite extensions, then $[L : K]_{\text{sep}} = [L : K']_{\text{sep}} \cdot [K' : K]_{\text{sep}}$. From this, it follows that $L|K$ is separable if and only if $L|K'$ and $K'|K$ are separable. In particular, every subextension of a finite separable extension is separable. So we may generalize our definition as follows: An arbitrary algebraic extension $L|K$ is called **separable** if every finite subextension $L_i|K$ is separable. If $L|K$ is separable and normal, then we will call it a **Galois extension**. Since we are concerned with infinite Galois theory, we do not require the property "finite" for Galois extensions. If $L|K$ is normal, then we shall use the notation $\operatorname{Gal} L|K$ for the group of all automorphisms of $L|K$, and we call it the **Galois group of** $L|K$, even if $L|K$ is not separable. We use the abbreviation $\operatorname{Gal} K$ for $\operatorname{Gal} \tilde{K}|K$ and call it the **absolute Galois group of** K.

Later, we will define the notion "separable" also for arbitrary, not necessarily algebraic extensions. So we will speak henceforth of **separable algebraic** extensions if we want to restrict our arguments to algebraic extensions. In contrast to this, **purely inseparable** and **normal** will always mean that the extension is algebraic.

Let 1 denote the multiplicative unit in the field K and $n \cdot 1$ the result of adding it up n times. If there is a smallest number p such that $p \cdot 1 = 0$, then p is a prime number; it is called the **characteristic** of K. If there is no such p, then we say that the characteristic of K is 0. If the characteristic of K is $p > 0$, then the smallest field contained in K is $\mathbb{F}_p$, the field with p elements. If the characteristic of K is 0, then the smallest subfield of K

is $\mathbb{Q}$, the rationals. The smallest subfield of K is called the **prime field of** K. A field extension $L|K$ is called **finitely generated** if $L = K(x_1, \ldots, x_n)$ for suitable elements $x_1, \ldots, x_n \in L$. Every finitely generated algebraic extension is a finite extension. The field K is called a **finitely generated field** if it is a finitely generated field extension of its prime field.

Assume $L|K$ to be a Galois extension of degree p a prime which is not equal to the characteristic of K. If K contains a primitive p-th root of unity, then L contains an element b such that $c := b^p \in K$ and $L = K(b)$. The minimal polynomial of b over K is $X^p - c$. This follows from **Kummer Theory**; cf. [LANG3], Chapter VIII, §8. Now assume that $L|K$ is a Galois extension of degree $p = \operatorname{char} K$. Then L contains an element b such that $c := b^p - b \in K$ and $L = K(b)$. This assertion is a special case of a more general result that will be proved in Section 11.7; see Corollary 11.27. See also [LANG3], Chapter VIII, §8. The minimal polynomial of b over K is $X^p - X - c$. Such a polynomial is called an **Artin-Schreier polynomial**. The extension is said to be an **Artin-Schreier extension**, and b is an **Artin-Schreier root of** f and an **Artin-Schreier generator of** $L|K$. We will frequently use the abbreviation $\wp(X)$ for the polynomial $X^p - X$. This polynomial is **additive**, that is, $\wp(b + b') = \wp(b) + \wp(b')$ for all $b, b' \in L$. This is a consequence of the additivity of the **Frobenius endomorphism** $x \mapsto x^p$ of a field of characteristic p; see Section 6.7 below. Note in particular that if b is a root of $X^p - X - c$ and b' is a root of $X^p - X - c'$, then $b + b'$ is a root of $X^p - X - (c + c')$. From the special case $c' = 0$ we see that the roots of $X^p - X - c$ are of the form $b + b'$ with b' a root of $X^p - X$. We can observe at once that 0 and 1 are roots of $X^p - X$. But in view of the additivity, if 1 is a root, then also $n \cdot 1$ for every $n \in \mathbb{N}$. Since the characteristic of K was assumed to be p, the roots of $X^p - X$ are precisely $0, 1, \ldots, p - 1$. Consequently:

Lemma 6.2 *Let K be a field of characteristic $p > 0$ and $b \in \tilde{K}$ a root of the Artin-Schreier polynomial $X^p - X - c$ with $c \in K$. Then all roots of $X^p - X - a$ are given by $b, b + 1, \ldots, b + p - 1$. In particular, if $X^p - X - a$ has a root in K, then it splits over K into linear factors.*

We will also use the name "Artin-Schreier extension" for certain extensions of valued fields of characteristic 0; see Section 7.6 for the definition.

Let be given two subextensions $E|K$ and $F|K$ of an arbitrary field extension $\Omega|K$. The intersection $E \cap F$ is again a subfield of Ω containing K. The **field compositum of** E **and** F **(in** Ω**)**, denoted by $E.F$, is the smallest subfield of Ω which contains both E and F. If $E.F = \Omega$ and $E \cap F = K$, then E is called a **field complement for** F **in** Ω **over** K**.** Note that every two arbitrary field extensions of K can be embedded in a common extension field Ω (that is, the theory of fields has the amalgamation property, see Section 6.5). In this extension field, their field compositum can be defined as above. But it may depend on the embeddings (for instance, distinct embeddings of non-normal algebraic extensions may produce non-isomorphic composita). Every element in the field compositum can be written as $(x_1y_1 + \ldots + x_my_m)/(x'_1y'_1 + \ldots + x'_ny'_n)$ for suitable $m, n \in \mathbb{N}$ and $x_i, x'_i \in E$, $y_i, y'_i \in F$. If both extensions $E|K$ and $F|K$ are algebraic, then so is $E.F|K$. If they are normal in addition, then every automorphism of $\tilde{K}|K$ sends E and F and hence also $E.F$ into itself, showing that also $E.F|K$ is normal.

Let $L|K$ be an algebraic and $K'|K$ an arbitrary field extension. Then $\widetilde{K'}$ contains $\tilde{K} = \tilde{L}$ and thus, we can take the compositum $L.K'$ inside $\widetilde{K'}$ (again, it depends on the

embedding of L in $\tilde{K}$, and there are different composita if $L|K$ is not normal). Every basis of $L|K$ is also a system of generators of $L.K'|K'$, but it may not be a basis anymore. So if $L|K$ is finite, we have that

$$[L.K' : K'] \leq [L : K] .$$

In other words, the minimal polynomial of an element $b \in L$ may not be irreducible over K', but as we know already, it is divided by the minimal polynomial of b over K'. If $[L.K' : K'] = [L : K]$, then it follows that for every element $b \in L$ its minimal polynomial over K will be irreducible over K' and thus be the minimal polynomial of b over K'. Otherwise, we would have that $[K'(b) : K'] < [K(b) : K]$ which yields that $[L.K' : K'] = [L.K' : K'(b)] \cdot [K'(b) : K'] < [L.K' : K'(b)] \cdot [K(b) : K] \leq [L : K(b)] \cdot [K(b) : K] = [L : K]$ since $K'(b) = K(b).K'$. It follows that if $[L.K' : K'] = [L : K]$ and $L.K'|K'$ is separable resp. purely inseparable, then so is $L|K$.

Even in the case of an infinite algebraic extension $L|K$, every finite subextension $L'|K'$ of $L.K'|K'$ is contained in $L_i.K'|K'$ for some finite subextension $L_i|K$ of $L|K$. Indeed, L' is generated over K' by finitely many elements $b_1, \ldots, b_n \in L.K'$. To write down the elements $b_1, \ldots, b_n$, only finitely many elements $c_1, \ldots, c_k \in L$ (and finitely many elements in K') are needed. Hence, $L_i := L(c_1, \ldots, c_k)$ is a finite extension of L, and $L' \subset L_i.K'$. Now assume in addition that $L|K$ is separable. Then also $L_i|K$ is separable, and it is thus simple; let b be a primitive element. Then the minimal polynomial f of b over K has no multiple root. The same is true for the minimal polynomial of b over K' since it is a divisor of f. This shows that $K'(b)|K'$ and thus also its subextension $L'|K'$ is separable. In this way, it is shown that every finite subextension of $L.K'|K'$ is separable algebraic. We have thereby proved:

If $L|K$ is separable algebraic and $K'|K$ is an arbitrary field extension, then $L.K'|K'$ is separable algebraic.

It follows by induction that the compositum of finitely many separable algebraic extensions is again separable. In particular, if every extension $K(b)|K$ for $b \in L$ is separable, then so is every finite subextension $L|K$ since it is the compositum of finitely many simple subextensions. Conversely, if $L|K$ is separable, then so is the finite subextension $K(b)|K$. We have proved:

The extension $L|K$ is separable if and only if every $b \in L$ is separable over K.

We shall now prove the **transitivity of separable extensions**:

Lemma 6.3 *Let $L|K$ be an algebraic extension with subextension $K'|K$. Then $L|K$ is separable if and only if $L|K'$ and $K'|K$ are separable.*

The proof of this lemma uses a typical argument which connects finite with infinite extensions. For later use, we formulate this argument in more generality than we need it here.

Lemma 6.4 *Let $K'|K$ be an arbitrary and $L'|K'$ a finite extension. Then there exists a finitely generated subextension $K_0|K$ of $K'|K$ and a finite subextension $L_0|K_0$ of $L'|K_0$ such that $L' = L_0.K'$ and $[L' : K'] = [L_0 : K_0]$. It follows that if $L'|K'$ is separable resp. purely inseparable, then so is $L_0|K_0$.*

If $S_{K'}$ and $S_{L'}$ are any finite sets of elements in K' and L' respectively, then K_0 and L_0 can be chosen such that $S_{K'} \subset K_0$ and $S_{L'} \subset L_0$. If $K'|K$ is algebraic, then $K_0|K$ and $L_0|K$ are finite.

Proof: Without loss of generality, we can enlarge the set $S_{L'}$ by finitely many elements such that it generates the finite extension $L'|K'$. Write $S_{L'} = \{b_1, \ldots, b_n\}$. Now we form a finite set S_1 of elements in K' by adjoining to $S_{K'}$ the coefficients of the minimal polynomial of b_1 over K'. Every element of $K'(b_1, \ldots, b_k)$ is a rational function in $b_1, \ldots, b_k$ which involves finitely many coefficients from K'. So for $1 \leq k < n$, we form S_{k+1} by adjoining to S_k the finitely many elements of K' that appear as coefficients in the rational functions which are the coefficients of the minimal polynomial of b_{k+1} over $K'(b_1, \ldots, b_k)$. We obtain a finite set S_n of elements in K', and we set $K_0 := K(S_n)$. We obtain that $[K'(b_1) : K'] = [K_0(b_1) : K_0]$ and $[K'(b_1, \ldots, b_{k+1}) : K'(b_1, \ldots, b_k)] = [K_0(b_1, \ldots, b_{k+1}) : K_0(b_1, \ldots, b_k)]$ for $1 \leq k < n$. It follows that $L_0 := K_0(b_1, \ldots, b_n) = K_0(S_{L'})$ satisfies $L' = L_0.K'$ and $[L' : K'] = [L_0 : K_0]$. By construction, $S_{K'} \subset K_0$ and $S_{L'} \subset L_0$. Further, $K_0|K$ is a finitely generated subextension of $K'|K$. If the latter is algebraic, then also $K_0|K$ is algebraic and thus finite.

The assertion about separability resp. inseparability follows directly from the equality $[L' : K'] = [L_0 : K_0]$, as we have shown earlier. □

Now we are able to prove Lemma 6.3. Let $L|K$ be algebraic with subextension $K'|K$. Assume first that $L|K$ is separable. Then it follows directly from the definition that also $K'|K$ is separable. Let $L'|K'$ be a finite subextension of $L|K'$, and choose K_0 and L_0 as in the above lemma. Then $L_0|K$ is a finite extension, so it is separable since $L|K$ is supposed to be separable. By what we have shown earlier, $L' = L_0.K'$ is separable over K'. This proves that $L|K'$ is separable.

For the converse, assume that $L|K'$ and $K'|K$ are separable. Let $E|K$ be a finite subextension of $L|K$ and $L' = E.K'$. As a finite subextension of $L|K'$, the extension $L'|K'$ is separable and thus simple. Let b be a primitive element of $L'|K'$. Then its minimal polynomial over K' has no multiple roots. Choose K_0 and L_0 as in the above lemma such that L_0 contains b and some set of generators of the finite extension $E|K$. Then $E \subset L_0$. Since $K'|K$ is separable, so is its finite subextension $K_0|K$. Since $[L_0 : K_0] = [L' : K'] = [K'(b) : K']$, we see that b also generates L_0 over K_0 and that its minimal polynomial over K_0 is the same as over K'. Since this minimal polynomial has no multiple roots, it follows that $L_0|K_0$ is separable. By what we have proved already about finite extensions, $L_0|K$ and thus also $E|K$ is separable. This completes our proof.

If $E|K$ is an arbitrary subextension of a Galois extension $L|K$, then $L|E$ is separable and normal and thus again a Galois extension (since $L.E = L$). In this case, $\operatorname{Gal} L|E$ may be identified in a natural way with the subgroup $\{\sigma \in \operatorname{Gal} L|K \mid \forall x \in E : \sigma x = x\}$ of $\operatorname{Gal} L|K$. For $\sigma \in \operatorname{Gal} L|K$, the **restriction** of σ to E will be denoted by $res_E(\sigma)$. If $E|K$ is normal, then the restriction map

$$\operatorname{res}_E : \operatorname{Gal} L|K \longrightarrow \operatorname{Gal} E|K$$

is a group epimorphism since every automorphism of $E|K$ can be extended to an automorphism of $L|K$. Note that $\operatorname{Gal} L|E = \operatorname{res}_E^{-1}(1)$.

If $E|K$ and $F|K$ are separable algebraic extensions, then by what we have shown, $E.F|F$ is separable and thus, $E.F|K$ is separable. That is, *the compositum of two separable algebraic extensions is again separable algebraic.* Since we have shown the analogue also for normal extensions, we conclude that *the compositum of two Galois extensions is again a Galois extension.* It also follows that an algebraic extension $L|K$ generated by the elements

b_i , $i \in I$, is normal (resp. separable) if and only if every extension $K(b_i)|K$ is. Further, we see that the composition of all separable algebraic subextensions of an arbitrary extension $L|K$ is itself a separable algebraic subextension of $L|K$, and it is thus the maximal one. We will call it the **relative separable-algebraic closure of K in L** and denote it by $(L|K)^{\rm sep}$. In fact, $(L|K)^{\rm sep}$ consists of all elements $b \in L$ such that $K(b)|K$ is separable algebraic. The maximal separable algebraic extension of K is $(\tilde{K}|K)^{\rm sep}$; it will be called the **separable-algebraic closure of** K and denoted by $K^{\rm sep}$. It consists of all elements (in $\tilde{K}$) which are separable algebraic over K.

To every algebraic extension $L|K$ we can find a normal extension $L'|K$ containing L. The minimal one is found by adjoining, for every $b \in L$, all roots of the minimal polynomial of b over K. This field L' is called the **normal hull** of L over K. If $L|K$ is finite, then so is $L'|K$. An arbitrary algebraic extension $L|K$ is normal if and only if it contains the normal hull of every finite subextension $L_i|K$. If $L|K$ is normal, then L is the union over all finite normal subextensions; in Section 6.4, we shall describe the connection between their Galois groups and the Galois group of $L|K$.

For every algebraic extension $L|K$, every homomorphism σ of L into L over K is an automorphism of $L|K$. Indeed, it is injective since it is a nontrivial field homomorphism. If $b \in L$ then $\sigma^n b \in L$ for every $n \in \mathbb{N}$. Since b has only finitely many conjugates (and σ is invertible), there is some $n > 0$ such that $\sigma^n b = b$. Hence, $b = \sigma(\sigma^{n-1}b) \in \sigma L$, which proves that σ is surjective.

The following is a criterion for the existence of embeddings of infinite algebraic extensions:

Theorem 6.5 (Compactness Principle for Algebraic Extensions)
Let $L|K$ be an algebraic field extension and $L = \bigcup_{i\in I} L_i$ where the L_i run through all finite subextensions of $L|K$. Let $F|K$ be an arbitrary field extension. If for every $i \in I$, the field L_i admits an embedding ι_i in F over K, then there exists an embedding ι of L in F over K such that

$$\forall i \in I: \ \mathrm{res}_{L_i}(\iota) = \mathrm{res}_{L_i}(\iota_j) \text{ for some } j \in I \text{ such that } L_j \supset L_i \ . \tag{6.1}$$

If $L|K$ is normal then the assertion remains true if we let L_i run through all finite normal subextensions of $L|K$.

We will prove the theorem in the next section using a compactness principle for inverse limits. The following corollary illustrates the use of assertion (6.1). In fact, this assertion implies that the embedding ι inherits the universal properties (cf. Section 13.1) that all ι_i have in common.

Corollary 6.6 *Let $(L, v)|(K, v)$ be an algebraic and $(F, w)|(K, v)$ an arbitrary extension of valued fields. If every finite subextension of $L|K$ admits a valuation preserving embedding ι in (F, w) over (K, v), then so does (L, v). If in addition, vL admits an embedding ρ in wF over vK and $\overline{L}$ admits an embedding σ in $\overline{F}$ over $\overline{K}$ and if the given embeddings of the finite subextensions respect the corresponding restrictions of ρ and σ, then ι may be chosen as to respect ρ and σ.*

Proof: Since every $b \in L$ is contained in some L_i , the embbeding ι chosen according to the foregoing theorem will satisfy $w\iota b \geq 0 \Leftrightarrow w\iota_j b \geq 0 \Leftrightarrow vb \geq 0$, resp. $(\iota b)w = (\iota_j b)w = \sigma bv$ if $vb = 0$, resp. $w\iota b = w\iota_j b = \rho vb$. □

Exercise 6.1 *Give an example where $L|K'$ and $K'|K$ are finite normal extensions, but $L|K$ is not normal.*

6.2 Inverse limits

Suppose that $(I, \leq)$ is a partially ordered set. A family $\{\mathcal{X}_i, \pi_{ji} \mid i, j \in I\,,\ i \leq j\}$ consisting of topological spaces $\mathcal{X}_i$ and continuous maps $\pi_{ji} : \mathcal{X}_j \to \mathcal{X}_i$ is called an **inverse system** (or **projective system**) if it satisfies

(INV0) for all $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$,

(INV1) π_{ii} is the identity map on $\mathcal{X}_i$ for every $i \in I$,

(INV2) $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$ for all $i, j, k \in I$ such that $i \leq j \leq k$.

The **inverse limit** (or **projective limit**) of an inverse system $\{\mathcal{X}_i, \pi_{ji}\}$ is defined to be the subspace

$$\varprojlim \mathcal{X}_i := \{(x_i)_{i\in I} \in \prod_{i\in I} \mathcal{X}_i \mid \pi_{ji}(x_j) = x_i \text{ whenever } i \leq j\}$$

of the cartesian product $\prod_{i\in I} \mathcal{X}_i$ which is endowed with the product topology. Recall that the **product topology** is the coarsest topology such that for all $i \in I$, the projection pr_i onto the i-th component $\mathcal{X}_i$ is continuous. It has a basis consisting of all sets of the form $\prod_{i\in I} \mathcal{U}_i$ where $\mathcal{U}_i$ is an open subset of $\mathcal{X}_i$ for all $i \in I$ and $\mathcal{U}_i = \mathcal{X}_i$ for almost all i. If all $\mathcal{X}_i$ are hausdorff, then also the product topology is hausdorff, and $\mathcal{X} := \varprojlim \mathcal{X}_i$ is a closed subspace of the product. The latter is seen as follows. Let $(x_i)_{i\in I} \notin \mathcal{X}$. Then there are $i, j \in I$, $i \leq j$, such that $\pi_{ji}x_j \neq x_i$. If $\mathcal{X}_i$ is hausdorff, then we may choose disjoint open neighborhoods $\mathcal{U}_i$ and $\mathcal{U}_i'$ of x_i and $\pi_{ji}(x_j)$ respectively. Now $\mathcal{U}_i \times \pi_{ji}^{-1}(\mathcal{U}_i') \times \prod_{k\neq i,j} \mathcal{X}_k$ is an open neighborhood of $(x_i)_{i\in I}$ whose intersection with $\mathcal{X}$ is empty. This proves that the complement of $\mathcal{X}$ in the product is open. The argument also shows that for $i \leq j$ and $\mathcal{X}_i$ hausdorff, the set $\mathcal{X}_{ji} := \{(x_\ell)_{\ell\in I} \mid \pi_{ji}(x_j) = x_i\}$ is closed in $\prod \mathcal{X}_i$. Observe that

$$\mathcal{X} = \bigcap_{i\leq j} \mathcal{X}_{ji}\ .$$

Lemma 6.7 *The inverse limit $\mathcal{X}$ of an inverse system of nonempty compact hausdorff spaces $\mathcal{X}_i$ is a nonempty compact hausdorff space.*

Proof: By Tychonoff's Theorem, the product of the compact spaces $\mathcal{X}_i$ is compact. Since all $\mathcal{X}_i$ are assumed to be hausdorff, their product is also hausdorff, and $\mathcal{X}$ is a closed subspace of the product, as we have seen above. Hence, $\mathcal{X}$ is hausdorff and compact. It remains to show that $\mathcal{X}$ is nonempty. By the compactness of the product space, this will follow if we are able to show that the intersection of finitely many of the closed sets $\mathcal{X}_{ji}$ is nonempty. In such a finite intersection, only finitely many indeces $i, j \in I$ are involved, and by (INV0) we may choose $k \in I$ which is bigger than all of them. Since $\mathcal{X}_k$ is nonempty by assumption, the set $\{(x_i)_{i\in I} \mid x_k \in \mathcal{X}_k \text{ and } \pi_{ki}(x_k) = x_i \text{ whenever } i \leq k\}$ is a nonempty subset of the intersection. □

The restriction of the projection pr_i to $\mathcal{X}$ is a map from $\mathcal{X}$ into $\mathcal{X}_i$ and will be denoted by π_i. These maps satisfy $\pi_i = \pi_{ji} \circ \pi_j$ whenever $i \leq j$. Observe that axiom (INV0) plays a crucial role in the above proof. Using again this axiom, the reader may show that a basis for the topology of $\mathcal{X}$ is given by the sets $\pi_i^{-1}(\mathcal{U}_i)$ where $i \in I$ and $\mathcal{U}_i$ is an open subset of $\mathcal{X}_i$. (For the product space, the sets $\mathrm{pr}_i^{-1}(\mathcal{U}_i)$ will in general only form a subbasis.)

Note that the foregoing lemma immediately applies to the case of finite sets $\mathcal{X}_i$ because they are compact under the discrete topology. This is the case of most of our applications. In particular, we are now able to prove the Compactness Principle for Algebraic Extensions.

Proof of Theorem 6.5: Let the notations be as in that theorem. We order I by defining $i \leq j :\Leftrightarrow L_i \subset L_j$. Then (INV0) is satisfied since every two finite extensions L_i, L_j are contained in the finite extension $L_i . L_j$ which is equal to L_k for some $k \in I$. (The same argument works for the finite normal extensions.) For $i \in I$, we take $\mathcal{X}_i$ to consist of all embeddings of L_i in F over K which are the restriction of ι_j to L_i for some $j \geq i$. The map π_{ji} is just given by restricting an embedding of L_j to the subfield L_i. Then axioms (INV1) and (INV2) are satisfied. Since every $L_i|K$ is a finite extension, every $\mathcal{X}_i$ is finite; it is nonempty since it contains ι_i. An application of Lemma 6.7 now shows that the inverse limit of $\{\mathcal{X}_i , \pi_{ji}\}$ is nonempty. Let $(\tau_i)_{i\in I}$ be an element of it. We define an embedding of L in F over K as follows. For every $a \in L$ we find some $i \in I$ such that $a \in L_i$, and we set $\iota a = \tau_i a$. The fact that $(\tau_i)_{i\in I}$ is an element of the inverse limit, guarantees that ι is welldefined. We have $\mathrm{res}_{L_i}(\iota) = \tau_i$ which by definition of $\mathcal{X}_i$ is the restriction of ι_j for some $j \geq i$. □

Note that the proof also indicates the existence of a bijection between the set of all embeddings of L in F over K and the inverse limit of $\{\mathcal{X}_i , \pi_{ji}\}$ where $\mathcal{X}_i$ is the set of all embeddings of L_i in F over K. The above proof shows how to associate an embedding to every element of the inverse limit. Surjectivity follows from the fact that for ι an embedding of L, the restrictions $\pi_i\iota$ of ι to L_i are compatible with the restrictions π_{ji}, that is, $\pi_{ji}\pi_j\iota = \pi_i\iota$. Injectivity follows from the fact that every element of L is contained in some L_i.

Setting $F = L$ and assuming that $L|K$ is normal, we obtain a bijection between $\mathrm{Gal}\, L|K$ and the inverse limit of $\{\mathrm{Gal}\, L_i|K , \pi_{ji}\}$ where the L_i run through all finite normal subextensions of $L|K$ and the maps $\pi_{ji} : \mathrm{Gal}\, L_j|K \to \mathrm{Gal}\, L_i|K$ are again given by restriction. Note that in this case they are surjective since every automorphism of $L_i|K$ may be extended to an automorphism of $L_j|K$ if $L_i \subset L_j$. The inverse limit inherits the group structure of the groups $\mathrm{Gal}\, L_i|K$ by componentwise composition (since composition commmutes with restriction). With this group structure, the above bijection turns into a group isomorphism

$$\mathrm{Gal}\, L|K \cong \varprojlim \mathrm{Gal}\, L_i|K .$$

For every i, the map π_i is the restriction $\mathrm{res}_{L_i} : \mathrm{Gal}\, L|K \to \mathrm{Gal}\, L_i|K$.

6.3 Topological and profinite groups

A **topological group** is a group G endowed with a topology such that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. It follows that for every $g \in G$, the maps $x \mapsto gx$, $x \mapsto xg$ and $x \mapsto x^{-1}$ are homeomorphisms. Hence, a subgroup H of G is open if and only if the coset gH is open, and the same holds for "closed" in the place of "open". A closed subgroup H of finite index in G is open since its complement is the union of finitely many cosets $gH \neq H$ and every coset gH is a closed set. Conversely, every open subgroup H is closed since its complement is the union of the open cosets $gH \neq H$. In every compact topological group an open subgroup H must be of finite index. Indeed, the cosets gH form an open covering of G, and since it does not admit a proper subcovering, it must itself be

finite, that is, there are only finitely many cosets. Hence for compact topological groups, the open subgroups are precisely the closed subgroups of finite index.

By a **topological isomorphism** of topological groups we will mean a **continuous** group isomorphism with continuous inverse. In particular, a topological isomorphism is a homeomorphism. Note that a group homomorphism $f : G \to G_1$ of topological groups is continuous if and only if for every (open) neighborhood U_1 of $1 \in G_1$ there exists a neighborhood U of $1 \in G$ such that $f(U) \subset U_1$. Similarly, f is **open** if and only if for every (open) neighborhood U of $1 \in G$ there exists a neighborhood U_1 of $1 \in G_1$ such that $U_1 \subset f(U)$. This follows from the fact that for every element $g \in G$, the neighborhood filter of 1 is sent onto the neighborhood filter of g by the homeomorphism $x \mapsto gx$.

Let us require in addition for a topological group G that the subset $\{1\}$ be closed. Then for every $g \in G$, the subset $\{g\}$ is closed. With this proviso, topological groups have the following properties.

Every topological group is hausdorff. Indeed, let $a, b \in G$ with $a \neq b$. Then $G \setminus \{b\}$ is an open neighborhood of a. By the continuity of $(x, y) \mapsto xy$ there exist open neighborhoods U_a of a and U_1 of 1 such that $U_a \cdot U_1 \subset G \setminus \{b\}$. It follows that $U_a \cap bU_1^{-1} = \emptyset$. Since $x \mapsto x^{-1}$ is a homeomorphism, bU_1^{-1} is an open neighborhood of b.

The kernel of a continuous homomorphism of topological groups is a closed normal subgroup since it is the preimage of the closed set $\{1\}$. For the facts stated in the following, cf. [PON], Chapter III, §20. If H is a closed normal subgroup of G, then the quotient topology on G/H turns G/H into a topological group and the canonical epimorphism $G \to G/H$ will be continuous and open. Recall that the quotient topology is the finest topology on G/H such that $G \to G/H$ is continuous. If G admits a continuous open epimorphism f onto the topological group G_1, then its kernel H is a closed normal subgroup of G, and f induces a topological isomorphism between G/H and G_1. Note that if G is compact, then the epimorphism $f : G \to G_1$ is open already if it is continuous (cf. [PON]). Here, we shall prove this assertion under the assumption that in both groups G and G_1, the open normal subgroups of G form a basis for the neighborhood filter of 1 (which is always the case for profinite groups, as we will see later). It suffices to show that for every open normal subgroup H of G, its image $f(H)$ is an open subgroup of G_1. We know already that in compact topological groups, the open subgroups are precisely the closed subgroups of finite index. Hence, H is closed and thus compact, and since the continuous homomorphism f sends compact sets onto compact sets, also $f(H)$ is closed. On the other hand, $(G : H)$ is finite, and since f is an epimorphism, also $(G_1 : f(H))$ is finite. Therefore, $f(H)$ is an open subgroup of G_1 (in fact, it is also normal since f is an epimorphism). Note that under the same condition, a group homomorphism $f : G \to G_1$ is continuous already if for every open normal subgroup H_1 of G_1 there is an open normal subgroup H of G with $f(H) \subset H_1$.

Let H and N be closed subgroups of G. Then their intersection $H \cap N$ is again a closed subgroup of G. Their **group compositum** $H.N$ is defined to be the closure of the subgroup generated by H and N, i.e. the intersection of all closed subgroups containing H and N. If the topology on G is discrete, (which we will always assume if G is finite or no topology is specified), then this compositum coincides with the subgroup generated by H and N. In contrast to "$H.N$", the notation HN indicates the set of products $\{h \cdot n \mid h \in H\,,\, n \in N\}$. If N is a normal subgroup of G, then $HN = NH$ and the group compositum $H.N$ is equal to HN. Further, if $H.N = G$ and $H \cap N = \{1\}$, then H is called a **group complement**

of N in G.

Let H and N be closed subgroups of the compact topological group G. If N is normal in G, then there is a topological isomorphism between $H/(H \cap N)$ and HN/N. This is seen as follows. Let f be the restriction of the canonical epimorphism $HN \to HN/N$ to the subgroup H of HN. Then f is still continuous and surjective. Since H is compact, f is also open. Hence, it induces a topological isomorphism $H/\ker f \cong HN/N$. Since $\ker f = H \cap N$, this proves our assertion. Note that it is in general not true without the assertion that G be compact; for an example, see [BOU2], Chapter III, §2.7 . In contrast to this, if H is also a normal subgroup and contains N, then G/H and $(G/N)/(H/N)$ are topologically isomorphic, even if G is not compact (cf. [BOU2], Chapter III, §2.7, Corollary to Proposition 22).

Let G be the inverse limit of an inverse system $\{G_i, \pi_{ji}\}$ of topological groups G_i . Since the topology on G is the coarsest such that all projections $\pi_i : G \to G_i$ are continuous, the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ will be continuous on G if multiplication is defined componentwise. Hence, the inverse limit of an inverse system $\{G_i, \pi_{ji}\}$ of topological groups is again a topological group. Further, $\{1\} = \bigcap_i \pi_i^{-1}(1)$ is closed (since we have assumed that $\{1\}$ is closed in every G_i).

The inverse limit G of an inverse system of finite groups G_i (which are endowed with the discrete topology) is called a **profinite group**. By Lemma 6.7, G is hausdorff and compact. Therefore, for profinite groups all results hold that we stated for compact topological groups (with $\{1\}$ a closed subgroup). As we have noted in the last section, a basis for the topology of G is given by the sets $\pi_i^{-1}(\mathcal{U}_i)$ where $i \in I$ and $\mathcal{U}_i$ is an open subset of G_i . Since $\{g_i\}$ is open in G_i for every $g_i \in G_i$ and $\pi_i^{-1}(\mathcal{U}_i)$ is the union over all $\pi_i^{-1}(g_i)$ with $g_i \in \mathcal{U}_i$, we find that the sets $\pi_i^{-1}(g_i)$, $g_i \in G_i$, $i \in I$, form a basis of the topology on G. Since $\{g_i\}$ is also closed in G_i , the set $\pi_i^{-1}(g_i)$ is at the same time open and closed. If $g, h \in G$ and $g \neq h$, then there is $i \in I$ such that $\pi_i g \neq \pi_i h$ and consequently, g and h lie in the disjoint open and closed sets $\pi_i^{-1}(\pi_i g)$ and $\pi_i^{-1}(\pi_i h)$ respectively. This proves that the connected component of every $g \in G$ is just $\{g\}$, that is, G is **totally disconnected**.

The set $\pi_i^{-1}(1)$ is the kernel of the projection π_i , and thus it is an open normal subgroup of G. Since the sets $\pi_i^{-1}(g_i)$ form a basis of the topology, an open set containing 1 must contain at least one set of the form $\pi_i^{-1}(1)$. Consequently, the open normal subgroups of G form a basis for the neighborhood filter of $1 \in G$. Their intersection is $\{1\}$; indeed, for every $g \in G \setminus \{1\}$ there is some $i \in I$ such that $\pi_i g \neq 1 \in G_i$ and consequently, $g \notin \pi_i^{-1}(1)$.

Every closed subgroup H of G is equal to the intersection of all open subgroups containing H. This is seen as follows. Let $g \in G$ be an element which belongs to every open subgroup of G that contains H. For every open normal subgroup N of G, the group HN is closed, and since it contains N, it is also of finite index in G. Thus, HN is open. By assumption, $g \in HN$ and thus, $gN \cap H \neq \emptyset$. Since N and H are closed, so is $gN \cap H$. Since any finite intersection of open normal subgroups N_i is again an open normal subgroup N, any finite intersection $\bigcap_i (gN_i \cap H) = g \bigcap_i N_i \cap H = gN \cap H$ is nonempty. By compactness, the intersection of all of these sets, N running through all open normal subgroups of G, is nonempty. That is, there is some $h \in H$ such that $h \in gN$ for every such N. But the intersection of all open normal subgroups N is $\{1\}$, showing that $g = h \in H$. This completes the proof. If in addition also H is normal, then every HN is normal and our argument shows that H is equal to the intersection of all open normal subgroups containing H.

Let N_i , $i \in I$, be a family of open normal subgroups of the profinite group G. The

index set is partially ordered through $i \leq j \Leftrightarrow N_j \subset N_i$. If $i \leq j$, then we have the canonical epimorphism $\eta_{ji} : G/N_j \to G/N_i$. These projections satisfy properties (INV1) and (INV2). If the family contains *all* open normal subgroups of G, then also (INV0) holds, because then the family is a basis for the neighborhood filter of $1 \in G$. We ask whether we may identify G with $\varprojlim G/N_i$.

Lemma 6.8 *Let G be a compact topological group and let N_i, $i \in I$, be a family of open normal subgroups in G with the following properties:*

(1) the subgroups N_i form a basis for the neighborhood filter of 1,
(2) $\bigcap_{i\in I} N_i = \{1\}$.
Then the map $g \mapsto (gN_i)_{i\in I}$ is a topological isomorphism $H \cong \varprojlim G/N_i$, showing that H is a profinite group.

Proof: The canonical epimorphisms $G \to G/N_i$ are continuous. By the universal property of the product, the induced map $\iota : G \to \prod_{i\in I} G/N_i$ is continuous. The image of G lies in the subgroup $\varprojlim G/N_i$. If $g \neq 1$ then by condition (2) there is some N_i which does not contain g. Thus, $\iota g \neq 1$, showing that ι is injective. Let $(g_iN_i)_{i\in I}$ be an element in $\varprojlim G/N_i$. From condition (1) it follows that for every finite $J \subset I$ there is some $i \in I$ such that $N_i \subset \bigcap_{i\in J} N_j$. Then $g_iN_i \subset \bigcap_{j\in J} g_jN_j$, showing that the intersection is not empty. Since the sets g_iN_i are closed and G is compact, it follows that there exists $g \in \bigcap_{i\in I} g_iN_i$. This element satisfies $\iota g = (g_iN_i)_{i\in I}$. We have proved that ι is surjective. Condition (1) implies that the open normal subgroups of G form a basis for the neighborhood filter of 1. For groups with this property, we have shown that continuous epimorphisms are open. Hence ι is open and ι^{-1} is continuous. □

Corollary 6.9 *a) Every profinite group G is topologically isomorphic to $\varprojlim G/N_i$ where N_i runs through all open normal subgroups of G. That is, every profinite group is (up to topological isomorphism) the inverse limit of its finite quotients.*

b) Every closed subgroup of a profinite group G is again a profinite group.

c) If N is a closed normal subgroup of G, then G/N is a profinite group.

Proof: a): This follows directly from the foregoing lemma since we have shown earlier that the intersection of all open normal subgroups of a profinite group G is $\{1\}$.

b): If H is a closed subgroup of a profinite group G, then H is compact. We take N_i, $i \in I$, to be all intersections of H with open normal subgroups of G. Since the open normal subgroups of G form a basis of the neighborhood filter of 1 and their intersection is $\{1\}$, the same is true for the open normal subgroups N_i in H. Now our assertion follows from the foregoing lemma, applied to H in the place of G.

c): N is the intersection of open normal subgroups N_i of G. The family N_i/N of open normal subgroups of G/N satisfies the conditions of the foregoing lemma and again, the assertion follows from that lemma, applied to G/N in the place of G. □

Every compact, hausdorff, totally disconnected topological group is a profinite group (and vice versa, as we have already shown). This is proved as follows. One shows that in a compact hausdorff group G, the connected component of $g \in G$ is equal to the intersection of all closed open neighborhoods of g (cf. [RIBES], Lemma 2.3). If G is also totally

disconnected, then this intersection is $\{g\}$. Further, one shows that every closed open neighborhood of 1 contains an open normal subgroup (cf. [RIBES], Lemma 2.4). Hence, the open normal subgroups of G satisfy the conditions 1) and 2) of Lemma 6.8 and we obtain that G is indeed a profinite group.

For a collection of exercises on topological and profinite groups, see [FR–JA], at the end of Chapter 1.

6.4 Infinite Galois theory

We have already seen at the end of Section 6.2 that every Galois group $\operatorname{Gal} L|K$ is isomorphic to the inverse limit of the finite Galois groups $\operatorname{Gal} L_i|K$ where $L_i|K$ runs through all finite normal subextensions of $L|K$. (In fact, Lemma 6.8 shows that it is not necessary to take *all* of them; it suffices to take a family of subextensions of $L|K$ such that every finite normal subextension of $L|K$ is contained in some member of that family.) Taking all finite Galois groups to be endowed with the discrete topology, this representation of $\operatorname{Gal} L|K$ as an inverse limit of finite Galois groups turns $\operatorname{Gal} L|K$ into a profinite group, with projections $\pi_i = \operatorname{res}_{L_i}$. The induced topology on $\operatorname{Gal} L|K$ is called the **Krull topology**. Hence, Galois groups with the Krull topology have all properties that we have already stated for profinite groups and for compact topological groups (with $\{1\}$ a closed subgroup). Conversely, every profinite group is the Galois group of some Galois extension (cf. [FR–JA], Corollary 1.11).

A basis of the neighborhood filter of 1 is given by the subgroups $\operatorname{Gal} L|L_i = \operatorname{res}_{L_i}^{-1}(1)$ of $\operatorname{Gal} L|K$, where $L_i|K$ runs through all finite normal subextensions of $L|K$. (It follows from Theorem 6.10 below that these subgroups are precisely all open normal subgroups of $\operatorname{Gal} L|K$.) If $L|K$ is finite, then $\operatorname{Gal} L|K$ is a finite group and the Krull topology is discrete.

If $F|K$ is a finite subextension of $L|K$, then there exists $i \in I$ such that $F|K$ is a subextension of the finite normal subextension $L_i|K$. In this case, $\operatorname{Gal} L|F$ is an open and closed subgroup of $\operatorname{Gal} L|K$ since it coincides with the finite intersection of the open and closed sets $\operatorname{res}_{L_i}^{-1}(g_i)$ where the $g_i \in \operatorname{Gal} L_i|K$ run through all automorphisms of $L_i|K$ which fix every element of F. If $F|K$ is an arbitrary subextension of $L|K$, then it is the union of finite subextensions $F_j|K$ of $L|K$. Consequently, $\operatorname{Gal} L|F = \bigcap_j \operatorname{Gal} L|F_j$ is a closed subgroup of $\operatorname{Gal} L|K$.

Recall that the **closure** of a set S is the intersection of all closed sets containing S. Consequently, an element σ belongs to the closure of S if and only if for every open set U containing σ, the intersection $S \cap U$ is nonempty. For a profinite group G and $S \subset G$, this is the case if and only if for every open normal subgroup H of G, the intersection $S \cap \sigma H$ is nonempty (indeed, since the open subgroups H of G form a basis of the neighborhood filter of 1, the sets σH form a basis of the neighborhood filter of σ). If $G = \operatorname{Gal} L|K$, then σ lies in the closure of S if and only if $S \cap \sigma \operatorname{Gal} L|L_i \neq \emptyset$ for every finite normal subextension $L_i|K$ of $L|K$. But $S \cap \sigma \operatorname{Gal} L|L_i \neq \emptyset$ just means that $\operatorname{res}_{L_i}(\sigma) \in \operatorname{res}_{L_i}(S)$. We have thus proved: *The closure of a set $S \subset \operatorname{Gal} L|K$ consists of all $\sigma \in \operatorname{Gal} L|K$ such that $\operatorname{res}_{L_i}(\sigma) \in \operatorname{res}_{L_i}(S)$ for every finite normal subextension $L_i|K$ of $L|K$.*

For every subgroup H of $\operatorname{Gal} L|K$, we let $\operatorname{Fix}(L, H)$ denote the **fixed field of H in L**, consisting of all elements of L which are fixed by all automorphisms in H. This fixed field is a subfield of L containing K.

Now we are ready to state the **Galois correspondence** for (not necessarily finite) Galois extensions. Note that the Krull topology on a finite profinite group is discrete, which yields that every subgroup is closed. So for finite Galois extensions, the following indeed gives the usual Galois correspondence.

Theorem 6.10 *Let $L|K$ be a Galois extension. The map $F \mapsto \operatorname{Gal} L|F$ is a bijection from the set of all subextensions $F|K$ of $L|K$ onto the set of all closed subgroups of $\operatorname{Gal} L|K$. Its inverse is the map $H \mapsto \operatorname{Fix}(L,H)$. For this correspondence, the following rules hold (where E, F are subfields of L containing K, the groups G, H are closed subgroups of $\operatorname{Gal} L|K$, and $\sigma \in \operatorname{Gal} L|K$):*

(Gal1) $E \subset F \;\Leftrightarrow\; \operatorname{Gal} L|E \supset \operatorname{Gal} L|F$

(Gal1′) $G \subset H \;\Leftrightarrow\; \operatorname{Fix}(L,G) \supset \operatorname{Fix}(L,H)$

(Gal2) $\operatorname{Gal} L|(E\,.\,F) = \operatorname{Gal} L|E \cap \operatorname{Gal} L|F$

(Gal3) $\operatorname{Gal} L|(E \cap F) = \operatorname{Gal} L|E\,.\,\operatorname{Gal} L|F$

(Gal2′) $\operatorname{Fix}(L, G \cap H) = \operatorname{Fix}(L,G)\,.\,\operatorname{Fix}(L,H)$

(Gal3′) $\operatorname{Fix}(L, G\,.\,H) = \operatorname{Fix}(L,G) \cap \operatorname{Fix}(L,H)$

(Gal4) $\operatorname{Gal} L|E$ *is a group complement of* $\operatorname{Gal} L|F$ *in* $\operatorname{Gal} L|K$ *if and only if E is a field complement of F in L over K.*

(Gal5) $\operatorname{Gal} L|\sigma E = \sigma(\operatorname{Gal} L|E)\sigma^{-1}$

(Gal5′) $\operatorname{Fix}(L, \sigma G \sigma^{-1}) = \sigma \operatorname{Fix}(L,G)$

(Gal6) $E|K$ *is a Galois extension if and only if* $\operatorname{Gal} L|E$ *is normal in* $\operatorname{Gal} L|K$.

Proof: For the proof of these assertions in the case of a finite Galois extension $L|K$, see [LANG3], Chapter VIII, §1. Note that (Gal4) follows from (Gal2), (Gal3), (Gal2′), (Gal3′), and that (Gal6) follows from (Gal5), (Gal5′).

Let $E|K$ be a subextension of $L|K$ and $G = \operatorname{Gal} L|E$. Then $E \subset \operatorname{Fix}(L,G)$. Let $a \in \operatorname{Fix}(L,G)$. Since $L|E$ is again a Galois extension, a is contained in a finite Galois subextension $M|E$ of $L|E$. Since every $\sigma \in \operatorname{Gal} M|E$ can be extended to an automorphism of $L|E$, we find that $\sigma a = a$ for every such σ. By finite Galois Theory, $a \in E$, showing that $E = \operatorname{Fix}(L,G)$.

Conversely, let G be an arbitrary subgroup of $\operatorname{Gal} L|K$ and $E = \operatorname{Fix}(L,G)$. Then $G \subset \operatorname{Gal} L|E$. We show that $\operatorname{Gal} L|E$ is equal to the closure of G; then $G = \operatorname{Gal} L|E$ if G is closed. Let $\sigma \in \operatorname{Gal} L|E$; we have to show that $\operatorname{res}_{L_i}(\sigma) \in \operatorname{res}_{L_i}(G) =: G_i$ for every finite normal subextension $L_i|K$ of $L|K$. Since E is the fixed field of G in L, we find that E is also the fixed field of G_i in $E.L_i$. By finite Galois Theory, $G_i = \operatorname{Gal} E.L_i|E$. Hence, $\operatorname{res}_{L_i}(\sigma) \in G_i$.

The proof of all remaining assertions is left to the reader. □

Now let $L|K$ and $L'|K'$ be two Galois extensions such that $L \subset L'$ and $K \subset K'$. Then the restriction map $\operatorname{res}_L : \operatorname{Gal} L'|K' \to \operatorname{Gal} L|K$ is a group homomorphism. It is continuous (and hence open); indeed, if $L_i|K$ is a finite normal subextension of $L|K$, then also $L_i.K'|K'$ is a finite normal extension, and $\operatorname{res}_L(\operatorname{Gal} L'|L_i.K') \subset \operatorname{Gal} L|L_i$. Since K' is the fixed field of $\operatorname{Gal} L'|K'$ in L', we find that $L \cap K'$ is the fixed field of $\operatorname{res}_L(\operatorname{Gal} L'|K')$ in L. Further, $\operatorname{res}_L(\operatorname{Gal} L'|K')$ is compact and thus closed in $\operatorname{Gal} L|K$. From the foregoing

theorem, we may now infer that $\mathrm{res}_L(\mathrm{Gal}\, L'|K') = \mathrm{Gal}\, L \,|\, L \cap K'$. We have proved that the restriction is in fact a continuous epimorphism

$$\mathrm{res}_L : \mathrm{Gal}\, L'|K' \longrightarrow \mathrm{Gal}\, L \,|\, L \cap K' .$$

Its kernel is $\mathrm{Gal}\, L'|L.K'$. Note that for every subextension $E'|K'$ of $L'|K'$, the fixed field of $\mathrm{res}_L(\mathrm{Gal}\, L'|E')$ in L is $E' \cap L$. Conversely, if E is this fixed field, then $E' = E.K'$.

Now we apply our results to two special cases. The first is the case of $K = K'$ and $L \subset L'$, that is, the restriction to a subextension. The second is the case of $L' = L.K'$ where the kernel is trivial and the above epimorphism is an isomorphism. We obtain:

Theorem 6.11 *Let $L|K$ be a Galois extension. Then:*

(Gal7) *If $E|K$ is a Galois subextension of $L|K$, then the restriction of the automorphisms of $L|K$ to E is a continuous open epimorphism with kernel $\mathrm{Gal}\, L|E$, giving rise to a topological isomorphism $\mathrm{Gal}\, E|K \cong \mathrm{Gal}\, L|K \,/\, \mathrm{Gal}\, L|E$. If $F|K$ is a subextension of $L|K$, then the fixed field of $\mathrm{res}_E(\mathrm{Gal}\, L|F)$ in E is $E \cap F$.*

(Gal8) *If $K'|K$ is an arbitrary field extension, then $L.K'|K'$ is a Galois extension, and the restriction of the automorphisms of $L.K'|K'$ to L is a continuous open epimorphism giving rise to a topological isomorphism $\mathrm{Gal}\, L.K'|K' \cong \mathrm{Gal}\, L \,|\, L \cap K'$. If $E|K$ and $E'|K'$ are subextensions of $L|K$ and $L.K'|K'$ respectively and if $\mathrm{Gal}\, L|E$ is the image of $\mathrm{Gal}\, L.K'|E'$ under this isomorphism, then $E = E' \cap K$ and $E' = E.K'$.*

(Gal9) *If $K'|K$ is a Galois extension, then $L.K'|K$ is a Galois extension and the restrictions of the automorphisms of $L.K' \,|\, L \cap K'$ to L and K' give a topological isomorphism $\mathrm{Gal}\, L.K' \,|\, L \cap K' \cong \mathrm{Gal}\, L \,|\, L \cap K' \times \mathrm{Gal}\, K' \,|\, L \cap K'$, where the latter is endowed with the product topology. Further,*

$$\mathrm{Gal}\, L.K'|K \cong \{(\sigma, \tau) \in \mathrm{Gal}\, L|K \times \mathrm{Gal}\, K'|K \mid \mathrm{res}_{L\cap K'}(\sigma) = \mathrm{res}_{L\cap K'}(\tau)\}.$$

The proof of (Gal9) uses (Gal7) and the following fact whose proof we leave to the reader: If N_1, N_2 are normal subgroups of G, then

$$G/(N_1 \cap N_2) \cong G/N_1 \times G/N_2 . \tag{6.2}$$

From (Gal7), where we replace K by $L \cap K'$, we obtain epimorphisms from $\mathrm{Gal}\, L.K'|L \cap K'$ onto $\mathrm{Gal}\, L|L \cap K'$ and $\mathrm{Gal}\, K'|L \cap K'$ with kernels $N_1 := \mathrm{Gal}\, L.K'|K'$ and $N_2 := \mathrm{Gal}\, L.K'|L$ respectively. But $N_1 \cap N_2 = \{1\}$, which yields (Gal9).

6.5 Linearly disjoint and algebraically disjoint extensions

Let $L|K$ and $F|K$ be subextensions of some extension Ω of the field K. **Unless stated otherwise, we will always assume all fields to be contained in such a universal extension field Ω.** The elements $x_1, \ldots, x_n \in L$ are said to be **K-linearly independent** if $c_1x_1 + \ldots + c_nx_n = 0$ with $c_1, \ldots, c_n \in K$ implies that $c_i = 0$ for all i. In other words,

$x_1, \ldots, x_n$ are K-linearly dependent if there exists a nontrivial K-linear combination $\sum c_i x_i$ which equates to zero. If $x_i \neq 0$, then $c_i x_i = 0$ implies $c_i = 0$, hence the x_i are K-linearly independent if and only if they are K-independent (where L is viewed as a K-module). We say that $L|K$ **is linearly disjoint from** $F|K$ **(in** Ω**)** if for every $n \in \mathbb{N}$ and every choice of K-linearly independent elements $x_1, \ldots, x_n \in L$, these elements will also be F-linearly independent. We shall show that this property is symmetrical, that is, it implies that for every $n \in \mathbb{N}$ and every choice of K-linearly independent elements $y_1, \ldots, y_n \in F$, these elements will also be L-linearly independent. Hence, assume that $L|K$ is linearly disjoint from $F|K$ and suppose that $y_1, \ldots, y_n \in F$ are K-linearly independent elements which satisfy a linear dependence relation

$$x_1 y_1 + \ldots + x_n y_n = 0$$

where $x_i \in L$. Then our definition implies that the x_i must be K-linearly dependent. Without loss of generality, we may assume that there is $m < n$ such that $x_1, \ldots, x_m$ are K-linearly independent and that there are $c_{ij} \in K$ such that $x_i = \sum_{j=1}^{m} c_{ij} x_j$ for $m < i \leq n$. Then we may rewrite the dependence relation as follows:

$$\sum_{j=1}^{m} x_j y_j + \sum_{i=m+1}^{n} \left(\sum_{j=1}^{m} c_{ij} x_j \right) y_i = 0 \, .$$

Reorganizing the terms, we find

$$\sum_{j=1}^{m} \left(y_j + \sum_{i=m+1}^{n} c_{ij} y_i \right) x_j = 0 \, .$$

In this sum, the coefficient of every x_j is nonzero since the elements y_i were assumed to be K-linearly independent. Hence, the elements $x_1, \ldots, x_m$ are not F-linearly independent. On the other hand, $x_1, \ldots, x_m$ were assumed to be K-linearly independent, which contradicts our assumption that $L|K$ be linearly disjoint from $F|K$.

In view of the symmetry that we have just proved, we will also say that L **and** F **are** K**-linearly disjoint** if $L|K$ is linearly disjoint from $F|K$. Note that if L and F are linearly disjoint over K, then also for arbitrary subextensions $L'|K$ of $L|K$ and $F'|K$ of $F|K$, the fields L' and F' are K-linearly disjoint.

Assume that $\mathcal{B}$ is a K-basis of L. If L and F are K-linearly disjoint, then the elements of $\mathcal{B}$ remain F-linearly independent, and consequently, $\mathcal{B}$ is also an F-basis of $L.F$. On the other hand, if the elements of $\mathcal{B}$ remain F-linearly independent, then also the basis of every finite K-subvector space of L will remain F-linearly independent, which implies that $L|K$ is linearly disjoint from $F|K$. Consequently,

$L|K$ is linearly disjoint from $F|K$ if and only if every finitely generated subextension of $L|K$ is linearly disjoint from $F|K$.

An algebraic extension $L|K$ is linearly disjoint from $F|K$ if and only if $[L' : K] = [L'.F : F]$ for every finite subextension $L'|K$.

A finite extension $L|K$ is linearly disjoint from $F|K$ if and only if $[L : K] = [L.F : F]$.

The proof of the following transitivity property is left to the reader; it can also be found in [LANG3], Chapter X, §5.

Lemma 6.12 *Let $L|K$ and $F \supset E \supset K$ be field extensions, all contained in the common extension field Ω. Then $L|K$ is linearly disjoint from $F|K$ if and only if $L|K$ is linearly disjoint from $E|K$ and $L.E|E$ is linearly disjoint from $F|E$.*

Lemma 6.13 *Let $F|K$ be an arbitrary extension with K relatively algebraically closed in F. Then $F|K$ is linearly disjoint from every simple algebraic and every separable algebraic extension of K. Moreover, if $L|K$ is separable algebraic, then L is relatively algebraically closed in $L.F$.*

Proof: Let $K(b)|K$ be algebraic. Then b is also algebraic over F. All conjugates of b (over K and hence also over F) are algebraic over K. The same is thus true for the coefficients of the minimal polynomial $f \in F[X]$ of b over F since they are symmetric functions in the conjugates of b over F. Since K is assumed to be relatively algebraically closed in F, it follows that $f \in K[X]$. Hence, f is also the minimal polynomial of b over K. Thus, $[F(b) : F] = [K(b) : K]$, showing that $F|K$ is linearly disjoint from $K(b)|K$.

Now let $L|K$ be a separable algebraic extension. Then every finite subextension of $L|K$ is simple and thus linearly disjoint from $F|K$. This proves that $L|K$ itself is linearly disjoint from $F|K$. Let $b \in L.F$ be algebraic over L. Since $L|K$ is separable algebraic, so is $L.F|F$. Hence, b is separable algebraic over F. But we have already shown that the minimal polynomial of b over F coincides with that over K. This shows that b is separable algebraic over K. Consequently, $L(b)|K$ is a separable extension. From what we have just proved, we thus know that $L(b)|K$ is linearly disjoint from $F|K$. By Lemma 6.12, $L(b)|L$ is linearly disjoint from $L.F|L$. In particular, $b \in L.F$ implies $b \in L$. This proves that L is relatively algebraically closed in $L.F$. □

Now assume that $L|K$ is a Galois extension and that it is not linearly disjoint from $F|K$. Since every finite Galois extension is simple, it follows that there is some $b \in L \setminus K$ such that the minimal polynomial f of b over K does not remain irreducible over F. Then the minimal polynomial h of b over F is a factor of f of smaller degree than f and consequently, not all of its coefficients will lie in K. On the other hand, these coefficients are elementary symmetric functions in the roots of h, and these roots are all in L since $L|K$ was assumed to be normal. Consequently, the coefficients of h lie in $L \cap F$, showing that this is a proper extension of K. Conversely, if $L \cap F$ is a proper extension of K, then L and F can not be K-linearly disjoint. We have proved:

Lemma 6.14 *Let $L|K$ a Galois and $F|K$ an arbitrary field extension. Then L and F are K-linearly disjoint if and only if $L \cap F = K$. More precisely, L and F are $(L \cap F)$-linearly disjoint.*

For a related criterion, see Lemma 6.34 below. From this Lemma together with (Gal8), we obtain:

Corollary 6.15 *Let $L|K$ be a Galois extension which is linearly disjoint from the arbitrary field extension $K'|K$. Then $L.K'|K'$ is a Galois extension, and the restriction of the automorphisms of $L.K'|K'$ to L is a topological isomorphism* $\operatorname{Gal} L.K'|K' \cong \operatorname{Gal} L|K$.

Let $L|K$ be any field extension. We say that $t_1, \ldots, t_n \in L$ are **K-algebraically independent** or **algebraically independent over** K if there is no nontrivial polynomial

$f(X_1, \ldots, X_n)$ with coefficients in K such that $f(t_1, \ldots, t_n) = 0$. Infinitely many elements t_i, $i \in I$, are called K-algebraically independent if every finite subset of them is K-algebraically independent. A single element t is called **transcendental over** K if it is K-algebraically independent. The elements t_i, $i \in I$, are K-algebraically independent if and only if for every choice of elements a_i, $i \in I$, in some extension field of K there is (a uniquely determined) homomorphism from the ring $K[t_i \mid i \in I]$ onto the ring $K[a_i \mid i \in I]$ over K which sends t_i to a_i for every $i \in I$.

A **transcendence basis of** $L|K$ is a maximal set of elements of L which are K-algebraically independent. The cardinality of all transcendence bases of $L|K$ is equal; it is called the **transcendence degree** of $L|K$ and will be denoted by $\operatorname{trdeg} L|K$. From every set of generators of L over K, one can select a transcendence basis $\mathcal{T}$ of $L|K$; indeed, one may just take $\mathcal{T}$ to be a maximal K-algebraically independent subset, so all other generators and thus also L will be algebraic over $K(\mathcal{T})$. Consequently,

Lemma 6.16 *Every finitely generated field extension $K(x_1, \ldots, x_n)$ has finite transcendence degree $\leq n$. Every finitely generated field has finite transcendence degree over its prime field.*

If $\mathcal{T}$ is a transcendence basis of $L|K$ and $F|K$ is an arbitrary extension, then a transcendence basis of $L.F|F$ can be selected from $\mathcal{T}$. In particular, if $\operatorname{trdeg} L|K$ is finite, then

$$\operatorname{trdeg} L.F|F \leq \operatorname{trdeg} L|K \ .$$

Note that the transcendence degree is additive, that is,

$$\operatorname{trdeg} L|K = \operatorname{trdeg} L|K' + \operatorname{trdeg} K'|K$$

for arbitrary extensions $L|K'$ and $K'|K$. The proof of these facts and the basic properties of transcendence bases and transcendence degree can be found in [LANG3], Chapter X.

An extension field $K(t_1, \ldots, t_n)$ of K is called a **rational function field in** n **variables over** K if the elements $t_1, \ldots, t_n$ are K-algebraically independent. An extension field F of K is called an **algebraic function field in** n **variables over** K if it is a finite extension of a rational function field in n variables over K, or equivalently, if it is a finitely generated extension of K of transcendence degree n. We will also speak of the **algebraic function field** $F|K$.

We will say that $L|K$ is **algebraically disjoint from** $F|K$ or **free from** $F|K$ (in their common extension field Ω) if for every $n \in \mathbb{N}$ and every choice of K-algebraically independent elements $t_1, \ldots, t_n \in L$, these elements will also be F-algebraically independent. Observe that $L|K$ is algebraically disjoint from $F|K$ if and only if every finitely generated subextension $E|K$ of $L|K$ satisfies $\operatorname{trdeg} E|K = \operatorname{trdeg} E.F|F$. Note that the elements $t_1, \ldots, t_n \in L$ are K-algebraically independent if and only if the elements $t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n} \in L$, $\nu_1, \ldots, \nu_n \in \mathbb{N}$, are K-linearly independent. This implies:

Lemma 6.17 *If $L|K$ is linearly disjoint from $F|K$, then it is algebraically disjoint from $F|K$. Conversely, if $t_i \in L$, $i \in I$, are F-algebraically independent, then $K(t_i \mid i \in I)|K$ is linearly disjoint from $F|K$. If $t_i \in L$, $i \in I$, are K-algebraically independent, then $K(t_i \mid i \in I)|K$ is linearly disjoint from $\tilde{K}|K$ and in particular, K is relatively algebraically closed in $K(t_i \mid i \in I)$.*

(For the proof of the last assertion, the reader may show that K-algebraically independent elements t_i are also $\tilde{K}$-algebraically independent.)

Like linear disjointness, also algebraic disjointness is symmetrical: Suppose that $F|K$ is not algebraically disjoint from $L|K$, i.e. there exist $t_1, \ldots, t_n \in F$ which are K-algebraically independent but not L-algebraically independent. Since a dependence relation only requires finitely many coefficients from L, there is a finitely generated subextension $E|K$ of $L|K$ such that $t_1, \ldots, t_n \in F$ are E-algebraically dependent. That is, $n + \operatorname{trdeg} E|K > \operatorname{trdeg} E(t_1, \ldots, t_n)|K$. In view of $E(t_1, \ldots, t_n) = E.K(t_1, \ldots, t_n)$, and because of $n = \operatorname{trdeg} K(t_1, \ldots, t_n)|K$, we find that $\operatorname{trdeg} E.K(t_1, \ldots, t_n)|K(t_1, \ldots, t_n) < \operatorname{trdeg} E|K$. Hence, $\operatorname{trdeg} E.F|F < \operatorname{trdeg} E|K$. That is, $L|K$ is not algebraically disjoint from $F|K$.

In view of the symmetry that we have just proved, we will also say that L **and** F **are** K**-algebraically disjoint** if $L|K$ is algebraically disjoint from $F|K$. Note that if L and F are K-algebraically disjoint, then also for arbitrary subextensions $L'|K$ of $L|K$ and $F'|K$ of $F|K$, the fields L' and F' are K-algebraically disjoint. We leave it to the reader to prove the following analogue to Lemma 6.12:

Lemma 6.18 *Let $L|K$ and $F \supset E \supset K$ be field extensions, all contained in the common extension field Ω. Then $L|K$ is algebraically disjoint from $F|K$ if and only if $L|K$ is algebraically disjoint from $E|K$ and $L.E|E$ is algebraically disjoint from $F|E$.*

Every two field extensions $L|K$ and $F|K$ can be embedded in a common overfield in such a way that their images are K-algebraically disjoint. Indeed, if t_i, $i \in I$, is a transcendence basis of $F|K$, then we choose L-algebraically independent elements t'_i, $i \in I$, and define an embedding $K(t_i \mid i \in I) \to L(t'_i \mid i \in I)$ by $t_i \mapsto t'_i$. Since $F|K(t_i \mid i \in I)$ is algebraic, this embedding can be extended to an embedding of F in the algebraic closure of $L(t'_i \mid i \in I)$, and we are done. In contrast to this, it is not always possible to embed two given field extensions in a common overfield in such a way that their images are linearly disjoint. For example, if they are algebraic and not linearly disjoint, then this will remain true for every embedding. But in any case, we have shown that every two fields can be embedded in a common extension field, that is, that the theory of fields has the **amalgamation property**.

Lemma 6.19 *Let $L|K$ be an arbitrary subextension of $\Omega|K$ and let $K(\mathcal{T})|K$ be generated by a set $\mathcal{T} \subset \Omega$ of elements which are K-algebraically independent. If they are also L-algebraically independent, that is, if L and $K(\mathcal{T})$ are K-algebraically disjoint, and if K is relatively algebraically closed in L, then $K(\mathcal{T})$ is relatively algebraically closed in $L(\mathcal{T})$.*

Proof: Every element in $L(\mathcal{T})$ algebraic over $K(\mathcal{T})$ is already algebraic over $K(\mathcal{T}_0)$ for a suitable finite subset $\mathcal{T}_0$ of $\mathcal{T}$. Hence, it suffices to prove the assertion in the case of $\mathcal{T}$ finite. Induction on the number of elements of $\mathcal{T}$ shows that it suffices to prove the assertion in the case of $\mathcal{T}$ consisting of just one element t. Let $r(t) = cf(t)/g(t) \in L(t)$, with $c \in L$, $f(t), g(t) \in L[t]$ monic and f prime to g. If $r(t)$ is algebraic over $K(t)$, then there is an equation

$$h_n(t)r(t)^n + \ldots + h_0(t) = 0 \quad \text{with } h_i(t) \in K[t] \text{ and } h_0(t) \neq 0$$

such that $h_n(t), \ldots, h_0(t)$ have no common factor $\in K[t] \setminus K$. Multiplying by $g(t)^n$, we obtain

$$h_n(t)c^n f(t)^n + \ldots + h_0(t)g(t)^n = 0\,.$$

Since t is transcendental over L, we can substitute for it an arbitrary root $b \in \tilde{L}$ of f to obtain that $h_0(b)g(b)^n = 0$. Since f was assumed to be prime to g, we have $g(b) \neq 0$. Hence, $h_0(b) = 0$, which shows that every root of f is algebraic over K. Consequently, the coefficients of the monic polynomial f, being symmetric functions in the roots, are all algebraic over K. Since K is assumed to be relatively algebraically closed in L, it follows that $f(t) \in K[t]$. By a similar argument, it is shown that $g(t) \in K[t]$.

Further, we substitute for t a root $d \in \tilde{L}$ of $f - g$ in the above equation. With this choice, we have $f(d) = g(d) \neq 0$ (f being prime to g), and we obtain that

$$h_n(d)c^n + \ldots + h_0(d) = 0 \, .$$

Since $h_n(t), \ldots, h_0(t)$ have no common nontrivial factor, not all $h_i(d)$ can be zero. Then at least two of the $h_i(d)$ are nonzero. This shows c to be algebraic over K and thus to be an element of K. Altogether, we find that $r(t) \in K(t)$. □

Lemma 6.20 *Suppose that L and F are K-algebraically disjoint. If for every finitely generated subextension $E|K$ of $F|K$, E is relatively algebraically closed in $E.L$, then L and F are K-linearly disjoint.*

Proof: It suffices to show that $L|K$ is linearly disjoint from every finitely generated subextension $K(x_1, \ldots, x_n)|K$ of $F|K$. Suppose that we have already shown that L and $K' := K(x_1, \ldots, x_{i-1})$ are K-linearly disjoint, for some $i \leq n$. In view of Lemma 6.12, it suffices to prove that $L' := L(x_1, \ldots, x_{i-1})$ and $K(x_1, \ldots, x_i)$ are K'-linearly disjoint. By hypothesis, K' is relatively algebraically closed in L'. If x_i is algebraic over K' then in view of Lemma 6.13, we are done. So assume that x_i is transcendental over K'. Then by our assumption that L and F are K-algebraically disjoint, it follows that x_i is also transcendental over L'. In this case, we are done by virtue of Lemma 6.17. □

6.6 Supernatural numbers

The degree $[L : K]$ of a finite field extension $L|K$ gives information about the degrees of all possible subextensions; these degrees are just divisors of $[L : K]$. Analogous information is contained in the order of a finite group, or the finite index of a subgroup. If we would just set $[L : K] = \infty$ if $L|K$ is infinite, then we would loose this information. In the case of infinite algebraic field extensions (and analogously, of profinite groups), we would like to generalize the notion of degree (resp., of order and index) in a way that preserves the information of the finite case. To this end, we define a **supernatural number** to be a formal product

$$\mathbf{n} = \prod_{p \text{ prime}} p^{\mathbf{n}_p} \quad \text{with } \mathbf{n}_p \in \mathbb{N} \cup \{0, \infty\}$$

and define the product of two supernatural numbers $\mathbf{m}$, $\mathbf{n}$ by

$$\mathbf{m} \cdot \mathbf{n} = \prod_{p \text{ prime}} p^{\mathbf{m}_p + \mathbf{n}_p}$$

with the provision that $n + \infty = \infty + n = \infty + \infty = \infty$ for all $n \in \mathbb{N} \cup \{0\}$. Every natural number $n \neq 0$ can be viewed as a supernatural number $\mathbf{n}$ in a canonical way,

taking $\mathbf{n}_p = v_p n$, where v_p denotes the usual p-adic valuation. A supernatural number $\mathbf{n}$ is said to be a **power of** p if it is of the form p^ν with $\nu \in \mathbb{N} \cup \{0, \infty\}$.

If $\mathbf{m}$, $\mathbf{n}$ are supernatural numbers, then like for natural numbers, we will say that $\mathbf{m}$ **divides** $\mathbf{n}$, denoted by $\mathbf{m}|\mathbf{n}$, if there is a supernatural number ℓ such that $\mathbf{m} = \ell \cdot \mathbf{n}$. Hence, $\mathbf{m}$ divides $\mathbf{n}$ if and only if $\mathbf{m}_p \leq \mathbf{n}_p$ for all primes p. As for natural numbers, $\mathbf{m}|\mathbf{n} \wedge \mathbf{n}|\mathbf{m}$ implies $\mathbf{m} = \mathbf{n}$.

Observe that every subset of $\mathbb{N} \cup \{0, \infty\}$ has a supremum in $\mathbb{N} \cup \{0, \infty\}$. This gives us the possibility of defining the **least common multiple** of an arbitrary set $\mathcal{N}$ of supernatural numbers to be the supernatural number $\operatorname{lcm} \mathcal{N}$ given by

$$\operatorname{lcm} \mathcal{N}_p = \sup\{\mathbf{n}_p \mid \mathbf{n} \in \mathcal{N}\} .$$

It is clear from this definition that every $\mathbf{n} \in \mathcal{N}$ divides $\operatorname{lcm} \mathcal{N}$ and that this does not remain true if we replace $\operatorname{lcm} \mathcal{N}$ by any of its proper divisors. Note that it is the notion of a supernatural number that enables us to define a least common multiple for every infinite set of natural numbers. Similarly, the **greatest common divisor** $\gcd \mathcal{N}$ is defined by just replacing the supremum by the minimum. Then $\gcd \mathcal{N}$ divides every $\mathbf{n} \in \mathcal{N}$, but no proper multiple of $\gcd \mathcal{N}$ has this property. If $\mathcal{N} = \{\mathbf{m}, \mathbf{n}\}$, then we write $(\mathbf{m}, \mathbf{n})$ instead of $\gcd \mathcal{N}$. We say that $\mathbf{m}$ **is prime to** $\mathbf{n}$ if $(\mathbf{m}, \mathbf{n}) = 1$.

Now let $L|K$ be an algebraic field extension. Then we define

$$[L : K] := \operatorname{lcm} \{[E : K] \mid E|K \text{ a finite subextension of } L|K\} .$$

In the case of $L|K$ a finite extension, this coincides with the usual degree (just because $[E : K]$ is a divisor of $[L : K]$ for every subextension $E|K$).

If $L|K$ is an algebraic extension, linearly disjoint from the arbitrary extension $F|K$, then it follows from the definition of linear disjointness that $[L.F : F] = [L : K]$. But the converse is not true, if $L|K$ is not finite.

Lemma 6.21 *If $L|K$ is an algebraic extension and $K'|K$ a subextension of $L|K$, then*

$$[L : K] = [L : K'] \cdot [K' : K] .$$

Proof: In Section 6.1 we have shown that for every finite subextension $L_i'|K'$ of $L|K'$ there is a finite subextension $K_i|K$ of $K'|K$ and a finite subextension $L_i|K_i$ of $L|K_i$ such that $L_i' = L_i.K'$ and $[L_i : K_i] = [L_i' : K']$. Moreover, $K_i|K$ may be chosen as to contain a given arbitrary finite subextension of $K'|K$. We have

$$[L_i' : K'] \cdot [K_i : K] = [L_i : K_i] \cdot [K_i : K] = [L_i : K] \tag{6.3}$$

with $L_i|K$ a finite subextension of $L|K$. It follows that for every finite subextension $L_i'|K'$ of $L|K'$ and $K_i|K$ of $K'|K$, the product $[L_i' : K'][K_i : K]$ divides $[L : K]$. Consequently, $[L : K'][K' : K]$ divides $[L : K]$.

For the converse, let $E|K$ be a finite subextension of $L|K$. We set $L_i' := E.K'$ and choose K_i and L_i as above. Recall that these both fields may be chosen such that L_i contains E. Hence, $[E : K]$ divides $[L_i : K]$. In view of (6.3), this in turn divides $[L : K'][K' : K]$. Hence, $[L : K]$ divides $[L : K'][K' : K]$. □

Now let G be a profinite group and H a closed subgroup of G. Then we define the **index of H in G** to be

$$(G : H) := \operatorname{lcm} \{(G : HN) \mid N \text{ an open normal subgroup of } G\} .$$

(Note that HN is of finite index in G since already N is.) Further, we define the **order of G** to be $\#G := (G : 1)$. Hence,

$$\#G = \operatorname{lcm} \{(G : N) \mid N \text{ an open normal subgroup of } G\} .$$

If G is finite, then 1 is an open normal subgroup of G, and $\#G$ coincides with the usual order. If $(G : H)$ is finite, then H is an open subgroup of G, and there is an open normal subgroup $N \subset H$ of G, showing that $(G : H)$ coincides with the usual index.

Lemma 6.22 *Let G be a profinite group and $H \subset \mathcal{H}$ closed subgroups of G. Then*

$$(G : H) = (G : \mathcal{H}) \cdot (\mathcal{H} : H) .$$

Proof: In what follows, let N always run through all open normal subgroups of G. Then the groups $H \cap N$ form a basis of the open normal subgroups of H. We compute: $(G : H) = \operatorname{lcm}_N(G : HN) = \operatorname{lcm}_N(G : \mathcal{H}N)(\mathcal{H}N : HN) = \operatorname{lcm}_N(G : \mathcal{H}N)(\mathcal{H}N/N : HN/N) = \operatorname{lcm}_N(G : \mathcal{H}N)(\mathcal{H}/\mathcal{H} \cap N : H(\mathcal{H} \cap N)/\mathcal{H} \cap N) = \operatorname{lcm}_N(G : \mathcal{H}N)(\mathcal{H} : H(\mathcal{H} \cap N))$. Since the intersection of any two open normal subgroups of G is again an open normal subgroup of G, the latter is equal to $\operatorname{lcm}_N(G : \mathcal{H}N) \cdot \operatorname{lcm}_N(\mathcal{H} : H(\mathcal{H} \cap N)) = (G : \mathcal{H})(\mathcal{H} : H)$. □

This lemma can also be deduced from Lemma 6.21, using the fact that every profinite group is the Galois group of a Galois extension, together with the following lemma:

Lemma 6.23 *Let $L|K$ be a Galois extension. Then*

$$[L : K] = \#\operatorname{Gal} L|K .$$

The finite case of this assertion is proved in finite Galois Theory. The infinite case follows from the definitions of $[L : K]$ and $\#\operatorname{Gal} L|K$ by Galois correspondence (Theorem 6.10).

If A is an abelian torsion group, then $\#A$ is defined to be

$$\#A := \operatorname{lcm} \{\#B \mid B \text{ a finite subgroup of } A\} .$$

Again, if A is finite, then this coincides with the usual order. To give an example,

$$\#\mathbb{Q}/\mathbb{Z} = \prod_{p \text{ prime}} p^\infty .$$

For further information and examples for supernatural numbers, see Chapter I, §4 of [RIBES].

Exercise 6.2 *Define the product over arbitrary sets of supernatural numbers. Use this to compute the order of direct products of abelian torsion groups.*

Exercise 6.3 *Show that $(G : H) = \operatorname{lcm} \{(G : N) \mid N$ an open subgroup of G containing $H\}$.*

6.7 Separable and inseparable extensions

A field K is called **perfect** if every of its algebraic extensions is separable. By virtue of Lemma 6.3, this is the case if and only if the extension $\tilde{K}|K$ is separable. If $L|K$ is an algebraic extension, then $\tilde{K} = \tilde{L}$, and if $\tilde{L}|K$ is separable, then by Lemma 6.3, also $\tilde{L}|L$ is separable. The latter yields that L is perfect. Hence,

Lemma 6.24 *Every algebraic extension field of a perfect field is again perfect.*

The **characteristic exponent** of a field K is defined to be equal to the characteristic $\operatorname{char} K$ if this is nonzero (and hence a prime number), and to be equal to 1 otherwise. We will denote it by $\operatorname{charexp} K$. Let $p \geq 1$ be the characteristic exponent of K. Then K admits an injective endomorphism

$$\begin{array}{rccl} \varphi: & K & \to & K^p = \{a^p \mid a \in K\} \subset K \\ & a & \mapsto & a^p \, . \end{array}$$

(Recall that the binomial coefficients $\binom{p}{i}$ are divisible by p if and only if $1 \leq i \leq p-1$ or $p = 1$. In a field of characteristic $p > 0$ this just means that they vanish, showing that $(a+b)^p = a^p + b^p$, which proves the additivity of φ.) The endomorphism φ is called the **Frobenius endomorphism** or just **Frobenius of** K. It is surjective if and only if $K = K^p$. If $K = K^p$ holds, then one shows by induction that $K = K^{p^m}$ for all $m \geq 0$. Let us consider the Frobenius of the rational function field $\tilde{K}(X)$. Its restriction to the polynomial ring $\tilde{K}[X]$ is an endomorphism of $\tilde{K}[X]$. We have $(X-b)^{p^m} = X^{p^m} - b^{p^m}$ for every $b \in \tilde{K}$ and every $m \in \mathbb{N}$. We see that for every $a = b^{p^m} \in K$, the minimal polynomial $X^{p^m} - a$ of b over K admits only b as a root. That is, for every $a \in K$ there is a *unique* p^m-th root in $\tilde{K}$; we shall denote it by a^{1/p^m} or $a^{p^{-m}}$. We define K^{1/p^∞} to be the smallest algebraic extension field of K on which the Frobenius is surjective. It consists of all elements of $\tilde{K}$ which are a p^m-th root of some element in K, for some $m \in \mathbb{N}$; this collection of elements is indeed a field, as follows from the fact that the Frobenius (on $\tilde{K}$) is a homomorphism. Similarly, we obtain a field if we collect all elements of $\tilde{K}$ which are a p^m-th root of some element in K, for fixed m; this field is denoted by K^{1/p^m}. Note that K^{1/p^∞} is the union over all K^{1/p^m}, $n \in \mathbb{N}$, because for every element b in K^{1/p^∞} there is some integer $m \in \mathbb{N}$ such that $b^{p^m} \in K$. Observe that for an extension generated over K by the elements x_i, $i \in I$, we have

$$\begin{aligned} K(x_i \mid i \in I)^{1/p^m} &= K^{1/p^m}(x_i^{1/p^m} \mid i \in I) \, , \\ K(x_i \mid i \in I)^{p^m} &= K^{p^m}(x_i^{p^m} \mid i \in I) \, . \end{aligned}$$

Let $b \in \tilde{K}$ and f its minimal polynomial over K. We write $f = X^n + c_{n-1}X^{n-1} + \ldots + c_1 X + c_0$ where the c_i are elements of K. Assume that for $i = 0, \ldots, n-1$, the index i is divisible by p if $c_i \neq 0$. Then f may be written in the form $f = g^p$ where $g = X^{n/p} + c_{n-p}^{1/p} X^{n/p-1} + \ldots + c_0^{1/p} \in K^{1/p}[X]$. In this case, every root of g occurs p-times as a root of f, showing that f is not separable, and b is not separable over K. Conversely, it can be shown that if f is not separable, then it is of the form $f = g^p$ with $g \in K^{1/p}[X]$. (In short, the proof is as follows. Since f has multiple roots, it has a root in common with its derivative $f' \in K[X]$. If f' does not vanish identically, it then follows that both f and f' have a common factor in $K[X]$, which is consequently of smaller degree than f. But

this is a contradiction to the irreducibility of f. Consequently, $f' \equiv 0$ which can only be the case if $f = g^p$ with $g \in K^{1/p}[X]$.) Now if $K = K^p$, then all coefficients of g lie in K, which means that $g \in K[X]$, contradicting the irreducibility of f. This shows that if $K = K^p$, then every irreducible polynomial over K is separable. In particular, *every field of characteristic 0 is perfect.* On the other hand, if $K \neq K^p$, then there is some $a \in K$ such that $X^p - a$ is irreducible and not separable. We have proved:

Lemma 6.25 *The field K is perfect if and only if $K = K^p$, that is, if and only if the Frobenius is surjective. This in turn is equivalent to $K = K^{1/p^\infty}$.*

If L is a perfect extension field of K, then $L = L^{1/p^\infty}$ and thus, L contains K^{1/p^∞}. Since K^{1/p^∞} is perfect, it is consequently the smallest perfect extension field of K. Therefore, it is called the **perfect hull of** K.

If the polynomial $X^p - a$ is not irreducible over K, then it splits into linear factors. (Indeed, $X^p - a = (X - a^{1/p})^p$, and if $(X - a^{1/p})^n \in K[X]$ with $1 < n < p$, hence $a^{n/p} \in K$, then we choose integers r, s such that $rn + sp = 1$ and obtain that $a^{1/p} = (a^{n/p})^r \cdot a^s \in K$.) Hence, $[K(a^{1/p}) : K]$ is equal to p or to 1. It follows that $[K^{1/p} : K]$ and $[K^{1/p^\infty} : K]$ are powers of p. The reader may show that $[K^{1/p^\infty} : K] = p^\infty$ if K is not perfect.

Since the Frobenius is injective, it is an automorphism on every finite field; this shows that *every finite field is perfect.* Recall that for every power $q = p^m$ of p, there is precisely one finite field $\mathbb{F}_q$ with q elements and that for every n it admits precisely one extension of degree n, namely the field with p^{mn} elements. Every such extension is Galois, the Galois group being generated by φ^m (m is the smallest integer ≥ 1 such that φ^m fixes $\mathbb{F}_q$). For the proofs, see [LANG3], VII, §5, Theorems 10, 12 and 13).

Let $\sigma : K \to F$ be any embedding. The reader may show that σ admits an extension to an embedding $\tilde{\sigma} : \tilde{K} \to \tilde{F}$. For every subextension $E|K$ of $K^{1/p^\infty}|K$, there is a unique extension to an embedding $\sigma_E : E \to F^{1/p^\infty}$. Indeed, since for every $b \in K^{1/p^\infty}$ there is some $m \in \mathbb{N}$ such that $a := b^{p^m} \in K$, the extension σ_1 must satisfy $\sigma_1 b = (\sigma a)^{1/p^m}$. In particular, the only embedding of E in $\tilde{K}$ over K is the identity. Recall that an extension $E|K$ is called **purely inseparable** if it is algebraic and the only embedding of E in $\tilde{K}$ over K is the identity. If an extension is separable and purely inseparable, then it is trivial. Further, if $K \subset E \subset L$, then $L|K$ is purely inseparable if and only if $L|E$ and $E|K$ are. If $E|K$ is a purely inseparable subextension of a normal extension $L|K$, then every automorphism of $L|K$ will also fix E, hence we can identify $\operatorname{Gal} L|K$ and $\operatorname{Gal} L|E$. All subextensions of $K^{1/p^\infty}|K$ are purely inseparable. We are going to show that also the converse is true, that is, that every purely inseparable extension $E|K$ is a subextension of $K^{1/p^\infty}|K$.

Let f be an irreducible polynomial over K with coefficients $c_i \in K$. Choose p^m to be the highest power of p which divides all i for which $c_i \neq 0$. Then we can write $f = g^{p^m}$ with $g(X) = \sum_j d_j X^j \in K^{1/p^m}[X]$ such that $c_i = d_j^{p^m}$ for $i = jp^m$. Note that g is irreducible over K^{1/p^m}, because every factorization $g = g_1 g_2$ yields a factorization $f = g_1^{p^m} g_2^{p^m}$, but f is irreducible by assumption. By our choice of m, there is some j not divisible by p such that $d_j \neq 0$. Consequently, g is separable over K^{1/p^m}. Observe that $m > 0$ if and only if f is not separable over K. On the other hand, we can also write $f = h(X^{p^m})$ where $h(X) = \sum_{j=1}^n c_{jp^m}(X) \in K[X]$. Then h is irreducible over K since every nontrivial factorization of h would also be a nontrivial factorization of f. If $a_1, \ldots, a_n$ are the roots of g in $\tilde{K}$, then they are all distinct because g is irreducible and separable over K^{1/p^m}. The

Frobenius being injective, also $a_1^{p^m}, \ldots, a_n^{p^m}$ are all distinct. But these are the roots of h since $h(X^{p^m}) = f(X) = g(X)^{p^m} = \prod_{i=1}^n (X - a_i)^{p^m} = \prod_{i=1}^n (X^{p^m} - a_i^{p^m})$. This proves that also h is separable over K. We summarize:

Lemma 6.26 *Let $f \in K[X]$ be irreducible over K and p be the characteristic exponent of K. Then there is some integer $m \geq 0$ and $g \in K^{1/p^m}[X]$, irreducible over K^{1/p^m}, such that $f = g^{p^m}$. So if $a_1, \ldots, a_n$ are the roots of g in $\tilde{K}$, then they are all distinct, and*

$$f = g^{p^m} = \prod_{i=1}^n (X - a_i)^{p^m} = \prod_{i=1}^n (X^{p^m} - a_i^{p^m}) = h(X^{p^m})$$

with $h \in K[X]$ irreducible and separable over K. We have $\deg g = \deg h$ and $\deg f = p^m \deg g$. Further, f is separable over K if and only if $m = 0$.

Now let $E|K$ be purely inseparable. Let $b \in E$ and f its minimal polynomial over K. We write $f = g^{p^m}$ according to the foregoing lemma. Then g must be linear. Indeed, if g would admit at least two roots a_1, a_2, they would be distinct and one of them, say a_1, would be equal to b. Then the assignment $a_1 \mapsto a_2$ would induce an embedding of $K(b)$ in $\tilde{K}$ over K which is not the identity. Since it can be extended to an embedding of E in $\tilde{K}$ over K, this contradicts our assumption that $E|K$ be purely inseparable. This proves that g and thus also h is linear. But that means that $h(X) = X - b^{p^m}$ with $b^{p^m} \in K$. We have shown that for every $b \in E$, there is some integer $m \geq 0$ such that $b^{p^m} \in K$. That is, $E \subset K^{1/p^\infty}$. We have proved:

Lemma 6.27 *Let K be a field with characteristic exponent p. Then every purely inseparable extension $E|K$ is a subextension of $K^{1/p^\infty}|K$. In this sense, $K^{1/p^\infty}|K$ is the maximal purely inseparable extension of K. In particular, the compositum of two purely inseparable extensions of K is again a purely inseparable extension of K.*

If $E|K$ is a purely inseparable extension and $F|K$ is an arbitrary extension, then $E.F|F$ is again a purely inseparable extension, because it is contained in F^{1/p^∞}. Together with an earlier observation, this lemma also shows that

Lemma 6.28 *A given embedding of K in an arbitrary perfect field has a unique extension to every purely inseparable extension of K.*

Recall that an element $b \in \tilde{K}$ is called **purely inseparable over** K, if all roots in $\tilde{K}$ of its minimal polynomial over K are equal. The deduction of the above lemma has shown that b is purely inseparable over K if and only if $b \in K^{1/p^\infty}$. Hence, the lemma shows that an extension of K is purely inseparable if and only if it is generated by elements which are purely inseparable over K. If b is purely inseparable of degree p^m over K, then $b^{p^{m-1}}$ is purely inseparable of degree p over K, and b is purely inseparable of degree p^{m-1} over $K(b^{p^{m-1}})$. By induction, we obtain

Corollary 6.29 *Every finite purely inseparable extension is a tower of purely inseparable extensions of degree p.*

The maximal purely inseparable extension of an algebraic extension $L|K$ will be denoted by $(L|K)^{\text{ins}}$. By what we have shown, $K^{1/p^\infty} = (\tilde{K}|K)^{\text{ins}}$ and $(L|K)^{\text{ins}} = L \cap K^{1/p^\infty}$.

Let $L|K$ be an algebraic extension, $b \in \tilde{K}$ with minimal polynomial f over K, and let g be as in Lemma 6.26. If f is not separable over K, then the degree of g is smaller than that of f, which shows that $K(b)|K$ is not linearly disjoint from $K^{1/p^\infty}|K$. Therefore, if an algebraic extension $L|K$ is linearly disjoint from $K^{1/p^\infty}|K$, then each of its simple subextensions is separable and hence, $L|K$ is itself separable. To show the converse, assume that $K(b)|K$ is separable, that is, $K(b)$ has $[K(b) : K]$ distinct embeddings in $\tilde{K}$ over K. They extend to $[K(b) : K]$ distinct embeddings of $K(b).K^{1/p^\infty} = K^{1/p^\infty}(b)$ in $\tilde{K} = \widetilde{K^{1/p^\infty}}$ over K. But these embeddings are in fact embeddings over K^{1/p^∞} since they must be the identity on K^{1/p^∞}. This proves that $[K^{1/p^\infty}(b) : K^{1/p^\infty}] = [K(b) : K]$, showing that $K(b)$ and K^{1/p^∞} are K-linearly disjoint. In view of Lemma 6.27, we conclude:

Lemma 6.30 *Let $L|K$ be an algebraic extension. Then $L|K$ is separable if and only if it is linearly disjoint from $K^{1/p^\infty}|K$, and this holds if and only if $L|K$ is linearly disjoint from every purely inseparable extension of K.*

The degree $[K^{1/p} : K]$ of the purely inseparable extension $K^{1/p}|K$ is called the **p-degree of K**. It is equal to 1 if and only if K is perfect. Since φ is an endomorphism of $K^{1/p}$ with image K and an endomorphism of K with image K^p, the p-degree of K is equal to $[K : K^p]$. A basis of $K|K^p$ is called a **p-basis of K**. Note that φ^n sends a p-basis of K^{1/p^n} onto a p-basis of K, and it follows that the p-degree of K^{1/p^n} is equal to that of K.

Remark 6.31 Among model theoretic algebraists, the p-degree is also called **Ershov-invariant**. In the literature, it is usual that n is called the p-degree if $[K : K^p] = p^n$. This is the additive form of the p-degree. But for our purposes, the multiplicative notation has turned out to be more useful.

Lemma 6.32 *If $L|K$ is a separable algebraic extension, then $L^{1/p} = L.K^{1/p}$, and the p-degree of L is equal to that of K. If $L|K$ is an arbitrary finite extension, then again, the p-degree of L is equal to that of K.*

Proof: Let $\mathcal{B}$ be a basis of L over K. We know already that $L^{1/p} = K^{1/p}(b^{1/p} \mid b \in \mathcal{B})$.

Assume that $L|K$ is separable. Then for every $b \in \mathcal{B}$, we have $K(b) = K(b^p)$ since otherwise, the separable extension $K(b)|K$ would contain a nontrivial purely inseparable extension $K(b)|K(b^p)$, which is impossible. It follows that $L = K(\mathcal{B}) = K(b^p \mid b \in \mathcal{B})$, which gives $L^{1/p} = K^{1/p}(\mathcal{B}) = L.K^{1/p}$. Since L and $K^{1/p}$ are K-linearly disjoint, $L|K$ being separable, it now follows that $[L^{1/p} : L] = [K^{1/p} : K]$.

To prove our last assertion, assume now that $L|K$ is an arbitrary finite extension. We have that $[L^{1/p} : K^{1/p}][K^{1/p} : K] = [L^{1/p} : K] = [L^{1/p} : L][L : K]$. The Frobenius endomorphism sends $L^{1/p}$ onto L and $K^{1/p}$ onto K. Thus, $[L^{1/p} : K^{1/p}] = [L : K] < \infty$. If $[K^{1/p} : K]$ is finite, then this yields that $[K^{1/p} : K] = [L^{1/p} : L]$. If $[K^{1/p} : K]$ is infinite, then so is $[L^{1/p} : L]$, so both are equal to p^∞. □

Lemma 6.33 *If $L|K$ is an algebraic extension, then $[L : L^p] \leq [K : K^p]$.*

Proof: In view of the foregoing lemma, it suffices to prove our assertion in the case of a purely inseparable extension $L|K$. Assume first that both $[L : L^p]$ and $[K : K^p]$ are infinite cardinals. Then $[L : L^p]$ is equal to $[L^{1/p^\infty} : L]$ since $L^{1/p^\infty} = \bigcup_{I=1}^{\infty} L^{1/p^i}$ and $[L^{1/p^i} : L^{1/p^{i-1}}] = [L : L^p]$ for all i. The same holds for $[K : K^p]$, and in view of $L^{1/p^\infty} = K^{1/p^\infty}$, we obtain that $[L : L^p] = [L^{1/p^\infty} : L] \leq [K^{1/p^\infty} : K] = [K : K^p]$.

If $[K : K^p]$ is infinite and $[L : L^p]$ is finite, then there is nothing to prove. Now assume that $[K : K^p]$ is finite. Let $b_1, \ldots, b_n \in L^{1/p}$ be L-linearly independent. We set $E := K(b_1^p, \ldots, b_n^p) \subset L$. Then the elements $b_1, \ldots, b_n$ are also E-linearly independent, and they lie in $E^{1/p}$. But they are also algebraic over K and thus, $E|K$ is a finite extension. By the foregoing lemma, we find that $n \leq [K : K^p]$. This proves that $[L : L^p] \leq [K : K^p]$. □

If K is not perfect, then it can happen that the p-degree drops under infinite inseparable algebraic extensions, as the extension $K^{1/p^\infty}|K$ shows.

Let us now discuss the behaviour of the p-degree under transcendental extensions. Let t_i, $i \in I$, be K-algebraically independent. Then also the elements $t_i^{p^m}$, $i \in I$, are K-algebraically independent, for every m. But also the elements t_i^{1/p^m}, $i \in I$, are K-algebraically independent, for every m. (Indeed, a nontrivial algebraic dependence relation of them could be raised to the p^m-th power to obtain a nontrivial algebraic dependence relation of the elements t_i.)

The finite products $\prod_{i\in I} t_i^{\nu_i}$ with $0 \leq \nu_i < p$ form a basis of the extension

$$K(t_i \mid i \in I) \,|\, K(t_i^p \mid i \in I) \,.$$

This is seen as follows. Since the elements $1, t_j, t_j^2, \ldots, t_j^{p-1}$ form a basis of the extension $K(t_i^p \mid i \in I)(t_j)|K(t_i^p \mid i \in I)$ for every $j \in I$, the products $\prod_{i\in I} t_i^{\nu_i}$ generate the above extension. If they would not be $K(t_i^p \mid i \in I)$-linearly disjoint, then there would be a nontrivial $K(t_i^p \mid i \in I)$-linear combination of them which equates to zero. Multiply by the common denominator of the coefficients. Note that in this denominator, every appearing t_i will appear to the p^m-th power for some integer $m > 0$. This yields that the result of the multiplication is a *nontrivial* K-linear combination of distinct finite products $\prod_{i\in I} t_i^{\mu_i}$ with $\mu_i \geq 0$, which equates to zero. But this contradicts our hypothesis that the t_i be K-algebraically independent. This contradiction establishes our assertion. If I has n elements, then $[K(t_i \mid i \in I) : K(t_i^p \mid i \in I)] = p^n$, and if I is infinite, then this degree is p^∞.

As we have noted already, $K(t_i \mid i \in I)^p = K^p(t_i^p \mid i \in I)$. We find that $[K(t_i \mid i \in I) : K(t_i \mid i \in I)^p] = [K(t_i \mid i \in I) : K(t_i^p \mid i \in I)] \cdot [K(t_i^p \mid i \in I) : K^p(t_i^p \mid i \in I)]$. Since the elements t_i^p are K-algebraically independent, Lemma 6.17 shows that $K^p(t_i^p \mid i \in I)$ and K are K^p-linearly disjoint. Hence, $[K(t_i^p \mid i \in I) : K^p(t_i^p \mid i \in I)]$ is equal to $[K : K^p]$, the p-degree of K. We thus find that the p-degree of a rational function field $K(t_1, \ldots, t_n)$ is equal to $[K : K^p] \cdot p^n$, and that the p-degree of $K(t_i \mid i \in I)$ is p^∞ if I is infinite. By virtue of Lemma 6.32 it follows that the p-degree of an algebraic function field $F|K$ in n variables is $[K : K^p] \cdot p^n$.

If K is perfect, then the finite products $\prod_{i\in I} t_i^{\nu_i}$, $0 \leq \nu_i < p$, form a p-basis of the rational function field $K(t_i \mid i \in I)$. For example, $1, t, \ldots, t^{p-1}$ is a p-basis of $\mathbb{F}_p(t)$ if t is transcendental over $\mathbb{F}_p$. If a field L has this p-basis, then we can write

$$L = L^p \oplus tL^p \oplus \ldots \oplus t^{p-1}L^p$$

(as an L^p-vector space).

Let again $L|K$ be an arbitrary algebraic extension and $b \in L$ with minimal polynomial f over K. Let m and $h \in K[X]$ be as in Lemma 6.26, such that $f(X) = h(X^{p^m})$ and that h is irreducible and separable over K. Then $K(b^{p^m})|K$ is a separable subextension of $L|K$, and thus lies in $(L|K)^{\mathrm{sep}}$. On the other hand, $K(b)|K(b^{p^m})$ is purely inseparable. Consequently, $L|(L|K)^{\mathrm{sep}}$ *is a purely inseparable extension.* With the help of this fact, we can deduce a lemma which is very similar to Lemma 6.14:

Lemma 6.34 *Let $L|K$ a normal and $F|K$ an arbitrary algebraic field extension. Then L and F are K-linearly disjoint if and only if $L \cap F = K$. More precisely, L and F are $(L \cap F)$-linearly disjoint.*

Proof: Set $L_s := (L|K)^s$. Then $L_s|K$ is a Galois extension. Hence by Lemma 6.14, L_s and F are $(L_s \cap F)$-linearly disjoint. Since $F|K$ is separable, the same holds for $F.L_s|L_s$. Since $L|L_s$ is purely inseparable, it follows from Lemma 6.30 that L and $F.L_s$ are L_s-linearly disjoint. Hence by Lemma 6.12, L and F are $(L_s \cap F)$-linearly disjoint. Finally, observe that $L_s \cap F = L \cap F$. Indeed, since $F|K$ is separable, also $L \cap F \,|\, K$ is separable, which shows that $L \cap F$ is contained in L_s. □

We have already defined the **separable degree** $[L : K]_{\rm sep}$ **of** $L|K$ to be the number of distinct embeddings of L in $\tilde{K}$ over K. But every embedding of $(L|K)^{\rm sep}$ has a unique extension to an embedding of L since $L|(L|K)^{\rm sep}$ is purely inseparable. On the other hand, $(L|K)^{\rm sep}|K$ is separable by definition. Hence, we have that $[L : K]_{\rm sep} = [(L|K)^{\rm sep} : K]_{\rm sep} = [(L|K)^{\rm sep} : K]$. Analogously, we define the **inseparable degree of** $L|K$ to be $[L : K]_{\rm ins} := [L : (L|K)^{\rm sep}]$. By virtue of Lemma 6.27, $[L : K]_{\rm ins}$ is a power of the characteristic exponent of K. If infinite, these degrees are to be understood as supernatural numbers. We have that

$$[L : K] = [L : K]_{\rm sep} \cdot [L : K]_{\rm ins} .$$

Assume $E|K$ to be a subextension of $L|K$. Then $E.\,(L|K)^{\rm sep}|E$ is separable, showing that $(L|E)^{\rm sep}$ contains $E.\,(L|K)^{\rm sep}$. On the other hand, L is a purely inseparable extension of $E.(L|K)^{\rm sep}$ since it is already a purely inseparable extension of $(L|K)^{\rm sep}$. This shows that

$$(L|E)^{\rm sep} = E.(L|K)^{\rm sep} .$$

Applying this with $L = \tilde{K}$, we find that for every algebraic extension $E|K$,

$$E^{\rm sep} = E.K^{\rm sep} .$$

Lemma 6.35 *Let $L|K$ be an algebraic extension and $E|K$ a subextension of $L|K$. Then*

$$\begin{aligned} [L : K]_{\rm sep} &= [L : E]_{\rm sep} \cdot [E : K]_{\rm sep} , \\ [L : K]_{\rm ins} &= [L : E]_{\rm ins} \cdot [E : K]_{\rm ins} . \end{aligned}$$

Proof: We abbreviate $K' := (E|K)^{\rm sep}$ and $E' := (L|E)^{\rm sep}$. Set $L' := (L|K')^{\rm sep}$. Observe that $L' = (L|K)^{\rm sep}$ and that $E' = E.L'$. Since $E|K'$ is purely inseparable, it is linearly disjoint from $L'|K'$. Hence, $[L : K]_{\rm sep} = [L' : K] = [L' : K'][K' : K] = [E' : E][K' : K] = [L : E]_{\rm sep}[E : K]_{\rm sep}$, and $[L : K]_{\rm ins} = [L : L'] = [L : E'][E' : L'] = [L : E'][E : K'] = [L : E]_{\rm ins}[E : K]_{\rm ins}$. □

In the following, let $L|K$ be a normal extension. Let $b \in L$ and f its minimal polynomial over K. By Lemma 6.26, we choose m and $g \in K^{1/p^m}[X]$ such that $f = g^{p^m}$ and that g is irreducible and separable over K^{1/p^m}. All roots of g are roots of f. Since $L|K$ was assumed to be normal, all of them lie in L and consequently, all coefficients of g lie in $L \cap K^{1/p^\infty} = (L|K)^{\rm ins}$. This proves that b is separable over $(L|K)^{\rm ins}$. Consequently, *if $L|K$ is a normal extension, then $L|(L|K)^{\rm ins}$ is separable.* Consequently, also the extension $L \,|\, (L|K)^{\rm sep}.\,(L|K)^{\rm ins}$ is separable. On the other hand, $L \,|\, (L|K)^{\rm sep}$ and thus also

$L\,|\,(L|K)^{\rm sep}.(L|K)^{\rm ins}$ is purely inseparable. This shows that $L\,|\,(L|K)^{\rm sep}.(L|K)^{\rm ins}$ must be trivial. We have proved that

$$\begin{aligned} L|K \text{ normal} &\implies L\,|\,(L|K)^{\rm ins} \text{ separable} \\ L\,|\,(L|K)^{\rm ins} \text{ separable} &\implies L = (L|K)^{\rm sep}.(L|K)^{\rm ins} \ . \end{aligned}$$

Lemma 6.36 *Let $L|K$ be an algebraic extension. Assume that $L\,|\,(L|K)^{\rm ins}$ is separable (which is the case if $L|K$ is normal). Then $[L:(L|K)^{\rm ins}] = [L:K]_{\rm sep}$ and $[(L|K)^{\rm ins}:K] = [L:K]_{\rm ins}$.*

Proof: Since $L\,|\,(L|K)^{\rm ins}$ is assumed to be separable, we have that $L = (L|K)^{\rm sep}.(L|K)^{\rm ins}$. The purely inseparable extension $(L|K)^{\rm ins}\,|\,K$ is linearly disjoint from the separable algebraic extension $(L|K)^{\rm sep}\,|\,K$. Hence, we find that $[L:K]_{\rm ins} = [L:(L|K)^{\rm sep}] = [(L|K)^{\rm sep}.(L|K)^{\rm ins}:(L|K)^{\rm sep}] = [(L|K)^{\rm ins}:K]$, and $[L:K]_{\rm sep} = [(L|K)^{\rm sep}:K] = [(L|K)^{\rm sep}.(L|K)^{\rm ins}:(L|K)^{\rm ins}] = [L:(L|K)^{\rm ins}]$. □

Assume $E|K$ to be a subextension of $L|K$. Then $E.(L|K)^{\rm ins}|E$ is purely inseparable, showing that $(L|E)^{\rm ins}$ contains $E.(L|K)^{\rm ins}$. On the other hand, L is a separable extension of $E.(L|K)^{\rm ins}$ since it is already a separable extension of $(L|K)^{\rm ins}$. This shows that

$$(L|E)^{\rm ins} = E.(L|K)^{\rm ins} \ .$$

Applying this with $L = \tilde{K}$, we find that for every algebraic extension $E|K$,

$$E^{1/p^\infty} = E.K^{1/p^\infty} \ .$$

Lemma 6.37 *Let $K'|K$ be an arbitrary algebraic extension and $L'|K'$ a finite extension.*
a) There is a finite extension $L|K$ such that $L' = L.K'$. If $L'|K'$ is separable, then also $L|K$ can be chosen to be separable, and if in addition $K'|K$ is purely inseparable, then $[L:K] = [L':K']$.
b) If $L'|K'$ is purely inseparable, then there is a finite purely inseparable extension $E|K$ such that $L' \subset E.K'$.

Proof: a): We write $L' = K'(b_1,\ldots,b_n)$. Since $K'|K$ is algebraic, all b_i are algebraic over K and thus, $L := K(b_1,\ldots,b_n)$ is a finite extension which satisfies $L' = L.K'$. Now assume that $L'|K'$ is separable. Let $K_s := (K'|K)^{\rm sep}$ and $L_s := (L'|K_s)^{\rm sep}$. Then $(L'|K_s)^{\rm ins} = K'$ because $L'|K'$ is separable. For the same reason, we know that $L' = L_s.(L'|K_s)^{\rm ins} = L_s.K'$. Now we replace L' by L_s and K' by K_s in the above argument. Then every b_i is separable algebraic over K_s and hence also over K. Consequently, L will be a finite separable extension of K which satisfies $L.K' = L.K_s.K' = L_s.K' = L'$. If in addition $K'|K$ is purely inseparable, then by Lemma 6.30, $L|K$ is linearly disjoint from $K'|K$ and thus, $[L:K] = [L.K':K'] = [L':K']$.
b): Let again $L' = K'(b_1,\ldots,b_n)$. If $L'|K'$ is purely inseparable, then by Lemma 6.27, L' is contained in $K'^{1/p^\infty} = K'.K^{1/p^\infty}$. To write $b_1,\ldots,b_n$ as elements of the latter product, we need finitely many elements $c_1,\ldots,c_m \in K^{1/p^\infty}$. Then $E := K(c_1,\ldots,c_m)$ is a finite purely inseparable extension of K which satisfies $L' = K'(b_1,\ldots,b_n) \subset E.K'$. □

Let $L|K$ be a normal extension. The fixed field of $\mathrm{Gal}\,L|K$ contains $(L|K)^{\rm ins}$, and we may thus identify $\mathrm{Gal}\,L|K$ with $\mathrm{Gal}\,L|(L|K)^{\rm ins}$. On the other hand, $L|(L|K)^{\rm ins}$ is separable,

and Theorem 6.10 shows that the fixed field of $\mathrm{Gal}\, L|(L|K)^{\mathrm{ins}}$ is $(L|K)^{\mathrm{ins}}$. It follows that for every subextension $E|K$ of $L|K$, the fixed field of $\mathrm{Gal}\, L|E$ is $(L|E)^{\mathrm{ins}} = E.(L|K)^{\mathrm{ins}}$. From Theorem 6.10, we thus obtain

Theorem 6.38 *Let $L|K$ be a normal extension. The map $E \mapsto \mathrm{Gal}\, L|E$ is a bijection from the set of all subfields of L containing $(L|K)^{\mathrm{ins}}$ onto the set of all closed subgroups of $\mathrm{Gal}\, L|K$. Its inverse is the map $G \mapsto \mathrm{Fix}\,(L, G)$. For this correspondence, the rules (Gal1) – (Gal9) hold, for E, F subfields of L containing $(L|K)^{\mathrm{ins}}$, G, H closed subgroups of $\mathrm{Gal}\, L|K$ and $\sigma \in \mathrm{Gal}\, L|K = \mathrm{Gal}\, L|(L|K)^{\mathrm{ins}}$.*

If E, F are two subfields of L and if E contains $(L|K)^{\mathrm{ins}}$, then also $E.F$ contains $(L|K)^{\mathrm{ins}}$, and $E.F = E.(L|K)^{\mathrm{ins}}.F = (L|E)^{\mathrm{ins}}.(L|F)^{\mathrm{ins}}$. By virtue of (Gal2′), in the context of the foregoing theorem, we thus obtain the following version of (Gal2′):

Lemma 6.39 *Let $L|K$ be a normal extension with subextensions $E|K$ and $F|K$. If E is the fixed field of $\mathrm{Gal}\, L|E$ (or equivalently, if $L|E$ is separable), then $E.F$ is the fixed field of $\mathrm{Gal}\, L|E \cap \mathrm{Gal}\, L|F$.*

Lemma 6.30 gives rise to the following more general definition: An arbitrary extension $L|K$ is called **separable** if it is linearly disjoint from $K^{1/p^\infty}|K$. Hence, *if K is perfect, then every extension of K is separable.* If $L|K$ is even linearly disjoint from $\tilde{K}|K$, then $L|K$ is called **regular**. So *regularity implies separability.* Note that in these definitions, we may use a very natural amalgamation: K^{1/p^∞} and $\tilde{K}$ may both be chosen as subfields of $\tilde{L}$; it is left to the reader to show that the definition does actually not depend on the choice of the amalgamation. If $L|K$ is separable (resp. regular), then so is every subextension of $L|K$. A basic example for regular extensions is given by the following lemma, which follows directly from Lemma 6.17 and the definition of regularity:

Lemma 6.40 *If the elements x_i, $i \in I$, are K-algebraically independent, then the extension $K(x_i \mid i \in I)|K$ is regular.*

Lemma 6.41 *$L|K$ is separable if and only if it is linearly disjoint from $K^{1/p}|K$, and this is the case if and only if L^p is linearly disjoint from $K|K^p$. If $L|K$ is not separable, then there is some finite subextension $E|K$ of $K^{1/p}|K$ (which may be chosen to be linearly disjoint from $L|K$) such that E admits a nontrivial purely inseparable algebraic extension in $L.E$.*

Proof: If $L|K$ is separable, then by definition it is linearly disjoint from $K^{1/p^\infty}|K$ and thus also from $K^{1/p}|K$. For the converse, assume that $L|K$ is not linearly disjoint from $K^{1/p^\infty}|K$. Then there are K-linearly independent elements $x_1, \ldots, x_n \in L$ which are not K^{1/p^∞}-linearly independent. Choose m minimal such that there are $y_1, \ldots, y_n \in K^{1/p^m}$ with $\sum_i x_i y_i = 0$. Then $m \geq 1$, and $x_1^{p^{m-1}}, \ldots, x_n^{p^{m-1}}$ are K-linearly independent. (Otherwise, we would have a nontrivial relation $\sum_i x_i^{p^{m-1}} z_i = 0$ with $z_i \in K$, hence $\sum_i x_i z_i^{1/p^{m-1}} = 0$, contradicting the minimality of m.) But $x_1^{p^{m-1}}, \ldots, x_n^{p^{m-1}}$ are not $K^{1/p}$-linearly independent since $\sum_i x_i^{p^{m-1}} y_i^{p^{m-1}} = 0$ with $y_i^{p^{m-1}} \in K^{1/p}$. This proves that $L|K$ is not linearly disjoint from $K^{1/p}|K$.

The Frobenius is an isomorphism from $L^{1/p}$ onto L and sends the subfield L onto L^p, the subfield $K^{1/p}$ onto K and the subfield K onto $K^{1/p}$. This shows that L is linearly disjoint from $K^{1/p}|K$ if and only if L^p is linearly disjoint from $K|K^p$.

Now assume that $L|K$ is not separable, that is, not linearly disjoint from $K^{1/p}|K$. Then there exists a finite subextension $E'|K$ of $K^{1/p}|K$ which is not linearly disjoint from $L|K$. Since $E'|K$ is a finite purely inseparable extension, there are intermediate fields $K = E_0 \subset E_1 \subset \ldots \subset E_r = E'$ such that every $E_{i+1}|E_i$ is purely inseparable of degree p. Let $i \geq 0$ be the maximal index such that E_i and L are K-linearly disjoint. Then $i < r$, and it follows that E_{i+1} and $L.E_i$ are not E_i-linearly disjoint. But this means that $E_{i+1} \subset L.E_i$ because $E_{i+1}|E_i$ is of degree p. Setting $E := E_i$, we obtain that $E|K$ is linearly disjoint from $L|K$ and admits a purely inseparable extension of degree p in $L.E$. □

The p-degree does not drop under separable extensions:

Corollary 6.42 *Let K be a field of characteristic $p > 0$ and $L|K$ a separable extension. Then $[L : L^p] \geq [K : K^p]$.*

Proof: Since $L|K$ is assumed to be separable, it is linearly disjoint from $K^{1/p}|K$. Hence, $[L.K^{1/p} : L] = [K^{1/p} : K]$. Since $L^{1/p}$ contains $K^{1/p}$, we obtain that $[L^{1/p} : L] \geq [L.K^{1/p} : L] = [K^{1/p} : K]$. □

By virtue of Lemma 6.12, where we set $F = K^{1/p^\infty}$ resp. $F = \tilde{K}$, we obtain:

Lemma 6.43 *If $L|K'$ and $K'|K$ are separable resp. regular extensions, then $L|K$ is separable resp. regular.*

The converse is not true: If x is transcendental over K and $\operatorname{char} K = p > 0$, then $K(x)|K$ and $K(x^p)|K$ are regular, but $K(x)|K(x^p)$ is not separable.

Let $L|K$ be an arbitrary field extension. If it admits a transcendence basis $\mathcal{T}$ such that $L|K(\mathcal{T})$ is a separable-algebraic extension, then we say that $L|K$ is **separably generated**, and $\mathcal{T}$ is called a **separating transcendence basis**. If $L|K$ is separably generated, then it is separable; this follows from Lemma 6.40 and Lemma 6.43. For finitely generated extensions, the converse is also true:

Lemma 6.44 *If $L|K$ is a finitely generated separable extension, then a separating transcendence basis can be selected from every given set of generators.*

Proof: Let $x_1, \ldots, x_n$ be generators for L over K. Without loss of generality, we may assume that $x_1, \ldots, x_m$ are K-algebraically independent, and that $x_{m+1}, \ldots, x_n$ are algebraic over $K(x_1, \ldots, x_m)$. If $m = n$, there is nothing to prove. Otherwise, let $f(X_1, \ldots, X_{m+1})$ be a polynomial of lowest degree with coefficients in K such that $f(x_1, \ldots, x_{m+1}) = 0$. Then f is irreducible. Assume we could write $f(X_1, \ldots, X_{m+1}) = g(X_1, \ldots, X_{m+1})^p$ where g has coefficients in $K^{1/p}$. Then $g(x_1, \ldots, x_{m+1}) = 0$ which shows that the set of monomials in $x_1, \ldots, x_{m+1}$ of degree $\leq \deg g$ is $K^{1/p}$-linearly dependent. Since $L|K$ is assumed to be separable and thus, to be linearly disjoint from $K^{1/p}|K$, it follows that this set of monomials is already K-linearly dependent. But this means that there exists a polynomial $f_1(X_1, \ldots, X_{m+1})$ of degree $\leq \deg g < \deg f$ with coefficients in K such that $f(x_1, \ldots, x_{m+1}) = 0$, a contradiction to the minimality of f. Thus, we may assume without loss of generality that X_1 does not appear in every monomial of f to a power which is divisible by p. In view of the irreducibility of f, this yields that $f(X_1, x_2, \ldots, x_{m+1})$ is a separable minimal polynomial for x_1 over $K(x_2, \ldots, x_{m+1})$. Hence, $K(x_1, \ldots, x_{m+1})$ is separable over $K(x_2, \ldots, x_{m+1})$ and thus also over $K(x_2, \ldots, x_n)$. If $x_2, \ldots, x_n$ do not already form a transcendence basis, then we iterate our procedure until we have found a separating transcendence basis among these elements. □

Lemma 6.45 *Let $L|K$ be an arbitrary extension with K relatively algebraically closed in L. Let L and F be K-algebraically disjoint in a common extension field Ω. The relative algebraic closure of F in $L.F$ is always a purely inseparable (possibly trivial) extension of F. If $F|K$ is separable, then $L|K$ and $F|K$ are linearly disjoint, and F is relatively algebraically closed in $L.F$.*

Proof: Let $\mathcal{T}$ be a transcendence basis of $F|K$. Since K is assumed to be relatively algebraically closed in L, Lemma 6.19 shows that $K(\mathcal{T})$ is relatively algebraically closed in $L(\mathcal{T})$. To prove our first assertion, we may thus assume that $F|K$ is algebraic. Abbreviate $E = (F|K)^{\rm sep}$. Since K is relatively algebraically closed in L, we infer from Lemma 6.13 that L and $K^{\rm sep}$ are K-linearly disjoint. By Lemma 6.12 it follows that $L.E$ and $E^{\rm sep} = K^{\rm sep}$ are E-linearly disjoint. Since F is a purely inseparable extension of E, also the extension $L.F|L.E$ is purely inseparable and thus linearly disjoint from the separable extension $L.E^{\rm sep}|L.E$. Again by Lemma 6.12, we find that $E^{\rm sep}|E$ is linearly disjoint from $L.F|E$. Applying the same lemma a third time, we conclude that $L.F$ and $F.E^{\rm sep} = F^{\rm sep}$ are F-linearly disjoint and thus, $L.F \cap F^{\rm sep} = F$. That is, the relative algebraic closure of F in $L.F$ is a purely inseparable extension of F.

To prove the second assertion, assume that $F|K$ is separable. It suffices to show for every finitely generated subextension $E|K$ of $F|K$ that $L|K$ is linearly disjoint from $E|K$ and that E is relatively algebraically closed in $L.E$. By Lemma 6.44, $E|K$ admits a separating transcendence basis $\mathcal{T}'$. Since $F|K$ is assumed to be algebraically disjoint from $L|K$, we know that the elements of $\mathcal{T}'$ are also L-algebraically independent. By Lemma 6.17, $L|K$ is linearly disjoint from $K(\mathcal{T}')|K$, and by Lemma 6.19, $K(\mathcal{T}')$ is relatively algebraically closed in $L(\mathcal{T}')$. It follows from Lemma 6.13 that $L(\mathcal{T}')|K(\mathcal{T}')$ is linearly disjoint from the separable algebraic extension $E|K(\mathcal{T}')$. By Lemma 6.12, $L|K$ is linearly disjoint from $E|K$, and by Lemma 6.13, E is relatively algebraically closed in $L.E = L(\mathcal{T}).E$. □

Lemma 6.46 *Let $L|K$ be a separable extension and $F|K$ an arbitrary extension which is algebraically disjoint from $L|K$ in a common extension field Ω. Then $L.F|F$ is a separable extension. If in addition, K is relatively algebraically closed in L, then F is relatively algebraically closed in $L.F$.*

Proof: It suffices to prove the assertions under the assumption that $L|K$ be finitely generated. Then by Lemma 6.44, there exists a separating transcendence basis $\mathcal{T}$ of $L|K$. Since $F|K$ is algebraically disjoint from $L|K$, we know that $\mathcal{T}$ is also a transcendence basis of $L.F|F$. Since $L|K(\mathcal{T})$ is separable algebraic, the same holds for $L.F|F(\mathcal{T})$, proving that $L.F|F$ is separably generated and thus separable.

Assume in addition that K is relatively algebraically closed in L. Then by the foregoing lemma, the relative algebraic closure of F in $L.F$ is purely inseparable. But we have just proved that $L.F|F$ is separable. Hence, F is relatively algebraically closed in $L.F$. □

With the help of Lemma 6.43, we derive from the foregoing lemma:

Corollary 6.47 *If $L|K$ and $F|K$ are separable and algebraically disjoint in a common extension field Ω, then $L.F|K$ is separable. In particular, if $L|K$ is separable algebraic and $F|K$ is an arbitrary separable extension, then $L.F|K$ is separable.*

Let us have a closer look at regular extensions. A characterization of regularity reads as follows:

Lemma 6.48 *An extension $L|K$ is regular if and only if it is separable and K is relatively algebraically closed in L. If $L|K$ is not regular, then there exists a finite subextension $E|K$ of $K^{1/p}|K$ (which may be chosen to be linearly disjoint from $L|K$) such that E is not relatively algebraically closed in $L.E$.*

Proof: We have already remarked that regularity implies separability. Also, $L|K$ can not be regular if K is not relatively algebraically closed in L. Now assume that $L|K$ is separable and K is relatively algebraically closed in L. By Lemma 6.46, $L.K^{\rm sep}|K^{\rm sep}$ is separable. It is thus linearly disjoint from the purely inseparable extension $\tilde{K}|K^{\rm sep}$. On the other hand, we know from Lemma 6.13 that $L|K$ is linearly disjoint from $K^{\rm sep}|K$. Hence by Lemma 6.12, $L|K$ is linearly disjoint from $\tilde{K}|K$.

Now assume that $L|K$ is not regular. If K is not relatively algebraically closed in L, then we are done. Otherwise, $L|K$ is not separable (by what we have just proved), and our assertion follows from Lemma 6.41. □

Lemma 6.49 *Let $F|K$ be a regular extension and $L|K$ an arbitrary extension which is algebraically disjoint from $F|K$ in a common extension field Ω. Then $L|K$ is linearly disjoint from $F|K$, and $F.L|L$ is regular.*

Proof: By definition, $F|K$ is linearly disjoint from $\tilde{K}|K$. Hence by Lemma 6.12, $F.K'|K'$ is linearly disjoint from $\tilde{K'}|K'$ for every algebraic extension $K'|K$. That is, $F.K'|K'$ is regular. We take K' to be the relative algebraic closure of K in L. Since $F.K'|K'$ is separable by the foregoing lemma and since K' is relatively algebraically closed in L, we can infer from Lemma 6.46 that $F.K'|K'$ is linearly disjoint from $L|K'$. Since also $K'|K$ is linearly disjoint from $F|K$, we can deduce from Lemma 6.12 that $L|K$ is linearly disjoint from $F|K$.

The whole argument works equally well if we replace L by $\tilde{L}$. Indeed, if $L|K$ is algebraically disjoint from $F|K$, then so is $\tilde{L}|K$ since every transcendence basis of $L|K$ is a transcendence basis of $\tilde{L}|K$. We obtain that $F|K$ is linearly disjoint from $\tilde{L}|K$. By virtue of Lemma 6.12, this shows that $F.L|L$ is linearly disjoint from $\tilde{L}|L$, that is, $F.L|L$ is regular. □

With the help of Lemma 6.43, we derive from the foregoing lemma:

Corollary 6.50 *If $L|K$ and $F|K$ are regular and algebraically disjoint in a common extension field Ω, then $L.F|K$ is regular.*

From the above lemma, we can also deduce the following analogue to Lemma 6.19:

Corollary 6.51 *Let $F|K$ be a regular extension and $\mathcal{T}$ a set of elements which are F-algebraically independent (in some field extension of F). Then $F(\mathcal{T})|K(\mathcal{T})$ is regular.*

For the conclusion of this section, let note that there exist inseparable extensions $L|K$ where K is relatively algebraically closed in L:

Example 6.52 Let t, x, y be algebraically independent over $\mathbb{F}_p$ and set $L := \mathbb{F}_p(t, x, y)$. Define

$$s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t, s) \; .$$

Then K is relatively algebraically closed in L. To show this, let $b \in L$ be algebraic over K. Note that x is algebraic over K. Indeed, otherwise x and thus also y would be algebraic over K, so that $\operatorname{trdeg} L|\mathbb{F}_p = \operatorname{trdeg} K|\mathbb{F}_p \leq 2$ in contradiction to our assumption that t, x, y be algebraically independent over $\mathbb{F}_p$. The element b^p is algebraic over K and lies in $L^p = \mathbb{F}_p(t^p, x^p, y^p)$ and thus also in $K(x) = \mathbb{F}_p(t, x, y^p)$. Since x is transcendental over K, it follows by Lemma 6.17 that $b^p \in K$. Consequently, $b \in K^{1/p}$. Since $\mathbb{F}_p$ is perfect, we have that $K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p})$. Write $b = r_{p-1}s^{p-1} + \ldots + r_1 s + r_0$ with $r_i \in \mathbb{F}_p(t^{1/p}, s) = K(t^{1/p})$. By the definition of s,

$$b = r_{p-1}x^{p-1} + \ldots + r_1 x^{1/p} \; + \ldots + \; t^{(p-1)/p} r_{p-1} y^{p-1} + \ldots + t^{1/p} r_1 y \; + \; r_0$$

(in the middle, we have omitted the summands in which both x and y appear). Since x, y are algebraically independent over $\mathbb{F}_p$, the p-degree of K is p^2, and the elements $x^i y^j$, $0 \leq i < p$, $0 \leq j < p$, form a basis of $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$. Since t and $t^{1/p}$ are transcendental over $\mathbb{F}_p(x^p, y^p)$, we know that $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$ is linearly disjoint from $\mathbb{F}_p(t, x^p, y^p)|\mathbb{F}_p(x^p, y^p)$ and from $\mathbb{F}_p(t^{1/p}, x^p, y^p)|\mathbb{F}_p(x^p, y^p)$. This shows that the elements $x^i y^j$, $0 \leq i < p$, $0 \leq j < p$, form a basis of $L|\mathbb{F}_p(t, x^p, y^p)$ and are still $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linearly independent. Hence, b can also be written as a linear combination of these elements with coefficients in $\mathbb{F}_p(t, x^p, y^p)$, and this must coincide with the above $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linear combination which represents b. That is, all coefficients r_i *and* $t^{i/p} r_i$, $1 \leq i < p$, are in $\mathbb{F}_p(t, x^p, y^p)$. This is impossible unless they are zero. Since also r_0 has to lie in $\mathbb{F}_p(t, x^p, y^p)$, it follows that $b = r_0 \in K(t^{1/p})$. Assume that $b \notin K$. Then $[K(b) : K] = p$ and thus, $K(b) = K(t^{1/p})$ since also $[K(t^{1/p}) : K] = p$. But then $t^{1/p} \in K(b) \subset L$, a contradiction. This proves that K is relatively algebraically closed in L.

On the other hand, $t^{1/p} = y^{-1}(s^{1/p} - x) \in L(s^{1/p})$. Hence, $[L.K^{1/p} : L] = [L(s^{1/p}) : L] \leq p < p^2 = [K^{1/p} : K]$, that is, $L|K$ is not linearly disjoint from $K^{1/p}|K$ and thus not separable. Although being finitely generated, $L|K$ is consequently not separably generated; in particular, it is not a rational function field. At the same time, we have seen that $K(s^{1/p})$ admits a nontrivial purely inseparable algebraic extension in $L(s^{1/p})$ (namely, $K^{1/p}$). On the other hand, $K(s^{1/p})$ and L are K-linearly disjoint because $s^{1/p} \notin L$.

Let us also observe that s can be replaced by any other element of K which does not lie in K^p. Indeed, we know that $a^{1/p} \notin L$ since K is relatively algebraically closed in L. Hence, $[L(a^{1/p}) : L] = p = [L.K^{1/p} : L]$, showing that $L(a^{1/p}) = L.K^{1/p}$ and that $K(a^{1/p})$ admits the nontrivial purely inseparable algebraic extension $K^{1/p}$ in $L(a^{1/p})$. $\diamondsuit$

Exercise 6.4 *Let $L|K$ be separable. Given K-linearly independent elements $x_1, \ldots, x_n \in L$, prove that also $x_1^{p^\mu}, \ldots, x_n^{p^\mu}$ are K^{p^ν}-linearly independent, for every all integers μ, ν.*

Example 6.53 Show that if k is relatively separable-algebraically closed in K, then k^{1/p^∞} is relatively algebraically closed in K^{1/p^∞}. (Hint: Otherwise, $K^{1/p^\infty}|k$ would admit a proper separable algebraic subextension. But this is lineaarly disjoint from $K|k$ and thus also from $K^{1/p^\infty}|k$, contradiction.) Is the converse also true? $\diamondsuit$

6.8 (Pro-) p-groups and p'-groups

Let p be a prime. A (not necessarily finite) torsion group in which the orders of all elements are a power of p is called a **p-group**. For example, an abelian torsion group A is a p-group if and only if $\#A$ divides p^∞. A torsion group will be called a **p'-group** (read: "*p-prime*-group") if the orders of all elements are prime to p. A profinite group is called a **pro-p-group** if it is the inverse limit of an inverse system of p-groups (or equivalently, if all of its finite quotients are p-groups), and it is called a **pro-p'-group** if it is the inverse limit of an inverse system of p'-groups. A finite pro-p-group is a p-group (and similarly for p'). The property of being a pro-p-group (resp. a pro-p'-group) is inherited by closed subgroups and quotients by closed subgroups. A field extension is called a **p-extension** if it is Galois and its Galois group is a pro-p-group, and it is called a **p'-extension** if it is Galois and its Galois group is a pro-p'-group. A field extension is a p-extension if and only if each of its finite subextensions is a p-extension (and similarly for p').

Let us now collect some statements about p-groups.

Lemma 6.54 *If G is a finite p-group and H is a proper subgroup of G, then there exists a subgroup G_1 of G such that $H \lhd G_1$ and $(G_1 : H) = p$.*

Proof: Note that $G \neq \{1\}$ by the assumption on H. We claim that the center $C(G) = \{g \in G \mid \forall h \in G : \ hg = gh\}$ is nontrivial. Assume the contrary. Then the centralizer $C_G(h) = \{g \in G \mid hg = gh\}$ would be a proper subgroup of G for every element $h \in G \backslash \{1\}$. This would mean that 1 is the only element of G whose orbit under conjugation has a length which is not divisible by p; but this is impossible in a group whose order is a power of p (because the order of a finite group G is the sum of the lengths of the orbits in G). This contradiction proves our claim.

Assume that $C(G) \subset H$. Then by induction on the order of G, the assertion of our lemma follows for the groups $G/C(G)$ and $H/C(G)$. That is, there is a subgroup G_1 of G containing $C(G)$ such that $H/C(G) \lhd G_1/C(G)$ and $(G_1/C(G) : H/C(G)) = p$. It follows that $H \lhd G_1$ and $(G_1 : H) = p$.

Finally, assume that $C(G) \not\subset H$. Then there exists some $g \in C(G) \setminus H$ such that $g^p \in H$. Then $G_1 := \langle g \rangle H$ is the required subgroup of G since the assertions $H \lhd G_1$ and $(G_1 : H) = p$ follow from the fact that g commutes with H. □

The **Frattini subgroup** of an arbitrary finite group G is defined to be the intersection of all maximal proper subgroups of G and is denoted by $\Phi(G)$. A group G is called **elementary-abelian** if it is of the form $\mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_n\mathbb{Z}$ for (not necessarily distinct) prime numbers $p_1, \ldots, p_n$. Consequently, an elementary-abelian p-group is a finite product of copies of $\mathbb{Z}/p\mathbb{Z}$, that is, a finite dimensional $\mathbb{F}_p$-vector space.

Theorem 6.55 *Let G be any finite p-group.*
a) If H is a maximal proper subgroup of G, then $H \lhd G$ and $(G : H) = p$. Consequently, $G/\Phi(G)$ is elementary-abelian.
b) For every subgroup $H \subset G$ there exists a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $H_{i-1} \lhd H_i$ and $(H_i : H_{i-1}) = p$ for $i = 1, \ldots, n$. In particular, every finite p-group is solvable.

Proof: a) Let H be a maximal proper subgroup of G. We choose G_1 according to the foregoing lemma. Then $G_1 = G$ by the maximality of H. Hence, $H \lhd G$ and $(G : H) = p$.

Now let $H_1, \ldots, H_n$ be all maximal proper subgroups of G. By what we have shown, they are normal subgroups of G. Now we use the isomorphism (6.2), where N_1, N_2 are normal subgroups of G. By induction on the number of normal subgroups, we find

$$G/\Phi(G) = G/(H_1 \cap \ldots \cap H_n) \cong G/H_1 \times \ldots \times G/H_n .$$

By what we have already shown, the latter is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \ldots \times \mathbb{Z}/p\mathbb{Z}$, which is an elementary-abelian group.

b) The first assertion follows from the foregoing lemma by induction on $(G : H)$. The second assertion follows from the first by taking $H = \{1\}$. □

A profinite group is called **prosolvable** if it is the limit of an inverse system of finite solvable groups. From the foregoing theorem, we obtain the following analogue assertions for pro-p-groups.

Corollary 6.56 *Let G be any pro-p-group.*
a) If H is a maximal proper closed subgroup of G, then H is an open normal subgroup of G and $(G : H) = p$. Consequently, $G/\Phi(G)$ is an $\mathbb{F}_p$-vector space.
b) G is prosolvable. For every open subgroup $H \subset G$ there exists a chain of open subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $H_{i-1} \lhd H_i$ and $(H_i : H_{i-1}) = p$ for $i = 1, \ldots, n$.

Proof: a): The closed subgroup H is contained in some proper open subgroup. Hence, if it is a maximal proper closed subgroup of G, then it is open. It contains an open normal subgroup N. By the foregoing theorem, the finite p-group G/N admits a normal subgroup of index p. Its foreimage in G is a normal subgroup of index p in G, containing H. By the maximality of H, it is equal to H.

As in the finite case, one shows that there is an epimorphism from G onto $\prod_i G/H_i$, where H_i runs through all maximal proper closed subgroups, and that its kernel is $\Phi(G)$. Since every G/H_i is isomorphic to $\mathbb{F}_p$, the product is an $\mathbb{F}_p$-vector space.

b): By the foregoing theorem, every finite p-group is solvable. So by definition, a pro-p-group is prosolvable. The second assertion is shown similarly to the first assertion of a). □

Corollary 6.57 *Every finite subextension of a p-extension of a field K is a tower of Galois extensions of degree p (which are Artin-Schreier extensions if p is the characteristic of K).*

Proof: Let $L|K$ be a finite subextension of the p-extension $L_1|K$. Then L_1 contains the normal hull L_0 of L over K. Hence, $L_0|K$ is a finite p-extension. Hence, $G = \mathrm{Gal}\, L_0|K$ is a finite p-group. We apply part b) of Theorem 6.55 to its subgroup $H = \mathrm{Gal}\, L_0|L$ and set $K_i = \mathrm{Fix}\,(L_0, H_{n-i})$, $i = 0, \ldots, n$. Since $H_{n-i} \lhd H_{n-i+1}$ and $(H_{n-i+1} : H_{n-i}) = p$, we have that $K_i|K_{i-1}$ is a Galois extension of degree p, for every $i = 1, \ldots, n$.

We have already noted that a Galois extension of degree p of a field of characteristic p is an Artin-Schreier extension; see Corollary 11.27. □

A subgroup H of a profinite group G is called a **p-Sylow group of** G if it is a closed pro-p-subgroup of G such that p does not divide $(G : H)$. If H is a p-Sylow group of G, then for every open normal subgroup N of G we find that $H.N/N$ is a p-Sylow group of the finite group G/N. Indeed, $H.N/N$ is a p-group, and since p does not divide $(G : H)$, it does not divide $(G : H.N) = (G/N : H.N/N)$. The following is a generalization of the Sylow Theorems of finite group theory:

Theorem 6.58 *Let p be a prime and G a profinite group. Then for every pro-p-subgroup G_0 of G, there exists a p-Sylow group H of G containing G_0. All p-Sylow groups of G are conjugate.*

Proof: Let N_i, $i \in I$, be the family of all open normal subgroups of G and recall the discussion preceding to Lemma 6.8. For every $i \in I$, the subgroup $G_0.N_i/N_i$ of the finite group G/N_i is a p-group. By finite group theory, the set of p-Sylow groups of G/N_i containing $G_0.N_i/N_i$ is finite and nonempty; denote this set by $\mathcal{S}_i$. If $i, j \in I$ such that $N_j \subset N_i$, then by the Sylow Theorems of finite group theory, the canonical epimorphism $\eta_{ji} : G/N_j \to G/N_i$ maps a pro-p-group of G/N_j containing $G_0.N_j/N_j$ onto a pro-p-group of G/N_i containing $G_0.N_i/N_i$. Thus, η_{ji} induces a map $\pi_{ji}^{\mathcal{S}}$ from $\mathcal{S}_j$ into $\mathcal{S}_i$. The reader may verify that the maps $\pi_{ji}^{\mathcal{S}}$ satisfy (INV1) and (INV2). Since also I satisfies (INV0) for the partial ordering that we have introduced preceding to Lemma 6.8, Lemma 6.7 shows that the inverse limit over the sets $\mathcal{S}_i$ is nonempty. Let $(H_i)_{i\in I}$ be an element of it. Then via the isomorphism $G \cong \varprojlim G/N_i$, the profinite group $\varprojlim H_i$ is a subgroup of G. Since every H_i is a p-Sylow group containing $G_0.N_i/N_i$, it follows that H is a pro-p-group containing $G_0 \cong \varprojlim G_0.N_i/N_i$. Further, p does not divide $(G/N_i : H_i)$. But $H_i = H.N_i/N_i$ and consequently, $(G/N_i : H_i) = (G/N_i : H.N_i/N_i) = (G : H.N_i)$. Hence, p does not divide $\mathrm{lcm}_i(G : H.N_i) = (G : H)$, showing that H is a p-Sylow group of G.

Now let H and $\mathcal{H}$ be p-Sylow groups of G. Let $i, j \in I$. Then $H.N_i/N_i$ and $\mathcal{H}.N_i/N_i$ are p-Sylow groups of the finite group G/N_i. Hence by the Sylow Theorems of finite group theory, the set $\mathcal{C}_i := \{g_i \in G/N_i \mid g_i(H.N_i/N_i)g_i^{-1} = \mathcal{H}.N_i/N_i\}$ is nonempty, and it is finite. If $N_j \subset N_i$, then the canonical epimorphism η_{ji} induces a map $\pi_{ji}^{\mathcal{C}}$ from $\mathcal{C}_j$ into $\mathcal{C}_i$. The maps $\pi_{ji}^{\mathcal{C}}$ satisfy (INV1) and (INV2). Hence, Lemma 6.7 shows that the inverse limit over the sets $\mathcal{C}_i$ is nonempty. Let $g = (g_i)_{i\in I}$ be an element of it. Via the isomorphism $G \cong \varprojlim G/N_i$, it is in fact an element of G which satisfies $gHg^{-1} = \mathcal{H}$. □

If H is the unique p-Sylow group of the profinite group G, then H is a closed normal subgroup of G, and G/H is a pro-p'-group.

To conclude this section, we introduce a sort of groups which behave very much like Galois groups (and indeed, we will derive a connection to Galois groups while developing the ramification theory in Section 7.8). Given an arbitrary abelian torsion group Γ and a field k of characteristic exponent p, then $\mathrm{Hom}(\Gamma, k^\times)$ denotes the set of all homomorphisms from Γ into the multiplicative group $k^\times$ of k. Given $\chi, \chi' \in \mathrm{Hom}(\Gamma, k^\times)$ then we define the product $\chi \cdot \chi'$ by $\chi \cdot \chi'(g) = \chi(g)\chi'(g)$ for every $g \in \Gamma$. In this way, $\mathrm{Hom}(\Gamma, k^\times)$ becomes a group, called a **p-character group of** Γ. Its elements are called **characters**. Its identity element is the **trivial character** 1 which sends every g to $1 \in k^\times$. The inverse of χ sends every g to $\chi(g)^{-1}$.

If k is algebraically closed, then $\mathrm{Hom}(\Gamma, k^\times)$ is called the **full p-character group of** Γ. Since every element of Γ has finite order by assumption, $\chi(\Gamma)$ is contained in the group of roots of unity of the field k, for every character χ. So we see that the group does actually not depend upon the field k but rather upon the group of roots of unity contained in it. So one may actually replace k by the relative algebraic closure of its prime field in k. In particular, all algebraically closed fields of a fixed characteristic p have the same group of roots of unity, and so the full character group for a given group Γ only depends upon the characteristic of k.

We leave it to the reader to show the following:

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \implies \mathrm{Hom}(\Gamma, k^\times) = \mathrm{Hom}(\Gamma_1, k^\times) \oplus \mathrm{Hom}(\Gamma_2, k^\times) \ . \tag{6.4}$$

Let q be any prime or equal to 1. Every abelian torsion group Γ admits a decomposition $\Gamma = \Gamma_q \oplus \Gamma_{q'}$ where Γ_q is the subgroup of all elements of Γ whose order is a power of q, and $\Gamma_{q'}$ is the subgroup of all elements of Γ whose order is prime to q. Hence, Γ_q is an abelian q-group, and $\Gamma_{q'}$ is an abelian q'-group. We have

$$\mathrm{Hom}(\Gamma, k^\times) = \mathrm{Hom}(\Gamma_q, k^\times) \oplus \mathrm{Hom}(\Gamma_{q'}, k^\times) \ .$$

We know that over a field of characteristic exponent p, the polynomial $X^{p^n} - 1$ splits into linear factors, for every $n \in \mathbb{N}$. Hence, 1 is the only element of order a power of p in the group of roots of unity in a field k of characteristic exponent p. This yields that 1 is the only element of order a power of p in $\mathrm{Hom}(\Gamma, k^\times)$, that is, $\mathrm{Hom}(\Gamma, k^\times)$ is a p'-group and $\mathrm{Hom}(\Gamma_p, k^\times) = \{1\}$. In other words,

$$\mathrm{charexp}\, k = p \implies \mathrm{Hom}(\Gamma, k^\times) = \mathrm{Hom}(\Gamma_{p'}, k^\times) \ . \tag{6.5}$$

Lemma 6.59 *If Γ is finite and k is algebraically closed of characteristic exponent p, then* $\mathrm{Hom}(\Gamma, k^\times) \cong \Gamma_{p'}$.

Proof: By the Main Theorem on Finitely Generated Abelian Groups, we can write Γ as a finite direct sum of cyclic groups, all of them finite since Γ is finite. In view of (6.4) and (6.5), it remains to show that $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, k^\times) \cong \mathbb{Z}/m\mathbb{Z}$ for every natural number which is prime to p. But for every such m, the group $\mu_{k,m}$ of m-th roots of unity contained in the algebraically closed field k is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. On the other hand, if η is a generator of $\mathbb{Z}/m\mathbb{Z}$, then for every m-th root of unity ζ, there is precisely one character $\chi \in \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, k^\times)$ such that $\chi(\eta) = \zeta$. This proves that $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, k^\times) \cong \mu_{k,m} \cong \mathbb{Z}/m\mathbb{Z}$. □

Suppose that Δ is a subgroup of the abelian torsion group Γ. We leave it to the reader to prove the following facts. If $\mathrm{Hom}(\Gamma, k^\times)$ is the full p-character group of Γ, then every character in $\mathrm{Hom}(\Delta, k^\times)$ can be extended to a character in $\mathrm{Hom}(\Gamma, k^\times)$. Consequently, the restriction map $\mathrm{Hom}(\Gamma, k^\times) \to \mathrm{Hom}(\Delta, k^\times)$ is surjective. Its kernel is the subgroup of all characters of Γ which are trivial on Δ; we will denote it by $\mathrm{Hom}_\Delta(\Gamma, k^\times)$. Note that $\mathrm{Hom}_\Delta(\Gamma, k^\times)$ is canonically isomorphic to $\mathrm{Hom}(\Gamma/\Delta, k^\times)$. Since every element of Γ is already contained in some finite subgroup, $\mathrm{Hom}(\Gamma, k^\times)$ is the inverse limit of the inverse system of the finite groups $\mathrm{Hom}(\Delta_i, k^\times)$, where Δ_i runs through all finite subgroups of Γ. Hence, $\mathrm{Hom}(\Gamma, k^\times)$ is a profinite group. The subgroups $\mathrm{Hom}_{\Delta_i}(\Gamma, k^\times)$ form a basis of the neighborhood filter of the trivial character 1 of Γ.

Exercise 6.5 *Let G be a profinite group. Prove:*
a) If $\eta : G \to G_1$ is an epimorphism of profinite groups and if H is a p-Sylow group of G, then $\eta(H)$ is a p-Sylow group of H.
b) $\#G = \prod_q \#H_q$ where q runs through all primes and every H_q is a q-Sylow group of G.
What can be said about the number of p-Sylow groups of G?

Exercise 6.6 *Describe a Galois correspondence of character groups.*

6.9 The Taylor expansion of a polynomial

We will frequently need a Taylor expansion of polynomials which works in fields of arbitrary characteristic. For every $j \in \mathbb{N}$, we have

$$(X+Y)^j = \sum_{i=0}^{n} \binom{j}{i} X^{j-i}Y^i \ .$$

This is also true in fields of characteristic $p > 0$ since the binomial coefficients are natural numbers which then will be taken modulo p. For an arbitrary polynomial $f(X) = c_nX^n + c_{n-1}X^{n-1} + \ldots + c_0$, summation now gives

$$f(X+Y) = \sum_{j=0}^{n}\sum_{i=0}^{j} c_j \binom{j}{i} X^{j-i}Y^i = \sum_{i=0}^{n}\sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i}Y^i \ .$$

Setting

$$f_i(X) := \sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i} = \sum_{j=0}^{n-i} c_{j+i} \binom{j+i}{i} X^j \tag{6.6}$$

which we will call the **i-th derivative of f**, we obtain the **Taylor expansion**

$$f(X+Y) = \sum_{i=0}^{n} f_i(X)Y^i \ . \tag{6.7}$$

All i-th derivatives of f are defined also over fields of positive characteristic, but for certain i they may vanish identically (even if $i \leq n$). Over fields of characteristic 0, we have

$$f_i(X) = \sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i} = \sum_{j=i}^{n} c_j \frac{1}{i!}\frac{d^i}{dX^i}X^j = \frac{1}{i!}f^{(i)}(X)$$

which gives the well-known Taylor identity

$$f(X+Y) = \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(X)Y^i \ .$$

The same identity can be used over fields of characteristic $p > 0$ for $n < p$; for i not prime to p, the factor $\frac{1}{i!}$ makes no sense. But in any case, we have

$$f_1(X) = f^{(1)}(X) = f'(X) \quad \text{and} \quad f_0(X) = f(X) \ .$$

Further, let us compute the j-th derivative of an i-th derivative:

$$\begin{aligned}
(f_i)_j(X) &= \left(\sum_{\nu=i}^{n} c_\nu \binom{\nu}{i} X^{\nu-i}\right)_j = \sum_{\nu=i+j}^{n} c_\nu \binom{\nu}{i}\binom{\nu-i}{j} X^{\nu-i-j} \\
&= \sum_{\nu=i+j}^{n} c_\nu \frac{(i+j)!}{i!j!}\binom{\nu}{i+j} X^{\nu-(i+j)} = \frac{(i+j)!}{i!j!} f_{i+j}(X) \\
&= \binom{i+j}{i} f_{i+j}(X) = \binom{i+j}{j} f_{i+j}(X) \ .
\end{aligned}$$

Setting $j = k - i$, we find

$$(f_i)_{k-i}(X) = \left(\begin{array}{c} k \\ i \end{array} \right) f_k(X) \ ,$$

whence by virtue of (6.7),

$$f_i(X+Y) = \sum_{k=0}^{n-i} (f_i)_k(X) Y^k = \sum_{k=i}^{n} (f_i)_{k-i}(X) Y^{k-i} = \sum_{k=i}^{n} \left(\begin{array}{c} k \\ i \end{array} \right) f_k(X) Y^{k-i} \ .$$

Putting $Y := Z - X$, we derive the following versions of the Taylor expansion (6.7) and of the last equation:

$$f(Z) = \sum_{k=0}^{n} f_i(X)(Z-X)^i \tag{6.8}$$

$$f_i(Z) = \sum_{k=i}^{n} \left(\begin{array}{c} k \\ i \end{array} \right) f_k(X)(Z-X)^{k-i} \ . \tag{6.9}$$

From equations (6.6) and (6.8) we obtain:

Lemma 6.60 *Let R be a subring of an arbitrary field K, and let $f \in R[X]$. Then all derivatives of f lie in $R[X]$. Further, there exist $h(X,Z), \tilde{h}(X,Z) \in R[X,Z]$ such that*

$$f(Z) - f(X) = (Z-X)h(X,Z) = f'(X)(Z-X) + (Z-X)^2 \tilde{h}(X,Z) \ .$$

We will also need a multidimensional version of the last assertion of the foregoing lemma. In the following, let f be a polynomial in the n variables $X_1, \ldots, X_n$, with coefficients in a subring R of an arbitrary field K. We will write $\underline{X} = (X_1, \ldots, X_n)$ and $f = f(\underline{X})$. Given a second n-tuple $\underline{Y} = (Y_1, \ldots, Y_n)$ of variables, we consider the polynomial $f(\underline{X} + \underline{Y})$. It is the sum over monomials of the form

$$g(\underline{X} + \underline{Y}) = c(X_1 + Y_1)^{m_1} \cdot \ldots \cdot (X_n + Y_n)^{m_n} \ .$$

Viewing $f(\underline{X} + \underline{Y})$ and $g(\underline{X} + \underline{Y})$ as polynomials in the variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, we ask for the monomials which are linear in one single Y_j. Evaluating $g(\underline{X} + \underline{Y})$ by means of the binomial expansion, we find just one such monomial for every j, namely

$$c X_1^{m_1} \cdot \ldots \cdot m_j X_i^{m_j - 1} Y_j \cdot \ldots \cdot X_n^{m_n} \ .$$

This is in fact equal to

$$\frac{\partial g}{\partial X_j}(\underline{X}) \cdot Y_j \ ,$$

where the first factor is the partial derivative of $g(\underline{X})$ with respect to X_j. Summing up over all monomials g in f and using the fact that the partial derivative is additive, we find that

$$\frac{\partial f}{\partial X_j}(\underline{X}) \cdot Y_j$$

is the sum of all monomials of $f(\underline{X}+\underline{Y})$ which are linear in Y_j. Similarly, one finds that the monomials which contain no Y_j at all, just sum up to $f(\underline{X})$. Consequently, there are polynomials $\hat{h}_{jk}(\underline{X},\underline{Y})$ with coefficients in R such that

$$f(\underline{X}+\underline{Y}) = f(\underline{X}) + \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(\underline{X}) \cdot Y_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \hat{h}_{jk}(\underline{X},\underline{Y}) Y_j Y_k \ .$$

Note that also the partial derivatives have coefficients in R. Putting $Z_j = X_j + Y_j$ and $\underline{Z} = (Z_1, \ldots, Z_n)$ and defining $\tilde{h}_{jk}(\underline{X},\underline{Z}) := \hat{h}_{jk}(\underline{X}\,,\,\underline{Z}-\underline{X})$, we obtain that

$$f(\underline{Z}) - f(\underline{X}) = \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(\underline{X}) \cdot (Z_j - X_j) + \sum_{j=1}^{n}\sum_{k=1}^{n} \tilde{h}_{jk}(\underline{X},\underline{Z})(Z_j - X_j)(Z_k - X_k) \ . \quad (6.10)$$

Chapter 7

Valued field extensions

7.1 Immediate extensions

If we are given an extension $(L|K,v)$ of valued fields, then associated with it we have two other extensions: the one of their value groups $vL \supset vK$, and the one of their residue fields $\overline{L}|\overline{K}$. The study of valued field extensions consists to a great extent of the study of the relation between these three extensions. In Section 5.3, we have defined $(L|K,v)$ to be **immediate** if the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are trivial, or loosely speaking, if (K,v) and (L,v) have the same value group and the same residue field. We have already shown that $(L|K,v)$ is immediate if and only if the underlying extension of valued abelian groups is immediate, and that the latter holds if and only if the underlying extension of ultrametric spaces is immediate. Immediate extensions play an important role in valuation theory. Many of their basic properties are already properties of immediate extensions of ultrametric spaces, of valued abelian groups or modules, and they have been proved in the previous chapters. But for the convenience of the reader, we will summarize them here and give direct proofs.

Let $a \in L$ and $\mathbf{A} = \operatorname{appr}(a,K)$ the approximation type of a over (K,v). Recall that for every $\alpha \in vK$, we have $c \in \mathbf{A}_\alpha$ if and only if $v(a-c) \geq \alpha$, and $c \in \mathbf{A}^\circ_\alpha$ if and only if $v(a-c) > \alpha$. The set $\Lambda(a,K) \subset vK\infty$, called the **support** of $\operatorname{appr}(a,K)$, consists of all $\alpha \in vK\infty$ for which $\mathbf{A}_\alpha \neq \emptyset$. Hence, we can write

$$\Lambda(a,K) = \{\alpha \in vK\infty \mid \exists c \in K:\ v(a-c) \geq \alpha\}\ .$$

Note that $\Lambda(a,K)$ is an initial segment of $vK\infty$. Further, $\infty \in \Lambda(a,K)$ if and only if $a \in K$.

Lemma 7.1 *Let $(L|K,v)$ be an extension of valued fields.*

a) **(Transitivity.)** *Assume that $E|K$ is an arbitrary subextension of $L|K$. Then $(L|K,v)$ is immediate if and only if $(L|E,v)$ and $(E|K,v)$ are immediate.*

b) If $L|K$ is algebraic, then $(L|K,v)$ is immediate if and only if every finite subextension of $(L|K,v)$ is immediate.

c) If $(L|K,v)$ is immediate, then for every $a \in L$, the set $\Lambda(a,K)$ has no maximal element $\neq \infty$.

d) $(L|K,v)$ is immediate if and only if for every $a \in L$, its approximation type $\operatorname{appr}(a,K)$ *over (K,v) is immediate.*

e) $(L|K,v)$ *is immediate if and only if for every* $a \in L^\times$ *there is some* $c \in K$ *such that* $v(a-c) > va$.

Proof: a): This is trivial since $vK \subset vE \subset vL$ and $\overline{K} \subset \overline{E} \subset \overline{L}$ for every extension $(L|K,v)$.

e): Suppose that $(L|K,v)$ is immediate, and let $0 \neq a \in L$. Then $va \in vL = vK$ and thus, there is some $c' \in K$ such that $va = vc'$. Hence, $v(a/c') = 0$. Then $\overline{a/c'} \in \overline{L} = \overline{K}$ and thus, there is some $c'' \in K$ such that $\overline{a/c'} = \overline{c''}$. That is, $v(a/c' - c'') > 0$, which yields that $v(a - c'c'') > vc' = va$. Hence $c = c'c''$ is the element that we have looked for.

For the converse, let $\alpha \in vL$ and $a \in L$ such that $va = \alpha$. If there is $c \in K$ such that $v(a-c) > va$, then $\alpha = va = vc \in vK$. Now let $\zeta \in \overline{L}$ and $a \in L$ such that $\overline{a} = \zeta$. If there is $c \in K$ such that $v(a-c) > va = 0$, then $\zeta = \overline{a} = \overline{c} \in \overline{K}$.

b): Implication "$\Rightarrow$" follows from part a). For the converse, just employ part e) and observe that every element a in an algebraic extension of K already lies in the finite extension $K(a)$ of K.

c): If $a \in L \setminus K$ and $\Lambda(a,K)$ has a maximal element, say $v(a-c_0)$ with $c_0 \in K$, then $a - c_0 \neq 0$, but there is no $c \in K$ such that $v(a - c_0 - c) > v(a - c_0)$. In view of part e), this proves that $(L|K,v)$ can not be immediate.

d): Let $(L|K,v)$ be immediate and $a \in L$. For every $c \in K$, $\alpha := v(a-c) \in vL\infty = vK\infty$ and thus, $c \in \operatorname{appr}(a,K)_\alpha \setminus \operatorname{appr}(a,K)^\circ_\alpha$. This proves that $\operatorname{appr}(a,K)$ is value-immediate. Now let $\alpha \in vK$ such that $\operatorname{appr}(a,K)_\alpha \neq \emptyset$. That is, there is $c_0 \in K$ such that $v(a-c_0) \geq \alpha$. In view of part e), there is some $c \in K$ such that $v(a - c_0 - c) > v(a - c_0) \geq \alpha$, which means that $c_0 + c \in \operatorname{appr}(a,K)^\circ_\alpha$. This proves that $\operatorname{appr}(a,K)$ is residue-immediate. Altogether, we find that $\operatorname{appr}(a,K)$ is immediate.

For the converse, assume that $0 \neq a \in L$ and that $\operatorname{appr}(a,K)$ is immediate. In view of (ATVI), choose $\alpha \in vK\infty$ such that $0 \in \operatorname{appr}(a,K)_\alpha \setminus \operatorname{appr}(a,K)^\circ_\alpha$. That is, $va = v(a-0) = \alpha$. Now using (ATRI), choose $c \in \operatorname{appr}(a,K)^\circ_\alpha$. That is, $v(a-c) > \alpha = va$. By virtue of part b), this proves that if $\operatorname{appr}(a,K)$ is immediate for every $a \in L$, then $(L|K,v)$ is immediate. □

From the proof of part d), we can deduce: $\operatorname{appr}(a,K)$ *is immediate if and only if for every* $c \in K$, $c \neq a$, *there is some* $c' \in K$ *such that* $v(a-c') > v(a-c)$.

Part e) of this lemma can also be stated as follows: $(L|K,v)$ *is an immediate extension if and only if for every element* $a \in L$, $a \neq 0$ *implies that* $va < \operatorname{dist}(a,K)$.

Recall that (K,v) is said to be **dense** in (L,v) if for every $a \in L$ and all $\beta \in vL$ there is some $c \in K$ such that $v(a-c) > \beta$. It follows directly from part e) of the foregoing lemma that $(L|K,v)$ is immediate if (K,v) is dense in (L,v). The converse is not true, cf. Example 8.47.

Lemma 7.2 *Let* $(L|K,v)$ *be an extension of valued fields.*

a) **(Transitivity.)** *Assume that* $E|K$ *is an arbitrary subextension of* $L|K$. *Then* (K,v) *is dense in* (L,v) *if and only if* (E,v) *is dense in* (L,v) *and* (K,v) *is dense in* (E,v).

b) If $L|K$ *is algebraic, then* (K,v) *is dense in* (L,v) *if and only if it is dense in every finite subextension of* $(L|K,v)$.

c) If (K,v) *is dense in* (L,v), *then* $\Lambda(a,K) = vK$ *for every* $a \in L \setminus K$.

d) (K,v) *is dense in* (L,v) *if and only if* $\operatorname{appr}(a,K)$ *is a completion type for every* $a \in L$.

Proof: a): Implication "⇒" is trivial. Suppose that (K,v) is dense in (E,v) and (E,v) is dense in (L,v). Let $a \in L$ and $\beta \in vL$. Choose $c' \in E$ such that $v(a-c') > \beta$. Without loss of generality, we may assume that $\beta \geq va$. Then it follows that $va = vc' \in vE$. So we can choose $c \in K$ such that $v(c'-c) > \beta$. Then $v(a-c) \geq \min\{v(a-c'), v(c'-c)\} > \beta$. We have proved that (K,v) is dense in (L,v).

b): Similar to the proof of part b) of Lemma 7.1 (by use of the definition of "dense" in the place of part e) of Lemma 7.1).

c): Let (K,v) be dense in (L,v), $a \in L \setminus K$ and $\beta \in vK$. Then there is $c \in K$ such that $v(a-c) > \beta$. Hence, $\beta \in \Lambda(a,K)$. On the other hand, $\infty \notin \Lambda(a,K)$ since $a \notin K$. This shows that $\Lambda(a,K) = vK$.

d): Let (K,v) be dense in (L,v). Then $(L|K,v)$ is immediate and thus by part d) of Lemma 7.1, the approximation type $\operatorname{appr}(a,K)$ is immediate for every $a \in L$. By part c), $\Lambda(a,K) = vK$ or, if $a \in K$, $\Lambda(a,K) = vK\infty$. That is, the distance $\operatorname{dist}(a,K)$ is ∞ and by definition, $\operatorname{appr}(a,K)$ is a completion type. For the converse, let $(L|K,v)$ be an extension such that $\operatorname{appr}(a,K)$ is a completion type for every $a \in L$. Since every completion type is an immediate approximation type, it follows from part d) of Lemma 7.1 that $(L|K,v)$ is immediate. Now let $\beta \in vL = vK$. Since $\operatorname{appr}(a,K)$ is a completion type, that is, $\operatorname{dist}(a,K) = \infty$, we know that there is some $\gamma > \beta$ such that $\operatorname{appr}(a,K)_\gamma \neq \emptyset$. This means that there is $c \in K$ such that $v(a-c) \geq \gamma > \beta$. We have proved that (K,v) is dense in (L,v). □

Note that vK has no maximal element if v is nontrivial. Indeed, if $vc > 0$ then $vc^2 = 2vc > vc$. So if a is an element in an immediate extension (L,v) such that $\Lambda(a,K) = vK$ or $\Lambda(a,K) = vK\infty$, and if we have a second element $b \in L$ such that $v(a-b) \geq \Lambda(a,K)$, then it follows that $v(a-b) = \infty$, that is, $a = b$. (On the other hand, if v is trivial, then $\overline{K} = K$ (up to equivalence of places), and every immediate extension of (K,v) must thus be trivial.) Similarly, if two elements in an immediate extension have the same completion type over (K,v), then they are equal.

Recall that a valued field (K,v) is **complete** if every completion type over (K,v) is trivial. By part d) of the foregoing lemma, this means that (K,v) admits no proper extension in which it is dense. In Section 5.3 we have constructed the **completion** $(K,v)^c$ of (K,v). It is a complete valued field in which (K,v) is dense. Hence, every completion type over (K,v) is realized in $(K,v)^c$, and the approximation type of every element in $(K,v)^c$ is a completion type.

Recall that $(\mathbb{Q}_p, v_p)$, the field of p-adic numbers, is the completion of the rationals $(\mathbb{Q}, v_p)$, where v_p denotes the p-adic valuation. Similarly, $(\mathbb{F}_p((t)), v_t)$ is the completion of $(\mathbb{F}_p(t), v_t)$, where v_t denotes the t-adic valuation (cf. Section 5.3). Density plays a special role in the case of a **discrete valuation**, that is, if $vK \cong \mathbb{Z}$, as it is the case for $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$.

Lemma 7.3 *Let (K,v) be a valued field and $vK \cong \mathbb{Z}$. Then (K,v) is dense in every immediate extension.*

Proof: Let $(L|K,v)$ be an immediate extension and $vL = vK \cong \mathbb{Z}$. If $a \in L \setminus K$, then by part c) of Lemma 7.1, $\Lambda(a,K) \subset vK$ has no maximal element. Consequently, $\Lambda(a,K) = vK = vL$. This shows that (K,v) is dense in (L,v). □

Every immediate approximation type $\mathbf{A}$ over a field (K,v) with $vK \cong \mathbb{Z}$ satisfies $vK \subset \Lambda(\mathbf{A})$ because $\Lambda(\mathbf{A})$ has no maximal element. Thus, $\operatorname{dist}(\mathbf{A}) = \infty$, that is, $\mathbf{A}$ is a

completion type. On the other hand, every completion type over a complete field is trivial. In view of Lemma 2.40, this proves:

Lemma 7.4 *Let (K, v) be a complete field with $vK \cong \mathbb{Z}$. Then (K, v) is spherically complete. In particular, $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are spherically complete.*

We know already from Corollary 5.5 that the power series field $(\mathbb{F}_p((t)), v_t)$ is spherically complete. However, we now see that this does not depend on the fact that $\mathbb{F}_p((t))$ is a power series field (which $\mathbb{Q}_p$ is not).

Since spherically complete fields are maximal, the above lemma shows that $(\mathbb{Q}_p, v_p)$ is a maximal immediate extension of $(\mathbb{Q}, v_p)$, and that $(\mathbb{F}_p((t)), v_t)$ is a maximal immediate extension of $(\mathbb{F}_p(t), v_t)$. We will prove in Section 7.10 that the completion of a valued field (K, v) is unique up to valuation preserving isomorphism over K. From this it follows that $(\mathbb{Q}_p, v_p)$ is the unique maximal immediate extension of $(\mathbb{Q}, v_p)$ with the p-adic valuation, up to valuation preserving isomorphism over $\mathbb{Q}$, and that $(\mathbb{F}_p((t)), v_t)$ is the unique maximal immediate extension of $(\mathbb{F}_p(t), v_t)$ with the p-adic valuation, up to valuation preserving isomorphism over $\mathbb{F}_p(t)$.

7.2 Valuation independence

After having treated immediate extensions in the last section, we shall now consider the contrary case: one might call it the "anti-immediate case", where the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are the maximal possible. But we have to make precise what we mean by that. For extensions of valued modules we have developed a notion of valuation independence of elements. We will now transfer this notion to extensions of valued fields. In what follows, let $(L|K, v)$ be an arbitrary extension of valued fields. As usual in field theory, we may view L as a K-vector space. Even more, we may view (L, v) as a valued K-vector space. If v is nontrivial on K, then the scalar multiplication is not value preserving and also not component-compatible. On the other hand, (L, v) is an ordinary valued K-vector space: if a and b are elements of L having equal bones and if $r \in K$, then $v(a-b) > va$ and thus, $v(ra - rb) = v(a-b) + vr > va + vr = vra$, showing that ra and rb have equal bones. But this is not the only basic property of (L, v) as a valued K-vector space. We also see that every 1-dimensional subvector space of (L, v) is isomorphic to (K, v) as a valued K-vector space. Indeed, it is of the form (Ka, v) for some $a \in L$, and the map $\iota : c \mapsto ca$ is an isomorphism of the K-vector space K onto the K-vector space Ka, satisfying $vb < vc \;\Leftrightarrow\; v\iota b < v\iota c$ for every $b, c \in K$. This gives rise to the following definition: A valued K-vector space (V, v) will be called a (K, v)**-vector space** if every 1-dimensional subvector space of (V, v) is isomorphic to (K, v) as a valued K-vector space. We have seen:
If $(L|K, v)$ is an extension of valued fields, then (L, v) is a (K, v)-vector space.

In the following, let us use on (K, v) the coefficient map as defined on page 128, with factors $\zeta_{\alpha,\beta}$ as given in (5.4). Then the scalar multiplication acts on the bones as follows. Given a bone (α, c), write it as $(\alpha, c) = (va, \operatorname{co} a)$ for some $a \in L$. Then $r \cdot (\alpha, c) = (vra, \operatorname{co} ra) = (vr + va\,,\, \overline{rat^{-vra}}) = (vr + va\,,\, \operatorname{co} r \cdot \operatorname{co} a \cdot \zeta_{vr,va})$. So we have:

$$r \cdot (\alpha, c) = (vr + \alpha\,,\, \operatorname{co} r \cdot c \cdot \zeta_{vr,\alpha})\,. \tag{7.1}$$

Note that the factor $\zeta_{vr,\alpha}$ only depends on α and the value of r. If the coefficient map is multiplicative, then this factor is always equal to 1.

Given bones $(\alpha_1, \zeta_1), \ldots, (\alpha_n, \zeta_n)$, we may deduce from (7.1) the following facts. If the values $\alpha_1, \ldots, \alpha_n$ belong to distinct cosets modulo vK, then the bones are K-independent. On the other hand, if all α_i are equal and the elements ζ_i are $\overline{K}$-independent, then again, the bones are K-independent. More precisely:

Lemma 7.5 *Let $(L|K, v)$ be an extension of valued fields and (α_i, ζ_{ij}), $i \in I$, $j \in J_i$, bones in the skeleton of (L, v). If the values α_i, $i \in I$, belong to distinct cosets modulo vK and for every i, the elements ζ_{ij}, $j \in J_i$, are $\overline{K}$-linearly independent, then these bones are K-independent.*

Proof: Suppose that $\sum_{i,j} r_{ij}(\alpha_i, \zeta_{ij})$, $r_{ij} \in K$, is an admissible finite linear combination, that is, all values $vr_{ij} + \alpha_i$ with $r_{ij} \neq 0$ are equal. (without loss of generality, we can assume that all $r_i \neq 0$). Since all α_i are generating distinct cosets modulo vK, this yields that there is some i_0 such that for all $i \neq i_0$, we must have $r_{ij} = 0$. Further, all $r_{i_0 j} \neq 0$ will have the same value, which we will denote by β. Let $\alpha := \alpha_{i_0}$. Then our linear combination is equal to the bone $(\beta + \alpha\,,\ \zeta_{\beta,\alpha} \cdot \sum_j \operatorname{co} r_{i_0 j} \cdot \zeta_{i_0 j})$. Since all $\operatorname{co} r_{i_0 j} \in \overline{K}$ and since the elements $\zeta_{i_0 j}$, $j \in J_{i_0}$, are $\overline{K}$-independent by assumption, the sum $\sum_j \operatorname{co} r_{i_0 j} \cdot \zeta_{i_0 j}$ is not equal to 0. This proves that the bones (α_i, ζ_{ij}) are K-independent. □

From Lemma 4.18 we know that elements of L are K-valuation independent if and only if their bones are K-independent. Further, note that if $a_i \in L$, $i \in I$, are elements of equal value α whose coefficients are K-independent, then for arbitrary $a \in L$, the elements aa_i, $i \in I$, have coefficients $\zeta_{va,\alpha} \cdot \operatorname{co} a \cdot \operatorname{co} a_i$ which are still K-independent. In view of these facts, the foregoing lemma implies:

Corollary 7.6 *Let $(L|K, v)$ be an extension of valued fields and $\{z_i, u_j \mid i \in I\,,\ j \in J\} \subset L$ such that the values vz_i, $i \in I$, belong to distinct cosets modulo vK and that u_j, $j \in J$, are elements of $\mathcal{O}_L^\times$ whose residues $\overline{u_j}$ are $\overline{K}$-independent. Then the elements*

$$z_i u_j \quad i \in I\,,\ j \in J$$

are K-valuation independent.

For the convenience of the reader, we give a direct proof for this corollary. Let $r_{ij} \in K$. Consider the summands of least value α in the (nontrivial) sum

$$\sum_{i,j} r_{ij} z_i u_j\ .$$

We have to show that α is also the value of the whole sum. So let i_0, j_0 be such that $r_{i_0 j_0} z_{i_0} u_{j_0}$ is a summand of least value α. Then $\alpha \in vz_{i_0} + vK$ and by hypothesis on the elements z_i, we find $\alpha \notin vz_i + vK$ for $i \neq i_0$. Hence, all summands $r_{ij} z_i u_j$ of least value must satisfy $i = i_0$. So if we are able to show that the value of the sum $\sum_j r_{i_0 j} z_{i_0} u_j$ is α, then we are done since the remaining summands are all of value $> \alpha$. After a division by the element $r_{i_0 j_0} z_{i_0}$ (which is of value α), it remains to show that the sum $a := \sum_j r_{i_0 j_0}^{-1} r_{i_0 j} u_j$ is of value 0. By construction, all $s_j := r_{i_0 j_0}^{-1} r_{i_0 j}$ are of value ≥ 0, and $s_{j_0} = 1$. Hence, $\overline{a} = \sum_j \overline{s_j}\,\overline{u_j}$ is a nontrivial linear combination with $\overline{s_j} \in \overline{K}$, and by our assumption on the elements u_j, it must be nonzero. That is, $va = 0$ as required.

Consider an arbitrary extension $(L|K, v)$ of valued fields. We choose elements $z_i \in L$ such that vz_i, $i \in I$, is a system of representatives for the cosets of vL modulo vK. Hence, $|I| = |vL/vK|$ (which may be a finite or infinite cardinal). Further, we may choose elements $u_j \in L$ such that $\overline{u_j}$, $j \in J$, is a $\overline{K}$-basis of $\overline{L}$. Hence, $|J| = \dim_{\overline{K}} \overline{L}$ (which again may be a finite or infinite cardinal). Then by the last corollary, the elements $z_i u_j$, $i \in I$, $j \in J$, are K-valuation independent. Hence, $[L : K]$ is an upper bound for the number $|I| \cdot |J|$ of these elements $z_i u_j$. This proves:

Lemma 7.7 *If $(L|K, v)$ is a finite extension of valued fields, then $\overline{L}|\overline{K}$ is finite and the quotient vL/vK is finite and hence a torsion group. Moreover,*

$$[L : K] \geq (vL : vK) \cdot [\overline{L} : \overline{K}] \; . \tag{7.2}$$

The (finite or infinite) cardinal $e(L|K, v) := (vL : vK)$ is called the **ramification index of** $(L|K, v)$, and $f(L|K, v) := [\overline{L} : \overline{K}]$ is called the **inertia degree of** $(L|K, v)$. Writing shortly $n = [L : K]$, $e = e(L|K, v)$ and $f = f(L|K, v)$, equation (7.2), called the **fundamental inequality**, can be expressed in the following mnemonic form:

$$n \geq e \cdot f \; .$$

Note that the extension $(L|K, v)$ is immediate if and only if $e = 1$ and $f = 1$.

Let us observe that ramification index and inertia degree are multiplicative, since the same holds for the degree of group and field extensions.

Lemma 7.8 *Let $(L|K, v)$ be a valued field extension and $(E|K, v)$ a subextension. Then*

$$e(L|K, v) = e(L|E, v) \cdot e(E|K, v) \quad \text{and} \quad f(L|K, v) = f(L|E, v) \cdot f(E|K, v) \; .$$

Now let $(L|K, v)$ be an algebraic extension, not necessarily finite. For every value $\alpha \in vL$ and every residue $c \in \overline{L}$ we may pick $a, b \in L$ such that $va = \alpha$ and $\overline{b} = c$. Since $L|K$ is algebraic, $K(a, b)|K$ is a finite extension. Consequently, $(vK + \mathbb{Z}\alpha)/vK$ is a torsion group and $\overline{K}(c)|\overline{K}$ is finite, by virtue of the Lemma 7.7. This shows:

Corollary 7.9 *If $(L|K, v)$ is an algebraic extension of valued fields, then also $\overline{L}|\overline{K}$ and $vL|vK$ are algebraic extensions.*

If $b_i \in L$ are K-valuation independent elements, then we will also say that they are **(linearly) valuation independent over** (K, v). This is not meant in the sense that they are K-valuation independent over the subvector space K. This would for instance not be true for the elements that we have constructed preceding to Lemma 7.7. Indeed, one of the values represents the coset vK, and some linear combination of the residues is an element of $\overline{K}$. In this case, however, we only have to omit one suitably chosen element b_i to obtain a set of elements which are K-valuation independent over the subvector space K. Vice versa, adding the element 1 to a set of the latter type, we obtain again a set of K-valuation independent elements. So there is no real danger of confusion in our abuse of notation. In the same spirit, a K-valuation basis of (L, v) (over the zero vector space) will also be called a **valuation basis of** (L, v) **over** (K, v) (or **of** $(L|K, v)$). Hence if the valued K-vector space (L, v) admits a K-valuation basis over (K, v), then the valued field extension $(L|K, v)$ admits a valuation basis.

Observe that in our construction preceding to Lemma 7.7, we may take the representative of the zero coset vK just to be 0 and the element of value 0 just to be 1. Similarly, we may choose a $\overline{K}$-basis of $\overline{L}$ which contains 1, and the element having residue 1 may also taken to be 1. Then, 1 will be an element of our valuation independent set of elements $z_i u_j$. A valuation independent set of this form will be called a **standard valuation independent set of** (L, v) **over** (K, v). If it is finite, we will assume in addition that it is of the form $\{z_i u_j, \mid 1 \leq i \leq e(L|K, v)\,,\ 1 \leq j \leq f(L|K, v)\}$, numbered such that $z_1 = 1$ and $u_1 = 1$. If a standard valuation independent set is a basis of $L|K$, then it will be called a **standard valuation basis of** (L, v) **over** (K, v). By the definition of a standard valuation independent set $\mathcal{B}$ of (L, v) over (K, v), the K-subvector space (V, v) generated by $\mathcal{B}$ has the same skeleton as the valued K-vector space (L, v). That is, the extension $(V, v) \subset (L, v)$ is immediate. We have $V = L$ if and only if $\mathcal{B}$ is a standard valuation basis of (L, v) over (K, v). If the latter holds, then the set $\mathcal{B} \setminus \{1\}$ is a K-valuation basis of the vector space (L, v) over the subspace (K, v), and Corollary 4.61 shows that $(K, v) \subset (L, v)$ is thus a defectless extension of valued K-vector spaces, that is, every finitely generated subextension admits itself a K-valuation basis over the subspace (K, v).

An extension $(L|K, v)$ of valued fields will be called **defectless** if $(K, v) \subset (L, v)$ is a defectless extension of valued K-vector spaces. Alternatively, the name **separated pair** was introduced by W. Baur in [BAUR1], [BAUR2] and used by F. Delon in [DEL5].

In view of the fact that valued fields are ordinary valued vector spaces, the following **transitivity of defectless extensions** is a direct consequence of Lemma 4.64.

Lemma 7.10 *Let $(L|K, v)$ be an extension of valued fields and $E|K$ an arbitrary subextension of $L|K$. If $(L|K, v)$ is a defectless extension, then so is $(E|K, v)$. Conversely, if $(L|E, v)$ and $(E|K, v)$ are defectless extensions, then so is $(L|K, v)$.*

But note that if $(L|K, v)$ is a defectless extension and $E|K$ is not finite, then it will in general not follow that $(L|E, v)$ is a defectless extension. The examples of Section 8.4 will show that there are fields (K, v) admitting algebraic extensions (L, v) such that $(\tilde{K}|K, v)$ is a defectless extension (for every extension of the valuation), while $(\tilde{K}|L, v)$ is not a defectless extension.

Finite defectless extensions are characterized as follows:

Lemma 7.11 *Let the extension $(L|K, v)$ be finite. Then the following conditions are equivalent:*

1) $(L|K, v)$ is a defectless extension,
2) $(L|K, v)$ admits a standard valuation basis,
3) $(L|K, v)$ admits a valuation basis,
4) $[L : K] = (vL : vK) \cdot [\overline{L} : \overline{K}]$.

Proof: Let again V denote the vector space generated by a standard valuation independent set of (L, v) over (K, v). We have seen above that the equality $V = L$ implies condition 2), and that condition 2) implies condition 1).

Now assume that the extension $L|K$ is finite. Then V is a K-vector space of dimension $e(L|K, v) \cdot f(L|K, v)$. Hence $V = L$ if and only if $[L : K] = e(L|K, v) \cdot f(L|K, v)$, which is condition 4). If $V \neq L$, then $(V, v) \subset (L, v)$ is a proper immediate extension of valued K-vector spaces, and Lemma 4.60 shows that the valued K-vector space (L, v) can not

admit a K-valuation basis over (K,v). This shows that condition 1) implies $V = L$. We have thus proved that conditions 1), 2) and 4) are equivalent.

Since 2)⇒3) is trivial, it now suffices to prove 3)⇒1). But a valuation basis of $(L|K,v)$ is a K-valuation basis of the valued K-vector space (L,v) over the zero vector space. Now Lemma 4.60 shows that (L,v) also admits a K-valuation basis over (K,v). By definition, this is condition 1). □

Let us remark that every K-valuation independent set in a valued field extension $(L|K,v)$ can be transformed by multiplication with elements from K into a valuation independent set where every two elements have already equal value if their values belong to the same coset modulo vK.

Algebraic defectless extensions may be characterized as follows:

Lemma 7.12 *An algebraic extension $(L|K,v)$ of valued fields is defectless if and only if every finite subextension $(E|K,v)$ of $(L|K,v)$ is defectless.*

Proof: The implication "⇒" is just a special case of Lemma 7.10, and the converse holds since every finite K-subvector space of an algebraic extension $L|K$ already lies in a finite subextension $E|K$. □

We will show later that a defectless algebraic extension $(L|K,v)$ admits only one extension of v from K to L (cf. Lemma 8.6). In the case of an extension $(L|K,v)$ which admits a standard valuation basis, we can show the following result here:

Lemma 7.13 *Let $(L|K,v)$ be an algebraic extension which admits a standard valuation basis $\{z_i, u_j \mid i \in I\,,\ j \in J\}$ (where the values $\alpha_i = vz_i\,,\ i \in I$, form a set of representatives for the cosets of vL modulo vK, and the residues $\zeta_j = \overline{u}_j\,,\ j \in J$, form a basis of $\overline{L}$ over $\overline{K}$). Then the valuation v on L is uniquely determined by its restriction to K and the data*

$$vz_i = \alpha_i \ (i \in I) \quad \text{and} \quad u_j v = \zeta_j \ (j \in J)\,. \tag{7.3}$$

Proof: Every element of L is a sum $\sum_{i,j} c_{ij} z_i u_j$. If a valuation w satisfies $w = v$ on K and (7.3), then the value of this sum is equal to that of the summands of minimal value. But the value of a summand $c_{ij} z_i u_j$ is uniquely determined by vc_{ij} and vz_i. □

At the beginning of this section, we have mentioned the expression "anti-immediate". The defectless extensions are indeed anti-immediate in the sense that they do not admit nontrivial immediate approximation types:

Lemma 7.14 *Let $(L|K,v)$ be a defectless extension and $a \in L \setminus K$. Then* appr$\,(a,K)$ *is not immediate.*

Proof: It follows from the definition of defectless extensions that the K-subvector space $(K \oplus Ka, v)$ of (L,v) admits a K-valuation basis. Corollary 4.14 shows that the set $\{v(a-c) \mid c \in K\}$ has a maximum, say $v(a-c_0)$. Since $a \notin K$, we know that $c_0 \neq a$. Now Lemma 2.21 (or the observation succeeding to Lemma 7.1) shows that appr$\,(a,K)$ is not immediate. □

The defectless extensions are anti-immediate also in the sense that they are linearly disjoint from immediate extensions. Recall that if $b_1, \ldots, b_n$ are K-valuation independent elements in a valued field extension of (K, v), then they are also K-linearly independent (cf. page 89).

Lemma 7.15 *Let $(\Omega|K, v)$ be an arbitrary valued field extension with $(L|K, v)$ a defectless and $(F|K, v)$ an immediate subextension. Then $L|K$ is linearly disjoint from $F|K$. Moreover, if $b_1, \ldots, b_n \in L$ are K-valuation independent, then they are also F-valuation independent.*

Proof: We have to show that for every choice of finitely many K-linearly independent elements $x_1, \ldots, x_n \in L$, these elements are also F-linearly independent. Since $(L|K, v)$ is assumed to be defectless, the K-vector space generated by $x_1, \ldots, x_n$ admits a valuation basis. This must consist of n elements, say $b_1, \ldots, b_n \in L$. Since they generate the same vector space as the x_i also over F, it now suffices to show that $b_1, \ldots, b_n$ are F-linearly independent. This in turn will follow if we can show that the elements $b_1, \ldots, b_n$ are F-valuation independent. So let $a_1, \ldots, a_n \in F$ such that $a_i \neq 0$ for at least one i. Since $(F|K, v)$ is assumed to be immediate, we can choose $c_1, \ldots, c_n \in K$ such that $c_i = 0$ if $a_i = 0$, and $v(a_i - c_i) > va_i$ otherwise. Then we have that $va_i = vc_i$ and

$$v\left(\sum_{i=1}^{n} c_i b_i\right) = \min vc_i b_i < \min v(a_i - c_i)b_i$$

and therefore,

$$v\left(\sum_{i=1}^{n} a_i b_i\right) = v\left(\sum_{i=1}^{n} c_i b_i \; + \; \sum_{i=1}^{n} (a_i - c_i)b_i\right) = v\left(\sum_{i=1}^{n} c_i b_i\right) = \min vc_i b_i = \min va_i b_i \; .$$

This proves that the elements $b_1, \ldots, b_n$ are F-valuation independent. □

So far, we have slipped through without a general characterization of valuation independence. But we will need it later, so we have to append it here.

Lemma 7.16 *Let $(L|K, v)$ be a valued field extension and $\mathcal{B} \subset L$. Then $\mathcal{B}$ is valuation independent over (K, v) if and only if the following holds: For every choice of distinct elements $b_1, \ldots, b_n \in \mathcal{B}$ such that the values $vb_1, \ldots, vb_n$ belong to the same coset modulo vK, there exist elements $c_2, \ldots, c_n \in K$ such that the elements $1, c_2 b_2 b_1^{-1} \ldots, c_n b_n b_1^{-1}$ are of value 0 and their residues are $\overline{K}$-linearly independent.*

Proof: "⇒": Let $b_1, \ldots, b_n \in \mathcal{B}$ such that the values $vb_1, \ldots, vb_n$ belong to the same coset modulo vK. Then there exist $c_2 \ldots, c_n \in K$ such that the elements $b_1, c_2 b_2 \ldots, c_n b_n$ are all of the same value, namely vb_1. Since $b_1, \ldots, b_n$ are assumed to be valuation independent over (K, v), we have that for all elements $c_1', \ldots, c_n' \in \mathcal{O}_{\mathbf{K}}$, not all of them in $\mathcal{M}_{\mathbf{K}}$, the value of $c_1' b_1 + c_2' c_2 b_2 + \ldots + c_n' c_n b_n$ is equal to vb_1, i.e., the value of $c_1' + c_2' c_2 b_2 b_1^{-1} + \ldots + c_n' c_n b_n b_1^{-1}$ is 0. This shows that for every choice of elements $\zeta_i \in \overline{K}$, not all of them zero, $\zeta_1 + \zeta_2 \overline{c_2 b_2 b_1^{-1}} + \ldots + \zeta_n \overline{c_n b_n b_1^{-1}} \neq 0$. We have proved that the residues of $1, c_2 b_2 b_1^{-1} \ldots, c_n b_n b_1^{-1}$ are $\overline{K}$-linearly independent.

"⇐": Let $b_1, \ldots, b_n \in \mathcal{B}$ and $d_1 \ldots, d_n \in K$. We have to show that $v(d_1 b_1 + \ldots + d_n b_n) = \min_i vd_i b_i$. If this holds under the assumption that all values $vd_i b_i$ are equal,

then it also holds in the general case. But if all vd_ib_i are equal, then the values of all b_i lie in the same coset modulo vK. So by assumption, there are $c_2 \ldots, c_n \in K$ such that the elements $1, c_2b_1^{-1}b_2 \ldots, \underline{c_nb_nb_1^{-1}}$ are of value 0 and their residues are $\overline{K}$-linearly independent. In particular, $1 + \overline{d_2d_1^{-1}c_2^{-1}}\,\overline{c_2b_2b_1^{-1}} + \ldots + \overline{d_nd_1^{-1}c_n^{-1}}\,\overline{c_nb_nb_1^{-1}} \neq 0$ (note that $d_id_1^{-1}c_i^{-1} \in K$ and that $v(d_id_1^{-1}c_i^{-1}) = 0$ since $vd_1b_1 = vd_ib_i$ and $vc_ib_ib_1^{-1} = 0$ imply that $vd_i = vd_1b_1b_i^{-1} = vd_1c_i$). It follows that the value of $1 + d_2b_2d_1^{-1}b_1^{-1} + \ldots + d_nb_nd_1^{-1}b_1^{-1}$ is zero. In other words, $v(d_1b_1 + \ldots + d_nb_n) = vd_1b_1 = \min_i vd_ib_i$, as required. □

How far is a finite valued field extension from having a valuation basis? This question probably makes no sense in general, but there is a situation where this question is the key to the proof of a nice result. Let us first observe (cf. Lemma 4.12 and Lemma 2.10):

Lemma 7.17 *Let (K, v) be a valued field. If the finite dimensional (K, v)-vector space (V, v) admits a K-valuation basis $\{b_1, \ldots, b_n\}$, then as an ultrametric space, (V, v) is the product $\prod_{i=1}^n (Kb_i, v)$ of the ultrametric spaces (Kb_i, v). Every (Kb_i, v) is isomorphic to (K, v) as a (K, v)-vector space. Hence if (K, v) is complete resp. spherically complete, then so is (V, v).*

We will later show that every finite extension of a spherically complete field admits a valuation basis and is thus again spherically complete. Unfortunately, not every extension of a complete field admits a valuation basis. Indeed, there are even complete fields of rank 1 which admit immediate extensions (cf. Example 8.51). Nevertheless, every finite extension of a complete field is again complete. This is because the extension of the valuation is "not too far" from having a valuation basis. We need the following observation whose proof is left to the reader.

Lemma 7.18 *Let (K, v) be a valued field, (V, v) a (K, v)-vector space of arbitrary dimension and $\mathcal{B}$ an arbitrary K-basis of V. Define*

$$v_{\mathcal{B}}(c_1b_1 + \ldots c_nb_n) = \min_{1 \leq i \leq n} vc_i$$

for every choice of basis elements $b_1, \ldots, b_n \in \mathcal{B}$ and coefficients $c_1, \ldots, c_n \in K$. Then $(V, v_{\mathcal{B}})$ is a (K, v)-vector space with valuation basis $\mathcal{B}$.

Theorem 7.19 *Let (K, v) be a complete valued field, (V, v) a finite dimensional (K, v)-vector space and $\mathcal{B}$ any basis of V. Then there are elements $\beta, \gamma \in vL$ such that*

$$v_{\mathcal{B}}a + \beta \leq va \leq v_{\mathcal{B}}a + \gamma \tag{7.4}$$

for every $a \in L$. Furthermore, (V, v) is complete.

Proof: Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $a = c_1b_1 + \ldots + c_nb_n \in L$. Then $va \geq \min_{1 \leq i \leq n} vc_ib_i \geq \min_{1 \leq i \leq n} vc_i + \min_{1 \leq i \leq n} vb_i = v_{\mathcal{B}}a + \min_{1 \leq i \leq n} vb_i$. Hence $\beta := \min_{1 \leq i \leq n} vb_i$ satisfies the lower inequality.

For the upper inequality, we proceed by induction on n. For $n = 1$, the assertion trivially follows with $\gamma := vb_1$. So let us assume that $n > 1$ and that the assertion of our theorem is already proved for dimension $n - 1$. For every $i \in \{1, \ldots, n\}$ we let V_i denote the K-subvector space of V generated by $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$. Then $b_i \notin V_i$, hence appr (b_i, V_i) is nontrivial by virtue of Lemma 2.27. By induction hypothesis, (V_i, v) is complete. Hence

by Lemma 2.40, every completion type over (V_i, v) is trivial. Consequently, appr (b_i, V_i) is not a completion type. In view of Corollary 2.44 we conclude that dist $(b_i, V_i) < \infty$ or vV_i is not cofinal in $\{v(b_i - d) \mid d \in V_i\} \cup vV_i$. If the former holds, then there is some $\gamma_i \in vV_i$ such that $v(b_i - d) \leq \gamma_i$ for every $d \in V_i$. If the latter holds, then there is some $d_i \in V_i$ such that $v(b_i - d_i) > vV_i$ and consequently, $\gamma_i := v(b_i - d_i) = \max\{v(b_i - d) \mid d \in V_i\}$. Hence in both cases, $v(b_i + \sum_{j \neq i} c_j b_j) \leq \gamma_i$ for every choice of $c_j \in K$. It follows that $v(\sum_{1 \leq j \leq n} c_j b_j) \leq vc_i + \gamma_i \leq vc_i + \max_{1 \leq \ell \leq n} \gamma_\ell$ for all i and all $c_j \in K$. Hence with $\gamma := \max_{1 \leq \ell \leq n} \gamma_\ell$, we find that $v(\sum_{1 \leq j \leq n} c_j b_j) \leq \min_{1 \leq i \leq n} vc_i + \gamma = v_{\mathcal{B}} a + \gamma$.

Now we wish to prove that (V, v) is complete. Let $\mathbf{A}$ be a nontrivial completion type over (V, v). Hence $\Lambda(\mathbf{A}) = vV$. As usual, we let $\mathbf{A}_\alpha = B_\alpha(a_\alpha)$ for all $\alpha \in vV$ (the balls are taken with respect to v). We distinguish two cases:

Case 1: vV is included in the convex hull of vK. It follows by construction that also β and γ lie in this convex hull. This implies that for every $\alpha \in vV$ there is some $\alpha' \in vK$ such that $\alpha' + \beta > \alpha$.

Now we consider the balls $B'_\alpha(a_{\alpha+\gamma})$ in $(V, v_{\mathcal{B}})$ for all $\alpha \in v_{\mathcal{B}}V = vK$ (the prime indicates that the balls are taken with respect to $v_{\mathcal{B}}$). The inequalities of (7.4) imply that $B_{\alpha+\gamma}(a_{\alpha+\gamma}) \subset B'_\alpha(a_{\alpha+\gamma}) \subset B_{\alpha+\beta}(a_{\alpha+\gamma})$, and it follows that the latter ball is equal to $B_{\alpha+\beta}(a_{\alpha+\beta})$. That is,

$$\mathbf{A}_{\alpha+\gamma} \subset B'_\alpha(a_{\alpha+\gamma}) \subset \mathbf{A}_{\alpha+\beta}$$

for all $\alpha \in vK$. From this it follows that the balls $B'_\alpha(a_{\alpha+\gamma})$ form a nest (since the same is true for the balls $\mathbf{A}_\alpha$). Since α runs through all of vK, this nest is a completion nest. Its intersection is nonempty because $(V, v_{\mathcal{B}})$ is complete by virtue of the two preceeding lemmata. On the other hand, we know that for every $\alpha \in vK$ there is some $\alpha' \in vK$ such that $\alpha' + \beta > \alpha$. This means that for every $\alpha \in vV$ there is some $\alpha' \in vV$ such that $B'_{\alpha'}(a_{\alpha'+\gamma}) \subset \mathbf{A}_{\alpha'+\beta} \subset \mathbf{A}_\alpha$. This shows that the nonempty intersection of the balls $B'_\alpha(a_{\alpha+\gamma})$ must lie in the intersection of the balls $\mathbf{A}_\alpha$. Hence the approximation type $\mathbf{A}$ is trivial, showing that (V, v) is complete.

Case 2: vV is not included in the convex hull of vK. Without loss of generality, we can assume that vb_1 is the minimal value among the values $vb_1, \ldots, vb_n$. Since completeness is preserved under isomorphisms (as the reader may show), we can replace (V, v) by the isomorphic (K, v)-vector space $(b_1^{-1}V, v)$. So we can assume that $vb_1 = 0$ and that no value vb_i is smaller than all values in vK. So also no K-linear combination of the b_i (that is, no element of V) has a value which is smaller than all values in vK. If our replacement has produced a vector space V whose value set lies in the convex hull of vK, then we proceed as in Case 1. Otherwise, there exist elements in V whose value is bigger than vK. Now we choose a basis $\{b'_1, \ldots, b'_n\}$ of V containing a maximal number of elements of value bigger than vK. This number can not be n since otherwise all elements of V would have a value bigger than vK, but by construction, V contains the element b_1 of value 0. Let us assume that $vb'_1, \ldots, vb'_m$ are bigger than vK, and that $vb'_{m+1}, \ldots, vb'_n$ lie in the convex hull of vK. Then $1 \leq m < n$ by what we have shown. If a K-linear combination of the b'_i has a value bigger than vK, then the coefficients of all b'_i with $i > m$ must be zero since otherwise, some b'_i with $i > m$ could be replaced by this linear combination, in contradiction to the maximality of m.

Now for every $\alpha \in vV$, $\alpha > vK$, consider the representation of the elements of the ball $\mathbf{A}_\alpha$ as linear combinations of the basis elements b'_i. Since $v(a - b) \geq \alpha > vK$ for every two

elements $a, b \in \mathbf{A}_\alpha$, we find that if $a = c_1 b'_1 + \ldots + c_n b'_n$ and $a = \tilde{c}_1 b'_1 + \ldots + \tilde{c}_n b'_n$, then $c_i = \tilde{c}_i$ for $i = m+1, \ldots, n$. Hence, subtraction of $d := c_{m+1} b'_{m+1} + \ldots + c_n b'_n$ transfers the nest $\{\mathbf{A}_\alpha \mid \alpha > vK\}$ onto the nest $\{B_\alpha(a_\alpha - d) \mid \alpha > vK\}$. The latter is a nest of balls in the (K, v)-subvector space of V which is generated by $b'_1, \ldots, b'_m$. Since this is of dimension less than n, it is complete by our induction hypothesis. Consequently, the nest $\{B_\alpha(a_\alpha - d) \mid \alpha > vK\}$ and thus also the nest $\{\mathbf{A}_\alpha \mid \alpha > vK\}$ has a nonempty intersection. But this means that $\mathbf{A}$ is realized in (V, v), showing that (V, v) is complete. □

The proof (with the exception of Case 2) is due to P. Roquette ([ROQ1], Lemma 2) and is based on lectures given by E. Artin. Since for an algebraic extension $(L|K, v)$ the value group vL lies in the divisible hull of vK by virtue of Corollary 7.9, the first case of the proof suffices to deduce:

Corollary 7.20 *Let (K, v) be a complete valued field and $(L|K, v)$ a finite extension of valued fields. Then also (L, v) is complete.*

Lemma 7.21 *Let $(L|K, v)$ be a finite extension of valued fields and $(K, v)^c = (K^c, v)$ the completion of (K, v). Then there is a unique extension of v from K^c to $L.K^c$ which is also an extension of v from L to $L.K^c$. With this extension, the valued field $(L.K^c, v)$ is the completion of (L, v). So we can write*

$$L^c = L.K^c \,.$$

Proof: Since $L|K$ is finite, so is $L.K^c|K^c$. Hence by the foregoing corollary, $(L.K^c, v)$ is complete for every extension of v from K^c to $L.K^c$. Let $a \in L.K^c$ and $va < \alpha \in v(L.K^c)$. Choose any K-basis $\{b_1, \ldots, b_n\}$ of L. Since these elements also generate $L.K^c$ over K^c, we can write $a = c_1 b_1 + \ldots + c_n b_n$ with $c_i \in K^c$. Since $L.K^c|K^c$ is finite, Corollary 7.9 shows that $v(L.K^c)$ lies in the divisible hull of vK. Hence, there is some $\beta \in vK$ such that $\beta \geq \alpha - \min_i vb_i$. Since (K, v) is dense in its completion $(K, v)^c$, we can find elements $c'_i \in K$ such that $v(c'_i - c_i) \geq \beta$. We set $a' := c'_1 b_1 + \ldots + c'_n b_n \in L$. Then it follows that $v(a - a') \geq \min_i v(c_i b_i - c'_i b_i) \geq \beta + \min_i vb_i \geq \alpha$. This proves that $va = va'$, so v is uniquely determined by its restriction to L. Moreover, since α was arbitrarily large, we have proved that (L, v) is dense in $(L.K^c, v)$. □

Corollary 7.22 *Let $(K, v) = (k((t)), v_t)$ and $(K(s)|K, v)$ an algebraic extension such that $s^n = t$ for some natural number n. Then $(K(s), v) = (k((s)), v_s)$.*

Proof: By the foregoing lemma, $K(s)$ is the completion of $(k(t, s), v) = (k(s), v_s)$. But this completion is just $(k((s)), v_s)$. □

The main notions introduced in this and the last section are invariant under isomorphisms. We leave it to the reader to prove the following facts:

Lemma 7.23 *Let $(L|K, v)$ an extension of valued fields and $\iota : L \to \iota L$ a field isomorphism. Then ι is an isomorphism from (L, v) onto $(\iota L, v\iota^{-1})$ and $\mathrm{res}_K(\iota)$ is an isomorphism from (K, v) onto $(\iota K, v\iota^{-1})$. For the valued field extension $(\iota L|\iota K, v\iota^{-1})$, the following facts hold:*

a) $(\iota L|\iota K, v\iota^{-1})$ *is defectless if and only if* $(L|K,v)$ *is,*
b) $(\iota L|\iota K, v\iota^{-1})$ *is immediate if and only if* $(L|K,v)$ *is,*
c) $(\iota K, v\iota^{-1})$ *is dense in* $(\iota L, v\iota^{-1})$ *if and only if* (K,v) *is dense in* (L,v)*,*
d) If $L|K$ *is finite, then so is* $\iota L|\iota K$*, and*

$$e(\iota L|\iota K, v\iota^{-1}) = e(L|K,v) \quad \textit{and} \quad (\iota L|\iota K, v\iota^{-1}) = f(L|K,v)\ ,$$

e) $\mathcal{B}$ *is a set of* K*-valuation independent elements in* (L,v) *if and only if* $\iota\mathcal{B} = \{\iota b \mid b \in \mathcal{B}\}$ *is a set of* ιK*-valuation independent elements in* $(\iota L, v\iota^{-1})$*,*
f) $\mathcal{B}$ *is a valuation basis of* $(L|K,v)$ *if and only if* $\iota\mathcal{B}$ *is a valuation basis of* $(\iota L|\iota K, v\iota^{-1})$*.*

Exercise 7.1 *Show how a finite valuation basis can be transformed into a standard valuation basis.*

Exercise 7.2 *Show that isomorphisms of valued fields preserve the properties of being a spherically complete or a maximal field.*

Exercise 7.3 *Let* b_i*,* $i \in I$*, be a valuation basis of the extension* $(L|K,v)$*. Show that if* $a_i \in L$ *are elements of value* $va_i > vb_i$*, then also* $b_i + a_i$*,* $i \in I$*, is a valuation basis of* $(L|K,v)$*.*

Exercise 7.4 *Define the notion of a* **direct sum of ultrametric spaces** *and apply it to infinite valued field extensions with valuation bases. What can be said about completeness and spherical completeness?*

7.3 Algebraic valuation independence

In field theory, we have the notions of linear independence and algebraic independence. In valuation theory, the notion of valuation independence is the analogue to the first. We shall now introduce the notion of algebraic valuation independence, which is the analogue to algebraic independence. Let $(L|K,v)$ be an extension of valued fields. A subset $\mathcal{T} \subset L$ is called **algebraically valuation independent over** (K,v), if for every choice of finitely many distinct elements $t_1, \ldots, t_n \in \mathcal{T}$, the value of every polynomial f in $K[t_1, \ldots, t_n]$ is equal to the value of a summand of f of minimal value. That means,

$$v\left(\sum_{\underline{\nu}} c_{\underline{\nu}} t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}\right) = \min_{\underline{\nu}} \left(v c_{\underline{\nu}} t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}\right) \tag{7.5}$$

where $\underline{\nu} = (\nu_1, \ldots, \nu_n)$ runs over all n-tuples of integers ≥ 0 and only finitely many $c_{\underline{\nu}}$ are nonzero. Observe that if at least one coefficient is nonzero, then the value of the polynomial is less or equal to the value of this coefficient and thus $< \infty$, that is, the polynomial does not equate to zero. This proves:

Lemma 7.24 *Let* $(L|K,v)$ *be a valued field extension. If* $\mathcal{T} \subset L$ *is algebraically valuation independent over* (K,v)*, then the elements of* $\mathcal{T}$ *are algebraically independent over* K*.*

If a transcendence basis of $L|K$ is algebraically valuation independent over (K,v), then we call it a **valuation transcendence basis of** $(L|K,v)$. Note that the elements $t_1, \ldots, t_n$ are algebraically valuation independent over (K,v) if and only if the elements $t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}$, $\nu_1, \ldots, \nu_n \in \mathbb{N}$, are linearly valuation independent over (K,v). From this, we deduce:

Lemma 7.25 *Let $(L|K,v)$ be a valued field extension. If $\mathcal{T} \subset L$ is algebraically valuation independent over (K,v), then the subextension $(K(\mathcal{T})|K,v)$ of $(L|K,v)$ is defectless.*

Proof: Let $V \subset K(\mathcal{T})$ be a K-vector space of finite dimension and containing K. We have to show that (V,v) admits a valuation basis over the subspace (K,v). Every K-basis of V over K consists of finitely many elements of $K(\mathcal{T})$, which are rational functions in the elements of $\mathcal{T}$. If we take $a \in K[\mathcal{T}]$ to be the common denominator of these rational functions, we find that $aV \subset K[\mathcal{T}]$. On the other hand, if $a \neq 0$ is any element of $K(\mathcal{T})$, then $\mathcal{B}$ is a valuation basis of (V,v) over the subspace (K,v) if and only if $a\mathcal{B}$ is a valuation basis of (aV,v). So we can assume from the start that $V \subset K[\mathcal{T}]$. We choose again a K-basis of V over K and collect the finitely many products of the form $t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}$ with $n, \nu_1, \ldots, \nu_n \in \mathbb{N}$ and $t_i \in \mathcal{T}$ that appear in the elements of that basis. These products form a valuation basis of a valued K-vector space $(V',v) \subset (K(\mathcal{T}),v)$ which contains (V,v). By Lemma 4.60 it follows that also (V,v) admits a valuation basis. □

If the elements $t_1, \ldots, t_n$ have values which are rationally independent over vK, then no two summands of the sum appearing on the left hand side of (7.5) will have equal value. This yields that the equality in (7.5) holds. Indeed, if two distinct summands $c_{\underline{\nu}} t_1^{\nu_1} \cdot \ldots \cdot t_n^{\nu_n}$ and $c_{\underline{\nu'}} t_1^{\nu'_1} \cdot \ldots \cdot t_n^{\nu'_n}$ would have equal value, we would obtain that $vc_{\underline{\nu}} - vc_{\underline{\nu'}} + (\nu_1 - \nu'_1)vt_1 + \ldots + (\nu_n - \nu'_n)vt_n = 0$, contradicting our assumption that the values $vt_1, \ldots, vt_n$ are rationally independent over vK.

Now assume that $n = r + s$, that $t_i = x_i$ have values which are rationally independent over vK, $1 \leq i \leq r$, and that $t_{r+j} = y_j$ have residues which are algebraically independent over $\overline{K}$, $1 \leq j \leq s$. Suppose that we have a polynomial $f \in K[t_1, \ldots, t_n]$ which is a counterexample to (7.5). Then it will still be a counterexample if we omit all summands that are not of minimal value; hence without loss of generality, we may assume that all summands in f have equal value. But then it follows from our above arguments that every element x_i has to appear to the same power ν_i in every of these summands. So dividing f by $x_i^{\nu_i}$ for $1 \leq i \leq r$, we obtain a polynomial $g \in K[y_1, \ldots, y_s]$ with summands of equal value, which is still a counterexample to (7.5). Since the elements y_j all have value zero, we find that all nonzero coefficients must have equal value. After dividing by one of them, we can assume that all summands have value zero, but that the value of g is > 0. Passing to the residue field through the residue map, we obtain a nontrivial polynomial $\overline{g}(X_1, \ldots, X_s) \in \overline{K}[X_1, \ldots, X_s]$ such that $\overline{g}(\overline{y_1}, \ldots, \overline{y_s}) = 0$. But this contradicts our assumption that the residues $\overline{y_1}, \ldots, \overline{y_s}$ be algebraically independent over $\overline{K}$. We have proved:

Lemma 7.26 *Let $(L|K,v)$ be an extension of valued fields. Assume that $x_i \in L$, $i \in I$, are elements whose values vx_i, $i \in I$, are rationally independent over vK. Assume further that $y_j \in L$, $j \in J$, are elements whose residues $\overline{y}_j$, $j \in J$, are algebraically independent over $\overline{K}$. Then $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ is algebraically valuation independent over (K,v) and thus also algebraically independent over K.*

Given any extension $(L|K,v)$ of valued fields, we can choose a special set of algebraically valuation independent elements as follows. We choose elements $y_j \in L$, $j \in J$, such that their residues $\overline{y}_j$ form a transcendence basis of $\overline{L}|\overline{K}$. For the extension $vL|vK$ of abelian groups, one can show by means of Zorn's Lemma that there exist maximal sets of elements in vL which are rationally independent over vK. This is analogous to the

existence of transcendence bases for arbitrary field extensions. In view of this analogy, such a maximal subset of vL will be called a **transcendence basis of** $vL|vK$. Note that elements $\alpha_i \in vL$ are rationally independent over vK if and only if their cosets $\alpha_i + vK$ are rationally independent. That is, the cardinality of a transcendence basis of $vL|vK$ is equal to the rational rank $\operatorname{rr} vL/vK$. We choose elements $x_i \in L$, $i \in I$, such that their values vx_i form a transcendence basis of $vL|vK$. By the foregoing lemma, $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ is algebraically valuation independent over (K,v) and thus also algebraically independent over K. A set $\mathcal{T}$ of this form will be called a **standard algebraically valuation independent set of** $(L|K,v)$. We obtain:

Lemma 7.27 *Let $(L|K,v)$ be an extension of valued fields. Then*

$$\operatorname{trdeg} L|K \geq \operatorname{rr} vL/vK \,+\, \operatorname{trdeg} \overline{L}|\overline{K}\;. \tag{7.6}$$

(If one of these numbers is infinite, the inequality is to be understood as an inequality of cardinal numbers.)

If $\operatorname{trdeg} L|K$ is finite, then the lemma shows that both $\operatorname{rr} vL/vK$ and $\operatorname{trdeg} \overline{L}|\overline{K}$ are finite, and that

$$\operatorname{trdef}(L|K,v) \;:=\; \operatorname{trdeg} L|K - \operatorname{rr} vL/vK - \operatorname{trdeg} \overline{L}|\overline{K}$$

is a nonnegative integer. We will call it the **transcendence defect of** $(L|K,v)$. If $\operatorname{trdef}(L|K,v) = 0$, then we call $(L|K,v)$ an **extension without transcendence defect**.

The transcendence degree of field extensions is additive, as we have stated in Section 6.5. The same holds for extensions of abelian groups, that is, $\operatorname{rr} C/A = \operatorname{rr} C/B + \operatorname{rr} B/A$ if $A \subset B \subset C$. Hence it follows that also the transcendence defect is additive:

$$\operatorname{trdef}(L|K,v) = \operatorname{trdef}(L|E,v) \,+\, \operatorname{trdef}(E|K,v)$$

if E is an intermediate field of $L|K$. Consequently:

Lemma 7.28 *Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. If $E|K$ is any subextension of $L|K$, then $(L|K,v)$ is an extension without transcendence defect if and only if both $(L|E,v)$ and $(E|K,v)$ are extensions without transcendence defect.*

In view of this lemma, we can generalize our definition to extensions of infinite transcendence degree as follows. We will say that an arbitrary extension $(L|K,v)$ is an **extension without transcendence defect** if every subextension $(E|K,v)$ of finite transcendence degree is an extension without transcendence defect.

If a standard algebraically valuation independent set of $(L|K,v)$ is a transcendence basis of $L|K$, then we call it a **standard valuation transcendence basis of** $(L|K,v)$. Now we can prove:

Lemma 7.29 *Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following assertions are equivalent:*

1) $(L|K,v)$ is an extension without transcendence defect.
2) $(L|K,v)$ admits a standard valuation transcendence basis,
3) $(L|K,v)$ admits a valuation transcendence basis.

Proof: 1)⇒2): As we have done above, we choose a standard algebraically valuation independent set of $(L|K,v)$. Its elements are algebraically independent over K, and its cardinality is equal to $\operatorname{rr} vL/vK + \operatorname{trdeg} \overline{L}|\overline{K}$. Hence if $(L|K,v)$ has transcendence defect 0, then its cardinality is equal to $\operatorname{trdeg} L|K$, which yields that it is a transcendence basis of $L|K$.

2)⇒3) is trivial.

3)⇒1): Let $\mathcal{T} = \{t_1, \ldots, t_n\}$ be a valuation transcendence basis of $(L|K,v)$. Hence $n = \operatorname{trdeg} L|K$. We can assume that the numeration is such that for some $r \geq 0$, the values $vt_1, \ldots, vt_r$ are rationally independent over vK and the values of every $r+1$ elements in $\mathcal{T}$ are rationally dependent over vK. That is, for every j such that $0 < j \leq s := n - r$, there are integers $\nu_j > 0$ and ν_{ij}, $1 \leq i \leq r$, and a constant $c_j \in K$ such that the element

$$t'_j \;:=\; c_j t_{r+j}^{\nu_j} \prod_{i=1}^{r} t_i^{\nu_{ij}}$$

has value 0. Observe that $r \leq \operatorname{rr} vL/vK$ and that $r + s = \operatorname{trdeg} L|K$.

Now assume that $(L|K,v)$ has nontrivial transcendence defect. Then

$$s = \operatorname{trdeg} L|K - r \geq \operatorname{trdeg} L|K - \operatorname{rr} vL|vK > \operatorname{trdeg} \overline{L}|\overline{K}\ .$$

This yields that the residues $\overline{t'_1}, \ldots, \overline{t'_s}$ are not $\overline{K}$-algebraically independent. Hence, there is a nontrivial polynomial $g(X_1, \ldots, X_s) \in \mathcal{O}_{\mathbf{K}}^{\times}[X_1, \ldots, X_s]$ such that $\overline{g}(\overline{t'_1}, \ldots, \overline{t'_s}) = 0$. Hence, $vf(t'_1, \ldots, t'_s) > 0$. After multiplying with sufficiently high powers of every element t_i, $1 \leq i \leq r$, we obtain a polynomial f in $t_1, \ldots, t_n$ which violates (7.5). But this contradicts our assumption that $\mathcal{T}$ be a valuation basis. Consequently, $(L|K,v)$ can not have a nontrivial transcendence defect. □

Corollary 7.30 *If a valued field extension admits a valuation transcendence basis, then it is an extension without transcendence defect.*

Proof: Let $(L|K,v)$ be an extension with valuation transcendence basis $\mathcal{T}$. Given a subextension $(E|K,v)$ of finite transcendence degree, we have to show that it is without transcendence defect. Let $\mathcal{T}'$ be a transcendence basis of $E|K$. Since $\mathcal{T}'$ is finite, there exists a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ such that all elements of $\mathcal{T}'$ are algebraic over $K(\mathcal{T}_0)$. Now let F be the relative algebraic closure of $K(\mathcal{T}_0)$ in L. Then $E \subset F$. The extension $(F|K,v)$ has valuation transcendence basis $\mathcal{T}_0$, because F it is algebraic over $K(\mathcal{T}_0)$. By the foregoing lemmy, it is an extension without transcendence defect. In view of Lemma 7.28 we conclude that also $(E|K,v)$ is an extension without transcendence defect. □

Equality (7.5) allows us to determine the value group and residue field of extensions which are generated by algebraically valuation independent elements. When we write "$vK \oplus \mathbb{Z}vx$" then we mean the direct sum of vK with the group generated by vx as abelian groups; the symbol gives no infromation about the ordering on this direct sum. The ordering will usually be given by the ordering on the value group of a valued field extension of (K,v) which contains x. Extensions of the form $vK \oplus \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \,|\, vK$ are the abelian group analogues of rational function fields $K(x_1 \ldots, x_n)|K$ of transcendence degree n (resp. of similar extensions of $\overline{K}$).

Lemma 7.31 *Let $(L|K,v)$ be an extension of valued fields containing a standard valuation independent set $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\} \subset L$ such that the values $vx_i\,,\ i \in I$, are rationally independent over vK, and that the residues $\overline{y}_j\,,\ j \in J$, are algebraically independent over $\overline{K}$. Then*

$$v\left(\sum_{\underline{\nu}} c_{\underline{\nu}} x_1^{\nu_1} \cdot \ldots \cdot x_r^{\nu_r} \cdot y_1^{\nu_{r+1}} \cdot \ldots \cdot y_s^{\nu_{r+s}}\right) = \min_{\underline{\nu}} vc_{\underline{\nu}} + \nu_1 vx_1 + \ldots \nu_r vx_r\ ,$$

and

$$\begin{aligned} vK(\mathcal{T}) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i \\ \overline{K(\mathcal{T})} &= \overline{K}(\overline{y}_j \mid j \in J)\ . \end{aligned}$$

If J is finite, then $\overline{K(\mathcal{T})}$ is a rational function field over $\overline{K}$. If P is the place associated with v, then we can write $K(\mathcal{T})P = KP(\mathcal{T}P)$, where $\mathcal{T}P = \{tP \mid t \in \mathcal{T}\}$.

Further, v is uniquely determined on $K(\mathcal{T})$ by its restriction to K, the prescription of the values vx_i, and the condition that the residues $\overline{y}_j$ be algebraically independent over $\overline{K}$. If in addition, the residues $\overline{y}_j$ are prescribed, then also the residue map is uniquely determined on $K(\mathcal{T})$.

Proof: Every element of $K(\mathcal{T})$ is the quotient of two polynomials in finitely many elements $t_1, \ldots, t_n \in \mathcal{T}$. According to (7.5), every such polynomial has a value of the form

$$v\left(c_{\underline{\nu}} x_1^{\nu_1} \cdot \ldots \cdot x_r^{\nu_r} \cdot y_1^{\nu_{r+1}} \cdot \ldots \cdot y_s^{\nu_n}\right) = vc_{\underline{\nu}} + \nu_1 vx_1 + \ldots + \nu_r vx_r \in vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$$

(the latter sum is direct by our assumption on the values vx_i). Since the value of a quotient f/g is $vf - vg$, we see that also these values lie in $vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$. By the choice of quotients of suitable polynomials, one shows that every value in $vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$ appears as a value of some element in $K(\mathcal{T})$. Observe that the prescription of the values vx_i determines which monomials in a given polynomial are the ones of least value, and that their value is computed by the above formula only by use of the restriction of v to K and the values vx_i. Hence, v is uniquely determined on $K(\mathcal{T})$ by these data.

Now assume that the value of some quotient f/g is zero; we want to determine its residue. Since $vf = vg$, the summands of minimal value in f and g all contain x_i to the same power ν_i, for every $i \in \{1, \ldots, r\}$. Dividing numerator and denominator by $c\prod_{i=1}^r x_i^{\nu_i}$ with a suitable constant $c \in K$, we obtain a quotient where numerator and denominator have value zero and where the summands of minimal value are polynomials in $K[y_1, \ldots, y_s]$. Since the residue of numerator and denominator are only depending on the residues of these summands of minimal value 0, our assertion now follows from the fact that the residue of every polynomial in $\mathcal{O}_{\mathbf{K}}[y_1, \ldots, y_s]$ lies in $\overline{K}(\overline{y}_j \mid j \in J)$. The prescribed values vx_i determine uniquely the summands of least value 0, and the prescribed residues $\overline{y}_j$ determine uniquely their residues. Hence, the residue map is uniquely determined on $K(\mathcal{T})$ by these data. □

The following corollary shows that the value group of a rational place of a function field in one variable is isomorphic to $\mathbb{Z}$. Note that if a valuation v is trivial on a field K, then we can assume that the residue map is the identity on K, so that we can write $\overline{K} = K$.

Corollary 7.32 *If $(L|K,v)$ is a finitely generated extension without transcendence defect, then also the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are finitely generated. In particular, if $(F|K,v)$ is a valued algebraic function field of transcendence degree 1 such that v is trivial on K but nontrivial on F, then $vF \cong \mathbb{Z}$ and $\overline{F}|K$ is finite.*

Proof: By Lemma 6.16 the finitely generated extension $L|K$ has finite transcendence degree. Hence by virtue of Lemma 7.29, we can choose a (finite) standard valuation transcendence basis $\mathcal{T}$ of $(L|K,v)$. The foregoing lemma shows that the extensions $vK(\mathcal{T})|vK$ and $\overline{K(\mathcal{T})}|\overline{K}$ are finitely generated. Since $L|K$ is finitely generated and $\mathcal{T}$ is a transcendence basis of $L|K$, it follows that $L|K(\mathcal{T})$ is a finite extension. From Lemma 7.7 we can thus infer that $vL|vK(\mathcal{T})$ and $\overline{L}|\overline{K(\mathcal{T})}$ are finite. Consequently, also the extensions $vL|vK$ and $\overline{L}|\overline{K}$ are finitely generated.

Assume that $(F|K,v)$ is a valued algebraic function field such that v is trivial on K. Then $vK = \{0\}$ and $\overline{K} = K$. If v is nontrivial on F, then $vF \neq \{0\}$, and there is some $t \in F$ such that $vt \neq 0$. Now $\{t\}$ is algebraically valuation independent and thus also algebraically independent over K. If $\operatorname{trdeg} F|K = 1$, then $\{t\}$ is a transcendence basis and thus also a valuation transcendence basis over K. In particular, $F|K(t)$ is finite. By Lemma 7.7, $vF|vK(t)$ and $\overline{F}|\overline{K(t)}$ are finite. On the other hand, the foregoing lemma shows that $vK(t) = \mathbb{Z}vt \cong \mathbb{Z}$ and $\overline{K(t)} = \overline{K}$. The latter implies that $\overline{F}|K$ is finite. Finally, since $vF|vK(t)$ is finite, we conclude that also vF is isomorphic to $\mathbb{Z}$. □

As a further consequence of Lemma 7.31, we prove the following embedding lemma:

Lemma 7.33 *Let the assumptions be as in Lemma 7.31. Further, let $(K,v) \subset (F,w)$ be a valued field extension and assume that there are embeddings*

$$\rho : vK(\mathcal{T}) \longrightarrow wF$$

over vK and

$$\sigma : \overline{K(\mathcal{T})} \longrightarrow Fw$$

over $\overline{K}$. For every $i \in I$, choose some $x_i' \in F$ such that $wx_i' = \rho vx_i$, and for every $j \in J$, choose some $y_j' \in F$ such that $y_j'w = \sigma\overline{y}_j$. Then $\mathcal{T}' = \{x_i', y_j' \mid i \in I, j \in J\}$ is a standard valuation independent set in (F,w) such that the values wx_i', $i \in I$, are rationally independent over vK, and that the residues $y_j'w$, $j \in J$, are algebraically independent over $\overline{K}$. Further, the assignment $x_i \mapsto x_i'$, $y_j \mapsto y_j'$ induces an embedding of $(K(\mathcal{T},v)$ in (F,w) over K which respects ρ and σ.

Proof: The assertions about the values wx_i', $i \in I$ and the residues $y_j'w$, $j \in J$ follow from the assumption that ρ is an embedding (over vK) and σ is an embedding (over $\overline{K}$). Now it follows from Lemma 7.26 that both $\mathcal{T}$ and $\mathcal{T}'$ are algebraically independent over K. Hence, the assignment $x_i \mapsto x_i'$, $y_j \mapsto y_j'$ induces an embedding ι of $K(\mathcal{T}$ in F over K of fields. Through this embedding, a valuation $w\iota$ and a residue map $a \mapsto (\iota^{-1}a)w$ are induced on $K(\mathcal{T})$. By the choice of the elements x_i' and y_j', the induced valuation satisfies $w\iota x_i = wx_i' = \rho vx_i$, and the induced residue map satisfies $(\iota y_j)w = y_j'w = \sigma\overline{y_j}$. By the uniqueness assertion of Lemma 7.31, we now obtain that $w\iota a = \rho va$ holds for all $a \in K(\mathcal{T})$, and $(\iota a)w = \sigma\overline{a}$ holds for all $a \in K(\mathcal{T})$ of value ≥ 0. That is, ι respects ρ and σ. □

If t is an element of a standard algebraically valuation independent set of $(L|K, v)$, then
– either its residue $\bar{t}$ is transcendental over $\overline{K}$,
– or its value vt is rationally independent over vK.
In the first case, we call t **residue-transcendental**, in the second case **value-transcendental over** K. If one of these cases holds for t, we call t **valuation-transcendental**.

In the residue-transcendental case, the valuation obtained on $K(t)$ is called the **functional valuation** or **Gauß valuation associated with** t. For a polynomial

$$f(x) = c_0 + c_1 t + \ldots + c_n t^n \in K[t]$$

we have

$$vf(t) = \min_{0 \le i \le n} vc_i$$

and consequently,

$$vK(t) = vK \quad \text{and} \quad \overline{K(t)} = \overline{K}(\bar{t}) \;, \tag{7.7}$$

with $\overline{K}(\bar{t})$ a rational function field in one variable over $\overline{K}$. The valuation v on $K(t)$ is uniquely determined by its restriction to K and the prescription that t be valuation-transcendental.

In the value-transcendental case, given a polynomial $f(t)$ like above, we have that

$$v(f(t)) = \min_{0 \le i \le n} (vc_i + ivt) \;,$$

and consequently,

$$vK(t) = vK \oplus \mathbb{Z}vt \quad \text{and} \quad \overline{K(t)} = \overline{K} \;. \tag{7.8}$$

The valuation v on $K(t)$ is uniquely determined by its restriction to K and the rationally independent value of t (which in turn is described by the cut induced by vt in the divisible hull $\widetilde{vK}$ of the value group vK).

Exercise 7.5 *Following the arguments of the proof of implication 3)⇒1) of Lemma 7.29, give criteria for algebraic valuation independence, by looking at the values and residues of given elements.*

Exercise 7.6 *Let $(L|K, v)$ be an extension which admits a valuation transcendence basis. Prove that $|L| \le \max\{\aleph_0 \,,\, |K| \,,\, |vL| \,,\, |\overline{L}|\}$ and that equality holds if L is infinite.*

7.4 Existence of places and their extensions

We have to answer two questions: Is every homomorphism of a subring of a given field extendable to a place of that field? Does every valuation of a given field admit an extension to a valuation of a given extension field? These questions are closely connected, and the key to their answers will be the following theorem:

Theorem 7.34 *Let $\mathcal{R}$ be a subring of the field K containing 1, and let $\mathcal{I}$ be a proper ideal of $\mathcal{R}$. Then there exists a valuation ring $\mathcal{O} \supset \mathcal{R}$ of K whose maximal ideal $\mathcal{M}$ contains $\mathcal{I}$.*

Proof: We consider the set S of all subrings R of K which contain $\mathcal{R}$ and satisfy $R\mathcal{I} \neq R$. We have $\mathcal{R} \in S$, hence S is a nonempty set which is partially ordered by inclusion. Given a totally ordered subset S_0 of S, the cardinality of its union R_0 is bounded by $|K|$, hence

R_0 is a ring (this follows from Lemma 13.7 and the fact that rings can be axiomatized by universal axioms). If we are able to show that $R_0 \in S$, then it will follow by Zorn's Lemma that S contains maximal elements. The condition $R_0 \supset \mathcal{R}$ is trivially satisfied. Now suppose that $R_0\mathcal{I} = R_0$. Then there are elements $r_1, \ldots, r_n \in R_0$ and $a_1, \ldots, a_n \in \mathcal{I}$ such that $\sum_{1 \leq i \leq n} r_i a_i = 1$. But there is a ring R_1 in S_0 which contains already all r_i, hence $R_1\mathcal{I} = R_1$ contradicting $R_1 \in S$. We have shown that $R_0\mathcal{I} \neq R_0$ and consequently, $R_0 \in S$.

Now it remains to prove that a maximal element $\mathcal{O}$ of S is a valuation ring with the required properties. Since $\mathcal{O} \in S$, we have $\mathcal{O} \supset \mathcal{R}$ and $\mathcal{O}\mathcal{I} \neq \mathcal{O}$. It suffices to show that $\mathcal{O}$ is a valuation ring of K since then, its unique maximal ideal $\mathcal{M}$ will automatically contain the proper ideal $\mathcal{O}\mathcal{I}$. We have to show that $\mathcal{O}$ satisfies (VR), i.e. that $x \in \mathcal{O} \vee \frac{1}{x} \in \mathcal{O}$ for every nonzero $x \in K$. By the lemma following this proof, it follows that $\mathcal{O}[x]\mathcal{I}$ is a proper ideal of the ring $\mathcal{O}[x]$ or $\mathcal{O}[\frac{1}{x}]\mathcal{I}$ is a proper ideal of the ring $\mathcal{O}[\frac{1}{x}]$. But this contradicts the maximality of $\mathcal{O}$, unless $x \in \mathcal{O}$ or $\frac{1}{x} \in \mathcal{O}$. □

Lemma 7.35 *Let $\mathcal{R}$ be a subring of the field K containing 1, and let $\mathcal{I}$ be a proper ideal of $\mathcal{R}$. Then for every $x \in K$, $\mathcal{R}[x]\mathcal{I}$ is a proper ideal of the ring $\mathcal{R}[x]$ or $\mathcal{R}[\frac{1}{x}]\mathcal{I}$ is a proper ideal of the ring $\mathcal{R}[\frac{1}{x}]$.*

Proof: To deduce a contradiction, assume that $\mathcal{R}[x]\mathcal{I} = \mathcal{R}[x]$ and $\mathcal{R}[\frac{1}{x}]\mathcal{I} = \mathcal{R}[\frac{1}{x}]$. Then there are elements $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathcal{I}$ such that $1 = \sum_{0 \leq i \leq n} a_i x^i$ and $1 = \sum_{0 \leq j \leq m} b_j x^{-j}$. Assume that m and n are the smallest numbers admitting such equations. Let $m \leq n$; for $m \geq n$, the proof is symmetrical. Multiplying the first equation by $1 - b_0$ and the second equation by $a_n x^n$, we obtain

$$\begin{aligned} 1 - b_0 &= (1 - b_0)a_0 + \ldots + (1 - b_0)a_n x^n \\ (1 - b_0)a_n x^n &= a_n b_1 x^{n-1} + \ldots + a_n b_m x^{n-m} \, . \end{aligned}$$

It follows that

$$1 = b_0 + (1 - b_0)a_0 + \ldots + (1 - b_0)a_{n-1}x^{n-1} + a_n b_1 x^{n-1} + \ldots + a_n b_m x^{n-m} \, .$$

The coefficients on the right hand side are all elements of $\mathcal{I}$. Hence, we have found a representation of 1 in $\mathcal{R}[x]\mathcal{I}$ of length $n - 1$, contradicting the minimality of n. This is the desired contradiction. □

The following theorem answers our initial questions.

Theorem 7.36 *Let $\mathcal{R}$ be a subring of the field K containing 1. Then every nontrivial homomorphism of $\mathcal{R}$ can be extended to a place of K. In particular, if k is a subfield of K, then every place of k admits an extension to a place of K.*

Proof: Let φ be a nontrivial homomorphism of $\mathcal{R}$ and $\mathcal{I}$ its kernel. Note that the kernel is a prime ideal; it is proper since φ is assumed to be nontrivial. Now let $\mathcal{R}_\mathcal{I}$ be the localization of $\mathcal{R}$ with respect to $\mathcal{I}$ which has the unique maximal ideal $\mathcal{R}_\mathcal{I}\mathcal{I}$ (it is a proper ideal since $\mathcal{I}$ is). Note that $\mathcal{R}_\mathcal{I}\mathcal{I} \cap \mathcal{R} = \mathcal{I}$. Now by the previous theorem, there is a valuation ring $\mathcal{O} \supset \mathcal{R}_\mathcal{I}$ whose maximal ideal $\mathcal{M}$ contains $\mathcal{R}_\mathcal{I}\mathcal{I}$. Since $\mathcal{R}_\mathcal{I} \setminus \mathcal{R}_\mathcal{I}\mathcal{I}$ only contains units and since $\mathcal{M}$ does not contain units, we find that $\mathcal{M} \cap \mathcal{R}_\mathcal{I} = \mathcal{R}_\mathcal{I}\mathcal{I}$. Consequently, $\mathcal{M} \cap \mathcal{R} = \mathcal{R}_\mathcal{I}\mathcal{I} \cap \mathcal{R} = \mathcal{I}$. This shows that the place P which is given by the canonical epimorphism $P : \mathcal{O} \to \mathcal{O}/\mathcal{M}$ is an extension of the canonical epimorphism

$\varphi' : \mathcal{R} \to \mathcal{R}/\mathcal{I}$. But since $\mathcal{I}$ is the kernel of φ on $\mathcal{R}$, we know that $\varphi = \sigma' \circ \varphi'$ for a suitable isomorphism of $\mathcal{R}/\mathcal{I}$ (which we may view in a natural way as a subring of $\mathcal{O}/\mathcal{M}$). We may extend σ' to an isomorphism σ of $\mathcal{O}/\mathcal{M}$. Then the place $\sigma \circ P$ (which is equivalent to P) is an extension of the homomorphism φ.

If φ is a place of k with valuation ring $\mathcal{R}$ and valuation ideal $\mathcal{I}$, then because of $\mathcal{M} \cap \mathcal{R} = \mathcal{I}$, $\mathcal{M}$ lies above $\mathcal{R}$ and we obtain that P is an extension of φ. □

Using the interchangeability of valuations and places, we obtain:

Corollary 7.37 *Let (K, v) be a valued field and $L|K$ a field extension. Then there is always an extension of v to a valuation of L.*

A further application of Theorem 7.36 is the following

Corollary 7.38 *Let $K|k$ be an arbitrary field extension. There are nontrivial places of $K|k$ if and only if* $\operatorname{trdeg} K|k > 0$.

Proof: If P is a place of $K|k$ then we know that the associated valuation ring contains k. The valuation ring is integrally closed in K by Lemma 5.1. If $\operatorname{trdeg} K|k = 0$ then $K|k$ is algebraic and every element of K is integrally dependent of k. This yields $\mathcal{O} = K$, that is, P is trivial on K.

Now assume that $x \in K$ is transcendental over k. Then the polynomial ring $k[x]$ admits a homomorphism φ onto k which sends x to 0. From Theorem 7.36 we infer the existence of a place P of K which extends φ. Hence $xP = 0$, showing that P is not trivial on K. On the other hand, its valuation ring contains k which yields that P is a place of $K|k$. □

"Almost all" fields admit nontrivial valuations or places:

Corollary 7.39 *The only fields which do not admit nontrivial places are precisely the algebraic extensions of finite fields. In particular, if v is a valuation of the field K of characteristic $p > 0$, then its restriction to the relative algebraic closure of $\mathbb{F}_p$ in K is trivial.*

Proof: Let $\operatorname{char} K = 0$. Then K contains $\mathbb{Q}$. We have shown in Section 5.2 that $\mathbb{Q}$ admits nontrivial places (the ones associated with p-adic valuations), hence by Theorem 7.36, so does K. Now let $\operatorname{char} K = p > 0$. The field $\mathbb{F}_p$ has no proper nontrivial subring, hence by Corollary 7.38, no algebraic extension of $\mathbb{F}_p$ admits a nontrivial place. On the other hand, if $\operatorname{trdeg} K|\mathbb{F}_p > 0$, then Corollary 7.38 shows that K admits a nontrivial place. □

The following is an important characterization of the integral closure:

Theorem 7.40 *Let $\mathcal{R}$ be a subring of the field K. Then the intersection of all valuation rings of K containing $\mathcal{R}$ is the integral closure of $\mathcal{R}$ in K.*

Proof: Since every valuation ring of K is integrally closed (Lemma 5.1), a valuation ring of K containing $\mathcal{R}$ will also contain the integral closure of $\mathcal{R}$. Hence, the intersection of all valuation rings of K containing $\mathcal{R}$ contains the integral closure of $\mathcal{R}$. For the converse, assume that $x \in K$ is not integrally dependent on $\mathcal{R}$; we want to show that there is a valuation ring $\mathcal{O}$ of K containing $\mathcal{R}$ such that $x \notin \mathcal{O}$. We consider the ring $\mathcal{R}[x^{-1}]$. Then x^{-1} is not a unit in this ring. Otherwise, there would be elements $a_0, \ldots, a_n \in \mathcal{R}$

such that $a_n x^{-n} + \ldots + a_1 x^{-1} + a_0 = (x^{-1})^{-1} = x$. Multiplying by x^n, we would find $x^{n+1} - a_0 x^n - \ldots - a_n = 0$ in contradiction to our assumption that x be not integrally dependent on $\mathcal{R}$. Since x^{-1} is not a unit in $\mathcal{R}[x^{-1}]$, it follows that $x^{-1}\mathcal{R}[x^{-1}]$ is a proper ideal of this ring. By Theorem 7.34, there exists a valuation ring $\mathcal{O}$ of K containing $\mathcal{R}[x^{-1}]$, whose maximal ideal $\mathcal{M}$ contains $x^{-1}\mathcal{R}[x^{-1}]$. Consequently, $\mathcal{O}$ contains $\mathcal{R}$ and $\mathcal{M}$ contains x^{-1}, showing that $x \notin \mathcal{M}$. □

Taking $\mathcal{R}$ to be the valuation ring of a given valued field (K, v), we can apply this theorem to an arbitrary field extension L of K, in the place of K. Then we obtain:

Corollary 7.41 *Let (K, v) be a valued field, $L|K$ a field extension and $x \in L$. Then $wx \geq 0$ for every extension w of v to L if and only if x is an element of the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L.*

7.5 Polynomials over valued fields

Let $\mathbf{K} = (K, v)$ be an arbitrary valued field and

$$f = c_n X^n + c_{n-1} X^{n-1} + \ldots + c_1 X + c_0 \in K[X] \tag{7.9}$$

an arbitrary polynomial. In this section, we will consider the valuation theoretical relation between the coefficients and the roots of f. Since the roots may not lie in K, we extend v to $\tilde{K}$ and call this extension again v. To begin with, we treat the case of a monic polynomial f, that is, $c_n = 1$. Write

$$f = \prod_{i=1}^{n} (X - a_i) \quad \text{with} \quad a_i \in \tilde{K} \; .$$

Recall that the coefficients c_i are, up to the sign, elementary symmetric polynomials in the roots a_i. In fact, if $0 < j \leq n$ then $(-1)^j c_{n-j}$ is equal to the j-th elementary symmetric polynomial in $a_1, \ldots, a_n$:

$$s_j(a_1, \ldots, a_n) := \sum_{1 \leq i_1 < \ldots < i_j \leq n} a_{i_1} \cdot \ldots \cdot a_{i_j} \; .$$

In particular, we observe that

$$vc_{n-1} = v(a_1 + \ldots + a_n) \geq \min_i va_i \quad \text{and} \quad vc_0 = v(a_1 \cdot \ldots \cdot a_n) = va_1 + \ldots + va_n \; .$$

Further, if all roots a_i lie in $\mathcal{O}_{\tilde{\mathbf{K}}}$, then all coefficients c_i lie in $\mathcal{O}_{\mathbf{K}}$ (observe that the latter does not depend on the chosen extension of v from K to $\tilde{K}$). More precisely, if $va_i \geq \alpha$ for all i, then $vc_{n-j} \geq j\alpha$ for $0 < j \leq n$.

Without loss of generality, we may assume that the roots a_i are numbered such that $va_1 \leq va_2 \leq \ldots \leq va_n$. Suppose that $a_1, \ldots, a_\ell$ are all roots of value < 0 (which means that f has precisely $n - \ell$ integral roots). Then the value of $a_1 \cdot \ldots \cdot a_\ell$ is smaller than the value of any other product of at most ℓ distinct roots a_i. This yields that $a_1 \cdot \ldots \cdot a_\ell$ is the unique summand of minimal value in the ℓ-th elementary symmetric polynomial in $a_1, \ldots, a_n$. Consequently, $vc_{n-\ell} = va_1 \cdot \ldots \cdot a_\ell = va_1 + \ldots + va_\ell$. It also yields that this value is smaller than the value of any summand in the j-th elementary symmetric polynomial,

for every $j < \ell$. Consequently, $vc_{n-\ell} < vc_{n-j}$ for $j < \ell$. Moreover, $va_1 \cdot \ldots \cdot a_\ell$ is smaller or equal to the value of any other product of distinct roots a_i. This yields that $vc_{n-\ell} \leq vc_{n-j}$ for $j \geq \ell$. Hence, among all coefficients of smallest value, $vc_{n-\ell}$ is the one with maximal index. Since $c_n = 1$, this also holds if all roots are integral, that is, if $\ell = 0$.

Suppose that precisely $a_1, \ldots, a_m$ are of minimal value among the elements $a_1, \ldots, a_n$. Then $a_1 \cdot \ldots \cdot a_m$ is the unique summand of minimal value in the m-th elementary symmetric polynomial in $a_1, \ldots, a_n$. Consequently, $vc_{n-m} = va_1 \cdot \ldots \cdot a_m = mva_1$, and c_{n-m} is integral if and only if $a_1, \ldots, a_m$ are integral. For every $j \leq m$, every product of j distinct roots a_i has value at least jva_1. On the other hand, for $j > m$ every product of j distinct roots a_i has value $> jva_1$ since it has at least one factor of value $> va_1$. Consequently, $vc_{n-j} \geq jva_1$ for $j \leq m$, and $vc_{n-j} > jva_1$ for $j > m$. Hence, among all coefficients for which $\frac{1}{j}vc_{n-j}$ is minimal, vc_{n-m} is the one with minimal index. Further, we see that $\min_i va_i = va_1 = \min_j \frac{1}{j}vc_{n-j}$.

If $0 < m < n$ is such that $va_m < va_{m+1}$, then as before, $a_1 \cdot \ldots \cdot a_m$ is the unique summand of minimal value in the m-th elementary symmetric polynomial in $a_1, \ldots, a_n$. This yields that $vc_{n-m} = va_1 \cdot \ldots \cdot a_m = va_1 + \ldots + va_m$. If moreover $m < \mu < n$ is such that $va_\mu < va_{\mu+1}$, then $vc_{n-\mu} - vc_{n-m} = va_{m+1} + \ldots + va_\mu$. In particular, if $1 < m < n$ and $va_{m-1} < va_m < va_{m+1}$, then $va_m = vc_{n-m} - vc_{n-m+1} \in vK$. A similar formula holds for the roots a_1 and a_n if they are the unique root of minimal resp. maximal value. Indeed, if $va_1 < va_2$, then $va_1 = vc_{n-1}$, which we can write as $vc_{n-1} - vc_n \in vK$ because of $c_n = 1$. If $va_{n-1} < va_n$, then by what we have shown, $vc_1 = va_1 + \ldots + va_{n-1}$. Since $vc_0 = va_1 + \ldots + va_n$, it follows that $va_n = vc_0 - vc_1 \in vK$.

Now assume that f is not monic. Then we divide by c_n to obtain a monic polynomial which has the same roots as f. We have to replace vc_i by $vc_i - vc_n$, but this does not change the order between the values of the coefficients. Let us summarize what we have proved:

Lemma 7.42 *Let $\mathbf{K} = (K, v)$ and $f \in K[X]$ a polynomial of the form (7.9) with roots $a_1, \ldots, a_n$. Then:*

a) Assume that f is monic. Then $f \in \mathcal{O}_\mathbf{K}[X]$ if and only if all roots a_i are integral.

b) Suppose that among all coefficients of smallest value, vc_j is the one with maximal index. Then f has precisely j integral roots. If in addition, the roots $a_1, \ldots, a_{n-j}$ are the roots which are not integral, then $vc_j - vc_n = va_1 + \ldots + va_{n-j}$.

c) The values of the roots are bounded from below by the following formula:

$$\min_i va_i = \min_{1 \leq j \leq n} \frac{vc_{n-j} - vc_n}{j} .$$

d) If $m \in \{1, \ldots, n\}$ such that $va_j \neq va_m$ for all $j \neq m$, then $va_m \in vK$.

Consider a polynomial of the form (7.9). If all coefficients c_i are integral, that is, if $f \in \mathcal{O}_\mathbf{K}[X]$, then $\overline{f}$ will denote the polynomial $\overline{c}_n X^n + \ldots + \overline{c}_1 X + \overline{c}_0 \in \overline{K}[X]$ which we obtain from f by replacing its coefficients by their residues. We will also use the notation fv if it is necessary to indicate the valuation. We call $\overline{f}$ the **reduction of** f. Since the residue map is a ring homomorphism, we have $\overline{f(a)} = \overline{c_n a^n + \ldots + c_1 a + c_0} = \overline{c}_n \, \overline{a}^n + \ldots + \overline{c}_1 \, \overline{a} + \overline{c}_0 = \overline{f}(\overline{a})$ for $a \in \mathcal{O}_\mathbf{K}$. In particular, if a is a root of f then $\overline{a}$ is a root of $\overline{f}$. The converse is certainly

not true if the valuation v is nontrivial on K, since then for every $\overline{a} \in \overline{K}$, the set $a + \mathcal{M}_{\mathbf{K}}$ of elements with residue $\overline{a}$ is infinite and thus, not all of these elements can be zeros of f.

Our definition of the reduction of a polynomial is generalized in the obvious way to polynomials in several variables. When one works with the notions of "polynomial ring" and "rational function field", then variables are viewed as transcendental elements (this ambiguity is also used in the model theory of fields, as we will also do in Chapter 13). Here, this observation leads to the question whether the reduction of a polynomial is actually the application of a place of the rational function field, for which the elements of the polynomial ring are integral. This is in fact true. We use the following lemma.

Lemma 7.43 *Let (K, v) be a valued field and ζ_j, $j \in J$, algebraically independent over $\overline{K}$. Choose elements y_j, $j \in J$ which are algebraically independent over K. Then v and its associate residue map extend in a unique way from K to the rational function field $K(y_j \mid j \in J)$ such that $\overline{y}_j = \zeta_j$ for every $j \in J$. For this extension,*

$$vK(y_j \mid j \in J) \,=\, vK \quad \textit{and} \quad \overline{K(y_j \mid j \in J)} \,=\, \overline{K}(\zeta_j \mid j \in J)\ .$$

On $K[y_j \mid j \in J]$, the valuation v is given by

$$v\left(\prod_{\underline{\nu}} c_{\underline{\nu}}\, y_1^{\nu_1} \cdot \ldots \cdot y_n^{\nu_n}\right) = \min_{\underline{\nu}}\, vc_{\underline{\nu}}\ ,$$

and on $\mathcal{O}_{\mathbf{K}}[y_j \mid j \in J]$, the residue map is given by

$$\overline{\prod_{\underline{\nu}} c_{\underline{\nu}}\, y_1^{\nu_1} \cdot \ldots \cdot y_n^{\nu_n}} = \prod_{\underline{\nu}} \overline{c_{\underline{\nu}}}\, \overline{y}_1^{\nu_1} \cdot \ldots \cdot \overline{y}_n^{\nu_n}\ .$$

Proof: Let P be the place associated with v on K. It is a homorphism from $\mathcal{O}_{\mathbf{K}}$ onto $\overline{K}$. Since the elements y_j, $j \in J$, are algebraically independent over K and the elements ζ_j, $j \in J$, are algebraically independent over $\overline{K}$, we can extend P to a homomorphism from $\mathcal{O}_{\mathbf{K}}[y_j \mid j \in J]$ onto $\overline{K}[\zeta_j \mid j \in J]$ by setting $y_j P = \zeta_j$. Via the rule $(f/g)P = fP/gP$, this homomorphism extends to a place of the quotient field $K(y_j \mid j \in J)$ of $\mathcal{O}_{\mathbf{K}}[y_j \mid j \in J]$ (cf. also Theorem 7.36). We have constructed an extension of P from K to $K(y_j \mid j \in J)$, hence the valuation associated with P on the latter field is also an extension of v.

The formula for the value of polynomials is just a special case of (7.5) since $vy_j = 0$ for all $j \in J$. The uniqueness of v on $K(y_j \mid j \in J)$ was already stated in Lemma 7.31. □

Consider the polynomial ring $K[X_1, \ldots, X_n]$. Although it is not quite formally correct, we let $X_1, \ldots, X_n$ represent elements which are algebraically independent over K as well as over $\overline{K}$. So we take $y_j = X_j = \zeta_j$ and apply the foregoing lemma. We obtain a residue map on $K(X_1, \ldots, X_n)$ which satisfies

$$\overline{f(X_1, \ldots, X_n)} = \overline{f}(X_1, \ldots, X_n) \quad \text{for every } f \in \mathcal{O}_{\mathbf{K}}[X_1, \ldots, X_n]\ .$$

So it coincides with our definition of the reduction of a polynomial. The associated valuation on $K[X_1, \ldots, X_n]$ assigns to f the minimal value of its coefficients. For a polynomial f in one variable, given in the form (7.9), this reads as follows:

$$vf = \min_{1 \le i \le n} vc_i\ .$$

In particular, $vf \geq 0$ if all coefficients are integral. That is,

$$K[X_1, \ldots, X_n] \cap \mathcal{O}_{(K(X_1,\ldots,X_n),v)} \;=\; \mathcal{O}_{\mathbf{K}}[X_1, \ldots, X_n] \,.$$

Note that a monic polynomial always has value ≤ 0 and that a monic polynomial with integral coefficients has value 0. A polynomial $f \in K[X_1, \ldots, X_n]$ is called **primitive** if $vf = 0$, that is, if the coefficients of minimal value are units in $\mathcal{O}_{\mathbf{K}}$. The latter is the classical definition of "primitive". But with the help of the Gau valuation, we see at once that *the product of two primitive polynomials is again primitive.* We also get the so-called **Gau' Lemma** for granted:

Lemma 7.44 *Let $f, g, h \in K[X_1, \ldots, X_n]$ such that $f = gh$.*
a) **(Gau' Lemma, first form)** *If f and g are primitive, then so is h.*
b) **(Gau' Lemma, second form)** *If f and g are monic and f has integral coefficients, then also h is monic, and g and h have integral coefficients.*

The next lemma tells about the relation of the roots of f and the roots of its reduction $\overline{f}$.

Lemma 7.45 *Let (K, v) be an arbitrary valued field and let v be extended to $\tilde{K}$. Let $f \in \mathcal{O}_{\mathbf{K}}[X]$ such that $\overline{f} \neq 0$. Then the residue map establishes a bijection from the integral roots of f onto the roots of $\overline{f}$ (counted with multiplicities). If all roots of f have value < 0, then $\overline{f} \in \overline{K}$.*

Proof: Let f be given in the form (7.9). If all roots of f have value < 0, then according to part b) of Lemma 7.42, c_0 is the unique coefficient of f of smallest value. Hence $\overline{f} = \overline{c_0} \in \overline{K} \setminus \{0\}$ does not admit any roots.

Now assume that $\overline{f}$ admits a root. Write $f = c_n \prod_{i=1}^{n}(X - a_i)$ with $a_i \in \tilde{K}$. Define $g := \prod_{i\in I}(X - a_i) \in \mathcal{O}_{(\tilde{K},v)}[X]$ where $I = \{i \mid 1 \leq i \leq n \wedge va_i \geq 0\}$. We have that $f = gh$ with $h \in \mathcal{O}_{(\tilde{K},v)}[X]$ by virtue of Gau' Lemma. By definition, all roots of h have value < 0, showing that $\overline{h}$ is a constant. It is a nonzero element in $\overline{K}$ since it is the leading coefficient of $\overline{f} \neq 0$. Consequently, with $\overline{c} := \overline{h} \in \overline{K}$,

$$\overline{f} \;=\; \overline{c} \prod_{va_i \geq 0} (X - \overline{a_i})$$

which shows that the residue map establishes the asserted bijection. □

We are now going to prove what is called the "continuity of roots". We need some easy observations.

Lemma 7.46 *Let a be a root of the polynomial (7.9). Then $vf - vc_n \leq 0$, and*

$$vc_0 - (n-1)(vf - vc_n) \;\geq\; va \;\geq\; \min_{1\leq j\leq n} \frac{vc_{n-j} - vc_n}{j} \;\geq\; vf - vc_n \,.$$

Proof: Since $vf - vc_n = c_n^{-1} f$ is the value of a monic polynomial, it is ≤ 0. So if $\frac{1}{j}(vc_{n-j} - vc_n) \geq 0$, then $\frac{1}{j}(vc_{n-j} - vc_n) \geq vf - vc_n$. If $\frac{1}{j}(vc_{n-j} - vc_n) < 0$, then $\frac{1}{j}(vc_{n-j} - vc_n) > vc_{n-j} - vc_n \geq vf - vc_n$ by definition of vf. Now the lower bounds for va follow from part c) of Lemma 7.42. Now let $a_1 = a, a_2, \ldots, a_n$ be all roots of f. By what we have just proved, $va_i \geq vf - vc_n$ for all i. Consequently, $vc_0 = v\prod_{i=1}^{n} a_i = \sum_{i=1}^{n} va_i \geq va + (n-1)(vf - vc_n)$. This gives the upper bound for va. □

Lemma 7.47 *Let a be a root of the monic polynomial $f \in K[X]$. Further, let $g \in K[X]$ be of degree $n = \deg f$. Then*

$$vg(a) \ \geq \ nvf + v(f-g) \ .$$

If $v(f-g) > 0$, then there is a root $b \in \tilde{K}$ of g such that

$$v(a-b) \ \geq \ vf + \frac{v(f-g)}{n} \ . \tag{7.10}$$

Proof: Write $g(X) = \sum_{i=1}^{n} c_i' X^i$. Then $vg(a) = v(g(a) - f(a)) = v\sum_{i=1}^{n}(c_i' - c_i)a^i \geq \min_i(v(c_i'-c_i)+iva) \geq \min_i v(c_i'-c_i)+\min_i iva \geq v(g-f)+\min_{0\leq i\leq n} ivf = v(f-g)+nvf$ because $vf \leq 0$.

Now we write $g(X) = c_n' \prod_{i=1}^{n}(X-b_i)$ with $b_i \in \tilde{K}$. By what we have just proved, $nvf + v(f-g) \leq vg(a) = vc_n' \prod_{i=1}^{n}(a-b_i) = vc_n' + \sum_{i=1}^{n} v(a-b_i)$. If $v(f-g) > 0$, then $v(c_n - c_n') > 0$ and since $c_n = 1$, this implies $vc_n' = 0$. It follows that at least one of the summands $v(a-b_i)$ is $\geq \frac{1}{n}(nvf + v(f-g)) = vf + \frac{1}{n}v(f-g)$. We choose such an i and set $b := b_i$. □

Lemma 7.48 *Assumptions as in the foregoing lemma. Let $v(f-g) > 0$ and choose b as in that lemma. Then*

$$v\left(\frac{f(X)}{X-a} - \frac{g(X)}{X-b}\right) \ \geq \ 2vf + \frac{v(f-g)}{n} \ .$$

Proof: Since the value of the normed polynomial $(X-a)(X-b)$ is ≤ 0, we have that

$$\begin{aligned} v\left(\frac{f(X)}{X-a} - \frac{g(X)}{X-b}\right) &= v\left(f(X)(X-b) - g(X)(X-a)\right) - v(X-a)(X-b) \\ &\geq v\left(f(X)(X-b) - g(X)(X-a)\right) \\ &\geq \min\{v\left((f(X)-g(X))X\right), v\left(f(X)b - g(X)a\right)\} \ . \end{aligned}$$

In view of $vX = 0$, we find that $v\left((f(X)-g(X))X\right) = v(f-g)$. Further, let us write $f(X)b - g(X)a = f(X)b - f(X)a + f(X)a - g(X)a$. Consequently, $v(f(X)b - g(X)a) \geq \min\{v\left(f(X)b - f(X)a\right), v\left(f(X)a - g(X)a\right)\} = \min\{vf + v(b-a), v(f-g) + va)\}$. In view of the foregoing lemmata, we thus obtain that

$$v\left(\frac{f(X)}{X-a} - \frac{g(X)}{X-b}\right) \geq \min\{v(f-g), vf + vf + \frac{v(f-g)}{n}, v(f-g) + vf\} = 2vf + \frac{v(f-g)}{n}$$

since $v(f-g) > 0$ by assumption and $vf \leq 0$ because f is monic. □

Theorem 7.49 (Continuity of roots)
Let (K, v) be an arbitrary valued field, $0 \leq \alpha \in vK$ and $f, g \in K[X]$. Let $\deg f = \deg g = n$ and c_n , c_n' be the leading coefficients of f and g respectively. If

$$v(f-g) > n^n\alpha - 3n^n(vf - vc_n) + vc_n \ , \tag{7.11}$$

then we can write $f = c_n \prod_{i=1}^{n}(X-a_i)$ and $g = c_n' \prod_{i=1}^{n}(X-b_i)$ with $a_i, b_i \in \tilde{K}$ in such a way that $v(a_i - b_i) > \alpha$ for $1 \leq i \leq n$.

Proof: Condition (7.11) can also be read as $v(c_n^{-1}f - c_n^{-1}g) > n^n\alpha - 3n^n vc_n^{-1}f$, with $c_n^{-1}f$ a monic polynomial. Since the roots do not change when the polynomial is multiplied with a constant, it thus suffices to prove our assertion under the assumption that f be monic. Then in particular, $vc_n = 0$ and $vf \leq 0$. We use induction on some enumeration of the roots of f. We set $f_1 := f$ and $g_1 := g$ (in this proof, the indeces will *not* indicate derivatives). After taking the first root a_1 and finding a root b_1 of g such that $v(a_1 - b_1) > \alpha$, we repeat the procedure with $f_2 := f_1/(X - a_1)$ and $g_2 := g_1/(X - b_1)$. Having found the pair a_i, b_i of roots of the polynomials f_i, g_i, we define $f_{i+1} := f_i/(X - a_i)$ and $g_{i+1} := g_i/(X - b_i)$. We continue until we arrive at the linear polynomials f_n, g_n. Note that $vf_{i+1} = vf_i - v(X - a_i) \geq vf_i$. Since $\deg f_i = n - i + 1$, the foregoing lemma shows that $v(f_{i+1} - g_{i+1}) \geq 2vf_i + \frac{v(f_i - g_i)}{n-i+1} \geq 2vf + \frac{v(f_i - g_i)}{n-i+1}$. By induction on i, we find that $vf_i \geq vf$ and that

$$v(f_{i+1} - g_{i+1}) \geq 2vf + \frac{2vf}{n-i+1} + \ldots + \frac{2vf}{n \cdot \ldots \cdot n - i + 1} + \frac{v(f-g)}{n \cdot \ldots \cdot n - i + 1}\ .$$

Since $vf \leq 0$ and $v(f - g) \geq 0$, all of these values are $\geq 2nvf + \frac{v(f-g)}{n!}$.

Note that every f_i is monic. So we may employ Lemma 7.47 to determine a condition which guarantees that we find a root b_i of g_i which satisfies $v(a_i - b_i) > \alpha$. This condition is that $vf_i + \frac{1}{n-i+1}v(f_i - g_i)$ be $> \alpha$ and that $v(f_i - g_i)$ be > 0. If the latter holds, then the former is satisfied if $vf_i + \frac{1}{n}v(f_i - g_i) > \alpha$. Now we can use the lower bounds that we have computed for vf_i and $v(f_i - g_i)$. We find that our condition is satisfied if $vf + \frac{1}{n}(2nvf + \frac{v(f-g)}{n!}) > \alpha$. This is equivalent to $v(f - g) > n!(n\alpha - 3nvf)$. If this holds, then we also have that $v(f_i - g_i) \geq 2nvf + \frac{v(f-g)}{n!} > n\alpha - nvf \geq n\alpha \geq 0$, because $vf \leq 0$ and α was assumed to be ≥ 0. Since $n^n \geq n! \cdot n$ for every n and $\alpha - 3vf \geq \alpha \geq 0$, we have that $n^n\alpha - 3n^n vf \geq n!(n\alpha - 3nvf)$. This proves that if $v(f - g) > n^n\alpha - 3n^n vf$, then at every induction step we can find a root b_i of g such that $v(a_i - b_i) > \alpha$, as required. □

Our condition (7.11) may be quite coarse. However, its meaning is that it gives a value which lies in the convex subgroup of vK generated by the values α, vf and vc_n. This will not essentially change under better estimates. For large enough α, the condition (7.11) can be replaced by "$v(f - g) > n^n\alpha$".

The following result has a long history. See N. Shell [SHE], §15, Theorem 1, for a list of contributors.

Corollary 7.50 *Let (K, v) be an algebraically closed valued field. Then its completion $(K, v)^c$ is also algebraically closed.*

Proof: It suffices to show that $(K, v)^c$ is dense in its algebraic closure since then it follows that both are equal. So let a be an element of this algebraic closure and f its minimal polynomial over K^c. Let $\alpha \in \widetilde{vK^c}$ be given. From Corollary 7.9 we know that $\widetilde{vK^c}|vK^c$ is algebraic, that is, $\widetilde{vK^c}$ lies in the divisible hull of $vK^c = vK$. Hence after enlarging α if necessary, we may assume that $\alpha \in vK$ (and that $\alpha \geq 0$). Since f has coefficients in K^c, we can find a polynomial g with coefficients in K, monic and of the same degree as f, such that $v(f - g)$ is bigger than $n^n\alpha - 3n^n(vf - vc_n) + vc_n$. Then it follows from Lemma 7.49 that there is a root b of g such that $v(a - b) > \alpha$. But $b \in K$ since K is algebraically closed by assumption. Hence, K and thus also K^c is dense in $\widetilde{K^c}$. □

For the conclusion of this section, let us discuss the case of two polynomials f, g not having the same degree. For this case, our above proof does not work. But we can trace this case back to the equal degree case by inverting the variables. Assume that f is given in the form (7.9) and that $g = c'_m X^m + \ldots + c'_0$ with $m \geq n$ and $c'_m \neq 0$. We can also assume that both c_0 and c'_0 are nonzero; otherwise, we just replace them by any constants of value $> n^n\alpha - 3n^n(vf - vc_n) + vc_n$. We can do this also to change the value of c_0 if vc_0 is higher than this value (we will use this fact later). Now we set

$$\begin{aligned} \tilde{f} &:= c_0 Y^m + c_1 Y^{m-1} + \ldots + c_n Y^{m-n} \\ \tilde{g} &:= c'_0 Y^m + c'_1 Y^{m-1} + \ldots + c'_n Y^{m-n} + \ldots + c'_m \end{aligned}$$

with $Y := 1/X$. Then $v\tilde{f} = vf$ and $v(\tilde{f} - \tilde{g}) = v(f - g)$. Further, a is a root of f if and only if $a \neq 0$ and $1/a$ is a root of $\tilde{f}$, and b is a root of g if and only if $b \neq 0$ and $1/b$ is a root of $\tilde{g}$. Note that $\tilde{g}$ does not admit 0 as a root, while 0 appears precisely $m - n$ times as a zero of $\tilde{f}$. Assume that $v\left(\frac{1}{a} - \frac{1}{b}\right) > v\frac{1}{a}$. Then it follows that $v\frac{1}{a} = v\frac{1}{b}$, that is, $va = vb$. Using the lower bound $vf - vc_n$ for the value $va = vb$ of the root a of f, we find that

$$v(a - b) = v\left(\frac{1}{a} - \frac{1}{b}\right) + va + vb \geq v\left(\frac{1}{a} - \frac{1}{b}\right) + 2(vf - vc_n) \; .$$

Now let

$$\alpha' := \alpha - 2(vf - vc_n)$$

and assume that

$$v(f - g) = v(\tilde{f} - \tilde{g}) > n^n\alpha' - 3n^n(vf - vc_0) + vc_0 \; . \tag{7.12}$$

Then by Theorem 7.49, $\tilde{f} = c_0 Y^{m-n} \prod_{i=1}^{n}(Y - \frac{1}{a_i})$ and $\tilde{g} = c'_0 \prod_{i=1}^{m}(Y - \frac{1}{b_i})$ such that $v\left(\frac{1}{a_i} - \frac{1}{b_i}\right) > \alpha'$ for $1 \leq i \leq n$, and $v\left(0 - \frac{1}{b_i}\right) = v\frac{1}{b_i} > \alpha'$ for $n + 1 \leq i \leq m$. It follows that $vb_i < 2(vf - vc_n) - \alpha$ for $n + 1 \leq i \leq m$. If $v\left(\frac{1}{a_i} - \frac{1}{b_i}\right) > v\frac{1}{a_i}$ is satisfied, then it also follows that $v(a_i - b_i) > \alpha$, for $1 \leq i \leq n$. On the other hand, this condition is satisfied whenever $\alpha - 2(vf - vc_n) > -va_i$ for $1 \leq i \leq n$. Using again the lower bound $vf - vc_n$ for the value of the roots of f, we find that the condition is satisfied whenever $\alpha - 2(vf - vc_n) > -(vf - vc_n)$, that is, if $\alpha > vf - vc_n$. But for this, it suffices to assume that $\alpha > 0$.

Hence, we have shown that (7.12) yields that we obtain pairs of roots with $v(a_i - b_i) > \alpha$. We have also mentioned that we can replace c_0 by another constant of value $> n^n\alpha - 3n^n(vf - vc_n) + vc_n =: \alpha_f$ if vc_0 is higher than α_f. Hence we can substitute something larger than α_f, say $\alpha + \alpha_f$ for vc_0 on the right hand side of (7.12). After this substitution, the right hand side of (7.12) will only depend on α, vf and vc_0 and will lie in the convex subgroup generated by them. We have thus proved:

Corollary 7.51 *Let (K, v) be an arbitrary valued field and $f, g \in K[X]$. Let $n := \deg f > m := \deg g$ and c_n , c'_m be the leading coefficients of f and g respectively. Then for every $\alpha \in vK$, $\alpha > 0$, there is a value β depending on α, vf and vc_0 and lying in the convex subgroup generated by these values, such that the following holds: If $v(f - g) > \beta$, then we can write $f = c_n \prod_{i=1}^{n}(X - a_i)$ and $g = c'_m \prod_{i=1}^{m}(X - b_i)$ with $a_i, b_i \in \tilde{K}$ in such a way that $v(a_i - b_i) > \alpha$ for $1 \leq i \leq n$, and $vb_i < 2(vf - vc_n) - \alpha$ for $n + 1 \leq i \leq m$.*

Exercise 7.7 *Let* $\mathbf{K} = (K, v)$ *be an arbitrary valued field and let* a *be algebraic with minimal polynomial* f *over* K*. Prove that*

a) $v\mathrm{Tr}_{K(a)|K}(a) \geq \min_i va_i$ *and* $v\mathrm{N}_{K(a)|K}(a) = va_1 + \ldots + va_n$,

c) $va \in vK$ *if* a *is separable over* K *and the value of* a *is different from the values of the other conjugates of* a*.*

Exercise 7.8 *Let the assumptions be as in Lemma 7.47, with the exception that we allow* f *to be not monic and* $m := \deg g$ *to be bigger than* $\deg f$*. Let* c_n , c'_m *be the leading coefficients of* f *and* g *respectively. Show that* $vg(a) \geq m(vf - vc_n) + v(f - g)$ *and that there is a root* b *of* g *such that* $v(a - b) \geq vf - vc_n + \frac{1}{m}v(f - g) - \frac{1}{m}vc'_m$ *. Explain why the latter formula in case of* $m > n$ *is not appropriate to describe the growth of* $v(a - b)$ *in dependence on* $v(f - g)$*.*

Exercise 7.9 *Let* f *be a polynomial of the form (7.9). In order to compute the value of all roots from the value of the coefficients, repeat the procedure that we have applied to the roots* $a_1, \ldots, a_m$ *of minimal value, preceding to Lemma 7.42. Cf. the method of* **Newton polygons** *as described e.g. in [KOB].*

Exercise 7.10 *Let* (K, v) *be a valued field and extend* v *to* $\tilde{K}$*. Let* $(K(a)|K, v)$ *be a finite defectless extension and* f *the minimal polynomial of* a *over* K*. Show that if* $g \in K[X]$ *of degree* $\deg g = \deg f$ *and if* $b \in \tilde{K}$ *is a root of* g*, then also* $(K(b)|K, v)$ *is a finite defectless extension with* $e(K(b)|K, v) = e(K(a)|K, v)$ *and* $f(K(b)|K, v) = f(K(a)|K, v)$ *if* $v(f - g)$ *is large enough (where* v *is the Gau valuation on* $K(X)$ *). Discuss also the case of* $\deg g > \deg f$*. (Hint: choose a standard valuation basis of* $(K(a)|K, v)$ *and use the idea of Exercise 7.3.)*

7.6 Construction of valued fields

In this section, we will deal with the following problem. Suppose that (K, v) is a valued field and that $\Gamma|vK$ is an extension of ordered abelian groups and $k|\overline{K}$ is a field extension. Does there exist an extension $(L|K, v)$ of valued fields such that $vL = \Gamma$ and $\overline{L} = k$? We include the case of (K, v) being trivially valued (like the finite field $\mathbb{F}_p$); this amounts to the construction of a valued field with given value group and residue field. We know from Section 7.4 that there are extensions of the valuation from (K, v) to every extension field, but there was nothing said about their value groups and residue fields. We need some lemmata. Let us first note the following almost trivial observation:

Lemma 7.52 *Let* (K, v) *be a valued field and* $L = K(t_i \mid i \in I)$ *an extension field of* K*. Assume that* w *is a map from* L *into an ordered abelian group* Γ *which extends* v*. Then* w *is a valuation on* L *if and only if it satisfies axioms* (V0), (VT), (VH) *on the ring* $K[t_i \mid i \in I]$*, and* $w(f/g) = wf - wg$ *for every* $f, g \in K[t_i \mid i \in I]$*.*

Proof: Suppose that w satisfies the above conditions. We show that it then satisfies the triangle inequality (VT) on all of L; the other details are left to the reader. Let $f_1, f_2, g_1, g_2 \in K[t_i \mid i \in I]$. Then $v(f_1/g_1 - f_2/g_2) = v(f_1g_2 - f_2g_1) - vg_1g_2 \geq \min\{vf_1g_2, vf_2g_1\} - vg_1g_2 = \min\{v(f_1/g_1), v(f_2/g_2)\}$. □

Lemma 7.53 *Let* (K, v) *be a valued field and* α_i , $i \in I$*, elements in some ordered abelian group extension* Γ *of* vK *which are rationally independent over* vK*. Choose elements* x_i , $i \in I$ *algebraically independent over* K*. Then* v *extends in a unique way from* K *to the rational function field* $K(x_i \mid i \in I)$ *such that* $vx_i = \alpha_i$ *for every* $i \in I$*. Moreover,*

$$vK(x_i \mid i \in I) \;=\; vK \oplus \bigoplus_{i \in I} \mathbb{Z}\alpha_i \quad \textit{and} \quad \overline{K(x_i \mid i \in I)} \;=\; \overline{K}\ .$$

Proof: In view of the foregoing lemma, it suffices to define the extension of v on the polynomial ring $K[x_i \mid i \in I]$ and to show that it satisfies axioms (V0), (VT), (VH) there; its definition on $K(x_i \mid i \in I)$ is then canonically given by the rule $v(f/g) = vf - vg$. Every polynomial $f \in K[x_i \mid i \in I]$ is a finite sum of monomials of the form $c\prod_{i\in I} x_i^{\nu_i}$, with $c \in K$ and integers $\nu_i \geq 0$, only finitely many of them nonzero. Since the values α_i, $i \in I$, are assumed to be rationally independent over K, we have that $vc+\sum_{i\in I}\nu_i\alpha_i \neq vc'+\sum_{i\in I}\nu_i'\alpha_i$ whenever $(\nu_i)_{i\in I} \neq (\nu_i')_{i\in I}$, independently of the choice of $c, c' \in K$. So we can define the value of f to be the minimum of the values $vc + \sum_{i\in I}\nu_i\alpha_i$, taken over all monomials $c\prod_{i\in I} x_i^{\nu_i}$ appearing in f. This value is equal to ∞ if and only if all coefficients c are zero. This prove (V0). If we multiply two polynomials f and g, then the monomial of minimal value in the product fg is just the product of the unique monomials of minimal value in f and in g. Let us denote them by $c\prod_{i\in I} x_i^{\nu_i}$ and $c'\prod_{i\in I} x_i^{\nu_i'}$. The value of their product is $v\left(cc'\prod_{i\in I} x_i^{\nu_i+\nu_i'}\right) = vcc' + \sum_{i\in I}(\nu_i+\nu_i')\alpha_i = vc + \sum_{i\in I}\nu_i\alpha_i + vc' + \sum_{i\in I}\nu_i'\alpha_i = vf + vg$. This proves (VH). For the proof of the triangle inequality (VT), we observe that if we add two polynomials f and g, then any two monomials $c\prod_{i\in I} x_i^{\nu_i}$ and $c'\prod_{i\in I} x_i^{\nu_i'}$ can be added if and only if $\nu_i = \nu_i'$ for all $i \in I$. If this is the case, then their sum is just $(c+c')\prod_{i\in I} x_i^{\nu_i}$ whose value is $v(c+c') + \sum_{i\in I}\nu_i\alpha_i \geq \min\{vc, vc'\} + \sum_{i\in I}\nu_i\alpha_i = \min\{vc + \sum_{i\in I}\nu_i\alpha_i, vc' + \sum_{i\in I}\nu_i'\alpha_i\} \geq \min\{vf, vg\}$. So we see that the value of every monomial in $f+g$ is $\geq \min\{vf, vg\}$, which by definition yields that $v(f+g) \geq \min\{vf, vg\}$. This shows that v satisfies (VT) on $K[x_i \mid i \in I]$.

The uniqueness of v on $K(x_i \mid i \in I)$ was already stated in Lemma 7.31. □

The analogue to this lemma for the residue-transcendental case was already stated in Lemma 7.43. Before we continue with algebraic extensions, let us adjust the following notion to our purposes. Usually, when one speaks of an **Artin-Schreier extension** then one means an extension of a field K generated by a root of an irreducible polynomial of the form $X^p - X - c$, provided that $p = \operatorname{char} K$. When working with a valued field, we will replace the condition "$p = \operatorname{char} K$" by the weaker condition "$p = \operatorname{char} \overline{K}$". In fact, such extensions also play an important role in the mixed characteristic case, where $\operatorname{char} K = 0$. This will be seen when we deal with the defect of valued field extensions. See also R. E.MacKenzie and G. Whaples [MCK–WHA].

Every Artin-Schreier polynomial $X^p - X - c$ is separable since its derivative does not vanish. The following is a simple but very useful observation:

Lemma 7.54 *Let (K, v) be a valued field and $c \in K$ such that $vc < 0$. If an element a in some valued field extension (L, v) of (K, v) satisfies $v(a^p - a - c) \geq 0$, then $v(a^p - c) = va > pva = vc$. In particular, if $a \in \tilde{K}$ such that $a^p - a = c$, then this holds for every extension of v from K to $K(a)$.*

Proof: Necessarily, $va < 0$ since otherwise, $0 \leq v(a^p - a - c) = \min\{pva, va, vc\} = vc$, a contradiction. Since $va < 0$, it follows that $va^p = pva < va$ and thus, $vc = v(a^p - a) = \min\{pva, va\} = pva$. Moreover, $v(a^p - c) = \min\{v(a^p - a - c), va\} = va$. □

Lemma 7.55 *Let (K, v) be a valued field, p a prime and α an element of the divisible hull of vK such that $p\alpha \in vK$, $\alpha \notin vK$. Choose an element $a \in \tilde{K}$ such that $a^p \in K$ and $va^p = p\alpha$.*

Then v extends in a unique way from K to $K(a)$. It satisfies $va = \alpha$ and

$$vK(a) = vK + \mathbb{Z}\alpha \quad \textit{and} \quad \overline{K(a)} = \overline{K} \ .$$

If $\operatorname{char} K = \operatorname{char} \overline{K} = p > 0$*, then this extension $K(a)|K$ is purely inseparable. On the other hand, if* $\operatorname{char} \overline{K} = p > 0$*, then there is always a separable Artin-Schreier extension $K(a)|K$ with the same properties (if $\alpha < 0$, then a itself can be chosen to be an Artin-Schreier root).*

Proof: Choose some $c \in K$ such that $vc = p\alpha$. Further, choose some $a \in \tilde{K}$ such that $a^p = c$. By Corollary 7.37, there is an extension of v from K to $K(a)$. For this extension, we must have that $pva = vc = p\alpha$ by virtue of Lemma 7.54. Consequently, $va = \alpha$ and $(vK(a) : vK) \geq (vK + \mathbb{Z}\alpha : vK) = p$. On the other hand, the fundamental inequality (7.2) shows that

$$p = [K(a) : K] \geq (vK(a) : vK) \cdot [\overline{K(a)} : \overline{K}] \geq (vK(a) : vK) \geq p \ .$$

Hence, equality holds everywhere, and we find that $(vK(a) : vK) = p$ and $[\overline{K(a)} : \overline{K}] = 1$. That is, $vK(a) = vK + \mathbb{Z}\alpha$ and $\overline{K(a)} = \overline{K}$. Further, $1, a, \ldots, a^{p-1}$ is a valuation basis of $(K(a)|K, v)$ since the values of these elements belong to distinct cosets modulo vK. The uniqueness of v on $K(a)$ thus follows from Lemma 7.13.

Now suppose that p is equal to the characteristic of K. Then the polynomial $X^p - c$ is purely inseparable. To construct a separable extension, assume that $vc = p\alpha < 0$; otherwise, just replace c by $1/c$ in the following. By the foregoing lemma, every root a of the Artin-Schreier polynomial $X^p - X - c$ must satisfy $pva = vc$. Now we replace a by $1/a$ if $\alpha > 0$ (but note that $1/a$ is not an Artin-Schreier root). □

Lemma 7.56 *Let (K, v) be a valued field and ζ an element of the algebraic closure of $\overline{K}$. Choose a monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ whose reduction $\overline{f}$ is the minimal polynomial of ζ over $\overline{K}$. Further, choose a root $b \in \tilde{K}$ of f. Then v extends in a unique way from K to $K(b)$. The associated residue map can be extended such that $\overline{b} = \zeta$. Further,* $[K(b) : K] = [\overline{K}(\zeta) : \overline{K}]$*, and*

$$vK(b) = vK \quad \textit{and} \quad \overline{K(b)} = \overline{K}(\zeta) \ .$$

In all cases, f can be chosen to be separable, provided that the valuation v is nontrivial. On the other hand, if $\operatorname{char} K = \operatorname{char} \overline{K} = p > 0$ *and ζ is purely inseparable over $\overline{K}$, then b can be chosen to be purely inseparable over K.*

If v is not trivial on K, $\operatorname{char} \overline{K} = p > 0$ *and $\zeta^p \in \overline{K}$, $\zeta \notin \overline{K}$, then there is also an Artin-Schreier extension $K(b)|K$ such that $[K(b) : K] = [\overline{K}(\zeta) : \overline{K}]$, $vK(b) = vK$ and $\overline{K(b)} = \overline{K}(\zeta)$.*

Proof: By Corollary 7.37, there is an extension of v from K to $K(b)$. Since f has integral coefficients, b must also be integral for this extension, and $\overline{b}$ must be a root of $\overline{f}$. We may compose the residue map with an isomorphism in $\operatorname{Gal} \overline{K}$ which sends this root to ζ. Doing so, we obtain a residue map (still associated with v) that satisfies $\overline{b} = \zeta$. Now $\zeta \in \overline{K(b)}$ and consequently, $[\overline{K(b)} : \overline{K}] \geq [\overline{K}(\zeta) : \overline{K}] = \deg \overline{f} = \deg f$. On the other hand, the fundamental inequality (7.2) shows that

$$\deg f = [K(b) : K] \geq (vK(b) : vK) \cdot [\overline{K(b)} : \overline{K}] \geq [\overline{K(b)} : \overline{K}] \geq \deg f \ .$$

Hence, equality holds everywhere, and we find that $[\overline{K(b)} : \overline{K}] = [\overline{K}(\zeta) : \overline{K}] = [K(b) : K]$ and $(vK(b) : vK) = 1$. That is, $vK(b) = vK$ and $\overline{K(b)} = \overline{K}(\zeta)$. Further, $1, b, \ldots, b^{\deg f-1}$ is a valuation basis of $(K(b)|K, v)$ since the residues of these elements are $\overline{K}$-linearly independent. The uniqueness of v on $K(a)$ thus follows from Lemma 7.13.

If $\overline{f}$ is separable, then so is f. If $\overline{f}$ is not separable and v is nontrivial on K, then f can always chosen to be separable since we can add a summand cX with $0 \neq c \in \mathcal{M}_{\mathbf{K}}$ without changing the reduction of f. On the other hand, if $\overline{f} = X^{p^\nu} - \overline{c}$ is purely inseparable, then we can choose $f = X^{p^\nu} - c$ which also is purely inseparable if $\operatorname{char} K = p$.

Now suppose that $\operatorname{char} \overline{K} = p > 0$ and $\zeta^p \in \overline{K}$, $\zeta \notin \overline{K}$. Choose $c \in K$ such that $\overline{c} = \zeta^p$. To construct an Artin-Schreier extension, choose any $d \in K$ with $vd < 0$, and let b be a root of the Artin-Schreier polynomial $X^p - X - d^p c$. Since $vd^p c = pvd < 0$, Lemma 7.54 shows that $v(b^p - d^p c) = vb > vb^p$. Consequently, $v((b/d)^p - c) > v(b/d)^p = vc = 0$, and $\overline{b/d} = \overline{c}^{1/p} = \zeta$. As before, it follows that $vK(b) = vK$ and $\overline{K(b)} = \overline{K}(\zeta)$. □

Theorem 7.57 *Let (K, v) be an arbitrary valued field. For every extension $\Gamma|vK$ of ordered abelian groups and every field extension $k|\overline{K}$, there is an extension (L, v) of (K, v) such that $vL = \Gamma$ and $\overline{L} = k$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then $L|K$ can be chosen to be algebraic.*

In general, L can be chosen to be a separable extension of K (provided that $\Gamma \neq \{0\}$) and such that $(L|K, v)$ admits a standard valuation transcendence basis. On the other hand, if Γ/vK is a p-group with p the characteristic exponent of $\overline{K}$, and if $k|\overline{K}$ is a purely inseparable extension, then L can be chosen to be a purely inseparable extension of K.

Proof: For the proof, we assume that $\Gamma \neq \{0\}$ (the other case is trivial). Let α_i, $i \in I$, be a transcendence basis of $\Gamma|vK$. Then by Lemma 7.53 there is an extension $(K_1, v) := (K(x_i \mid i \in I), v)$ of (K, v) such that $vK_1 = vK \oplus \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\overline{K}_1 = \overline{K}$. Next, choose a transcendence basis ζ_j, $j \in J$, of $k|\overline{K}$. Then by Lemma 7.43 there is an extension $(K_2, v) := (K_1(y_j \mid j \in J), v)$ of (K_1, v) such that $vK_2 = vK_1$ and $\overline{K}_2 = \overline{K}(\zeta_j \mid j \in J)$. Note that $\{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ is a standard valuation transcendence basis of $(K_2|K, v)$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then $K_2 = K$.

If we are given an ascending chain of valued fields whose value groups are subgroups of Γ and whose residue fields are subfields of k, then the union over this chain is again a valued field whose value group is a subgroup of Γ and whose residue field is a subfield of k. So a standard argument using Zorn's Lemma together with the transitivity of separable extensions shows that there are maximal separable algebraic extension fields of (K_2, v) with these properties. Choose one of them and call it (L, v). We have to show that $vL = \Gamma$ and $\overline{L} = k$. Since already Γ/vK_2 is a torsion group and $k|\overline{K}_2$ is algebraic, the same holds for Γ/vL and $k|\overline{L}$. If vL is a proper subgroup of Γ, then there is some prime p and some element $\alpha \in \Gamma \setminus vL$ such that $p\alpha \in vL$. But then, Lemma 7.55 shows that there exists a proper separable algebraic extension (L', v) of (L, v) such that $vL' = vL + \mathbb{Z}\alpha \subset \Gamma$ and $\overline{L'} = \overline{L} \subset k$, which contradicts the maximality of L. If $\overline{L}$ is a proper subfield of k, then there is some element $\zeta \in k \setminus \overline{L}$, and ζ is algebraic over $\overline{L}$. But then, Lemma 7.56 shows that there exists a proper separable algebraic extension (L', v) of (L, v) such that $vL' = vL \subset \Gamma$ and $\overline{L'} = \overline{L}(\zeta) \subset k$, which again contradicts the maximality of L (here we have used $\Gamma \neq \{0\}$, which implies that v is not trivial on L). This proves that $vL = \Gamma$ and $\overline{L} = k$, and (L, v) is the required extension of (K, v). Since K_2 is generated over K by a set of elements which are algebraically independent over K, we know from Lemma 6.40 that

$K_2|K$ is separable. Since also $L|K_2$ is separable, we find that (L,v) is a separable extension of (K,v). Since $L|K_2$ is algebraic, $\{x_i\,,\ y_j \mid i \in I\,,\ j \in J\}$ is a valuation transcendence basis of $(K_2|K,v)$. If $\Gamma|vK$ and $k|\overline{K}$ are algebraic, then L is an algebraic extension of $K = K_2\,$.

In the case of Γ/vK being a p-group and $k|\overline{K}$ being a purely inseparable extension, we set $K_2 = K$ and consider the purely inseparable extensions of (K,v) whose value group is a subgroup of Γ and whose residue field is a subfield of k. The same arguments as above work, with "purely inseparable" in the place of "separable". We then obtain a purely inseparable extension of (K,v) with the required properties. □

Every ordered abelian group is an extension of the trivial group $\{0\}$ as well as of the ordered abelian group $\mathbb{Z}$. Every field of characteristic 0 is an extension of $\mathbb{Q}$, and every field of characteristic $p > 0$ is an extension of $\mathbb{F}_p$. Let Γ be an ordered abelian group and k a field. If $\operatorname{char} k = 0$, then $\mathbb{Q}$ endowed with the trivial valuation v will satisfy $v\mathbb{Q} = \{0\} \subset \Gamma$ and $\overline{\mathbb{Q}} = \mathbb{Q} \subset k$. If $\operatorname{char} k = p > 0$, then we can choose v to be the p-adic valuation on $\mathbb{Q}$ (see Section 5.2) to obtain that $v\mathbb{Q} = \mathbb{Z} \subset \Gamma$ and $\overline{\mathbb{Q}} = \mathbb{F}_p \subset k$. But also $\mathbb{F}_p$ endowed with the trivial valuation v will satisfy $v\mathbb{F}_p = \{0\} \subset \Gamma$ and $\overline{\mathbb{F}_p} = \mathbb{F}_p \subset k$. An application of the foregoing theorem now proves:

Corollary 7.58 *Let Γ be an ordered abelian group and k a field. Then there is a valued field (L,v) with $vL = \Gamma$ and $\overline{L} = k$. If $\operatorname{char} k = p > 0$, then L can be chosen to be of characteristic 0 (mixed characteristic case) or of characteristic p (equal characteristic case).*

We can also derive information about extensions of the form $(\tilde{K}|K,v)$ and $(K^{\rm sep}|K,v)$. From Corollary 7.9 we know that $v\tilde{K}|vK$, $vK^{\rm sep}|vK$, $\overline{\tilde{K}}|\overline{K}$ and $\overline{K^{\rm sep}}|\overline{K}$ are algebraic extensions. On the other hand, Lemma 7.55 shows that the value group of a separable-algebraically closed field must be divisible. Similarly, it follows from Lemma 7.56 that the residue field of a separable-algebraically closed field must be algebraically closed. This proves:

Lemma 7.59 *Let (K,v) be a valued field and extend v to $\tilde{K}$. Then the value groups $v\tilde{K}$ and $vK^{\rm sep}$ are equal to the divisible hull of vK, and the residue fields $\overline{\tilde{K}}$ and $\overline{K^{\rm sep}}$ are equal to the algebraic closure of $\overline{K}$.*

Corollary 7.60 *Let (K,v) be a valued field with divisible value group and algebraically closed residue field. Then every maximal immediate extension and every maximal immediate algebraic extension is algebraically closed. In particular, every maximal immediate extension of an algebraically closed valued field is again algebraically closed.*

Proof: If K is algebraically closed, then by the foregoing lemma, vK is divisible and $\overline{K}$ is algebraically closed. Hence it suffices to prove the first assertion. So let vK be divisible and $\overline{K}$ be algebraically closed, and let (L,v) be an immediate extension of (K,v). Then by the foregoing lemma, the extension $(\tilde{L}|L,v)$ must be immediate for every extension of v from L to $\tilde{L}$. Hence if (L,v) admits no immediate algebraic extension, then this extension is trivial, that is, L is algebraically closed. □

A valued field (K,v) of residue characteristic $p>0$ will be called **Artin-Schreier closed** if every Artin-Schreier polynomial with coefficients in K admits a root in K. Recall that if $\operatorname{char} K = p$, then this means that every Artin-Schreier polynomial with coefficients in K splits into linear factors over K. As a corollary to Lemma 7.56 and Lemma 7.55, we obtain:

Corollary 7.61 *Every Artin-Schreier closed valued field of residue characteristic $p>0$ has p-divisible value group and perfect residue field.*

Similarly, Lemma 7.56 and Lemma 7.55 show that the value group of the perfect hull of a valued field (K,v) of characteristic $p>0$ is p-divisible, and that its residue field is perfect. On the other hand, if $\alpha \in vK^{1/p^\infty}$ and $a \in K^{1/p^\infty}$ such that $va=\alpha$, then $a^{p^m} \in K$ and thus $p^m\alpha \in vK$ for some integer $m \geq 0$. This shows that vK^{1/p^∞} is contained in the p-divisible hull of vK. Similarly, if ζ is an element of $\overline{K^{1/p^\infty}}$ and $b \in K^{1/p^\infty}$ is such that $\overline{b}=\zeta$, then $b^{p^m} \in K$ and thus $\zeta^{p^m} \in \overline{K}$ for some integer $m \geq 0$. This shows that $\overline{K^{1/p^\infty}}$ is contained in $\overline{K}^{1/p^\infty}$. This proves:

Lemma 7.62 *Let (K,v) be a valued field of characteristic $p>0$ and extend v to K^{1/p^∞}. Then vK^{1/p^∞} is the p-divisible hull of vK, and $\overline{K^{1/p^\infty}}$ is the perfect hull of $\overline{K}$.*

We leave it to the reader to prove the following equalities along the lines of our earlier arguments:

$$vK^{1/p} = \frac{1}{p}vK \quad \text{and} \quad \overline{K^{1/p}} = \overline{K}^{1/p}$$

and

$$vK^p = pvK \quad \text{and} \quad \overline{K^p} = \overline{K}^p \ .$$

7.7 Extension of places to algebraic field extensions

In this section, we let $L|K$ be an algebraic field extension. We want to determine all possible extensions of a given valuation v from K to L. If $\tilde{v}$ is an arbitrary extension of v from K to $\tilde{K}$, then for every embedding ι of L in $\tilde{K}$ over K, the valuation $\tilde{v}\iota$ (which shall denote the more correct expression $\tilde{v}|_{\iota L}\iota$) is an extension of v from K to L. We shall show that these are all possible extensions (see Theorem 7.68 below).

We need a basic approximation theorem:

Theorem 7.63 (Chinese Remainder Theorem)
Let $\mathcal{R}$ be a commutative ring with 1 and $\mathcal{I}_1,\ldots,\mathcal{I}_n$ ideals of $\mathcal{R}$. Assume that these ideals are **pairwise comaximal**, *that is, $\mathcal{I}_i+\mathcal{I}_j=\mathcal{R}$ whenever $1 \leq i < j \leq n$. Then for every choice of $r_1,\ldots,r_n \in \mathcal{R}$ there is some $r \in \mathcal{R}$ such that $r \equiv r_i$ modulo $\mathcal{I}_i$ for $1 \leq i \leq n$.*

Proof: In the case $n=2$, there is an equation $1=s_1+s_2$ with $s_i \in \mathcal{I}_i$, and $r=r_2s_1+r_1s_2$ is the required element.

Now assume that we have proved our theorem for $n-1$ ideals. For every $i \geq 2$ we choose elements $s_i' \in \mathcal{I}_1$ and $s_i \in \mathcal{I}_i$ such that $1=s_i'+s_i$. The product

$$\prod_{i=2}^{n}(s_i'+s_i) \in \mathcal{I}_1 + \prod_{i=2}^{n}\mathcal{I}_i$$

is equal to 1. That is, the ideals $\mathcal{I}_1$ and $\prod_{i=2}^{n}\mathcal{I}_i$ of $\mathcal{R}$ are comaximal. By the first part of our proof, we may choose an element $t_1 \in \mathcal{R}$ such that

$$t_1 \equiv 1 \pmod{\mathcal{I}_1} \quad \text{and} \quad t_1 \equiv 0 \pmod{\prod_{i=2}^{n}\mathcal{I}_i}\,.$$

By a similar procedure we find elements $t_1, \ldots, t_n$ such that we obtain

$$t_i \equiv 1 \pmod{\mathcal{I}_i} \quad \text{and} \quad t_j \equiv 0 \pmod{\mathcal{I}_j} \quad \text{for } j \neq i\,.$$

Now $r = r_1t_1 + \ldots + r_nt_n$ is the required element. □

Now let us fix the following situation:

$$\left.\begin{array}{rl} \mathbf{K} & = (K, v) \text{ a valued field with} \\ \mathcal{O}_{\mathbf{K}} & \text{its valuation ring, which has} \\ \mathcal{M}_{\mathbf{K}} & \text{as its maximal ideal,} \\ L|K & \text{an algebraic field extension,} \\ \mathcal{R} & \text{the integral closure of } \mathcal{O}_{\mathbf{K}} \text{ in } L. \end{array}\right\} \tag{7.13}$$

We need a first statement about the prime ideals of $\mathcal{R}$.

Lemma 7.64 *Let $\mathcal{R} \supset \mathcal{R}'$ be an extension of commutative rings with 1 and assume that $\mathcal{R}$ is integral over $\mathcal{R}'$. Let $\mathcal{I}$ be a prime ideal of $\mathcal{R}$ and set $\mathcal{I}' = \mathcal{I} \cap \mathcal{R}'$. Then $\mathcal{I}'$ is a maximal ideal if and only if $\mathcal{I}$ is.*

Proof: Assume that $\mathcal{I}'$ is maximal. Then $\mathcal{R}'/\mathcal{I}'$ is a field. Since $\mathcal{I}$ is a prime ideal satisfying $\mathcal{I}' = \mathcal{I} \cap \mathcal{R}'$, we have that $\mathcal{R}/\mathcal{I}$ is an entire ring extending $\mathcal{R}'/\mathcal{I}'$. Further, $\mathcal{R}/\mathcal{I}$ is integral over $\mathcal{R}'/\mathcal{I}'$ because $\mathcal{R}$ is integral over $\mathcal{R}'$. Hence, every element $a \in \mathcal{R}/\mathcal{I}$ is algebraic over the field $\mathcal{R}'/\mathcal{I}'$, which yields that $\mathcal{R}'/\mathcal{I}'[a]$ is a field. Consequently, every element of $\mathcal{R}/\mathcal{I}$ is invertible, that is, $\mathcal{R}/\mathcal{I}$ is a field and $\mathcal{I}$ is a maximal ideal.

For the converse, assume that $\mathcal{I}$ is a maximal ideal. Then $\mathcal{R}/\mathcal{I}$ is field, and as before, it is integral over the subring $\mathcal{R}'/\mathcal{I}'$. Suppose that $\mathcal{I}'$ is not maximal, that is, $R' := \mathcal{R}'/\mathcal{I}'$ is not a field. Then it admits a nonzero maximal ideal I'. By Theorem 7.34 there exists a valuation ring R of $\mathcal{R}/\mathcal{I}$ containing R' such that the maximal ideal of R contains I' and is thus nonzero. But by virtue of Theorem 7.40, R must contain the integral closure of R' in $\mathcal{R}/\mathcal{I}$ which is $\mathcal{R}/\mathcal{I}$ itself. Hence, R is a field and can not have a proper nonzero ideal. This contradiction shows that $\mathcal{I}'$ must be maximal. □

Lemma 7.65 *Assume that $\mathcal{O}$ is a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}$, and let $\mathcal{I}$ be the prime ideal $\mathcal{M} \cap \mathcal{R}$ of $\mathcal{R}$. Then $\mathcal{I}$ is a maximal ideal and $\mathcal{O}$ is the localization $\mathcal{R}_{\mathcal{I}}$.*

Proof: Since $\mathcal{O}$ lies above $\mathcal{O}_{\mathbf{K}}$, we have $\mathcal{I} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M} \cap \mathcal{R} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M} \cap \mathcal{O}_{\mathbf{K}} = \mathcal{M}_{\mathbf{K}}$. Hence by Lemma 7.64, $\mathcal{I}$ is a maximal ideal. From $\mathcal{I} = \mathcal{M} \cap \mathcal{R}$ it follows that $\mathcal{R}_{\mathcal{I}} \subset \mathcal{O}$. Now let $x \in \mathcal{O}$; we want to show that $x \in \mathcal{R}_{\mathcal{I}}$. Since $L|K$ is assumed to be algebraic, there is an equation $a_nx^n + \ldots + a_0 = 0$ with coefficients $a_i \in K$, not all 0. Let m be the maximal index for which va_m is the minimum of the values of these coefficients. Hence,

the elements $b_i := a_i/a_m$ are all contained in $\mathcal{O}_\mathbf{K}$. Dividing our equation by $a_m x^m$, and defining

$$y := b_n x^{n-m} + \ldots + b_{m+1}x + 1 \in \mathcal{O}_\mathbf{K}[x], \quad z = b_{m-1} + \ldots + b_1 \left(\frac{1}{x}\right)^{m-2} + b_0 \left(\frac{1}{x}\right)^{m-1} \in \mathcal{O}_\mathbf{K}\left[\frac{1}{x}\right],$$

we obtain $y + \frac{1}{x}z = 0$. Now let $\mathcal{O}'$ be any valuation ring of L containing $\mathcal{O}_\mathbf{K}$. If $\mathcal{O}'$ contains x, then $y \in \mathcal{O}_\mathbf{K}[x] \subset \mathcal{O}'$, and $z = -yx \in \mathcal{O}'$. If $\mathcal{O}'$ contains $\frac{1}{x}$, then $z \in \mathcal{O}_\mathbf{K}[\frac{1}{x}] \subset \mathcal{O}'$, and $y = -\frac{1}{x}z \in \mathcal{O}'$. Since $\mathcal{O}'$ satisfies (VR), we see that it will always contain y and z. From Theorem 7.40 we deduce $y, z \in \mathcal{R}$.

By the minimality of m, we have $b_n, \ldots, b_{m+1} \in \mathcal{M}_\mathbf{K} \subset \mathcal{I}$. This shows $y \in 1 + \mathcal{I}$, yielding that $y \notin \mathcal{I}$. Consequently, $x = -z/y \in \mathcal{R}_\mathcal{I}$. □

We will now study how the automorphisms of $L|K$ act on the maximal ideals of $\mathcal{R}$. Note that every automorphism of $L|K$ maps $\mathcal{R}$ into $\mathcal{R}$. Indeed, if $\sigma \in \operatorname{Gal} L|K$ and $b \in \mathcal{R}$ then σb satisfies the same equation with coefficients from $\mathcal{O}_\mathbf{K}$ as b and is consequently also integral over $\mathcal{O}_\mathbf{K}$. Further, σ sends prime ideals of $\mathcal{R}$ onto prime ideals of $\mathcal{R}$, and maximal ideals onto maximal ideals.

Lemma 7.66 *Let $L|K$ be a finite normal extension. Assume that $\mathcal{I}$ and $\mathcal{J}$ are maximal ideals of $\mathcal{R}$ and that $\mathcal{I}$ contains $\mathcal{M}_\mathbf{K}$. Then there exists $\sigma \in \operatorname{Gal} L|K$ such that $\sigma\mathcal{J} = \mathcal{I}$.*

Proof: Suppose that $\mathcal{I} \neq \sigma\mathcal{J}$ for all $\sigma \in \operatorname{Gal} L|K$. Then $\tau\mathcal{I} \neq \sigma\mathcal{J}$ for all $\sigma, \tau \in \operatorname{Gal} L|K$. Since $\mathcal{J}$ and $\mathcal{I}$ are maximal ideals of $\mathcal{R}$, the same holds for all $\sigma\mathcal{J}$ and $\sigma\mathcal{I}$, $\sigma \in \operatorname{Gal} L|K$. Hence, the sum of any two distinct of them is equal to $\mathcal{R}$. So we may apply the Chinese Remainder Theorem 7.63. According to this theorem, there exists $b \in \mathcal{R}$ such that for all $\sigma \in \operatorname{Gal} L|K$, $b \equiv 0$ modulo $\sigma\mathcal{J}$ and $b \equiv 1$ modulo $\sigma\mathcal{I}$. Let $[L:K]_\text{insep} = p^\nu$ where p is the characteristic exponent of K. The norm $\mathrm{N}_{L|K}(b) = \prod_{\sigma \in \operatorname{Gal} L|K} \sigma b^{p^\nu}$ lies in $\mathcal{R} \cap K$. Since $\mathcal{O}_\mathbf{K}$ is integrally closed by Lemma 5.1, we have $\mathcal{R} \cap K = \mathcal{O}_\mathbf{K}$, showing that $\mathrm{N}_{L|K}(b) \in \mathcal{O}_\mathbf{K}$.

Since $b \in \mathcal{J}$ by our choice of b, it follows that $a \in \mathcal{J} \cap \mathcal{O}_\mathbf{K}$. Since $\mathcal{J}$ is a proper ideal of $\mathcal{R}$, we know that $\mathcal{J} \cap \mathcal{O}_\mathbf{K}$ is a proper ideal of $\mathcal{O}_\mathbf{K}$ and thus contained in $\mathcal{M}_\mathbf{K}$. By assumption, $\mathcal{I} \cap \mathcal{O}_\mathbf{K} = \mathcal{M}_\mathbf{K}$, hence $\mathrm{N}_{L|K}(b) \in \mathcal{I}$. But $b \notin \sigma\mathcal{I}$ for all $\sigma \in \operatorname{Gal} L|K$ by our choice of b, that is, $\sigma b \notin \mathcal{I}$ for all $\sigma \in \operatorname{Gal} L|K$. This contradicts our assumption that $\mathcal{I}$ be a prime ideal. This contradiction completes the proof of our lemma. □

Theorem 7.67 *Let $L|K$ be a normal algebraic field extension and $\mathcal{O}_\mathbf{K}$ a valuation ring of K with maximal ideal $\mathcal{M}_\mathbf{K}$. Let $\mathcal{R}$ be the integral closure of $\mathcal{O}_\mathbf{K}$ in L. For every two valuation rings $\mathcal{O}, \mathcal{O}'$ of L lying above $\mathcal{O}_\mathbf{K}$, with maximal ideals $\mathcal{M}, \mathcal{M}'$ respectively, there is an automorphism $\sigma \in \operatorname{Gal} L|K$ such that $\sigma\mathcal{O}' = \mathcal{O}$ and $\sigma\mathcal{M}' = \mathcal{M}$. The valuation rings of L lying above $\mathcal{O}_\mathbf{K}$ are precisely all the localizations of $\mathcal{R}$ with respect to its maximal ideals. Moreover, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{O}_\mathbf{K}$.*

Proof: According to Lemma 7.65, with the maximal ideals $\mathcal{I} = \mathcal{M} \cap \mathcal{R}$ and $\mathcal{I}' = \mathcal{M}' \cap \mathcal{R}$, we have $\mathcal{O} = \mathcal{R}_\mathcal{I}$ and $\mathcal{O}' = \mathcal{R}_{\mathcal{I}'}$. Suppose that we are able to show for every maximal ideal $\mathcal{J}$ the existence of some $\sigma \in \operatorname{Gal} L|K$ such that $\sigma\mathcal{J} = \mathcal{I}$. Then because of $\sigma\mathcal{R} = \mathcal{R}$, we obtain $\sigma\mathcal{R}_\mathcal{J} = \mathcal{R}_{\sigma\mathcal{J}} = \mathcal{R}_\mathcal{I} = \mathcal{O}$. Applying this to $\mathcal{J} = \mathcal{I}'$, we obtain $\sigma\mathcal{O}' = \mathcal{O}$. But then, σ will also send the unique maximal ideal $\mathcal{M}'$ of $\mathcal{O}'$ onto the unique maximal ideal $\mathcal{M}$ of $\mathcal{O}$. With $\mathcal{J}$ running through all maximal ideals of $\mathcal{R}$, also the last assertion of the theorem will follow: By Theorem 7.34 there is at least one valuation ring $\mathcal{O}$ of L lying above $\mathcal{O}_\mathbf{K}$; every

localization $\mathcal{R}_{\mathcal{J}}$ will thus be sent onto $\mathcal{O}$ by some $\sigma \in \operatorname{Gal} L|K$ and will therefore be itself a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. On the other hand, we have seen at the beginning that every valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$ is the localization $\mathcal{R}_{\mathcal{J}}$ with respect to some maximal $\mathcal{J}$ of $\mathcal{R}$.

In the case of a finite normal extension $L|K$, the existence of σ is assured by Lemma 7.66. We have to generalize to the infinite case. Since $L|K$ is a normal algebraic extension, L is the union over all finite normal subextensions $L_i|K$, $i \in I$. Let $\mathcal{R}_i := \mathcal{R} \cap L_i$ which is the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L_i. Further, let $\mathcal{I}_i := \mathcal{I} \cap \mathcal{R}_i$ and $\mathcal{J}_i := \mathcal{J} \cap \mathcal{R}_i$. Since $\mathcal{I}$ and $\mathcal{J}$ are maximal ideals, so are $\mathcal{I}_i$ and $\mathcal{J}_i$ by virtue of Lemma 7.64; moreover, $\mathcal{I}_i$ contains $\mathcal{M}_{\mathbf{K}}$. Hence by Lemma 7.66 there exists $\sigma_i \in \operatorname{Gal} L_i|K$ such that $\sigma_i \mathcal{J}_i = \mathcal{I}_i$. By the Compactness Principle for Algebraic Extensions (cf. Lemma 6.5 where we take $F = \tilde{L}$), there exists $\sigma \in \operatorname{Gal} L|K$ and a subset $J \subset I$ such that $L = \bigcup_{j\in J} L_j$ and $\operatorname{res}_{L_i}(\sigma) = \sigma_i$. Since $\mathcal{I} = \bigcup_{j\in J} \mathcal{I}_i$ and $\mathcal{J} = \bigcup_{j\in J} \mathcal{J}_i$, it follows that $\sigma\mathcal{J} = \mathcal{I}$.

It remains to prove the last assertion of the theorem. By Theorem 7.40, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{R}$. In view of what we have proved so far, it suffices to show that every valuation ring $\mathcal{O}_0$ of L lying above $\mathcal{R}$ contains a valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$. Let $\mathcal{M}_0$ be the maximal ideal of $\mathcal{O}_0$. Choose some maximal ideal $\mathcal{I}$ of $\mathcal{R}$ containing the prime ideal $\mathcal{I}_0 = \mathcal{O}_0 \cap \mathcal{R}$. Since the latter contains precisely all elements of $\mathcal{R}$ which are non-units in $\mathcal{O}_0$ we find that $\mathcal{O}_0$ must contain the localization $\mathcal{R}_{\mathcal{I}}$. But this is a valuation ring of L which lies above $\mathcal{O}_{\mathbf{K}}$. □

For $L|K$ an arbitrary algebraic extension, we are now able to determine all extensions of v from K to L. We have $\tilde{L} = \tilde{K}$, and $\tilde{L}|K$ is a normal algebraic extension containing $L|K$. Given two extensions w, w' of v from K to L, by virtue of Theorem 7.36 we may choose extensions $\tilde{w}, \tilde{w}'$ to $\tilde{L}$. Then by the preceding theorem, there is $\sigma \in \operatorname{Gal} \tilde{L}|K$ such that $\sigma\mathcal{O}_{\tilde{w}'} = \mathcal{O}_{\tilde{w}}$ and $\sigma\mathcal{M}_{\tilde{w}'} = \mathcal{M}_{\tilde{w}}$. Consequently, $\tilde{w}' = \tilde{w}\sigma$ showing that also $w' = \tilde{w}\operatorname{res}_L(\sigma)$. If $L|K$ is normal, then the embedding $\iota := \operatorname{res}_L(\sigma)$ of L in $\tilde{L}$ over K is an automorphism of L and we may write $w' = w\iota$. Even if ι is not an automorphism of L, we shall call w and $w' = \tilde{w}\iota$ **conjugate extensions of** v. One also says that the valuations w and w' resp. their associated places P_w and $P_{w'} = P^{\iota}$ are **conjugate over** K. Fixing an arbitrary extension $\tilde{v}$ of v from K to $\tilde{K}$, we may write $w = \tilde{v}|_L$ and $\tilde{w} = \tilde{v}$. Then we find that every extension w' of v from K to L is of the form $\tilde{v}\iota$. We have proved:

Theorem 7.68 *If $L|K$ is algebraic, then every two extensions of v from K to L are conjugate. Given an arbitrary extension $\tilde{v}$ of v from K to $\tilde{K}$, the set of all extensions of v from K to L is*

$$\{(\tilde{v}\iota)|_L \mid \iota \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\} .$$

As a corollary to this theorem and Corollary 7.37, we can note for the special case of the algebraic closure of a valued field:

Corollary 7.69 *Let (K, v) be an arbitrary valued field. Then there exists an extension $\tilde{v}$ of v from K to $\tilde{K}$, and the set of all such extensions is given by*

$$\{\tilde{v}\iota \mid \iota \in \operatorname{Gal} K\} .$$

If the algebraic extension $L|K$ is normal and we pick one extension of v to L and denote it again by v, then the set of all extensions of v from K to L can be written as

$$\{v\iota \mid \iota \in \operatorname{Gal} L|K\} \ .$$

In view of part c) of Lemma 7.23, this yields:

Corollary 7.70 *If $(L|K, v)$ is a finite normal extension, then for every extension w of v from K to L,*

$$e(L|K, w) = e(L|K, v) \quad \textit{and} \quad f(L|K, w) = f(L|K, v) \ .$$

For a finite extension $L|K$, the number of distinct embeddings of L in $\tilde{L}$ over K is equal to the separable degree $[L : K]_{\rm sep}$. Hence, we may deduce from the foregoing theorem:

Corollary 7.71 *If $L|K$ is finite, then the number of distinct extensions of v from K to L is not greater than the separable degree $[L : K]_{\rm sep}$.*

As a special case, we obtain the following corollary:

Corollary 7.72 *If $L|K$ is a purely inseparable algebraic extension, then the extension w of v from K to L is uniquely determined.*

Let us give another proof of this corollary. Let $p = \operatorname{char} K$. For every $b \in L$ there is some nonzero $n \in \mathbb{N}$ such that $b^{p^n} = a \in K$. Hence, $wb = \frac{1}{p^n} va$ for every extension w.

We generalize the last two assertions of Theorem 7.67 to algebraic extensions which are not necessarily normal.

Lemma 7.73 *Let $L|K$ be an algebraic field extension and $\mathcal{O}_{\mathbf{K}}$ a valuation ring of K with maximal ideal $\mathcal{M}_{\mathbf{K}}$. Let $\mathcal{R}$ be the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L. Then the valuation rings of L lying above $\mathcal{O}_{\mathbf{K}}$ are precisely all the localizations of $\mathcal{R}$ with respect to its maximal ideals. Moreover, $\mathcal{R}$ is equal to the intersection of all valuation rings of L lying above $\mathcal{O}_{\mathbf{K}}$.*

Proof: It follows already from Lemma 7.65 that every valuation ring of L lying above $\mathcal{O}_{\mathbf{K}}$ is the localization of $\mathcal{R}$ with respect to some maximal ideal. For the converse, assume that $\mathcal{I}$ is a maximal ideal of $\mathcal{R}$. Since $\mathcal{R}$ is integral over $\mathcal{O}_{\mathbf{K}}$, Lemma 7.64 shows that $\mathcal{I} \cap \mathcal{O}_{\mathbf{K}}$ is a maximal ideal and thus equal to $\mathcal{M}_{\mathbf{K}}$. From Theorem 7.34 we infer the existence of a valuation ring $\mathcal{O}$ of L containing $\mathcal{R}$ whose maximal ideal $\mathcal{M}$ contains $\mathcal{I}$. Consequently, $\mathcal{O}$ contains $\mathcal{O}_{\mathbf{K}}$ and $\mathcal{M}$ contains $\mathcal{M}_{\mathbf{K}}$ which shows that $\mathcal{O}$ lies over $\mathcal{O}_{\mathbf{K}}$. By Lemma 7.65, $\mathcal{O}$ is the localization of $\mathcal{R}$ with respect to the maximal ideal $\mathcal{M} \cap \mathcal{R}$. Since this maximal ideal contains $\mathcal{I}$, it must be equal to $\mathcal{I}$.

The last assertion is shown as in the proof of Theorem 7.67. □

Two valuation rings $\mathcal{O}_1$, $\mathcal{O}_2$ resp. their associated valuations are called **comparable** if $\mathcal{O}_1 \subset \mathcal{O}_2$ or $\mathcal{O}_1 \supset \mathcal{O}_2$; otherwise, they are called **incomparable**. From the foregoing lemma, we obtain:

Corollary 7.74 *Every two distinct extensions of a valuation to an algebraic extension field are incomparable.*

Proof: Let the notation be as in the foregoing lemma. Given two extensions of v from K to L with valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, the foregoing lemma tells us that $\mathcal{O}_i = \mathcal{R}_{\mathcal{I}_i}$ with maximal ideals $\mathcal{I}_i$ of $\mathcal{R}$, $i = 1, 2$. Let us assume that $\mathcal{O}_1 \subset \mathcal{O}_2$. Then $\mathcal{I}_1 \subset \mathcal{I}_2$ which yields $\mathcal{I}_1 = \mathcal{I}_2$ because of their maximality. Consequently, $\mathcal{O}_1 = \mathcal{O}_2$. □

Let us exploit once more the Chinese Remainder Theorem 7.63:

Lemma 7.75 *Let (K, v) be a valued field, $L|K$ an algebraic extension and $v_1, \ldots, v_n$ distinct extensions of v from K to L. Given any elements $\zeta_1 \in L/v_1$, ... , $\zeta_n \in L/v_n$, there is some $a \in L$ such that $a/v_1 = \zeta_1$, ... , $a/v_n = \zeta_n$.*

Proof: Let $\mathcal{O}_v$ be the valuation ring of v in K and let $\mathcal{R}$ be its integral closure in L. Denote the valuation ring of v_i in L by $\mathcal{O}_i$, its maximal ideal by $\mathcal{M}_i$ and set $\mathcal{I}_i := \mathcal{M}_i \cap \mathcal{R}$. By Lemma 7.65, $\mathcal{O}_i = \mathcal{R}_{\mathcal{I}_i}$. Since $L/v_i = \mathcal{O}_i/\mathcal{M}_i = \mathcal{R}_{\mathcal{I}_i}/\mathcal{R}_{\mathcal{I}_i}\mathcal{I}_i = \mathcal{R}/\mathcal{I}_i$, we may choose $a_i \in \mathcal{R}$ such that $a_i/v_i = a_i/\mathcal{I}_i = \zeta_i$. Since the $\mathcal{I}_i$ are maximal ideals, we have $\mathcal{I}_i + \mathcal{I}_j = \mathcal{R}$ for $i \neq j$. So we may use the Chinese Remainder Theorem to find $a \in \mathcal{R}$ which satisfies $a \equiv a_i$ modulo $\mathcal{I}_i$ for $1 \leq i \leq n$. That is, $a/v_i = a/\mathcal{I}_i = a_i/\mathcal{I}_i = \zeta_i$. □

With Lemma 7.42 as a tool, we prove:

Lemma 7.76 *Let $(L|K, v)$ be a normal extension of valued fields. Then for every $\zeta \in \overline{L}$ and every $\xi \in \widetilde{\overline{K}}$ which is conjugate to ζ over $\overline{K}$, there exists some $a \in L$ and an automorphism $\sigma \in \operatorname{Gal} L|K$ such that $\overline{a} = \zeta$ and $\overline{\sigma a} = \xi$. In particular, $\overline{L}|\overline{K}$ is normal.*

Proof: Let g be the minimal polynomial of ζ over $\overline{K}$, so ζ, ξ are roots of g. By Lemma 7.75, there exists some $a \in L$ such that $\overline{a} = \zeta$ and $v_i a > 0$ for all extensions $v_i \neq v$ of v from K to L. In view of Theorem 7.68, this yields that all conjugates of a, hence all roots of f, are integral. Consequently, the minimal polynomial f of a over K has integral coefficients. Since a is integral and a root of f, we know that $\overline{a} = \zeta$ is a root of $\overline{f} \in \overline{K}[X]$. Consequently, g must divide $\overline{f}$, showing that also ξ is a root of $\overline{f}$. Then by part b) of Lemma 7.42, ξ is the residue of some root $b \in L$ of f. By our choice of f, there is some $\sigma \in \operatorname{Gal} L|K$ such that $b = \sigma a$. So $\xi = \overline{\sigma a}$, as contended. In particular, $\xi \in \overline{L}$. Since ζ was an arbitrary element of $\overline{L}$ and ξ was an arbitrary conjugate of ζ over $\overline{K}$, this proves that $\overline{L}|\overline{K}$ is normal. □

Exercise 7.11 *Let $\mathbf{K} = (K, v)$ be an arbitrary valued field and let a be algebraic with minimal polynomial f over K. Prove that $f \in \mathcal{O}_{\mathbf{K}}[X]$ if and only if a is integral for all extensions of v from K to $\tilde{K}$ (or to $K(a)$).*

7.8 Ramification theory

Let $(L|K, v)$ be a normal algebraic extension of valued fields, not necessarily finite. We shall investigate three distinguished subgroups of the Galois group $\operatorname{Gal} L|K$. The subgroup

$$G^d(L|K, v) := \{\sigma \in \operatorname{Gal} L|K \mid v\sigma = v \text{ on } L\} \tag{7.14}$$

of $\operatorname{Gal} L|K$ is called the **decomposition group of** $(L|K, v)$. The condition "$v\sigma = v$ on L" means that $\forall x \in L : v\sigma x = vx$.

Remark 7.77 In the literature, one finds a definition of the decomposition group which appears to be different from ours. In the place of $v\sigma = v$, the condition is used that $v\sigma$ and v be equivalent, that is, that they have the same valuation ring. This holds if and only if there is an isomorphism ρ of vL onto $v\sigma L$ over vK such that $v\sigma = \rho \circ v$. Since we are dealing with algebraic extensions $L|K$, Lemma 7.9 shows that vL lies in the divisible hull of vK. But then, ρ can only be the identity.

From infinite Galois Theory (Section 6.4), we know that $\sigma \in \operatorname{Gal} L|K$ lies in the closure of $G^d(L|K, v)$ if and only if $\operatorname{res}_{L_i}(\sigma) \in \operatorname{res}_{L_i}(G^d(L|K, v))$ for every finite normal subextension $L_i|K$ of $L|K$. But then, $v\sigma = v$ on every L_i, and since L is the union over all L_i, this yields that $v\sigma = v$ on L, that is, $\sigma \in G^d(L|K, v)$. This proves that $G^d(L|K, v)$ is a closed subgroup of $\operatorname{Gal} L|K$.

Let $\mathcal{O}_\mathbf{L}$ and $\mathcal{M}_\mathbf{L}$ be the valuation ring and valuation ideal of $\mathbf{L} = (L, v)$. Then for every $\sigma \in G^d(L|K, v)$ we have $\sigma\mathcal{O}_\mathbf{L} = \mathcal{O}_\mathbf{L}$ and consequently also $\sigma\mathcal{M}_\mathbf{L} = \mathcal{M}_\mathbf{L}$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_\mathbf{L}/\mathcal{M}_\mathbf{L} = \overline{L}$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. It may be called the **reduction of** σ. Since σ fixes K, it follows that $\overline{\sigma}$ fixes $\overline{K}$. Moreover, the map

$$G^d(L|K, v) \ni \sigma \;\mapsto\; \overline{\sigma} \in \operatorname{Gal} \overline{L}|\overline{K} \tag{7.15}$$

is a group homomorphism. Note that we have written "$\operatorname{Gal} \overline{L}|\overline{K}$" since by virtue of Lemma 7.76, our general hypothesis that $L|K$ be normal yields that also $\overline{L}|\overline{K}$ is normal.

The homomorphism (7.15) is continuous. Indeed, we only have to show that for every open subgroup of $\operatorname{Gal} \overline{L}|\overline{K}$ there is an open subgroup of $G^d(L|K, v)$ that is mapped into it. Now an open subgroup of $\operatorname{Gal} \overline{L}|\overline{K}$ is of the form $\operatorname{Gal} \overline{L}|\overline{K}_1$ where $\overline{K}_1$ is a finite extension of $\overline{K}$. Let $a_1, \ldots, a_n \in L$ such that $\overline{a_1}, \ldots, \overline{a_n} \in \overline{K}_1$ generate $\overline{K}_1$ over $\overline{K}$. Then $a_1, \ldots, a_n$ generate a finite extension K_2 of $(L|K, v)^d$ such that $\overline{K}_1 \subset \overline{K_2}$, and the open subgroup $\operatorname{Gal} L|K_2$ of $G^d(L|K, v)$ is mapped into $\operatorname{Gal} \overline{L}|\overline{K_2} \subset \operatorname{Gal} \overline{L}|\overline{K}_1$. Consequently, the kernel of the homomorphism (7.15) is a closed normal subgroup of $G^d(L|K, v)$; it is called the **inertia group of** $(L|K, v)$ and denoted by $G^i(L|K, v)$. We shall show that

$$\begin{aligned} G^i(L|K, v) &= \{\sigma \in \operatorname{Gal} L|K \mid \forall x \in \mathcal{O}_\mathbf{L} : \sigma x - x \in \mathcal{M}_\mathbf{L}\} \\ &= \{\sigma \in \operatorname{Gal} L|K \mid \forall x \in \mathcal{O}_\mathbf{L} : v(\sigma x - x) > 0\} \; . \end{aligned} \tag{7.16}$$

Let us abbreviate

$$G_\mathrm{T} := G^i(L|K, v) \quad \text{and} \quad G_\mathrm{Z} := G^d(L|K, v)$$

as long as we are working with our fixed extension $(L|K, v)$. Let $\sigma \in \operatorname{Gal} L|K$ such that $\sigma a - a \in \mathcal{M}_\mathbf{L}$ for all $a \in \mathcal{O}_\mathbf{L}$. Then $\sigma a \in \mathcal{O}_\mathbf{L}$ for every $a \in \mathcal{O}_\mathbf{L}$, because otherwise we would have $v(\sigma a - a) < 0$. This gives $\sigma\mathcal{O}_\mathbf{L} \subset \mathcal{O}_\mathbf{L}$. Since both $\sigma\mathcal{O}_\mathbf{L}$, $\mathcal{O}_\mathbf{L}$ are valuation rings lying above $\mathcal{O}_\mathbf{K}$, Corollary 7.74 shows that both must be equal. Hence, $v\sigma = v$ which proves that $\sigma \in G_\mathrm{Z}$. Now for $\sigma \in G_\mathrm{Z}$, we find that $\overline{\sigma}$ is the identity if and only if $\overline{\sigma a} = \overline{a}$ for all $a \in \mathcal{O}_\mathbf{L}$. But $\overline{\sigma a} = \overline{a}$ is equivalent to $\sigma a - a \in \mathcal{M}_\mathbf{L}$ and to $v(\sigma a - a) > 0$. This gives (7.16).

We will now consider a **pairing**, that is, a bilinear map

$$(.\,,.) : \; L^\times \times G_\mathrm{T} \;\longrightarrow\; \overline{L}^\times \quad \text{which sends } a \in L^\times \text{ and } \sigma \in G_\mathrm{T} \text{ to } \quad (a, \sigma) := \overline{\left(\frac{\sigma a}{a}\right)} \tag{7.17}$$

where $L^\times$ and $\overline{L}^\times$ denote the multiplicative groups of L and $\overline{L}$. Note that $\sigma \in G_\mathrm{T} \subset G_\mathrm{Z}$ implies $v\sigma a = va$ and thus, $\sigma a \,/\, a \in \mathcal{O}_\mathbf{L}$ and $\overline{\sigma a \,/\, a} \neq 0$. For $\sigma \in G_\mathrm{T}$, every $a \in L$ with $va = 0$ will satisfy $v(\sigma a - a) > 0$ and hence, $v(\frac{\sigma a}{a} - 1) > 0$. This shows

$$(a, \sigma) = 1 \quad \text{for all } a \in \mathcal{O}_\mathbf{L}^\times \, , \, \sigma \in G_\mathrm{T} \; . \tag{7.18}$$

For fixed $\sigma \in G_{\mathrm{T}}$, the map $(\,.\,,\sigma)$ is a homomorphism from $L^\times$ into $\overline{L}^\times$ since $\frac{\sigma ab}{ab} = \frac{\sigma a}{a}\frac{\sigma b}{b}$. For fixed $a \in L^\times$, the map $(a,\,.\,)$ is a homomorphism from G_{T} into $\overline{L}^\times$. To show this, let also $\tau \in G_{\mathrm{T}}$. Then $\frac{\sigma\tau a}{a} = \frac{\sigma\tau a}{\sigma a}\frac{\sigma a}{a} = \sigma\left(\frac{\tau a}{a}\right)\frac{\sigma a}{a} = \left(\sigma\left(\frac{\tau a}{a}\right)\Big/\frac{\tau a}{a}\right)\frac{\sigma a}{a}\frac{\tau a}{a}$ showing that $(a,\sigma\tau) = (\frac{\tau a}{a},\sigma)(a,\sigma)(a,\tau)$. But $v\frac{\tau a}{a} = 0$, so (7.18) shows that $(\frac{\tau a}{a},\sigma) = 1$, and we have proved that $(a,\,.\,)$ is linear.

The bilinearity can be restated as follows: the maps

$$G_{\mathrm{T}} \longrightarrow \mathrm{Hom}(L^\times,\overline{L}^\times) \qquad \sigma \mapsto (\,.\,,\sigma) \tag{7.19}$$

$$L^\times \longrightarrow \mathrm{Hom}(G_{\mathrm{T}},\overline{L}^\times) \qquad a \mapsto (a,\,.\,) \tag{7.20}$$

are group homomorphisms. The kernel of (7.19) is a normal subgroup of G_{T}, called the **ramification group of** $(L|K,v)$. It is denoted by $G^r(L|K,v)$, and for the present discussion we will abbreviate it by G_{V}. It consists of all $\sigma \in G_{\mathrm{T}}$ for which $(\,.\,,\sigma) = 1$, or in other words, $\overline{\sigma a\,/\,a} = 1$ for all $a \in L^\times$. But $\overline{\sigma a\,/\,a} = 1$ is equivalent to $v(\frac{\sigma a}{a} - 1) > 0$, which is the same as $v(\sigma a - a) > va$. Note that if $\sigma \in \mathrm{Gal}\,L|K$ satisfies $v(\sigma a - a) > va$ and hence $v(\sigma a - a) > 0$ for all $a \in \mathcal{O}_{\mathbf{L}}$, then $\sigma \in G_{\mathrm{T}}$. We have proved that

$$\begin{aligned} G^r(L|K,v) &= \{\sigma \in \mathrm{Gal}\,L|K \mid \forall x \in \mathcal{O}_{\mathbf{L}} : \tfrac{\sigma x}{x} - 1 \in \mathcal{M}_{\mathbf{L}}\} \\ &= \{\sigma \in \mathrm{Gal}\,L|K \mid \forall x \in \mathcal{O}_{\mathbf{L}} : v(\sigma x - x) > vx\}\ . \end{aligned} \tag{7.21}$$

Now let $\sigma \in G_{\mathrm{V}}$, $\tau \in G_{\mathrm{Z}}$ and $a \in L^\times$. Since $G_{\mathrm{T}} \lhd G_{\mathrm{Z}}$, we have $\tau\sigma\tau^{-1} \in G_{\mathrm{T}}$. Setting $b := \tau^{-1}a$, we compute: $\overline{\tau\sigma\tau^{-1}a\,/\,a} = \overline{\tau\sigma b\,/\,\tau b} = \overline{\tau}\,\overline{\sigma b\,/\,b} = \overline{\tau}\,1 = 1$. This shows that $\tau\sigma\tau^{-1} \in G_{\mathrm{V}}$, proving that G_{V} is a normal subgroup also of G_{Z}. Further, G_{V} is closed in $\mathrm{Gal}\,L|K$; the proof of this fact is similar to that for the decomposition group.

The fixed field of $G^d(L|K,v)$ in $K_s := (L|K)^{\mathrm{sep}}$ is called the **decomposition field of** $(L|K,v)$ and will be denoted by $(L|K,v)^d$ or by $(L|K)^{d(v)}$. Similarly, the fixed field of $G^i(L|K,v)$ in K_s is called the **inertia field of** $(L|K,v)$ and will be denoted by $(L|K,v)^i$ or by $(L|K)^{i(v)}$. Finally, the fixed field of $G^r(L|K,v)$ in K_s is called the **ramification field of** $(L|K,v)$ and will be denoted by $(L|K,v)^r$ or by $(L|K)^{r(v)}$. By definition, these fields are separable subextensions of $L|K$. Note that in contrast to the common use in the literature, we define the decomposition field to be the fixed field of the decomposition group in the maximal separable subextension $K_s|K$ of $L|K$, and similarly we do for the inertia field and the ramification field. This has the consequence that the ramification field is a separable extension of K and that all inseparability is shifted "to the top", that is, to the extension $L\,|\,(L|K,v)^d$ (cf. the table on page 246). This version has significant advantages for the formulation of certain facts (see for instance Lemma 8.20 and the definition of tame extensions in Section 12.2).

Since decomposition group, inertia group and ramification group are closed subgroups of $\mathrm{Gal}\,L|K$, Theorem 6.10 shows that they are equal to $\mathrm{Gal}\,L\,|\,(L|K,v)^d$, $\mathrm{Gal}\,L\,|\,(L|K,v)^i$ and $\mathrm{Gal}\,L\,|\,(L|K,v)^r$ respectively.

We take over from J. Neukirch [NEU] the following description of the functorial properties of G^d, G^i and G^r. Let us assume that $(L'|K',v')$ is an arbitrary normal algebraic valued field extension and that τ is an embedding of (L,v) in (L',v') such that $\tau K \subset K'$. This embedding induces a homomorphism

$$\tau_* : \mathrm{Gal}\,L'|K' \to \mathrm{Gal}\,L|K\,, \quad \tau_*(\sigma') = \tau^{-1}\sigma'\tau\ .$$

Note that $\tau L|\tau K$ is normal since $L|K$ is; hence $\sigma'\tau L \subset \tau L$ for $\sigma' \in \operatorname{Gal} L'|K'$ in view of $\tau K \subset K'$. This shows that the expression $\tau^{-1}\sigma'\tau$ makes sense. Furthermore, τ_* is continuous and open since it is the composition of the continuous open restriction map $\operatorname{res}_{\tau L}$ with the topological isomorphism $\operatorname{Gal}\tau L|\tau K \to \operatorname{Gal} L|K$ which is induced by the isomorphism $\tau : L \to \tau L$.

Lemma 7.78 *The continuous homomorphism τ_* induces continuous homomorphisms*

$$\begin{aligned} G^d(L'|K', v') &\longrightarrow G^d(L|K, v) \\ G^i(L'|K', v') &\longrightarrow G^i(L|K, v) \\ G^r(L'|K', v') &\longrightarrow G^r(L|K, v)\ . \end{aligned}$$

Proof: By assumption, $v = v'\tau$ on L or equivalently, $v\tau^{-1} = v'$ on τL; we also have $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$. Let $\sigma' \in \operatorname{Gal} L'|K'$ and $\sigma = \tau_*(\sigma')$. If $\sigma' \in G^d(L'|K', v')$, then $v'\sigma' = v'$, hence

$$v\sigma = v\tau^{-1}\sigma'\tau = v'\sigma'\tau = v'\tau = v$$

on L, showing that $v \in G^d(L|K, v)$. If $\sigma' \in G^i(L'|K', v')$, then in view of $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$,

$$v(\sigma a - a) = v\tau^{-1}(\sigma'\tau a - \tau a) = v'(\sigma'\tau a - \tau a) > 0$$

for every $a \in \mathcal{O}_{\mathbf{L}}$, showing that $v \in G^i(L|K, v)$. If $\sigma' \in G^r(L'|K', v')$, then again in view of $\tau\mathcal{O}_{\mathbf{L}} \subset \mathcal{O}_{(L',v')}$,

$$v(\sigma a - a) = v\tau^{-1}(\sigma'\tau a - \tau a) = v'(\sigma'\tau a - \tau a) > v'\tau a = va\ ,$$

for every $a \in \mathcal{O}_{\mathbf{L}}$, showing that $v \in G^r(L|K, v)$. □

If $\tau : L \to L'$ and $\tau : K \to K'$ are isomorphisms, then so are τ_* and the above homomorphisms induced by τ_*. For $L = L'$ and $K = K'$, this yields the following corollary. In view of the fact that all extensions of a valuation to an algebraic extension field are conjugate (Theorem 7.68), it gives information about the decomposition field, inertia field and ramification field with respect to the other extensions of v from K to L:

Corollary 7.79 *Let $\iota \in \operatorname{Gal} L|K$. Then*

$$\begin{array}{rclcrcl} G^d(L|K, v\iota) &=& \iota^{-1}\, G^d(L|K, v)\, \iota & \text{and} & (L|K, v\iota)^d &=& \iota^{-1}(L|K, v)^d \\ G^i(L|K, v\iota) &=& \iota^{-1}\, G^i(L|K, v)\, \iota & \text{and} & (L|K, v\iota)^i &=& \iota^{-1}(L|K, v)^i \\ G^r(L|K, v\iota) &=& \iota^{-1}\, G^r(L|K, v)\, \iota & \text{and} & (L|K, v\iota)^r &=& \iota^{-1}(L|K, v)^r\ . \end{array}$$

The left hand sides of our assertions follow directly from Lemma 7.78, where we set $v' = v\iota$ and $\tau = \iota^{-1}$. The right hand sides follow from the left by Theorem 6.10.

For our present discussion of the given extension $(L|K, v)$, let us abbreviate $\mathrm{Z} = (L|K, v)^d$, $\mathrm{T} = (L|K, v)^i$ and $\mathrm{V} = (L|K, v)^r$. So we are studying the following situation:

$$\left.\begin{array}{l} (L|K, v) \text{ a normal algebraic extension of valued fields with} \\ G_{\mathrm{Z}} \text{ its decomposition group and } (\mathrm{Z}, v) \text{ its decomposition field,} \\ G_{\mathrm{T}} \text{ its inertia group and } (\mathrm{T}, v) \text{ its inertia field,} \\ G_{\mathrm{V}} \text{ its ramification group and } (\mathrm{V}, v) \text{ its ramification field,} \\ K_s|K \text{ the maximal separable subextension of } L|K, \\ p \text{ the characteristic exponent of the residue field } \overline{K}. \end{array}\right\} \qquad (7.22)$$

Remark 7.80 The abbreviations Z, T, V are a reference to the german words "Zerlegungskörper", "Trägheitskörper" and "Verzweigungskörper".

Let us summarize what we have already shown for Z, T and V:

Lemma 7.81 *In the situation (7.22),*

$$G_{\mathrm{V}} \subset G_{\mathrm{T}} \subset G_{\mathrm{Z}} \qquad \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z}$$

where the groups are the Galois groups in L of the respective fields. Moreover, the inertia group G_{T} and the ramification group G_{V} are normal subgroups of the decomposition group G_{Z} and thus, the inertia field T *and the ramification field* V *are Galois extensions of the decomposition field* Z*.*

Assume that $E|K$ is any subextension of $L|K$. With $L' = L$, $K' = E$ and $\tau = \mathrm{id}_L$, this constitutes another special case of Lemma 7.78; in this case, τ_* is just the inclusion of $\mathrm{Gal}\, L|E$ in $\mathrm{Gal}\, L|K$.

Lemma 7.82 *Assume (7.22). If $E|K$ is any arbitrary subextension of $L|K$, then*

$$\begin{array}{rclcrcl} G^d(L|E,v) & = & G_{\mathrm{Z}} \cap \mathrm{Gal}\, L|E & \quad\textit{and}\quad & (L|E,v)^d & = & (E.\mathrm{Z},v) \\ G^i(L|E,v) & = & G_{\mathrm{T}} \cap \mathrm{Gal}\, L|E & \quad\textit{and}\quad & (L|E,v)^i & = & (E.\mathrm{T},v) \\ G^r(L|E,v) & = & G_{\mathrm{V}} \cap \mathrm{Gal}\, L|E & \quad\textit{and}\quad & (L|E,v)^r & = & (E.\mathrm{V},v)\,. \end{array}$$

Proof: The inclusions "$\subset$" on the left hand side follow directly from Lemma 7.78. But actually, we do not need to employ this lemma, since already the equalities follow immediately from (7.14), (7.16) and (7.21).

If $L|K$ is Galois, then the right hand side follows from the left hand side by (Gal2′). If $L|K$ is not separable, then we have to proceed as follows. In that case, we have to take the fixed fields in $(L|E)^{\mathrm{sep}} = E.K_s$. Since $L|E.K_s$ is purely inseparable, we may view all subgroups of $\mathrm{Gal}\, L|K$ as subgroups of $\mathrm{Gal}\, E.K_s|K$. The extension $E.K_s|E$ is separable. Hence, the right hand side now follows from Lemma 6.39, applied to the extension $E.K_s|K$ with $F = \mathrm{Z}$, $F = \mathrm{T}$ and $F = \mathrm{V}$ respectively. □

A further special case of Lemma 7.78 is given when $E|K$ is a normal subextension of $L|K$. In the lemma, we have to replace L by $E_s = (E|K)^{\mathrm{sep}} = E \cap K_s$ and L' by K_s, and we set $K' = K$ and $\tau = \mathrm{id}_{E_s}$. In this case, τ_* is the restriction $\mathrm{res}_{E_s} : \mathrm{Gal}\, K_s|K \to \mathrm{Gal}\, E_s|K$. Using Lemma 7.78 and (Gal7), we obtain:

Corollary 7.83 *Assume (7.22). If $E|K$ is a normal subextension of $L|K$, then*

$$\begin{array}{rclcrcl} \mathrm{res}_E(G_{\mathrm{Z}}) & \subset & G^d(E|K,v) & \quad\textit{and}\quad & (E\cap\mathrm{Z},v) & \supset & (E|K,v)^d \\ \mathrm{res}_E(G_{\mathrm{T}}) & \subset & G^i(E|K,v) & \quad\textit{and}\quad & (E\cap\mathrm{T},v) & \supset & (E|K,v)^i \\ \mathrm{res}_E(G_{\mathrm{V}}) & \subset & G^r(E|K,v) & \quad\textit{and}\quad & (E\cap\mathrm{V},v) & \supset & (E|K,v)^r\,. \end{array}$$

The inclusions are in fact equalities, as we will show later. Note that the fixed fields of the restricted groups are $E_s \cap \mathrm{Z}$, $E_s \cap \mathrm{T}$, $E_s \cap \mathrm{V}$ respectively, by virtue of (Gal7). But since the extensions $\mathrm{Z}|K$, $\mathrm{T}|K$, $\mathrm{V}|K$ are separable, these intersections are equal to $E \cap \mathrm{Z}$, $E \cap \mathrm{T}$, $E \cap \mathrm{V}$ respectively.

Let us now study the properties of the decomposition field.

Lemma 7.84 *Assume (7.22). Then:*

a) The extension of v from Z to L is unique.

b) For $\sigma, \tau \in \mathrm{Gal}\, L|K$, the following assertions are equivalent:
1) $v\sigma = v\tau$ on $\tau^{-1}\mathrm{Z}$,
2) $v\sigma = v\tau$ on L,
3) $\sigma\tau^{-1} \in G_{\mathrm{Z}}$.

c) If the automorphism $\sigma \in \mathrm{Gal}\, L|K$ does not fix Z, then $v\sigma \neq v$ on Z.

d) If $E|K$ is a subextension of $L|K$, then the extension of v from E to L is unique if and only if $\mathrm{Z} \subset E$. In particular, the extension of v from K to L is unique if and only if $\mathrm{Z} = K$, that is, if and only if $G_{\mathrm{Z}} = \mathrm{Gal}\, L|K$.

e) If $\mathrm{Z}|K$ is finite, then the number $g(L|K, v)$ of distinct extensions of v from K to L is finite and equal to $(\mathrm{Gal}\, L|K : G_{\mathrm{Z}}) = [\mathrm{Z} : K]$.

f) For every $\iota \in \mathrm{Gal}\, L|K$, the restriction $\mathrm{res}_{\mathrm{Z}}(\iota^{-1})$ is the unique isomorphism over K sending (Z, v) onto $(L|K, v\iota)^d$.

Proof: a): From Theorem 7.68 we know that every two extensions of v from Z to L are conjugate. But by definition, every automorphism of $G_{\mathrm{Z}} = \mathrm{Gal}\, L|\mathrm{Z}$ fixes v.

b): If $\sigma, \tau \in \mathrm{Gal}\, L|K$ such that $v\sigma = v\tau$ on $\tau^{-1}\mathrm{Z}$ and hence $v\sigma\tau^{-1} = v$ on Z, then also $v\sigma\tau^{-1} = v$ on L by virtue of part a). Since $v\sigma\tau^{-1} = v$ on L if and only if $v\sigma = v\tau$ on L, this proves 1)⇒2). The converse is trivial. By definition of G_{Z}, $v\sigma\tau^{-1} = v$ holds on L if and only if $\sigma\tau^{-1} \in G_{\mathrm{Z}}$. This proves the equivalence of 2) and 3).

c): This is the implication 1)⇒3) of part b) with $\tau = \mathrm{id}_L$.

d): Every extension of v from E to L is an extension of v from K to L. Hence if $\mathrm{Z} \subset E$, then by part a), v admits a unique extension from E to L. For the conversely, assume that $\mathrm{Z} \not\subset E$. Then there is some $a \in \mathrm{Z} \setminus E$. Since $\mathrm{Z}|K$ is separable, a is separable algebraic over E. Thus, there is some $\sigma \in \mathrm{Gal}\, L|E$ which moves a. Hence $\sigma \notin G_{\mathrm{Z}}$, so the implication 2)⇒3) of part b) shows that $v \neq v\sigma$ on L. On the other hand, $v = v\sigma$ on E. This shows that v and $v\sigma$ are two distinct extensions of v from E to L.

e): By the equivalence 2)⇔3) of part b), $g(L|K, v)$ is equal to the number of cosets of $\mathrm{Gal}\, L|K$ modulo $G_{\mathrm{Z}} = \mathrm{Gal}\, L|\mathrm{Z}$. This in turn is equal to $[\mathrm{Z} : K]$.

f): It follows from Corollary 7.79 that the restriction of ι^{-1} is the required isomorphism. If there would be a second isomorphism, say σ^{-1}, then $v\sigma = v\iota$ on $\iota^{-1}\mathrm{Z}$, so by the implication 1)⇒3) of part b), ι^{-1} and σ^{-1} must coincide on Z. □

Lemma 7.85 *Assume (7.22) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\mathrm{res}_E(G_{\mathrm{Z}}) = G^d(E|K, v) \quad \text{and} \quad (E \cap \mathrm{Z}, v) = (E|K, v)^d\,.$$

Proof: In view of Corollary 7.83, we have to show that $\mathrm{res}_E : G_{\mathrm{Z}} \to G^d(E|K, v)$ is surjective. Let $\rho \in G^d(E|K, v)$ and $\sigma \in \mathrm{Gal}\, L|K$ such that $\rho = \mathrm{res}_E(\sigma)$. By assumption on ρ, we have $v = v\rho = v\sigma$ on E. Hence, by Theorem 7.68 there is some $\tau \in \mathrm{Gal}\, L|E$ such that $v\tau = v\sigma$ on L. That is, $\sigma\tau^{-1} \in G_{\mathrm{Z}}$ with $\rho = \mathrm{res}_E(\sigma\tau^{-1})$. □

The reader may prove that the number $g(L|K, v)$ of distinct extensions of v from K to L is multiplicative, for arbitrary finite extensions:

Lemma 7.86 *Let $(L|K,v)$ be a finite extension of valued fields and $E|K$ a subextension of $L|K$. Then $g(L|K,v) = g(L|E,v) \cdot g(E|K,v)$.*

We the help of Theorem 7.84, we prove:

Lemma 7.87 *Assume (7.22). Then $(\mathrm{Z}|K,v)$ is an immediate extension.*

Proof: Assume first that $L|K$ is finite. Then also $\mathrm{Z}|K$ is finite. Let $\zeta \neq 0$ be an element of the residue field of Z. By Lemma 7.75, there exists some $c \in \mathrm{Z}$ such that $\overline{c} = \zeta$ and $v'c > 0$ for all extensions $v' \neq v$ of v from K to Z. By Theorem 7.84 we know that $v\sigma \neq v$ on Z for every $\sigma \in \mathrm{Gal}\, L|K \setminus G_{\mathrm{Z}}$. Hence for all conjugates $\sigma c \neq c$ we have $v\sigma c > 0$. Now $\mathrm{Tr}_{\mathrm{Z}|K}(c) = c + \sum_{\sigma \in H} \sigma c$ where $H \subset \mathrm{Gal}\, L|K$ is a set of representatives of the cosets $\mathrm{Gal}\, L|K$ modulo G_{Z} which are different from G_{Z}. Hence $v\sigma c > 0$ for all $\sigma \in H$. Consequently, $\zeta = \overline{c} = \overline{\mathrm{Tr}_{\mathrm{Z}|K}(c)} \in \overline{K}$. We have proved that $\overline{\mathrm{Z}} = \overline{K}$.

Now let α be an element of the value group of Z. Choose $b \in \mathrm{Z}$ with $vb = \alpha$. Let c and H be as before, that is, $vc = 0$ and $v\sigma c > 0$ for all $\sigma \in H$. We choose some $m \in \mathbb{N}$ such that $vc^m b = vb \neq m \cdot v\sigma c + v\sigma b = v\sigma c^m b$ for all $\sigma \in H$. Set $a := c^m b$ and observe that $va = \alpha$. The conjugates of a over K are precisely the roots of the minimal polynomial f of a over K. By construction of a, it is the only root of value α. Thus, an application of part c) of Lemma 7.42 shows that $\alpha = va \in vK$. We have proved that $v\mathrm{Z} = vK$. This concludes the proof of our assertion for the case of finite $L|K$.

In the case of an infinite extension $L|K$, it suffices to show that every finite subextension $(\mathrm{Z}_1|K,v)$ of $(\mathrm{Z}|K,v)$ is immediate. Let L_1 be the normal hull of $\mathrm{Z}_1|K$; it lies in L since $L|K$ is assumed to be normal. Moreover, $L_1|K$ is finite. By the previous lemma, $(L_1|K,v)^d = L_1 \cap (L|K,v)^d \supset \mathrm{Z}_1$. By what we have already shown, $(L_1|K,v)^d$ is an immediate extension of (K,v), and the same is consequently true for (Z_1,v). □

Next, we investigate the extension $\mathrm{T}|\mathrm{Z}$.

Theorem 7.88 *Assume (7.22). The value group of the inertia field* T *is equal to $v\mathrm{Z} = vK$, and its residue field is equal to the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$ (and hence a Galois extension of $\overline{K}$). The homomorphism (7.15) (which sends σ to $\overline{\sigma}$) is onto and consequently,* $\mathrm{T}|\mathrm{Z}$ *is a Galois extension with Galois group*

$$\mathrm{Gal}\, \mathrm{T}|\mathrm{Z} \;\cong\; G_{\mathrm{Z}} \,/\, G_{\mathrm{T}} \;\cong\; \mathrm{Gal}\, \overline{\mathrm{T}}|\overline{\mathrm{Z}} \;\cong\; \mathrm{Gal}\, \overline{L}|\overline{K}$$

(these are topological isomorphisms).

For every subextension $K'|\mathrm{Z}$ of $\mathrm{T}|\mathrm{Z}$*, the image of* $\mathrm{Gal}\, \mathrm{T}|K'$ *under this isomorphism is equal to* $\mathrm{Gal}\, \overline{\mathrm{T}}|\overline{K'}$*. Hence if $K'|\mathrm{Z}$ is normal, the isomorphism induces an isomorphism of* $\mathrm{Gal}\, K'|\mathrm{Z}$ *onto* $\mathrm{Gal}\, \overline{K'}|\overline{\mathrm{Z}}$*. Moreover, $[K_1 : K_0] = [\overline{K_1} : \overline{K_0}]$ for every finite extension $(K_1|K_0,v)$ such that $\mathrm{Z} \subset K_0 \subset K_1 \subset \mathrm{T}$.*

Proof: By Lemma 7.87, $\overline{\mathrm{Z}} = \overline{K}$. Since $L|\mathrm{Z}$ is a Galois extension, Lemma 7.76 shows that $\overline{L}|\overline{\mathrm{Z}}$ is a normal extension. Now let $k|\overline{\mathrm{Z}}$ be a finite Galois subextension of $\overline{L}|\overline{\mathrm{Z}}$ and let ζ be a primitive element of it. Let $\tau \in \mathrm{Gal}\, \overline{L}|\overline{\mathrm{Z}}$; its restriction to k is uniquely determined by the conjugate $\tau\zeta$. From Lemma 7.76 we know that there is some $a \in L$ and an automorphism $\sigma \in \mathrm{Gal}\, L|\mathrm{Z}$ such that $\overline{a} = \zeta$ and $\overline{\sigma a} = \tau\zeta$. That is, $\overline{\sigma}$ coincides with τ on k.

If $L|K$ is finite, then also $\overline{L}|\overline{\mathrm{Z}}$ is finite by virtue of the fundamental inequality (7.2), and we may choose $k = (\overline{L}|\overline{\mathrm{Z}})^{\mathrm{sep}}$; then our argument shows that (7.15) is surjective. For

the infinite case, we proceed as follows. Let $\tau \in \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}}$ be given. We know that the maximal separable subextension of $\overline{L}|\overline{\mathrm{Z}}$ is the union of finite Galois subextensions $k_i|\overline{\mathrm{Z}}$, $i \in I$, and we have shown that for every $i \in I$, the restriction τ_i of τ to k_i coincides with that of $\overline{\sigma_i}$ for some $\sigma_i \in \operatorname{Gal} L|\mathrm{Z}$. From the Compactness Principle for Algebraic Extensions (Lemma 6.5), it now follows that there is $\sigma \in \operatorname{Gal} L|\mathrm{Z} = G_{\mathrm{Z}}$ such that $\overline{\sigma} = \tau$. This proves the surjectivity of (7.15) in the general case. Since G_{T} is defined to be the kernel of the continuous homomorphism (7.15), we thus obtain a topological isomorphism $G_{\mathrm{Z}}/G_{\mathrm{T}} \cong \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}} = \operatorname{Gal}\overline{L}|\overline{K}$. The topological isomorphism $\operatorname{Gal}\mathrm{T}|\mathrm{Z} \cong G_{\mathrm{Z}} \,/\, G_{\mathrm{T}}$ is inferred from infinite Galois theory.

Now let $K'|\mathrm{Z}$ be any subextension of $\mathrm{T}|\mathrm{Z}$. Then by Lemma 7.82, K' is the decomposition field and T is the inertia field of the normal extension $(L|K', v)$. What we have just proved may be applied to K' in the place of K, showing that the restriction of (7.15) to $\operatorname{Gal} L|K'$ induces an isomorphism from $\operatorname{Gal} L|K' \,/\, G_{\mathrm{T}} \cong \operatorname{Gal}\mathrm{T}|K'$ onto $\operatorname{Gal}\overline{L}|\overline{K'}$. On the one hand, this result yields that if $K'|\mathrm{Z}$ is normal, then the isomorphism induces an isomorphism of $\operatorname{Gal}\mathrm{T}|\mathrm{Z} \,/\, \operatorname{Gal}\mathrm{T}|K' \cong \operatorname{Gal} K'|\mathrm{Z}$ onto $\operatorname{Gal}\overline{L}|\overline{\mathrm{Z}} \,/\, \operatorname{Gal}\overline{L}|\overline{K'} \cong \operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}$. On the other hand, we apply this result to $K' = \mathrm{T}$ to find that $\operatorname{Gal}\overline{L}|\overline{\mathrm{T}}$ is trivial. That is, $\overline{L}|\overline{\mathrm{T}}$ must be a purely inseparable extension.

Let $K'|\mathrm{Z}$ be a finite normal subextension of the normal extension $\mathrm{T}|\mathrm{Z}$. We have already shown that $\operatorname{Gal} K'|\mathrm{Z} \cong \operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}$. Since $\mathrm{T}|\mathrm{Z}$ and thus also $K'|\mathrm{Z}$ is separable, we obtain $[K' : \mathrm{Z}] = |\operatorname{Gal} K'|\mathrm{Z}| = |\operatorname{Gal}\overline{K'}|\overline{\mathrm{Z}}| \leq [\overline{K'} : \overline{\mathrm{Z}}] \leq [K' : \mathrm{Z}]$, where the last inequality follows from the fundamental inequality (7.2). Thus, equality holds everywhere, showing that $[K' : \mathrm{Z}] = [\overline{K'} : \overline{\mathrm{Z}}]$ and that $\overline{K'}|\overline{\mathrm{Z}}$ is Galois. Moreover, from the fundamental inequality we infer that $vK' = v\mathrm{Z}$, which in view of Lemma 7.87 tells us that $vK' = vK$. Since every value of $v\mathrm{T}$ is already contained in the value group of a finite normal subextension $K'|\mathrm{Z}$, we find that $v\mathrm{T} = vK$. Similarly, since every element of $\overline{\mathrm{T}}$ is already contained in the residue field of a finite normal subextension $K'|\mathrm{Z}$, we see that $\overline{\mathrm{T}}|\overline{K}$ is separable. Since $\overline{L}|\overline{\mathrm{T}}$ is purely inseparable, we may conclude that $\overline{\mathrm{T}}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. From this, we also obtain the topological isomorphism of $\operatorname{Gal}\overline{L}|\overline{K} = \operatorname{Gal}\overline{L}|\overline{\mathrm{Z}}$ with $\operatorname{Gal}\overline{\mathrm{T}}|\overline{\mathrm{Z}}$.

Finally, let $\mathrm{Z} \subset K_0 \subset K_1 \subset \mathrm{T}$ with $K_1|K_0$ finite. We choose a finite normal subextension $K'|K_0$ of $\mathrm{T}|K_0$ such that $K_1 \subset K'$. From Lemma 7.82 we know that the decomposition field of $(L|K_0, v)$ is $K_0.\mathrm{Z} = K_0$ and that its inertia field is $K_0.\mathrm{T} = \mathrm{T}$, and similarly for K_1 in the place of K_0. What we have already shown may thus be applied to K_0 and K_1 in the place of Z to obtain that $[K' : K_0] = [\overline{K'} : \overline{K_0}]$ and $[K' : K_1] = [\overline{K'} : \overline{K_1}]$. This proves that $[K_1 : K_0] = [\overline{K_1} : \overline{K_0}]$. □

Lemma 7.89 *Assume (7.22) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\operatorname{res}_E(G_{\mathrm{T}}) = G^i(E|K, v) \quad \text{and} \quad (E \cap \mathrm{T}, v) = (E|K, v)^i \ .$$

Proof: In view of Corollary 7.83, we have to show that $\operatorname{res}_E : G_{\mathrm{T}} \to G^i(E|K, v)$ is surjective. Let $\rho \in G^i(E|K, v)$, which means that $\overline{\rho}$ is the identity on $\overline{E}$. It also implies that $\rho \in G^d(E|K, v)$, hence by Lemma 7.85 there exists $\sigma \in G_{\mathrm{Z}}$ such that $\rho = \operatorname{res}_E(\sigma)$. We see that $\overline{\sigma}$ fixes $\overline{E}$, so the surjectivity of (7.15) proved in Theorem 7.88 (applied to the extension $(L|E, v)$) shows that there is some $\tau \in G^d(L|E, v)$ such that $\overline{\tau} = \overline{\sigma}$. Consequently, $\sigma\tau^{-1} \in G_{\mathrm{T}}$ with $\rho = \operatorname{res}_E(\sigma\tau^{-1})$. □

Proceeding not too systematically may sometimes turn out to be of advantage. In this spirit, let us jump to the extension $L|\mathrm{V}$. The next theorem will describe its main properties. Beforehand, we need a lemma.

Lemma 7.90 *Let $(K_1|K_0, v)$ be an algebraic extension of valued fields. If $K_1|K_0$ is a p-extension, then vK_1/vK_0 is a p-group. The same is true if $K_1|K_0$ is purely inseparable.*

Proof: Assume that $K_1|K_0$ is a p-extension. Let Z_0 be the decomposition field of $(K_1|K_0, v)$. From Lemma 7.87 we know that $vK_0 = v\mathrm{Z}_0$. Since $K_1|K_0$ is a p-extension, the same is true for $K_1|\mathrm{Z}_0$. Hence, it suffices to show our assertion for the extension $K_1|\mathrm{Z}_0$.

Let $a \in K_1$. Since $\mathrm{Gal}\, K_1|\mathrm{Z}_0$ is a pro-p-group, the number of conjugates of a over Z_0 is a power of p, say p^n. Since the extension of v from Z_0 to K_1 is unique, all conjugates have the same value. The norm $\mathrm{N}_{\mathrm{Z}_0(a)|\mathrm{Z}_0}(a) \in \mathrm{Z}_0$, being the product of these conjugates, thus has value $p^n va$. This shows that $p^n va \in v\mathrm{Z}_0$. We have proved that $vK_1/v\mathrm{Z}_0 = vK_1/vK_0$ is a p-group.

Now assume that $K_1|K_0$ is purely inseparable. Then for every $a \in K_1$ there is some $n \in \mathbb{N}$ such that $a^{p^n} \in K_0$ and thus again, $p^n va \in vK_0$. As before we find that vK_1/vK_0 is a p-group. □

Theorem 7.91 *Assume (7.22). The ramification group G_V is a pro-p-group. Hence $K_s|\mathrm{V}$ is a p-extension, and $[L : \mathrm{V}]$ is a (possibly infinite) power of p. In particular, $G_\mathrm{V} = 1$ and $\mathrm{V} = L$ if $\mathrm{char}\, \overline{K} = 0$.*

The factor group $vL/v\mathrm{V}$ is a p-group, and the residue field extension $\overline{L}|\overline{\mathrm{V}}$ is purely inseparable.

Proof: Assume that G_V is not a pro-p-group, that is, that there exists a prime $q \neq p$ and a finite normal subextension $E|V$ of $K_s|V$ whose Galois group contains an element σ of order q. Let K_0 be the fixed field in E of the cyclic group generated by σ. Then $E|K_0$ is a Galois extension of degree q. Let a be a primitive element for $E|K_0$ and $f = X^q + c_{q-1}X^{q-1} + \ldots + c_0$ its minimal polynomial over K_0. Note that $-c_{q-1}$ is equal to the trace $\mathrm{Tr}_{E|K_0}(a)$. Replacing a by $a + c_{q-1}/q$ (which is possible since $q \neq 0$ in E), we may assume from the start that this trace of a is zero. On the other hand, let $\sigma_i a$ be all conjugates of a, with a suitable choice of $\sigma_1, \ldots, \sigma_q \in \mathrm{Gal}\, K_s|K_0 \subset G_\mathrm{V}$. Then $\overline{\sigma_i a/a} = 1$ for all i, and the element $0 = a^{-1}\mathrm{Tr}_{E|K_0}(a) = \sum_{1 \leq i \leq q} \sigma_i a/a$ has residue q, but $q \neq p$ is not zero in $\overline{K_1}$. This contradiction shows that G_V must be a pro-p-group. Since it is the Galois group of the separable extension $K_s|\mathrm{V}$, this extension is a p-extension. On the other hand, $L|K_s$ is purely inseparable by definition of K_s, so both degrees $[L : K_s]$ and $[K_s : \mathrm{V}]$ and consequently also $[L : \mathrm{V}]$ are powers of p. (Recall that $\mathrm{char}\, K = \mathrm{char}\, \overline{K} = p$ if $\mathrm{char}\, K \neq 0$.)

From the preceding lemma it now follows that vL/vK_s and $vK_s/v\mathrm{V}$ are p-groups. Hence also $vL/v\mathrm{V}$ is a p-group. Finally, the residue field extension $\overline{L}|\overline{\mathrm{V}}$ must be purely inseparable since it was asserted by Theorem 7.88 that $\overline{\mathrm{T}}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. □

In view of Corollary 6.29 and Corollary 6.57, the foregoing theorem yields:

Corollary 7.92 *Assume (7.22) with $p > 1$ and let $\mathrm{V} \subset K_0 \subset K_1 \subset L$. Then $K_1|K_0$ is a tower of normal extensions of degree p, the separable ones being Artin-Schreier extensions.*

We will now examine the extension V|T. To this end, we return to our pairing (7.17) and ask for the kernel of the homomorphism (7.20). It contains $\mathrm{T}^\times$ since for all $a \in \mathrm{T}$ and all $\sigma \in G_\mathrm{T}$ we have $\sigma a = a$ and consequently, $(a, \sigma) = 1$. By (7.18), the kernel also contains $\mathcal{O}_\mathbf{L}^\times$. It follows that every element $a \in L$ with $va \in vK$ lies in the kernel since it can be written as $a = bc$ with $b \in K$, $vb = va$, so that $c \in \mathcal{O}_\mathbf{L}^\times$.

Now we see that the value of (a, σ) only depends on the coset of va modulo vK and on the coset of σ modulo G_V (since the latter was defined to be the kernel of (7.19)). But $G_\mathrm{T}/G_\mathrm{V} \cong \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. So the pairing (7.17) is in fact a pairing

$$(.\,,.)' : \; vL/vK \times \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\longrightarrow\; \overline{L}^\times \tag{7.23}$$

between the additive group vL/vK and the Galois group $\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. For $\overline{\alpha} := \alpha + vK \in vL/vK$ and $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$, it satisfies

$$(\overline{\alpha}, \sigma)' = (a, \sigma_L) \tag{7.24}$$

where $a \in L$ is an arbitrary element such that $va = \alpha$, and $\sigma_L \in G_\mathrm{T}$ is an arbitrary automorphism such that $\mathrm{res}_\mathrm{V}(\sigma_L) = \sigma$. If $\sigma \in G_\mathrm{T}$, then we will simply write "$(\overline{\alpha}, \sigma)'$" instead of "$(\overline{\alpha}, \mathrm{res}_\mathrm{V}(\sigma))'$". Since every element in vL/vK has finite order, the same is true for every element $(\overline{\alpha}, \sigma)' \in \overline{L}^\times$. That is, the range of the pairing $(.\,,.)'$ (which is equal to that of $(.\,,.)$) lies in the subgroup of all torsion elements in $\overline{L}^\times$. This is in fact the subgroup of all roots of unity in $\overline{L}$. It is an abelian torsion group.

Since $\mathrm{char}\,\overline{L} = p$, the group of roots of unity in $\overline{L}$ is a p'-group. Consequently, $(\overline{\alpha}, \sigma)' = 1$ for arbitrary $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$ if $\alpha \in vL$ is an element whose order modulo vK is a power of p. We write $vL/vK = (vL/vK)_p \oplus (vL/vK)_{p'}$ where $(vL/vK)_p$ is an abelian p-group and $(vL/vK)_{p'}$ is an abelian p'-group. We find that $((vL/vK)_p\,,\, \mathrm{Gal}\,\mathrm{V}|\mathrm{T})' = \{1\}$, so the above pairing can be rewritten as

$$(vL/vK)_{p'} \times \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\longrightarrow\; \overline{L}^\times \;.$$

Given any extension $\Delta \subset \Delta'$ of abelian groups, the **(relative) p'-divisible hull of Δ in Δ'** is defined to be the subgroup $\{\alpha \in \Delta' \mid \exists n :\; (p, n) = 1 \wedge n\alpha \in \Delta\} \subset \Delta'$ of all elements in Δ' whose order modulo Δ is prime to p. It is equal to the foreimage of $(\Delta'/\Delta)_{p'}$ under the canonical epimorphism $\Delta' \to \Delta'/\Delta$. Since $vL/v\mathrm{V}$ is a p-group by virtue of Lemma 7.91, the p'-divisible hull of vK in vL lies already in $v\mathrm{V}$. Hence, $(vL/vK)_{p'} = (v\mathrm{V}/vK)_{p'}$. In view of Theorem 7.88 we may also replace vK by $v\mathrm{T}$. Furthermore, observe that we may replace $\overline{L}^\times$ by $\overline{\mathrm{V}}^\times$. Indeed, since the group of all roots of unity in a field of characteristic exponent p is a p'-group, it remains unchanged under purely inseparable extensions. On the other hand, Theorem 7.91 shows that $\overline{L}|\overline{\mathrm{V}}$ is purely inseparable. So the groups of roots of unity in $\overline{L}$ and in $\overline{\mathrm{V}}$ are equal, and the above pairing can be rewritten as

$$(v\mathrm{V}/v\mathrm{T})_{p'} \times \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\longrightarrow\; \overline{\mathrm{V}}^\times \;. \tag{7.25}$$

The homomorphism (7.19) with kernel G_V turns into an embedding

$$\mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\longrightarrow\; \mathrm{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\, \overline{\mathrm{V}}^\times\right) \tag{7.26}$$

of $\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$ in the p-character group $\mathrm{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)$. Since the latter is an abelian group, this shows that also the Galois group $\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$ is abelian.

Now let H be any pro-p-Sylow group of G_{T} containing the pro-p-subgroup G_{V}, and let L_0 be its fixed field in K_s. Then $\mathrm{T} \subset L_0 \subset \mathrm{V}$, and $V|L_0$ is a p-extension. From Lemma 7.82 we infer that L_0 itself is the inertia field of $(L|L_0, v)$ and that V is its ramification field. Hence, in (7.26) we can just replace T by L_0 to obtain an embedding

$$H/G_{\mathrm{V}} \cong \mathrm{Gal}\,\mathrm{V}|L_0 \;\longrightarrow\; \mathrm{Hom}\left((v\mathrm{V}/vL_0)_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)\;.$$

But Lemma 7.90 shows that $(v\mathrm{V}/vL_0)_{p'} = 0$ since $V|L_0$ is a p-extension. Consequently, $\mathrm{Hom}\left((v\mathrm{V}/vL_0)_{p'}\,,\,\overline{\mathrm{V}}^{\times}\right)$ and thus also H/G_{V} are trivial, showing that $H = G_{\mathrm{V}}$. Hence, G_{V} is a pro-p-Sylow group in G_{T}; it is the only one since it is a normal subgroup of G_{T}. We have proved:

Lemma 7.93 *Assume (7.22). The ramification group G_{V} is the unique pro-p-Sylow group in the inertia group G_{T}, and $\mathrm{Gal}\,\mathrm{V}|\mathrm{T} \cong G_{\mathrm{T}}/G_{\mathrm{V}}$ is an abelian pro-p'-group.*

Now we can state the main properties of the extension V|T:

Theorem 7.94 *Assume (7.22). Then $\mathrm{V}|\mathrm{T}$ is an abelian p'-extension. The value group of the ramification field V is the p'-divisible closure of vK in vL, that is, $v\mathrm{V}/vK = (vL/vK)_{p'}$. Its residue field is equal to the residue field of the inertia field T. The homomorphism (7.26) is onto and consequently, the Galois group of V over T is*

$$G_{\mathrm{T}}\,/\,G_{\mathrm{V}} \;\cong\; \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \;\cong\; \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}\right)\;.$$

This is isomorphic to $v\mathrm{V}/v\mathrm{T}$ if $\mathrm{V}|\mathrm{T}$ is finite. The isomorphisms are topological isomorphisms, and $\mathrm{Hom}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times})$ is the full p-character group of $v\mathrm{V}/v\mathrm{T}$.

For every subextension $K'|\mathrm{T}$ of $\mathrm{V}|\mathrm{T}$, the image of $\mathrm{Gal}\,\mathrm{V}|K'$ under this isomorphism is equal to $\mathrm{Hom}_{vK'/v\mathrm{T}}(v\mathrm{V}/v\mathrm{T}\,,\,\overline{\mathrm{T}}^{\times}) \cong \mathrm{Hom}(v\mathrm{V}/vK'\,,\,\overline{\mathrm{T}}^{\times})$. If $K'|\mathrm{T}$ is normal, then this isomorphism induces an isomorphism of $\mathrm{Gal}\,K'|\mathrm{T}$ onto $\mathrm{Hom}(vK'/v\mathrm{T}\,,\,\overline{\mathrm{V}}^{\times})$. If $K'|\mathrm{T}$ is also finite, then $\mathrm{Gal}\,K'|\mathrm{T}$ is isomorphic to $vK'/v\mathrm{T} = vK'/vK$. Moreover, $[K_1 : K_0] = (vK_1 : vK_0)$ for every finite extension $(K_1|K_0, v)$ such that $\mathrm{T} \subset K_0 \subset K_1 \subset \mathrm{V}$.

Proof: According to the previous lemma, the Galois group of the Galois extension $\mathrm{V}|\mathrm{T}$ is a p'-group, that is, $\mathrm{V}|\mathrm{T}$ is a p'-extension. Assume that $K'|\mathrm{T}$ is a finite normal subextension of $\mathrm{V}|\mathrm{T}$. Since $\mathrm{V}|\mathrm{T}$ is a p'-extension, the same is true for the subextension $K'|\mathrm{T}$. Theorem 7.91 shows that the ramification group of a p'-extension must be trivial; hence, the ramification field of $(K'|\mathrm{T}, v)$ must be equal to K'. On the other hand, Lemma 7.82 shows that the inertia field of $(L|\mathrm{T}, v)$ is T. Together with Lemma 7.89, this in turn yields that the inertia field of $(K'|\mathrm{T}, v)$ is again T. In (7.26), we can thus replace V to obtain an embedding of $\mathrm{Gal}\,K'|\mathrm{T}$ in the p-character group $\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\overline{K'}^{\times})$. This in turn is a subgroup of the full p-character group $\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\widetilde{k}^{\times})$, where $k = \overline{K'}$. For k a field of characteristic exponent p and a finite abelian p'-group Δ, the full p-character group $\mathrm{Hom}(\Delta, \tilde{k})$ is isomorphic to Δ (cf. Lemma 6.59). Using also the inequality $|vK'/v\mathrm{T}| \leq [K' : \mathrm{T}]$ which we infer from the fundamental inequality (7.2), we compute

$$\begin{aligned}
[K' : \mathrm{T}] &= \left|\,\mathrm{Gal}\,K'|\mathrm{T}\,\right| \leq \left|\,\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\overline{K'}^{\times})\,\right| \\
&\leq \left|\,\mathrm{Hom}((vK'/v\mathrm{T})_{p'}\,,\,\widetilde{k}^{\times})\,\right| = \left|(vK'/v\mathrm{T})_{p'}\right| \\
&\leq |vK'/v\mathrm{T}| \leq [K' : \mathrm{T}]\;.
\end{aligned}$$

Hence, equality holds everywhere. In particular, $(vK'/v\mathrm{T})_{p'} = vK'/v\mathrm{T}$ which means that $vK'/v\mathrm{T}$ is a p'-group, and

$$\mathrm{Gal}\, K'|\mathrm{T} \;\cong\; \mathrm{Hom}(vK'/v\mathrm{T}\,,\, \overline{K'}^{\times}) \;\cong\; vK'/v\mathrm{T}\;.$$

Note that we also obtain that $\mathrm{Hom}(vK'/v\mathrm{T}\,,\, \overline{K'}^{\times})$ is already the full character group of $vK'/v\mathrm{T}$. Furthermore, we see that $[K' : \mathrm{T}] = (vK' : v\mathrm{T})$, and in view of the fundamental inequality (7.2), we may conclude that $\overline{K'} = \overline{\mathrm{T}}$. Since $K'|\mathrm{T}$ was an arbitrary finite normal subextension of $\mathrm{V}|\mathrm{T}$, we have now proved that $v\mathrm{V}/v\mathrm{T}$ is a p'-group, and that $\overline{\mathrm{V}} = \overline{\mathrm{T}}$. Consequently,

$$\mathrm{Hom}\left((v\mathrm{V}/v\mathrm{T})_{p'}\,,\, \overline{\mathrm{V}}^{\times}\right) = \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\, \overline{\mathrm{T}}^{\times}\right)\;.$$

If $\mathrm{V}|\mathrm{T}$ is finite, we may choose $K' = \mathrm{V}$; then our argument shows that the embedding (7.26) is surjective, and in this case, it follows that $\mathrm{Gal}\,\mathrm{V}|\mathrm{T} \cong v\mathrm{V}/v\mathrm{T}$. To show the surjectivity of (7.26) in the infinite case, we proceed as follows. Let $\chi \in \mathrm{Hom}(v\mathrm{V}/v\mathrm{T}\,,\, \overline{\mathrm{T}}^{\times})$ be given. The extension $\mathrm{V}|\mathrm{T}$ is the union of finite Galois extensions $K_i|\mathrm{T}$, $i \in I$ and consequently, $v\mathrm{V}$ is the union of the groups vK_i. By the surjectivity in the finite case that we have proved above, we know that for every $i \in I$, the restriction χ_i of χ to vK_i is of the form $(\,.\,,\sigma_i)'$ for some $\sigma_i \in \mathrm{Gal}\, K_i|\mathrm{T}$. From the Compactness Principle for Algebraic Extensions (Lemma 6.5) it now follows that there is $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$ such that $(\,.\,,\sigma)' = \chi$. This proves the surjectivity of (7.26) in the general case. In this argument, $\mathrm{V}|\mathrm{T}$ can be replaced by an arbitrary normal subextension $K'|\mathrm{T}$, showing that the induced embedding $\mathrm{Gal}\, K'|\mathrm{T} \to \mathrm{Hom}(vK'/v\mathrm{T}\,,\, \overline{\mathrm{T}}^{\times})$ is onto.

Now let $K'|\mathrm{T}$ be an arbitrary subextension of $\mathrm{V}|\mathrm{T}$. By Lemma 7.82, the ramification field of $(L|K', v)$ is V, and its inertia field is K'. The pairing associated with this extension is obtained by restricting (7.17) to the group $G^i(L|K', v) = G_{\mathrm{T}} \cap \mathrm{Gal}\, L|K'$. By what we have shown already, the restricted pairing yields an isomorphism of $\mathrm{Gal}\,\mathrm{V}|K'$ onto $\mathrm{Hom}_{vK'/v\mathrm{T}}(v\mathrm{V}/v\mathrm{T}\,,\, \overline{\mathrm{T}}^{\times}) \cong \mathrm{Hom}(v\mathrm{V}/vK'\,,\, \overline{\mathrm{T}}^{\times})$ (note that $\overline{\mathrm{T}} = \overline{K'}$).

Finally, let $\mathrm{T} \subset K_0 \subset K_1 \subset \mathrm{V}$ with $K_1|K_0$ finite. We choose a finite normal subextension $K'|K_0$ of $\mathrm{T}|K_0$ such that $K_1 \subset K'$. From Lemma 7.82 we know that the inertia field of $(L|K_0, v)$ is $K_0.\mathrm{T} = K_0$ and that its ramification field is $K_0.\mathrm{V} = \mathrm{V}$, and similarly for K_1 in the place of K_0. What we have already shown may thus be applied to K_0 and K_1 in the place of T to obtain that $[K' : K_0] = (vK' : vK_0)$ and $[K' : K_1] = (vK' : vK_1)$. This proves that $[K_1 : K_0] = (vK_1 : vK_0)$. □

Lemma 7.95 *Assume (7.22) and let $E|K$ be a normal subextension of $L|K$. Then*

$$\mathrm{res}_E(G_{\mathrm{V}}) = G^r(E|K, v) \quad \textit{and} \quad (E \cap \mathrm{V}, v) = (E|K, v)^r\;.$$

Proof: In view of Corollary 7.83, we have to show that $\mathrm{res}_E : G_{\mathrm{V}} \to G^r(E|K, v)$ is surjective. Let $(\,.\,,.\,)'_{L|K}$, $(\,.\,,.\,)'_{E|K}$ and $(\,.\,,.\,)'_{L|E}$ denote the pairing (7.23) for the extension $L|K$, $E|K$ and $L|E$ respectively. Assume that $\rho \in G^r(E|K, v)$, which means that $(\,.\,,\rho)'_{E|K}$ is the trivial character of vE/vK. It also implies that $\rho \in G^i(E|K, v)$, hence by Lemma 7.89 there exists $\sigma \in G_{\mathrm{T}}$ such that $\rho = \mathrm{res}_E(\sigma)$. It follows that $(vE/vK, \sigma)'_{L|K} = (vE/vK, \rho)'_{E|K} = 1$, so $(\,.\,,\sigma)'_{L|K}$ is in fact a character of vL/vE. Now the surjectivity proved in Theorem 7.94 shows that there is some $\tau \in G^i(L|E, v)$ such that $(\,.\,,\tau)'_{L|E} = (\,.\,,\sigma)'_{L|K}$. It follows that also $(\,.\,,\tau)'_{L|K} = (\,.\,,\sigma)'_{L|K}$. Consequently, $(\,.\,,\sigma\tau^{-1})'_{L|K} = 1$, showing that $\sigma\tau^{-1} \in G_{\mathrm{V}}$ with $\rho = \mathrm{res}_E(\sigma\tau^{-1})$. □

From the foregoing theorem together with Theorem 7.88, we deduce:

Corollary 7.96 *Assume (7.22). If the maximal separable subextension $\overline{T}|\overline{K}$ of $\overline{L}|\overline{K}$ is a p'-extension, then also* T|Z *and* V|Z *are p'-extensions.*

Corollary 7.97 *Assume (7.22). Let $(K_1|K_0, v)$ be a finite normal extension such that $K \subset K_0 \subset K_1 \subset \mathrm{V}$. Then*

$$[K_1 : K_0] = e \cdot f \cdot g$$

where $e = (vK_1 : vK_0)$, $f = [\overline{K_1} : \overline{K_0}]$ and g is the number of distinct extensions of v from K_0 to K_1. If $\mathrm{Z} \subset K_0$, then $g = 1$.

Proof: From Lemma 7.95 it follows that the ramification field of $(K_1|K, v)$ is K_1. Together with Lemma 7.82, this in turn yields that the ramification field of $(K_1|K_0, v)$ is again K_1. In view of

$$[K_1 : K_0] = [(K_1|K, v)^r : (K_1|K, v)^i] \cdot (K_1|K, v)^i : (K_1|K, v)^d] \cdot [(K_1|K, v)^d : K_0] \ ,$$

our first assertion follows from Theorem 7.94, Theorem 7.88 and Theorem 7.84. If $\mathrm{Z} \subset K_0$, then by Lemma 7.95 and Lemma 7.82, K_0 is the decomposition field of $(K_1|K_0, v)$, and Theorem 7.84 shows that $g = 1$. □

Remark 7.98 The isomorphism of Theorem 7.94 induces a topology on the group $\mathrm{Hom}(v\mathrm{V}/v\mathrm{T}, \overline{\mathrm{T}}^\times)$ which shows that it is in fact a profinite group. Further, Theorem 7.94 shows that the subgroups $\mathrm{Hom}(v\mathrm{V}/vK', \overline{\mathrm{T}}^\times)$ are closed for every subextension $K'|\mathrm{T}$ of V|T, and that they are open if and only if $K'|\mathrm{T}$ is finite. See O. Endler [END2], §20 for further details.

We summarize our main results of this section in the following table:

Galois group	**field**		**value group**	**residue field**
	(L,v)		vL	$\overline{L}$
	purely inseparable			
1	$(L\|K,v)^{\mathrm{sep}}$	maximal separable subextension	$(vL/vK)_p$	purely inseparable
	p-extension			
$G^r(L\|K,v)$	$(L\|K,v)^r$	ramification field	$\frac{1}{p'^\infty}vK \cap vL$	$(\overline{L}\|\overline{K})^{\mathrm{sep}}$
Char	p'-extension		$(vL/vK)_{p'}$	$\|\|$
$G^i(L\|K,v)$	$(L\|K,v)^i$	inertia field	vK	$(\overline{L}\|\overline{K})^{\mathrm{sep}}$
$\mathrm{Gal}\,\overline{L}\|\overline{K}$	unramified		$\|\|$	Galois
$G^d(L\|K,v)$	$(L\|K,v)^d$	decomposition field	vK	$\overline{K}$
	immediate		$\|\|$	$\|\|$
$\mathrm{Gal}\,L\|K$	(K,v)		vK	$\overline{K}$

where $\frac{1}{p'^\infty}vK \cap vL$ is the relative p'-divisible closure of vK in vL, and Char denotes the character group

$$\mathrm{Hom}\left(vL/vK\,,\,\overline{L}^\times\right) \cong \mathrm{Hom}\left((vL/vK)_{p'}\,,\,((\overline{L}|\overline{K})^{\mathrm{sep}})^\times\right)\ ,$$

which is also isomorphic to the character group appearing in Theorem 7.94.

Let us return to the pairing (7.25). Lemma 7.94 has shown that $v\mathrm{V}/v\mathrm{T}$ is a p'-group, hence $(v\mathrm{V}/v\mathrm{T})_{p'} = v\mathrm{V}/v\mathrm{T}$. Further, $\overline{\mathrm{V}} = \overline{\mathrm{T}}$ by Theorem 7.94. Therefore, the pairing (7.25) induces a homomorphism

$$v\mathrm{V}/v\mathrm{T} \longrightarrow \mathrm{Hom}\left(\mathrm{Gal}\,\mathrm{V}|\mathrm{T}\,,\,\overline{\mathrm{T}}^\times\right) \tag{7.27}$$

of $v\mathrm{V}/v\mathrm{T}$ into the character group $\mathrm{Hom}\left(\mathrm{Gal}\,\mathrm{V}|\mathrm{T}\,,\,\overline{\mathrm{T}}^\times\right)$. We prove:

Lemma 7.99 *The homomorphism (7.27) is an embedding.*

Proof: Assume that $b \in \mathrm{V}^\times$ is such that $\overline{\sigma b\,/\,b} = 1$ for every $\sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$. Then in particular, all conjugates $\sigma_i b$ of b over T can be written as $\sigma_i b = b(1+c_i)$ with $c_i \in \mathcal{M}_{(\mathrm{V},v)}$ $(1 \le i \le n := [\mathrm{T}(b) : \mathrm{T}])$. Hence,

$$\mathrm{Tr}_{\mathrm{T}(b)|\mathrm{T}}(b) = b(n+c) \qquad \text{with } \; c = \sum_{1\le i\le n} c_i \in \mathcal{M}_{(\mathrm{V},v)}\ .$$

Since V|T is a p'-extension according to Lemma 7.94, n is prime to p. That is, $vn = 0$ showing that $vb = vb(n+c) = v\mathrm{Tr}_{\mathrm{T}(b)|\mathrm{T}}(b) \in v\mathrm{T}$. This yields our assertion. □

Corollary 7.100 *The pairing*

$$(.\,,.)' :\ v\mathrm{V}/v\mathrm{T} \times \mathrm{Gal}\,\mathrm{V}|\mathrm{T} \ \longrightarrow\ \overline{\mathrm{T}}^\times \tag{7.28}$$

is faithful: If $1 \neq \sigma \in \mathrm{Gal}\,\mathrm{V}|\mathrm{T}$*, then* $(\,.\,,\sigma)'$ *is a nontrivial character of* $v\mathrm{V}/v\mathrm{T}$*. If* $0 \neq \overline{\alpha} \in v\mathrm{V}/v\mathrm{T}$*, then* $(\overline{\alpha},\,.\,)'$ *is a nontrivial character of* $\mathrm{Gal}\,\mathrm{V}|\mathrm{T}$*.*

Finally, let us return once more to the "functorial discussion" which was initiated by Lemma 7.78. Let us now consider the following case. Let $(K'|K, v)$ be an arbitrary extension, $(L'|K', v)$ a normal algebraic extension and $(L|K, v)$ a normal algebraic subextension of $(L'|K, v)$. In this case, τ_* is the restriction map res_L, which is a topological epimorphism from $\mathrm{Gal}\,L.K'|K'$ onto $\mathrm{Gal}\,L \,|\, L \cap K'$.

For the purposes of ramification theory, we take fixed fields in the maximal separable subextension of a given normal algebraic extension. So let $L_s := (L \,|\, L \cap K')^{\mathrm{sep}}$. Then also $L_s.K'|K'$ is separable, so $L_s.K'$ and thus also L_s are contained in $L'_s := (L'|K')^{\mathrm{sep}}$. Further, $L_s \cap K' = L \cap K'$ by definition of L_s. Now τ_* is the restriction map res_{L_s}, which again gives the topological epimorphism from $\mathrm{Gal}\,L'|K'$ onto $\mathrm{Gal}\,L \,|\, L \cap K'$ since $\mathrm{Gal}\,L'|K' = \mathrm{Gal}\,L'_s|K'$ and $\mathrm{Gal}\,L \,|\, L \cap K' = \mathrm{Gal}\,L_s \,|\, L_s \cap K'$. From Lemma 7.78, we obtain:

Lemma 7.101 *Let* $(K'|K, v)$ *be an arbitrary extension,* $(L'|K', v)$ *a normal algebraic extension and* $(L|K, v)$ *a normal algebraic subextension of* $(L'|K, v)$*. Then*

$$\begin{array}{lll}
\mathrm{res}_L(G^d(L'|K', v)) \subset G^d(L \,|\, L \cap K', v) & \textit{and} & (L \cap (L'|K', v)^d, v) \supset (L \,|\, L \cap K', v)^d \\
\mathrm{res}_L(G^i(L'|K', v)) \subset G^i(L \,|\, L \cap K', v) & \textit{and} & (L \cap (L'|K', v)^i, v) \supset (L \,|\, L \cap K', v)^i \\
\mathrm{res}_L(G^r(L'|K', v)) \subset G^r(L \,|\, L \cap K', v) & \textit{and} & (L \cap (L'|K', v)^r, v) \supset (L \,|\, L \cap K', v)^r\ .
\end{array}$$

Later, we will determine criteria for the above inclusions to be equalities. One important case will be met in the next section.

Exercise 7.12 *Show that* $\mathrm{Hom}((v\mathrm{V}/v\mathrm{T})_{p'}, \overline{\mathrm{V}}^\times)$ *is a pro-*p'*-group, without using that it is isomorphic to some Galois group.*

7.9 Henselian fields and henselizations

To the lazy mathematician it may appear uncomfortable to work with too many valuations, so we take the occasion to fix a very handy class of valued fields: (K, v) will be called **henselian** if it admits a unique extension of v to *every* algebraic extension. If (K, v) is henselian and if it is clear that the symbol "v" refers to the valuation v on K, then we will

also say that v is a **henselian valuation** and that its associated place P_v is a **henselian place**. Henceforth, if we are working with a henselian field (K, v), we will automatically assume the valuation extended to every algebraic field extension and to be called v again; since the extension is unique (and since it is again henselian, as we will see below), this can't cause confusion.

Let $L|K$ be an arbitrary algebraic extension of fields and $E|K$ a subextension of $L|K$. Let v be a valuation on K. Suppose that v_1 and v_2 are two distinct extensions of v to E. By Corollary 7.37, v_1 and v_2 can be extended to valuations v_1 and v_2 of L, and we will have $v_1 \neq v_2$ on L since already $v_1 \neq v_2$ on E. Hence K admits a unique extension of v to L if and only if it admits a unique extension of v to every intermediate field E.

Now let v be extended to E and call this extension again v. Every extension of v from E to L is also an extension of v from K to L. Hence if K admits a unique extensions of v to L, then so does E.

If K admits two distinct extensions v_1, v_2 of v to $\tilde{K}$, then in view of Corollary 7.72, already their restrictions to the separable-algebraic closure $(L|K)^{\mathrm{sep}}$ will be distinct. But then, there is some a separable over K such that $v_1 a \neq v_2 a$. So we have found a finite separable subextension $K(b)|K$ of $L|K$ such that v_1 and v_2 are distinct on $K(b)$. If in addition, $L|K$ is normal, then we can pass to the normal hull N of $K(a)$ over K, and we have found a finite Galois extension of $N|K$ such that v_1 and v_2 are distinct on N.

With $L = \tilde{K}$, these considerations prove:

Lemma 7.102 *A valued field (K, v) is henselian if and only if it admits a unique extension of v to $\tilde{K}$, and this holds if and only if (K, v) admits a unique extension of v to every finite Galois extension. In particular, every separable-algebraically closed valued field is henselian. Every algebraic extension of a henselian field is again henselian.*

The property of being henselian is preserved under isomorphisms of valued fields. To see this, let $\iota : \mathbf{K} \to \mathbf{K}'$ be such an isomorphism. In particular, $\iota\mathcal{O}_{\mathbf{K}} = \mathcal{O}_{\mathbf{K}'}$ (cf. Exercise 5.2). The field isomorphism ι can be extended to an isomorphism $\tilde{\iota} : \tilde{K} \to \tilde{K}'$. If $\mathbf{K}'$ is not henselian, then there are two distinct valuation rings $\mathcal{O}'_1$ and $\mathcal{O}'_2$ of $\tilde{K}'$ lying above $\mathcal{O}_{\mathbf{K}'}$. Then the valuation rings $\tilde{\iota}^{-1}(\mathcal{O}'_1)$ and $\tilde{\iota}^{-1}(\mathcal{O}'_2)$ of $\tilde{K}$ are distinct and lie above $\mathcal{O}_{\mathbf{K}}$. This shows that also $\mathbf{K}$ is not henselian.

Setting $L = \tilde{K}$ in Theorem 7.84, we infer that $(K, v)^h := (\tilde{K}|K, v)^d$ is henselian. This valued field is called the **henselization of** (K, v) **in** $(\tilde{K}, v)$. By definition, it is a separable algebraic extension of K. To denote the underlying field of $(K, v)^h$, we will also write K^h or $K^{h(v)}$. It should be pointed out that our definition depends on the extension of v from K to $\tilde{K}$ that we have chosen. A different extension leads to a different henselization, but all of these henselizations are isomorphic over (K, v), as we will see now.

Lemma 7.103 *Choose an extension of the valuation v from K to $\tilde{K}$ and call it again v. Denote by (K^h, v) the henselization of (K, v) in $(\tilde{K}, v)$. Then for every $\iota \in \mathrm{Gal}\, K$, the field $(\iota^{-1}K^h, v\iota)$ is the henselization $(K^{h(v\iota)}, v\iota)$ of (K, v) in $(\tilde{K}, v\iota)$, and (K^h, v) is isomorphic over K to $(\iota^{-1}K^h, v\iota)$ via the uniquely determined isomorphism $\mathrm{res}_{K^h}(\iota^{-1})$.*

Proof: The assertion follows from the definition of the henselization together with Corollary 7.79; the uniqueness of $\mathrm{res}_{K^h}(\iota^{-1})$ is stated in part f) of Theorem 7.84. □

Lemma 7.104 *Let w be any extension of v from K to $\tilde{K}$, and let E be an algebraic extension of K such that (E,w) is henselian. Then (E,w) contains the henselization $(K^{h(w)},w)$ of (K,v) in $(\tilde{K},w)$. Further, there is a unique embedding of (K^h,v) over K in (E,w), and its image is $(K^{h(w)},w)$.*

Proof: Since (E,w) is assumed to be henselian, the extension of w to $\tilde{E}=\tilde{K}$ is unique. Applying part d) of Theorem 7.84 to the extension $(\tilde{K}|K,w)$, we find that (E,w) contains $(\tilde{K}|K,w)^d$. But the latter is the henselization $(K^{h(w)},w)$ of (K,v) in $(\tilde{K},w)$. By virtue of Theorem 7.68, $w=v\iota$ for some $\iota\in\operatorname{Gal}K$. By the foregoing lemma, $\operatorname{res}_{K^h}(\iota^{-1})$ is the unique isomorphism of (K^h,v) onto $(K^{h(w)},w)$ over K.

If σ is any embedding of (K^h,v) over K in (E,w), then its image $\sigma(K^h,v)$ is henselian. So by what we have just proved, it must contain $(K^{h(w)},w)$. Conversely, $\sigma^{-1}(K^{h(w)},w)$ is a henselian subfield of $(\tilde{K},v)$, and thus, it contains (K^h,v). This proves that $\sigma(K^h,v)=(K^{h(w)},w)$ and that $\operatorname{res}_{K^h}(\iota^{-1})=\operatorname{res}_{K^h}(\sigma^{-1})$ is the unique embedding of (K^h,v) in (E,w) over K. □

As we will later work with a universal extension that is fixed once and for all it is convenient to choose the notation $(K,v)^h$ as introduced above. As the henselization is a henselian field, the following is a direct consequence of the foregoing lemma:

Corollary 7.105 *(K,v) is henselian if and only if $(K,v)^h=(K,v)$.*

For the generalization of Lemma 7.104 to the case of arbitrary (not necessarily algebraic) extensions, we need the following corollary to Lemma 7.101:

Corollary 7.106 *The relative separable-algebraic closure of a valued field in a henselian extension field is again henselian.*

Proof: Let (F,v) be a henselian field and (K,v) a subfield. We apply Lemma 7.101 with $K'=F$, $L'=F^{\rm sep}$ and $L=K^{\rm sep}$ to find that $K^{\rm sep}\cap(F^{\rm sep}|F,v)^d\supset(K^{\rm sep}|K^{\rm sep}\cap F,v)^d$. But $(F^{\rm sep}|F,v)^d=(F,v)^h=(F,v)$ by the foregoing corollary because (F,v) is henselian by assumption. Consequently, $(K^{\rm sep}\cap F,v)^h=(K^{\rm sep}|K^{\rm sep}\cap F,v)^d\subset(K^{\rm sep}\cap F,v)$, showing that the latter is henselian. But $K^{\rm sep}\cap F$ is just the separable-algebraic closure of K in F. □

We are now able to prove the following universal property of the henselization:

Theorem 7.107 *For every henselian extension field (F,w) of (K,v), there is a unique embedding of $(K,v)^h$ into (F,w) over K. Consequently, there is a henselization of (K,v) in every henselian extension field of (K,v).*

Proof: If (F,w) is henselian, then by the preceding corollary, the same holds for the relative separable-algebraic closure of (K,v) in (F,w). Since the henselization is an algebraic extension, every embedded image of $(K,v)^h$ must be contained in this relative algebraic closure. Hence, it suffices to prove the theorem for the case of F being algebraic over K. But for this case, it is already asserted in Lemma 7.104. □

The following property of the henselization is a consequence of Lemma 7.82, applied with $L=\tilde{K}$.

Corollary 7.108 *Let $(K,v)\subset(E,v)\subset(\tilde{K},v)$. Then $(E,v)^h=(E.K^h,v)$.*

This assertion can also be proved as follows. Since $(E.K^h, v)$ is henselian according to Lemma 7.102, it contains $(E, v)^h$ by virtue of Lemma 7.104. Conversely, E^h must contain K^h and E, so we have $(E, v)^h = (E.K^h, v)$.

If there are more than one extension of the valuation in a normal field extension $L|K$, then the henselization K^h allows us to shift the setting to a normal extension $L.K^h|K^h$ which admits a unique extension of the valuation (this is the so-called **local case**). The ramification theory of both extensions is connected as follows.

Theorem 7.109 *Let $(L|K, v)$ be a normal subextension of $(\tilde{K}|K, v)$ and (K^h, v) the henselization of (K, v) in $(\tilde{K}, v)$. Then*

$$\begin{array}{lll} G^d(L|K,v) \cong \operatorname{Gal} L.K^h|K^h & \text{and} & (L|K,v)^d = (L \cap K^h, v) \\ G^i(L|K,v) \cong G^i(L.K^h|K^h, v) & \text{and} & (L|K,v)^i = (L \cap (L.K^h|K^h, v)^i, v) \\ G^r(L|K,v) \cong G^r(L.K^h|K^h, v) & \text{and} & (L|K,v)^r = (L \cap (L.K^h|K^h, v)^r, v)\ . \end{array}$$

(the isomorphisms being induced by the restriction map res_L, which is a topological isomorphism from $\operatorname{Gal} L.K^h|K^h$ onto $\operatorname{Gal} L \,|\, L \cap K^h$). Further, $L \,|\, L \cap K^h$ is linearly disjoint from $K^h \,|\, L \cap K^h$. In particular, the extension of v from K to L is unique if and only if $L|K$ is linearly disjoint from $K^h|K$.

Proof: Since $K^h|K$ is separable and since $L|K$ is normal by assumption, it follows from Lemma 6.34 that $L \,|\, L \cap K^h$ is linearly disjoint from $K^h \,|\, L \cap K^h$, and that $L \cap K^h = K$ if and only if $L|K$ is linearly disjoint from $K^h|K$. It is asserted by (Gal8) that res_L is a topological isomorphism from $\operatorname{Gal} L.K^h|K^h$ onto $\operatorname{Gal} L \,|\, L \cap K^h$. From Lemma 7.85 we know that $\operatorname{res}_L(G^d(\tilde{K}|K, v)) = G^d(L|K, v)$. By the definition of K^h, $G^d(\tilde{K}|K, v) = \operatorname{Gal} K^h$, so we have $G^d(L|K, v) = \operatorname{res}_L(G^d(\tilde{K}|K, v)) = \operatorname{res}_L(\operatorname{Gal} K^h) = \operatorname{res}_L(\operatorname{res}_{L.K^h}(\operatorname{Gal} K^h)) = \operatorname{res}_L(\operatorname{Gal} L.K^h|K^h)$.

From Lemma 7.89 we know that $\operatorname{res}_L(G^i(\tilde{K}|K, v)) = G^i(L|K, v)$. Since $G^i(\tilde{K}|K, v) \subset G^d(\tilde{K}|K, v) = \operatorname{Gal} K^h$, Lemma 7.82 shows that $G^i(\tilde{K}|K^h, v) = G^i(\tilde{K}|K, v) \cap \operatorname{Gal} K^h = G^i(\tilde{K}|K, v)$. Consequently, we have $G^i(L|K, v) = \operatorname{res}_L(G^i(\tilde{K}|K, v)) = \operatorname{res}_L(G^i(\tilde{K}|K^h, v)) = \operatorname{res}_L(\operatorname{res}_{L.K^h}(G^i(\tilde{K}|K^h, v))) = \operatorname{res}_L(G^i(L.K^h|K^h, v))$, where the first and the last equality are inferred from Lemma 7.89. An analogous argument works for the ramification groups, by use of Lemma 7.95. The equalities on the right hand side follow by (Gal8), applied to the maximal separable algebraic subextension as explained preceding to Lemma 7.101.

The last assertion of our theorem is a consequence of part d) of Theorem 7.84 and Lemma 6.34. □

The next important property of the henselization follows from its definition and Lemma 7.87

Theorem 7.110 *The henselization is an immediate separable algebraic extension.*

Remark 7.111 Our deduction of the fact that the henselization is an immediate extension seems to be not commonly known. Most authors and lecturers have a hard time proving that the value group of the henselization (resp. of the decomposition field of an extension $(L|K, v)$) is equal to vK. The main ingredient in many proofs is some version of the Strong Approximation Theorem (cf. [ZA–SA2]). The strikingly simple method used in the proof of Lemma 7.87 is to be found in the appendix of J. Ax' paper [AX3]. This appendix gives a remarkably concise introduction to the valuation theory which is needed for the deduction of the Ax–Kochen–Ershov Theorem (Theorem 14.30).

In view of Corollary 7.105, the foregoing theorem enables us to provide important examples for henselian fields. Recall that by Lemma 7.4, $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are spherically complete.

Theorem 7.112 *Every maximal and thus also every spherically complete valued field is henselian. In particular, power series fields are henselian. For every prime p, the valued fields* $(\mathbb{Q}_p, v_p)$ *and* $(\mathbb{F}_p((t)), v_t)$ *are henselian.*

Finally, we want to state an easy but important lemma about approximation types over henselian fields.

Lemma 7.113 *Let* $(L|K, v)$ *be a normal extension of valued fields such that the extension of* v *from* K *to* L *is unique. Then for every* $a \in L$ *and every* $\iota \in \mathrm{Gal}\, L|K$,

$$v(a - \iota a) \geq \Lambda(a, K) \ .$$

Consequently, if (K, v) *is henselian and* f *is an irreducible polynomial over* K*, then all roots of* f *have the same approximation type over* K*.*

Proof: Let $(L|K, v)$, $a \in L$ and $\iota \in \mathrm{Gal}\, L|K$ be as in the assumption. Let $\alpha \in \Lambda(a, K)$ and $c \in K$ such that $v(a - c) \geq \alpha$. Since $v\iota = v$ on L, it follows that $v(a - c) = v\iota(a - c) = v(\iota a - c)$, showing that $v(a - \iota a) \geq \min\{v(a - c), v(\iota a - c)\} = \alpha$. This proves that $v(a - \iota a) \geq \Lambda(a, K)$.

If f is an irreducible polynomial over K, then there is a normal extension $L|K$ which contains all roots of f, and all these roots are conjugate over K. If (K, v) is henselian, then the extension of v from K to L is unique. Let a_1 and a_2 be two roots of f, and choose $\iota \in \mathrm{Gal}\, L|K$ such that $a_2 = \iota a_1$. As we have just shown, for every $c \in K$ we will have that $v(a_1 - c) = v(\iota a_1 - c) = v(a_2 - c)$. Hence $c \in \mathrm{appr}\,(a_1, K)_\alpha$ if and only if $c \in \mathrm{appr}\,(a_2, K)_\alpha$, and $c \in \mathrm{appr}\,(a_1, K)^\circ_\alpha$ if and only if $c \in \mathrm{appr}\,(a_2, K)^\circ_\alpha$. That is, $\mathrm{appr}\,(a_1, K) = \mathrm{appr}\,(a_2, K)$. □

As a consequence of this lemma, we obtain

Theorem 7.114 *A henselian field is relatively separable-algebraically closed in every extension in which it is dense, and in particular in its completion.*

Proof: Let (K, v) be dense in (L, v) and a be an element in $L \setminus K$ which is algebraic over K. If v is the trivial valuation, then $L = K$ and our assertion is trivial. So let v be nontrivial, hence in particular, vK has no maximal element.

By part c) of Lemma 7.2, $\Lambda(a, K) = vK$. By the foregoing lemma, every conjugate ιa of a over K satisfies $v(a - \iota a) \geq \Lambda(a, K)$. But since vK has no maximal element, this means that $v(a - \iota a) = \infty$, i.e. $a = \iota a$. Consequently, a is purely inseparable over K. This proves that K is separable-algebraically closed in L. □

We use this theorem to prove:

Corollary 7.115 *If* (K, v) *is henselian and* $vK \cong \mathbb{Z}$*, then every immediate algebraic extension of* (K, v) *is purely inseparable.*

Proof: Let $(L|K, v)$ be an immediate algebraic extension with (K, v) henselian and $vK \cong \mathbb{Z}$. By Lemma 7.3, (K, v) is dense in (L, v). By virtue of Theorem 7.114, $L|K$ must be purely inseparable. □

Exercise 7.13 *Let* $\mathbf{K}$ *be henselian, and let* a *be algebraic over* K*. Show that* $v\mathrm{Tr}_{K(a)|K}(a) \geq va$ *and* $v\mathrm{N}_{K(b)|K}(a) = [K(b) : K] \cdot va$*. Given an algebraic extension* $(L|K, v)$ *of degree* n*, conclude that* $va = \frac{1}{n} v\mathrm{N}_{L|K}(a)$ *for every* $a \in L$*.*

7.10 Immediate extensions (continued)

In this section, we shall develop the theory of immediate approximation types over a valued field (K, v). This will be the approximation type version of Ostrowski's and Kaplansky's theory of pseudo Cauchy sequences of algebraic and transcendental type. The main goal of this section is to show that every maximal valued field is spherically complete. This result is due to I. Kaplansky and was published in his famous paper "Maximal fields with valuations". Preliminary results in this direction had been achieved by W. Krull [KRU7] and by A. Ostrowski [OST3]. Krull proved the slightly weaker result that every maximal valued field is what he called **perfekt**. This notion had been introduced for valued fields of rank 1, and Krull generalized it to valued fields of arbitrary rank. For rank 1, it coincides with the notion **complete** as we use it. Seeing that completeness is too week for valued fields of arbitrary rank, Krull gave this stronger notion of "perfekt". It was later called **complete by stages** (in french: **complet par étages**). We refer the reader to the book of P. Ribenboim [RIB12] for a detailed treatment of it. The reader should note that the notion of a **perfect field** has nothing in common with the above notion. Actually, the english word "perfect" is the translation for the german word **vollkommen** which is the original (and still most appropriate) german term for perfect fields.

For our further work with approximation types, we introduce the following notation which is particularly useful in the immediate case. We introduce it in connection with valued fields, but its application to ultrametric spaces and other valued structures is similar. So let (K, v) be an arbitrary valued field and $\mathbf{A}$ an approximation type over (K, v). Further, let $\mathbf{F}$ be a formula with one free variable z (not necessarily, but usually in some expansion of the first order language of valued fields with constants from K). Then both expressions

$$c \nearrow \mathbf{A} \Longrightarrow \mathbf{F}(c)$$
$$\forall z \nearrow \mathbf{A} : \ \mathbf{F}(z)$$

will denote the assertion

$$\exists \alpha \in \Lambda(\mathbf{A}) \setminus \{\infty\} \ \forall z \in \mathbf{A}_\alpha : \ \mathbf{F}(z) \ .$$

In particular, this assertion includes the information that there exists an element $a \in K$ with $\mathbf{F}(a)$. If $\mathbf{A} = \mathrm{appr}(x, K)$, then we will also write

$$c \nearrow x \ , \ \ \forall z \nearrow x$$

in the place of "$c \nearrow \mathbf{A}$" and "$\forall z \nearrow \mathbf{A}$". Furthermore, if we are given a value $\gamma = \gamma(c) \in vK$ depending on $c \in K$ (e.g. the value $vf(c)$ of a polynomial $f \in K[X]$), then we will say that γ **increases for** $c \nearrow x$ if there exists some $\alpha \in \Lambda(x, K) \setminus \{\infty\}$, such that for every choice of $c' \in \mathrm{appr}(x, K)_\alpha$ with $x \neq c'$,

$$\gamma(c) > \gamma(c') \ \text{ for } \ c \nearrow x \ .$$

Note that the condition $x \neq c'$ is automatically satisfied if $\mathrm{appr}(x, K)$ is nontrivial.

In this chapter, we will consider an immediate approximation type $\mathbf{A}$ over the valued field (K, v). Recall that in this situation, the support $\Lambda(\mathbf{A})$ has a maximal element if and

only if $\mathbf{A}$ is trivial, that is, if $\mathbf{A}$ is the approximation type of an element in K (in which case the maximal element of $\Lambda(\mathbf{A})$ is ∞). In other words, the distance of $\mathbf{A}$ is not finitely assumed. For these facts, cf. Lemma 2.27 and Corollary 2.29. If $\mathbf{A} = \mathrm{appr}(x, K)$, we observe that $v(x - c)$ increases for $c \nearrow x$. This is seen as follows. For arbitrary $\alpha \in \Lambda(\mathbf{A})$ and $c' \in \mathbf{A}_\alpha$, there is some $\beta \in \Lambda(\mathbf{A})$ such that $c' \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$, because $\mathbf{A}$ is value-immediate. This means that $v(x - c') = \beta$. Since $\Lambda(\mathbf{A})$ has no maximal element, there is some $\gamma \in \Lambda(\mathbf{A})$, $\gamma > \beta$. Now for all $c \in \mathbf{A}_\gamma$, we have that $v(x - c) \geq \gamma > \beta = v(x - c')$.

Let $f \in K[X]$ be an arbitrary polynomial. We will say that $\mathbf{A}$ **fixes the value of** f if there is some $\alpha \in vK$ such that $vf(c) = \alpha$ for $c \nearrow \mathbf{A}$. We will call $\mathbf{A}$ a **transcendental approximation type** (**over** K) if $\mathbf{A}$ fixes the value of every polynomial $f(X) \in K[X]$. Otherwise, $\mathbf{A}$ is called an **algebraic approximation type** (**over** K). If there exists any polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, then there exists also a monic polynomial of the same degree having the same property (since this property is not lost by multiplication with nonzero constants from K). If $f(X)$ is a monic polynomial of minimal degree $\mathbf{d}$ such that $\mathbf{A}$ does not fix the value of f, then it will be called an **associated minimal polynomial** for $\mathbf{A}$, and $\mathbf{A}$ is said to be of **degree d**. We define the degree of a transcendental approximation type to be $\mathbf{d} = \infty$. According to this terminology, an approximation type over K of degree $\mathbf{d}$ fixes the value of every polynomial $f \in K[X]$ with $\deg f < \mathbf{d}$. Note that an associated minimal polynomial f for $\mathbf{A}$ is always irreducible over K. Indeed, if $g, h \in K[X]$ are of degree $< \deg f$, then $\mathbf{A}$ fixes the value of g and h and thus also of $g \cdot h$. Since every polynomial $g \in K[X]$ of degree $\mathbf{d}$ whose value is not fixed by $\mathbf{A}$, is just a multiple cf of an associated minimal polynomial f for $\mathbf{A}$ (with $c \in K^\times$), the irreducibility holds for every such polynomial too.

Lemma 7.116 *Over a valued field, every immediate approximation type of degree 1 is trivial, and every trivial approximation type is of degree 1 and immediate.*

Proof: Let $\mathbf{A}$ be an immediate approximation type of degree 1 over a valued field (K, v). Assume that $f(X) = X - c_0$ with $c_0 \in K$ such that $\mathbf{A}$ does not fix the value of f. Since $\mathbf{A}$ is value-immediate, there is some $\alpha \in vK\infty$ such that $c_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$. Assume that $\alpha < \infty$. By Corollary 2.29, $\Lambda(\mathbf{A})$ has no maximal element. So let $\alpha < \beta \in \Lambda(\mathbf{A})$. But then $\forall a \in \mathbf{A}_\beta : \ v(a - c_0) \leq \alpha$ since otherwise, c_0 would appear in $\mathbf{A}_\alpha^\circ$. On the other hand, $a, c_0 \in \mathbf{A}_\alpha$ implies $v(a - c_0) \geq \alpha$, so we have $\forall a \in \mathbf{A}_\beta : \ v(a - c_0) = \alpha$. Since this contradicts our hypothesis that $\mathbf{A}$ does not fix the value of $f(x)$, we have shown that $\alpha = \infty$. Hence $c_0 \in \mathbf{A}_\infty$, and Lemma 2.27 now shows that $\mathbf{A}$ is trivial.

Lemma 2.27 has also shown that every trivial approximation type is immediate. So it remains to prove that for $c_0 \in K$, the approximation type $\mathbf{A} = \mathrm{appr}(c_0, K)$ is of degree 1. Let $f(X) = X - c_0$. Then for all $\beta \in vK$, we can choose some $\gamma > \beta$ in vK and $c_\gamma \in K$ of value γ to obtain:

$$c_0 + c_\gamma \in \mathbf{A}_\beta \ \text{ and } \ vf(c_0 + c_\gamma) = \gamma \ .$$

This shows that $\mathbf{A}$ does not fix the value of f for $c \nearrow c_0$. □

Given a nontrivial immediate approximation type $\mathbf{A}$ over a valued field (K, v), the question arises whether there is an immediate extension (L, v) of (K, v) in which $\mathbf{A}$ is realized, that is, such that $\mathbf{A} = \mathrm{appr}(x, K)$ for some $x \in L$. It can quite easily be shown by model theoretic methods that there is at least an arbitrary extension in which $\mathbf{A}$ is realized (cf. Lemma 13.85). Hence we can assume from the start that $\mathbf{A} = \mathrm{appr}(x, K)$ for

some element x in some valued extension of (K, v). We will use this assumption since it facilitates our formulas, and in this way the formulas will serve a dual purpose. For the reader who does not believe in our model theoretic magician's hat, we will also give the full construction as it was done by Kaplansky. The same reader can replace the "hypothetic" element x in the formulas of Lemma 7.118 and Lemma 7.121 below by any element c' in $\mathbf{A}_\alpha$, where $\infty \neq \alpha \in \Lambda(\mathbf{A})$ is chosen such that $c \notin \mathbf{A}_\alpha$ (such an α exists for every $c \in K$ by virtue of Lemma 2.30, because $\mathbf{A}$ is immediate and nontrivial). With this substitution, all formulas will remain valid. (Expressed in the language of pseudo Cauchy sequences, c is an element with sufficiently high index, and c' is any of the subsequent elements of the sequence.)

We need the following lemma for ordered abelian groups, which is a reformulation of a lemma of Kaplansky [KAP1]. For archimedean ordered groups, it was proved by Ostrowski [OST3].

Lemma 7.117 *Let $\alpha_1, ..., \alpha_m$ be any elements of an ordered abelian group Γ and $\Upsilon \subset \Gamma$ an infinite subset without maximal element. Let $t_1, ..., t_m$ be distinct integers. Then there exists an element $\beta \in \Upsilon$ and a permutation σ of the indeces $1, \ldots, m$ such that for all $\gamma \in \Upsilon$, $\gamma \geq \beta$,*

$$\alpha_{\sigma(1)} + t_{\sigma(1)}\gamma > \alpha_{\sigma(2)} + t_{\sigma(2)}\gamma > \ldots > \alpha_{\sigma(m)} + t_{\sigma(m)}\gamma \ .$$

Proof: Assume that $1 \leq i, j \leq m$ with $i \neq j$ and that there exist $\gamma_1, \gamma_2 \in \Upsilon$ such that

$$\alpha_i + t_i\gamma_1 > \alpha_j + t_j\gamma_1 \ \text{ and } \ \alpha_i + t_i\gamma_2 < \alpha_j + t_j\gamma_2 \ . \tag{7.29}$$

By assumption, $t_i \neq t_j$, and w.l.o.g., we may assume $t_j - t_i > 0$. Since every ordered abelian group is a torsion free $\mathbb{Z}$-module, the element $\gamma_{i,j} = (\alpha_i - \alpha_j)/(t_j - t_i) \in \tilde{\Gamma}$ is the unique solution of the equation

$$\alpha_i + t_i X = \alpha_j + t_j X \ , \tag{7.30}$$

and from (7.29) we infer

$$\gamma_1 < \frac{\alpha_i - \alpha_j}{t_j - t_i} < \gamma_2 \ .$$

There are only finitely many elements in the divisible hull $\tilde{\Gamma}$ of Γ which solve an equation (7.30) for some pair $i \neq j$, $(1 \leq i, j \leq m)$. Consider the subset S of those elements among them which are exceeded by some element in Υ. Let α be the maximal element of this subset. Then there exists $\beta \in \Upsilon$, $\beta > \alpha$. Suppose that $\beta \leq \gamma_1, \gamma_2 \in \Upsilon$ such that (7.29) holds. Then by what we have shown above, there exists a solution $\gamma_{i,j}$ of equation (7.30) which lies between γ_1 and γ_2 and is thus contained in S. But this contradicts the definition of β. Consequently, if the inequalities given in our assertion are those which hold for β in the place of γ, then they will hold for all $\gamma \in \Upsilon$, $\gamma \geq \beta$. □

If the approximation type $\mathbf{A}$ is of degree $\mathbf{d}$ and $f \in K[X]$ is of degree $\leq \mathbf{d}$, then $\mathbf{A}$ fixes the value of every formal derivative f_i of f ($\deg f \geq i \geq 1$), since every such derivative has degree $< \mathbf{d}$. So we can define $\beta_i := vf_i(c)$ for $c \nearrow x$. In certain cases, a derivative may be identically 0. In this case, we have $\beta_i = \infty$. However, the Taylor expansion of f shows that not all derivatives vanish identically, and the vanishing ones will not play a role in our computations. By use of the foregoing lemma, we can now prove:

Lemma 7.118 *Let* $\mathbf{A} = \mathrm{appr}(x, K)$ *be a nontrivial immediate approximation type of degree* $\mathbf{d}$ *over* (K, v) *and* $f \in K[X]$ *a polynomial of degree* $\leq \mathbf{d}$*. Further, let* β_i *denote the fixed value* $vf_i(c)$ *for* $c \nearrow x$*. Then there is an integer* $\mathbf{h} = \mathbf{h}(f, \mathbf{A}) \leq \deg f$ *such that*

$$v(f(x) - f(c)) = \beta_{\mathbf{h}} + \mathbf{h}v(x - c) \quad \textit{for } c \nearrow x\ . \tag{7.31}$$

Consequently, if $\mathbf{A}$ *fixes the value of* f*, then*

$$v(f(x) - f(c)) > vf(x) = vf(c) \quad \textit{for } c \nearrow x\ ,$$

and if $\mathbf{A}$ *does not fix the value of* f*, then*

$$vf(x) > vf(c) = \beta_{\mathbf{h}} + \mathbf{h}v(x - c) \quad \textit{for } c \nearrow x\ .$$

Proof: Note that $c \neq x$ for every $c \in K$ since we assume $\mathrm{appr}(x, K)$ to be nontrivial. Let $n = \deg f$. We consider the Taylor expansion

$$f(x) - f(c) = f_1(c)(x - c) + ... + f_n(c)(x - c)^n$$

with $c \in K$. We have that $vf_i(c)(x - c)^i = vf_i(c) + iv(x - c)$. So we apply the foregoing lemma with $\alpha_i = \beta_i$ and $t_i = i$, and with $\Upsilon = \Lambda(\mathbf{A}) \subset vK$ (which has no maximal element since $\mathbf{A}$ is a nontrivial immediate approximation type). We find that there is an integer $\mathbf{h} \leq \deg f$ such that $\beta_{\mathbf{h}} + \mathbf{h}v(x - c) < \beta_i + iv(x - c)$ for $c \nearrow x$ and $i \neq \mathbf{h}$. This implies equation (7.31).

If $\mathbf{A}$ fixes the value of f, then $vf(x) \neq vf(c)$ is impossible for $c \nearrow x$ since otherwise, the left hand side of (7.31) would be equal to $\min\{vf(x), vf(c)\}$ and thus fixed while the right hand side of (7.31) increases for $c \nearrow x$. This proves that $vf(x) = vf(c)$ and thus also $v(f(x) - f(c)) \geq vf(x)$ for $c \nearrow x$. But since the right hand side increases, we find that $v(f(x) - f(c)) > vf(x)$ for $c \nearrow x$.

If $\mathbf{A}$ does not fix the value of f, then $vf(x) \neq vf(c)$ and thus $v(f(x) - f(c)) = \min\{vf(x), vf(c)\}$ for $c \nearrow x$. Since the right hand side increases for $c \nearrow x$ and $vf(x)$ is a constant, the minimum can only be $vf(c)$. □

In view of $\mathbf{A} = \mathrm{appr}(x, K)$, we will call $\mathbf{h}(f, \mathbf{A})$ the **relative approximation degree of** $f(x)$ **in** x (**over** K).

If $g \in K[X]$ has a degree smaller than the degree of $\mathbf{A}$, then by the foregoing lemma, the value of $g(x)$ in $(K(x), v)$ is given by $vg(x) = vg(c)$ for $c \nearrow x$. Since $g(c) \in K$, that means that the value of $g(x)$ is uniquely determined by $\mathbf{A}$ and the restriction of v to K. If g is a nonzero polynomial, then $g(c) \neq 0$ for $c \nearrow x$ (since every nonempty $\mathbf{A}_\alpha$, $\alpha < \infty$, contains infinitely many elements). Consequently, $g(x) \neq 0$, which shows that the elements $1, x, \ldots, x^{\mathbf{d}-1}$ are K-linearly independent.

We even know that $v(g(x) - g(c)) > vg(x)$ for $c \nearrow x$. This means that $(K, v) \subset (K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ is an immediate extension of valued vector spaces. If $\mathbf{d} = [K(x) : K] < \infty$, then $K(x) = K[x] = K + Kx + \ldots + Kx^{\mathbf{d}-1}$, and we find that the valued field extension $(K(x)|K, v)$ is immediate. If $\mathbf{d} = \infty$, then $(K, v) \subset (K[x], v)$ is immediate. But then again it follows that the valued field extension $(K(x)|K, v)$ is immediate. Indeed, if $v(g(x) - g(c)) > vg(x)$ and $v(h(x) - h(c)) > vh(x)$, then $vg(x) = vg(c)$, $vh(x) = vh(c)$

and

$$\begin{aligned} v\left(\frac{g(x)}{h(x)}-\frac{g(c)}{h(c)}\right) &= v\left(g(x)h(c)-g(c)h(x)\right)-vh(x)h(c) \\ &= v\left(g(x)h(c)-g(c)h(c)+g(c)h(c)-g(c)h(x)\right)-vh(x)h(c) \\ &> vg(x)h(x)-vh(x)h(x) \;=\; v\,\frac{g(x)}{h(x)}\,. \end{aligned}$$

We have proved:

Lemma 7.119 *Let* $\mathbf{A}=\mathrm{appr}(x,K)$ *be an immediate approximation type of degree* $\mathbf{d}$ *over* K*. Then the valuation on the valued* (K,v)*-subvector space* $(K+Kx+\ldots+Kx^{\mathbf{d}-1},v)$ *of* $(K(x),v)$ *is uniquely determined by* $\mathbf{A}$ *because*

$$vg(x)=vg(c) \quad \textit{for } c\nearrow x$$

for every $g(x)\in K+Kx+\ldots+Kx^{\mathbf{d}-1}$*. The elements* $1,x,\ldots,x^{\mathbf{d}-1}$ *are* K*-linearly independent. In particular,* x *is transcendental over* K *if* $\mathbf{d}=\infty$*.*

Moreover, the extension $(K,v)\subset(K+Kx+\ldots+Kx^{\mathbf{d}-1},v)$ *of valued vector spaces is immediate. In particular, if* $\mathbf{d}=\infty$ *or if* $\mathbf{d}=[K(x):K]<\infty$*, then* $(K[x]|K,v)$ *is immediate and the same is consequently true for the valued field extension* $(K(x)|K,v)$*.*

Now we are able to prove one of the main theorems of Kaplansky's paper "Maximal fields with valuations" [KAP1]:

Theorem 7.120 (cf. Theorem 2 of [KAP1])
For every immediate transcendental approximation type $\mathbf{A}$ *over* (K,v) *there exists a simple immediate transcendental extension* $(K(x),v)$ *such that* $\mathrm{appr}(x,K)=\mathbf{A}$*. If* $(K(y),v)$ *is another valued extension field of* (K,v) *such that* $\mathrm{appr}(y,K)=\mathbf{A}$*, then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *sending* x *to* y *is valuation preserving.*

Proof: According to Lemma 13.85, there exists an extension $(K(x),v)$ of (K,v) such that $\mathbf{A}=\mathrm{appr}(x,K)$. If we do not want to use Lemma 13.85, then we take $K(x)|K$ to be a transcendental extension, and we define the valuation on $K(x)$ as follows. In view of the rule $v(g/h)=vg-vh$, it suffices to define v on $K[x]$. Let $g\in K[X]$. By assumption, $\mathbf{A}$ fixes the value of g, that is, there is $\alpha\in vK$ such that $vg(c)=\alpha$ for $c\nearrow\mathbf{A}$. We set $vg(x)=\alpha$. If g is a constant in K, we just obtain the value given by the valuation v on K. Our definition implies that $vg\neq\infty$ for every nonzero g, showing that (V0) is satisfied. Further, (VH) and (VT) are satisfied since they are already satisfied in K: We have that $vg(x)h(x)=vg(c)h(c)=vg(c)+vh(c)=vg(x)+vh(x)$ and $v(g(x)+h(x))=v(g(c)+h(c))\geq\min\{vg(c),vh(c)\}=\min\{vg(x),vh(x)\}$ for $c\nearrow\mathbf{A}$. We have proved that our definition gives a valuation v on $K(x)$ which extends the valuation v of K. Under this valuation, we have that $\mathbf{A}=\mathrm{appr}\,(x,K)$. This is seen as follows. In view of Lemma 2.41, it suffices to prove that for every $\alpha\in\Lambda(\mathbf{A})$, we have that $v(x-c_\alpha)\geq\alpha$ for some $c_\alpha\in\mathbf{A}_\alpha$. But this follows directly from our definition of $v(x-c_\alpha)$ because $c\in\mathbf{A}_\alpha$ and thus $v(c-c_\alpha)\geq\alpha$ for $c\nearrow x$.

From Lemma 7.119, we now infer that $(K(x)|K,v)$ is a transcendental immediate extension. Given another element y in some valued field extension of (K,v) such that

$\mathbf{A} = \mathrm{appr}(y, K)$, we want to show that the epimorphism from $K[x]$ onto $K[y]$ induced by $x \mapsto y$, is valuation preserving. For this, we only have to show that $vg(x) = vg(y)$ for every $g \in K[X]$. By hypothesis, the degree of $\mathbf{A}$ is ∞. From Lemma 7.119 we can thus infer that $vg(x) = vg(c) = vg(y)$ holds for $c \nearrow x$; this proves the desired equality. Again from Lemma 7.119, we deduce that y is transcendental over K. Hence, the assignment $x \mapsto y$ induces an isomorphism from $K(x)$ onto $K(y)$. Since the valuation of $K(x)$ and $K(y)$ is uniquely determined by its restriction to $K[x]$ and $K[y]$ respectively, it follows from what we have already proved that this isomorphism is valuation preserving. □

To prove the analogue of this theorem for immediate algebraic approximation types, we need a further lemma.

Lemma 7.121 *Let $\mathbf{A} = \mathrm{appr}(x, K)$ be a nontrivial immediate algebraic approximation type over (K, v) and $f \in K[X]$ an associated minimal polynomial for $\mathbf{A}$. Further, let $g \in K[X]$ be an arbitrary polynomial and write*

$$g(X) = c_k(X)f(X)^k + \ldots + c_1(X)f(X) + c_0(X)$$

with polynomials $c_i \in K[X]$ of degree $< \deg f$. Then there is some integer $m < k$ and a value $\beta \in vK$ such that with $\mathbf{h} = \mathbf{h}(f, \mathbf{A})$,

$$v(g(c) - c_0(c)) = vc_m(c) + mvf(c) = \beta + m\mathbf{h}v(x - c) \quad \text{for } c \nearrow x\ . \tag{7.32}$$

Consequently, if $\mathbf{A}$ fixes the value of g, then

$$v(g(c) - c_0(c)) > vg(c) = vc_0(c) \quad \text{for } c \nearrow x\ ,$$

and if $\mathbf{A}$ does not fix the value of g, then

$$vc_0(c) > vg(c) = \beta + m\mathbf{h}v(x - c) \quad \text{for } c \nearrow x\ .$$

Proof: Since $\deg c_i(X) < \deg f(X) = \deg \mathbf{A}$, we have that $\mathbf{A}$ fixes the value of $c_i(X)$, for $0 \leq i \leq k$. We denote by γ_i the fixed value of $c_i(c)$ for $c \nearrow x$. Since f is an associated minimal polynomial for $\mathbf{A}$, we know that $\mathbf{A}$ does not fix the value of f. In view of Lemma 7.118 we find that the value of $c_i(c)f(c)^i$ is equal to $\gamma_i + i\beta_{\mathbf{h}} + i\mathbf{h}v(x - c)$. We apply Lemma 7.117 with $\alpha_i = \gamma_i + i\beta_{\mathbf{h}}$ and $t_i = i\mathbf{h}$ and $\Upsilon = \Lambda(\mathbf{A})$ to deduce that there is an integer $m > 0$ such that $vc_m(c)f(c)^m < vc_i(c)f(c)^i$ for $c \nearrow x$ and $0 < i \neq m$. Consequently, $v(g(c) - c_0(c)) = vc_m(c)f(c)^m = \gamma_m + m\beta_{\mathbf{h}} + m\mathbf{h}v(x - c)$. We set $\beta := \gamma_m + m\beta_{\mathbf{h}}$.

The proof of the further assertions of our lemma is similar to that of the corresponding assertions of Lemma 7.118. Here, one uses the fact that $vc_0(c)$ is fixed for $c \nearrow x$. □

Theorem 7.122 (cf. Theorem 3 of [KAP1])
For every immediate algebraic approximation type $\mathbf{A}$ over (K, v) of degree $\mathbf{d}$ with associated minimal polynomial $f(X) \in K[X]$ and y a root of f, there exists an extension of v from K to $K(y)$ such that $(K(y)|K, v)$ is an immediate extension and $\mathrm{appr}(y, K) = \mathbf{A}$. If $(K(z), v)$ is another valued extension field of (K, v) such that $\mathrm{appr}(z, K) = \mathbf{A}$, then any field isomorphism between $K(y)$ and $K(z)$ over K sending y to z will preserve the valuation. (Note that there exists such an isomorphism if and only if z is also a root of f.)

Proof: By Lemma 13.85 there exists an extension $(K(x), v)$ of (K, v) such that $\mathrm{appr}(x, K) = \mathbf{A}$. From Lemma 7.119 we infer that the (K, v)-subvector space $(K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ of $(K(x), v)$ is an immediate extension of (K, v), and that the elements $1, x, \ldots, x^{\mathbf{d}-1}$ are K-linearly independent. Sending x to a root y of f, we thus obtain an isomorphism from this vector space onto $K(y) = K + Ky + \ldots + Ky^{\mathbf{d}-1}$ (note that $[K(y) : K] = \deg f = \mathbf{d}$). This isomorphism induces a vector space valuation v on $K(y)$ such that $(K(y)|K, v)$ is an immediate extension. For this induced valuation, $\mathrm{appr}(y, K) = \mathrm{appr}(x, K) = \mathbf{A}$. Since $(K(y), v)$ is a valued vector space, it satisfies (V0) and (VT).

If we do not want to rely on Lemma 13.85, then we define the valuation on $K(y) = K[y]$ as follows. Let $g \in K[X]$ of degree $< [K(y) : K] = \mathbf{d}$. By assumption, $\mathbf{A}$ fixes the value of g, that is, there is $\alpha \in vK$ such that $vg(c) = \alpha$ for $c \nearrow \mathbf{A}$. We set $vg(y) = \alpha$. Like in the proof of Theorem 7.120 it is shown that the so defined map extends v from K to $K(y)$ and satisfies (V0) and (VT).

It remains to prove that $(K(y), v)$ satisfies (VH). Suppose that $g, h \in K[X]$ are of degree $< [K(y) : K] = \mathbf{d}$. Then $\mathbf{A}$ fixes the value of both g and h and thus also the value of gh. Let $f \in K[X]$ be an associated minimal polynomial for $\mathbf{A}$ and write $g(X)h(X) = c_k(X)f(X)^k + \ldots + c_1(X)f(X) + c_0(X)$. By our construction resp. definition, $vg(y)h(y) = vc_0(c)$ as well as $vg(y) = vg(c)$ and $vh(y) = vh(c)$ for $c \nearrow \mathbf{A}$. So it remains to show that $vc_0(c) = vg(c)h(c)$ for $c \nearrow \mathbf{A}$. But since $\mathbf{A}$ fixes the value of gh, this follows from the foregoing lemma.

The last assertion of our theorem is shown like the corresponding assertion of Theorem 7.120: if $\mathrm{appr}(y, K) = \mathrm{appr}(z, K)$ and $g \in K[X]$ with $\deg g < \mathbf{d}$ then $vg(y) = vg(c) = vg(z)$ for $c \nearrow x$. Hence an isomorphism over K sending y to z will preserve the valuation. □

If (K, v) admits no immediate extensions, then by the last two theorems, it admits no nontrivial immediate approximation types. On the other hand, if (K, v) admits no nontrivial immediate approximation types, then it admits no proper immediate extensions (cf. Corollary 2.28). This proves

Corollary 7.123 (cf. Theorem 4 of [KAP1])
A valued field (K, v) is maximal if and only if every immediate approximation type $\mathbf{A}$ over K is trivial, i.e. there exists an element $a \in K$ such that $\mathbf{A} = \mathrm{appr}(a, K)$.

We know that a valued field (K, v) is spherically complete if and only if every approximation type over (K, v) is trivial (cf. Lemma 2.40). By Theorem 5.7, every valued field admits a maximal immediate extension. Hence, we obtain:

Theorem 7.124 *A valued field is maximal if and only if it is spherically complete. Every valued field admits an immediate extension which is spherically complete.*

It follows from the last assertion of Theorem 7.120 that $\mathrm{appr}\,(x, K)$ can not be a transcendental immediate approximation type if x is algebraic over K. But we can even give a relation between $[K(x) : K]$ and the degree of $\mathrm{appr}\,(x, K)$. To this end, we have to deal again with polynomials whose degree exceeds the degree of $\mathrm{appr}\,(x, K)$, seeking for an analogue of Lemma 7.118. This time we use an approach which is different from that of Lemma 7.121.

Lemma 7.125 *Let* $\mathbf{A} = \mathrm{appr}(x, K)$ *be a nontrivial immediate approximation type over* (K, v) *and* $g \in K[X]$ *a polynomial of arbitrary degree. Then there is an integer* $\ell \leq \deg g$ *such that*

$$v(g(x) - g(c)) = vg_\ell(x) + \ell v(x - c) \quad \text{for } c \nearrow x \,. \tag{7.33}$$

Consequently, if $\mathbf{A}$ *fixes the value of* g*, then*

$$v(g(x) - g(c)) > vg(x) = vg(c) \quad \text{for } c \nearrow x \,,$$

and if $\mathbf{A}$ *does not fix the value of* g*, then*

$$vg(x) > vg(c) = vg_\ell(x) + \ell v(x - c) \quad \text{for } c \nearrow x \,.$$

Proof: Let $n = \deg f$. We consider the Taylor expansion

$$g(c) - g(x) = g_1(x)(c - x) + ... + g_n(x)(c - x)^n$$

with $c \in K$. We apply Lemma 7.117 with $\alpha_i = vg_i(x)$ and $t_i = i$, and with $\Upsilon = \Lambda(\mathbf{A}) \subset vK$. We find that there is an integer $\ell \leq \deg g$ such that $vg_\ell(x) + \ell v(x - c) < vg_i(x) + iv(x - c)$ for $c \nearrow x$ and $i \neq \ell$. This implies equation (7.33). The proof of the further assertions of our lemma is similar to that of the corresponding assertions of Lemma 7.118. □

If $g(x) = 0$, then we obtain from equation (7.33) that $vg(c)$ increases for $c \nearrow x$. This also holds for trivial approximation types, as we have proved in Lemma 7.116. We obtain:

Corollary 7.126 *Let* $\mathbf{A} = \mathrm{appr}(x, K)$ *be an immediate approximation type over* (K, v)*. Assume that* x *is algebraic over* K *with minimal polynomial* $g \in K[X]$*. Then* $\mathbf{A}$ *does not fix the value of* g*. Hence,* $\deg \mathbf{A} \leq [K(x) : K]$*, showing in particular that* $\mathbf{A}$ *is algebraic.*

The infinite case is included if we simply read $[K(x) : K] = \infty$ for an infinite extension $K(x)|K$ (instead of thinking of cardinalities or supernatural numbers). It is in general not true that $\deg \mathbf{A} = [K(x) : K]$, and if this does not hold, then the extension $(K(x)|K, v)$ need not be immediate. But it holds in the case of completion types:

Lemma 7.127 *Let* $\mathbf{A} = \mathrm{appr}(x, K)$ *be a completion type over* (K, v)*. Then every associated minimal polynomial for* $\mathbf{A}$ *is a minimal polynomial of* x *over* K *(so there is only one). Hence,* $\deg \mathbf{A} = [K(x) : K]$*, and* $(K(x)|K, v)$ *is immediate.*

Proof: Let f be an associated minimal polynomial for $\mathbf{A}$. By definition, $\mathbf{A}$ does not fix the value of f. Hence from Lemma 7.118 we know that $vf(x) > vf(c) = \beta_{\mathbf{h}} + \mathbf{h}v(x - c)$ for $c \nearrow x$. But the values $v(x - c)$ and thus also the values $vf(c)$ are cofinal in vK for $c \nearrow x$. Since we assume $K(x)|K$ to be algebraic, we know by Corollary 7.9 that $vK(x)$ lies in the divisible hull of vK. It follows that vK and thus also the values $vf(c)$ are cofinal in $vK(x)$. Consequently, $vf(x) = \infty$, that is, $f(x) = 0$. On the other hand, we know already that $\deg f = \deg \mathbf{A} \leq [K(x) : K]$. We conclude that $\deg f = [K(x) : K]$ and that f is the minimal polynomial of x over K. Finally, Lemma 7.119 shows that $(K(x)|K, v)$ is immediate. □

Now we are able to supply the proof of Lemma 5.11 (see the remark succeeding to that lemma).

Lemma 7.128 *If (K,v) is dense in (L,v), then (L,v) can be embedded over K in the completion $(K,v)^c$. In this sense, the completion is* the *maximal extension in which (K,v) is dense; it is unique up to valuation preserving isomorphism over K.*

Proof: Assume that (K,v) is dense in (L,v). By a standard argument using Zorn's Lemma, one shows that there is a maximal subextension $E|K$ of $L|K$ such that (E,v) admits an embedding in $(K,v)^c$ over K. We wish to show that $E=L$. We identify (E,v) with its image in $(K,v)^c$. Like (K,v), also (E,v) is dense in (L,v) and in $(K,v)^c$. Assume that $x \in L \setminus E$. By part d) of Lemma 7.2, $\mathrm{appr}\,(x,E)$ is a completion type. By Lemma 2.46, there is $y \in K^c$ such that $\mathrm{appr}\,(x,E) = \mathrm{appr}\,(y,E)$. If $\mathrm{appr}\,(x,E)$ is transcendental, then by Theorem 7.120 there is an isomorphism of $(E(x),v)$ and $E(y),v)$ over E, contradicting our choice of E. Suppose that $\mathrm{appr}\,(x,E)$ is algebraic and let f be an associated minimal polynomial for it. Then by the foregoing lemma, both x and y are roots of f. Hence, by Theorem 7.122 there is an isomorphism of $(E(x),v)$ and $E(y),v)$ over E, contradicting again our choice of E. We have thus proved that $E=L$.

Now if (K_1,v_1) and (K_2,v_2) are two complete fields in which (K,v) lies dense, then by what we have shown, (K_2,v_2) can be embedded in (K_1,v_1) over K. Like (K,v), also the image of (K_2,v_2) lies dense in (K_1,v_1). On the other hand, this image is complete since completeness is preserved under valuation preserving isomorphisms. Hence, (K_1,v_1) can not be a proper extension of it. This proves that (K_1,v_1) and (K_2,v_2) are isomorphic over K. □

In view of Lemma 7.3, this lemma shows:

Corollary 7.129 *Let (K,v) be a valued field with $vK \cong \mathbb{Z}$. Then $(K,v)^c$ is the maximal immediate extension of (K,v), uniquely determined up to valuation preserving isomorphism over K.*

In the special case $vK \cong \mathbb{Z}$, it was easy to prove the uniqueness of the maximal immediate extensions. One has to do much more effort to show it in other cases. See Theorem 8.31 for the case of valued fields of residue characteristic 0. The case of Kaplansky fields was treated by I. Kaplansky in [KAP1] (cf. also [KU–PA–ROQ]). In general, maximal immediate extensions are not unique. An example is given in [KU–PA–ROQ]. See also S. Warner [WA1], who considers a slightly stronger notion of uniqueness.

The corollary shows that $(\mathbb{Q}_p, v_p)$ is the unique maximal immediate extension of $(\mathbb{Q}, v_p)$, up to valuation preserving isomorphism over $\mathbb{Q}$, and that $(\mathbb{F}_p((t)), v_t)$ is the unique maximal immediate extension of $(\mathbb{F}_p(t), v_t)$, up to valuation preserving isomorphism over $\mathbb{F}_p(t)$.

The proof of Lemma 7.128 has shown that an element x lies in the completion of (K,v) (up to valuation preserving isomorphism over K) as soon as its approximation type over (K,v) is a completion type. From Lemma 7.114 we know that a henselian field is relatively separable-algebraically closed in its completion. Assume that $K(x)|K$ is separable. Hence if $\mathrm{appr}\,(x,K)$ is a completion type, then $x \in K$ or x is transcendental over K. If the latter holds, then also $\mathrm{appr}\,(x,K)$ is transcendental by virtue of Lemma 7.127. Consequently, if $\mathrm{appr}\,(x,K)$ is a nontrivial algebraic approximation type over the henselian field (K,v), then it can not be a completion type. On the other hand, if $K(x)|K$ is a proper algebraic immediate extension, then $\mathrm{appr}\,(x,K)$ is a nontrivial immediate approximation type and by Corollary 7.126, $\mathrm{appr}\,(x,K)$ is algebraic. This proves:

Corollary 7.130 *If (K,v) is henselian and $(K(x)|K,v)$ is a proper separable algebraic immediate extension, then* $\mathrm{dist}\,(x,K) < \infty$.

Exercise 7.14 *Compare the data ℓ and $vg_\ell(x)$ of Lemma 7.125 with the data $m\mathbf{h}$ and β of Lemma 7.121.*

7.11 Algebraically maximal fields

A valued field is called **algebraically maximal** if it admits no proper immediate algebraic extensions.

Theorem 7.131 *A valued field (K, v) is algebraically maximal if and only if every nontrivial immediate approximation type $\mathbf{A}$ over (K, v) is transcendental and hence fixes the value of every polynomial in $K[X]$.*

Proof: If (K, v) is algebraically maximal, then it does not admit a proper immediate algebraic extension. So if $\mathbf{A}$ is an immediate algebraic approximation type over (K, v), then by Theorem 7.122 it must be trivial.

For the converse, assume that (K, v) is not algebraically maximal and let $(L|K, v)$ be a proper immediate algebraic extension. Pick an element $x \in L \setminus K$. Then by Corollary 7.126, $\operatorname{appr}(x, K)$ is algebraic. But since $x \notin K$, it is nontrivial (cf. Lemma 2.27). □

Our goal now is to give an even better characterization of algebraically maximal fields.

Lemma 7.132 *Let $\mathbf{A}$ be a nontrivial algebraic approximation type over (K, v) with associated minimal polynomial $f \in K[X]$. Extend the valuation v to the algebraic closure $\tilde{K}$ of K. Then some root of f in $\tilde{K}$ will realize $\mathbf{A}$ in $(\tilde{K}, v)$. If in addition (K, v) is henselian, then every root of f in $\tilde{K}$ realizes $\mathbf{A}$.*

Proof: Write $f(X) = \prod_{i=1}^{n}(X - a_i)$ with $a_i \in \tilde{K}$. Suppose that no a_i realizes $\mathbf{A}$ in $(\tilde{K}, v)$. But then for every $i = 1, \ldots, n$ there are $\alpha_i \in \Lambda(\mathbf{A})$ such that $v(a_i - c)$ is fixed for all $c \in \mathbf{A}_i$ (cf. Corollary 2.34). Taking $\alpha \in \Lambda(\mathbf{A})$ to be the maximum of the α_i, we find that every value $v(a_i - c)$ and thus also $vf(c)$ is fixed for $c \in \mathbf{A}_\alpha$. But this contradicts our assumption that f be an associated minimal polynomial for $\mathbf{A}$. Hence, there is at least one root a_i which realizes $\mathbf{A}$.

Assume in addition that (K, v) is henselian. Then $v = v\sigma$ on $\tilde{K}$ for every $\sigma \in \operatorname{Gal} K$. If a_j is a second root of f and we choose σ such that $\sigma a_i = a_j$, then we obtain for every $c \in K$ that $v(a_j - c) = v(\sigma a_i - c) = v\sigma(a_i - c) = v(a_i - c)$, which shows that also a_j realizes $\mathbf{A}$. □

Lemma 7.133 *Let (K, v) be a valued field and $f \in K[X]$ a polynomial without a root in K. If there is no $c_{\max} \in K$ such that $vf(c_{\max}) \geq vf(c)$ for all $c \in K$, then there exists a nontrivial immediate approximation type of degree $\leq \deg f$ over (K, v).*

Proof: We choose a sequence c_ν of elements in K, indexed by ordinals $\nu < \lambda$, such that the values $vf(c_\nu)$ are increasing with ν and are cofinal in the set $\{vf(c) \mid c \in K\}$. By hypothesis, this sequence has no last element (so λ is a limit ordinal). We write $f(X) = \prod_{i=1}^{n}(X - a_i)$ with $a_i \in \tilde{K}$. Since $vf(c_\nu) = \sum_i v(c_\nu - a_i)$ increases, there must be some i and a subsequence of $(c_\nu)_{\nu<\lambda}$ such that $v(c_\nu - a_i)$ for c_ν from this subsequence. We replace $(c_\nu)_{\nu<\lambda}$ by this subsequence. Consider the balls $B_\nu := B_{\alpha_\nu}(c_\nu)$ in (K, v) with $\alpha_\nu := v(c_\nu - a_i)$. They form a nest of balls since for $\mu \geq \nu$ we have that $v(c_\mu - c_\nu) \geq \min\{v(c_\mu - a_i), v(c_\nu - a_i)\} = \alpha_\nu$, which yields that $c_\mu \in B_\nu$. Suppose that their intersection

is nonempty, and let $c \in K$ lie in this intersection. That is, $v(c-a_i) > v(c_\nu - a_i)$ for all ν. Let a_j another root of f. If $v(a_i - a_j) > v(c_\nu - a_i)$ for all ν, then also $v(c-a_j) \geq \min\{v(c-a_i), v(a_i-a_j)\} \geq v(c_\nu - a_i) = \min\{v(c_\nu - a_i), v(a_i - a_j)\} = v(c_\nu - a_j)$ for all ν. If on the other hand there is some $\nu_0 < \lambda$ $v(a_i - a_j) < v(c_\nu - a_i)$ for all $\nu \geq \nu_0$, then $v(c-a_j) \geq \min\{v(c-a_i), v(a_i-a_j)\} = v(a_i-a_j) = \min\{v(c_\nu - a_i), v(a_i-a_j)\} = v(c_\nu - a_j)$ for all $\nu \geq \nu_0$. In both cases, we find that $v(c-a_j) \geq v(c_\nu - a_j)$ for all j and all large enough ν. But this implies that $vf(c) \geq vf(c_\nu)$ for all ν, which is a contradiction. We have thus proved that the nest of balls B_ν has an empty intersection. By Lemma 2.39 we thus find that these balls give rise to a nontrivial immediate approximation type $\mathbf{A}$, which realized by a_i in $(\tilde{K}, v)$. From Corollary 7.126 we conclude that $\deg \mathbf{A} \leq \deg f$. □

Now we are prepared to give the following characterization of algebraically maximal fields:

Theorem 7.134 *The valued field (K,v) is algebraically maximal if and only if it is henselian and for every polynomial $f \in K[X]$,*

$$\exists x \in K \, \forall y \in K : \; vf(x) \geq vf(y) \; . \tag{7.34}$$

Proof: Assume that (K,v) is algebraically maximal. Then it is equal to its henselization since by Theorem 7.110, the henselization is an immediate algebraic extension. This proves that (K,v) is henselian. Further, from Theorem 7.131 we infer that there is no nontrivial algebraic approximation type over (K,v). By the foregoing lemma, this yields that (7.34) holds for every polynomial f without a root in K. But if f has a root a in K, then (7.34) also holds since $vf(a) = v0 = \infty \geq vf(c)$ for all $c \in K$.

For the converse, assume that (K,v) is henselian, and let $\mathbf{A}$ be an algebraic approximation type $\mathbf{A}$ over (K,v). Let $f \in K[X]$ be an associated minimal polynomial of $\mathbf{A}$. By Lemma 7.132, every root of f in $\tilde{K}$ realizes $\mathbf{A}$ in $(\tilde{K}, v)$. Now assume that there is $c_{\max} \in K$ such that $vf(c_{\max}) \geq vf(c)$ for every $c \in K$. It follows that for every $c \in K$ there is some i such that $v(c_m ax - a_i) \geq v(c-a_i)$, and since (K,v) is supposed to be henselian, the same argument as in the proof of Lemma 7.132 shows that this holds for all i. It follows that $v(c_{\max} - a_i) \geq \Lambda(\mathbf{A})$, and part b) of Lemma 2.36 shows that also $c_{\max}$ realizes $\mathbf{A}$. Since $c_{\max} \in K$, this means that $\mathbf{A}$ is trivial. □

Chapter 8

Defect or no defect?

8.1 The fundamental (in)equality

In this section, we will relate the degree of a finite valued field extension $(L|K, v)$ with the ramification indices and the inertia degrees of all extensions of the valuation. We know already from Corollary 7.71 that the number of distinct extensions does not exceed $[L : K]$. If v admits more than one extension from K to L, then the estimate given in Lemma 7.7 will never be sharp. To obtain a better inequality involving also the other extensions, we will lift the extension $L|K$ to the henselizations of L with respect to these extensions. We need two lemmata. Recall the following. If H_1 and H_2 are subgroups of a group G, then for $g \in G$, the set $H_1gH_2 = \{h_1gh_2 \mid h_1 \in H_1 \wedge h_2 \in H_2\}$ is called a **double coset of** G. Further, $g \sim h :\Leftrightarrow H_1gH_2 = H_1hH_2$ is an equivalence relation, and its equivalence classes are double cosets of the form H_1gH_2 . If $(G : H_1)$ is finite, then there are at most $(G : H_1)$ distinct such double cosets.

Lemma 8.1 *Let $L|K$ be a finite and $K'|K$ an arbitrary algebraic extension. Let $\iota_1, \ldots, \iota_m$ be representatives of the double cosets $\{\operatorname{Gal} K' \, \iota \operatorname{Gal} L \mid \iota \in \operatorname{Gal} K\}$ (it follows that $m \leq (\operatorname{Gal} K : \operatorname{Gal} L) \leq [L : K] < \infty$).*

a) An automorphism $\iota \in \operatorname{Gal} K$ lies in $\operatorname{Gal} K' \, \iota_i \operatorname{Gal} L$ if and only if the isomorphism $\operatorname{res}_L(\iota \, \iota_i^{-1}) : \iota_i L \to \iota L$ can be extended to an isomorphism of $\iota_i L.K'$ onto $\iota L.K'$ over K'.

b) For $1 \leq i \leq m$, let q_i denote the quotient $[L : K]_{\text{ins}}/[\iota_i L.K' : K']_{\text{ins}}$, which is a power of charexp K. *Then*

$$[L : K] = \sum_{1 \leq i \leq m} [\iota_i L.K' : K'] \cdot q_i \ . \tag{8.1}$$

c) If $L|K$ or $K'|K$ is separable, then $q_i = 1$ for $1 \leq i \leq m$.

Proof: a): Let $\iota \in \operatorname{Gal} K$. Then an automorphism in $\operatorname{Gal} K$ extends $\operatorname{res}_{\iota_i L}(\iota \, \iota_i^{-1})$ if and only if it lies in the coset $\iota \, \iota_i^{-1} \operatorname{Gal} \iota_i L$. This coset is equal to $\iota \, \iota_i^{-1} \iota_i \operatorname{Gal} L \, \iota_i^{-1} = \iota \operatorname{Gal} L \, \iota_i^{-1}$. Hence, there is an extension of $\operatorname{res}_{\iota_i L}(\iota \, \iota_i^{-1})$ to an isomorphism over K' if and only if $\iota \operatorname{Gal} L \, \iota_i^{-1} \cap \operatorname{Gal} K' \neq \emptyset$. But this is equivalent to $\iota \in \operatorname{Gal} K' \, \iota_i \operatorname{Gal} L$.

b): Let $K_s = (L|K)^{\text{sep}}$ be the maximal separable subextension of K in L. As a finite separable extension, $K_s|K$ is simple. Let $f \in K[X]$ be the minimal polynomial of some generator b of this extension. Let $\prod_i f_i$ be the decomposition of f into irreducible factors over K'. Then $\operatorname{res}_{\iota_i K_s}(\iota \, \iota_i^{-1}) : \iota_i K_s \to \iota K_s$ can be extended to an isomorphism of $\iota_i K_s.K'$

onto $\iota K_s.K'$ over K' if and only if $\iota_i b$ and ιb are roots of the same irreducible factor. By virtue of part a), applied to K_s in the place of L, there are m such factors, and we may enumerate them such that $\iota_i b$ is a root of f_i. Then $[\iota_i K_s.K' : K']$ is equal to the degree of f_i. Hence,

$$[L : K]_{\text{sep}} = [K_s : K] = \deg f = \sum_{1 \le i \le m} \deg f_i = \sum_{1 \le i \le m} [\iota_i K_s.K' : K'] \; . \tag{8.2}$$

Like $K_s|K$, every extension $\iota_i K_s|K$ is separable. It follows from Lemma 6.43 that also every extension $\iota_i K_s.K'|K'$ is separable. On the other hand, $\iota_i L \,|\, \iota_i K_s$ is purely inseparable like $L|K_s$. Consequently, $\iota_i L.K' \,|\, \iota_i K_s.K'$ is purely inseparable, showing that

$$[\iota_i K_s.K' : K'] = [\iota_i L.K' : K']_{\text{sep}} = [\iota_i L.K' : K'] \cdot [\iota_i L.K' : K']_{\text{ins}}^{-1} \; .$$

Thus, multiplying equation (8.2) with $[L : K]_{\text{ins}}$ gives equation (8.1).

c): If $L|K$ is separable, then so is $\iota_i L|K$ and hence also $\iota_i L.K'|K'$, for $1 \le i \le m$. Hence $[L : K]_{\text{ins}} = 1 = [\iota_i L.K' : K']_{\text{ins}}^{-1}$, which yields that $q_i = 1$.

Now assume that $K'|K$ is separable. Since we have already shown that $\iota_i K_s.K'|K'$ is separable, it follows that $\iota_i K_s.K'|K$ and hence also $\iota_i K_s.K' \,|\, \iota_i K_s$, are separable. Since $\iota_i L \,|\, \iota_i K_s$ is purely inseparable, it is linearly disjoint from $\iota_i K_s.K' \,|\, \iota_i K_s$, and $\iota_i L.K' \,|\, \iota_i K_s.K'$ is purely inseparable. This yields that $[\iota_i L : \iota_i K_s] = [\iota_i L.K' : \iota_i K_s.K']$ and that $\iota_i K_s.K'|K'$ is the maximal separable subextension of $\iota_i L.K'|K'$. Hence,

$$[L : K]_{\text{ins}} = [L : K_s] = [\iota_i L : \iota_i K_s] = [\iota_i L.K' : \iota_i K_s.K'] = [\iota_i L.K' : K']_{\text{ins}}$$

which yields that $q_i = 1$. □

Lemma 8.2 *Let (K, v) be a valued field and $L|K$ a finite extension. Choose an extension of the valuation v from K to $\tilde{K}$ and call it again v. Denote by (K^h, v) the henselization of (K, v). Let $\iota_1, \ldots, \iota_g \in \operatorname{Gal} K$ be representatives of the double cosets*

$$\{\operatorname{Gal} K^h \, \iota \operatorname{Gal} L \mid \iota \in \operatorname{Gal} K\} \; .$$

Then the distinct extensions of v from K to L are given by the restrictions of the valuations $v_i = v\iota_i$ to L, $1 \le i \le g$. Further, $(L.\iota_i^{-1}K^h, v\iota_i)$ is the henselization of (L, v_i) in $(\tilde{K}, v\iota_i)$, and it is isomorphic over K to $(\iota_i L.K^h, v)$ via ι_i .

Proof: By virtue of Lemma 7.103, $(\iota_i^{-1}K^h, v_i)$ is the henselization of (K, v) in $(\tilde{K}, v_i)$. From Corollary 7.108 it follows that $(L.\iota_i^{-1}K^h, v\iota_i)$ is the henselization of (L, v) in $(\tilde{K}, v_i)$. The restriction of ι_i is an isomorphism from $(L.\iota_i^{-1}K^h, v\iota_i)$ onto $(\iota_i L.K^h, v)$ over K.

Assume that $v\iota = v\iota_i$ on L. Then $v\iota$ and $v\iota_i$ are both extensions of the same valuation from L to $\tilde{K}$. From Theorem 7.68 we infer the existence of $\tau \in \operatorname{Gal} L$ such that $v\iota_i\tau = v\iota$ on $\tilde{K}$. Consequently, the restrictions of both ι^{-1} and $\tau^{-1}\iota_i^{-1}$ are embeddings of (K^h, v) in the henselian field $(\tilde{K}, v\iota) = (\tilde{K}, v\iota_i\tau)$. By Lemma 7.104, they must be equal, that is, $\sigma := \iota\tau^{-1}\iota_i^{-1}$ must be an element of $\operatorname{Gal} K^h$. So we find $\iota = \sigma\iota_i\tau \in \operatorname{Gal} K^h \cdot \iota_i \cdot \operatorname{Gal} L$.

For the converse, assume that ι lies in the double coset represented by ι_i , say $\iota = \sigma\iota_i\tau$ with $\sigma \in \operatorname{Gal} K^h = G^d(\tilde{K}|K, v)$ and $\tau \in \operatorname{Gal} L$. Then we have $v\sigma = v$ on $\tilde{K}$ and $\tau a = a$ for all $a \in L$. This yields that $v\iota a = v\sigma\iota_i\tau a = v\iota_i a$ for all $a \in L$, that is, $v\iota = v\iota_i$ on L. □

From this lemma together with part a) of Lemma 8.1 we can also deduce: *If $\iota, \iota' \in \operatorname{Gal} K$, then $v\iota = v\iota'$ on L if and only if $\iota L.K^h$ and $\iota' L.K^h$ are isomorphic over K^h.*

With $K^{h(v_i)}$ denoting the henselization of (K, v) in $(\tilde{K}, v_i)$, we have $K^{h(v_i)} = \iota_i^{-1} K^h$. This field lies in $L^{h(v_i)} = L.\iota_i^{-1} K^h$. Since the henselization is a separable extension, we can apply equation (8.1) of Lemma 8.1 with $q_i = 1$ to obtain:

Corollary 8.3 *Let (K, v) be a valued field and L a finite extension of K. Let $v_1, \ldots, v_g$ be the distinct extensions of v from K to L. Then*

$$[L : K] = \sum_{1 \le i \le g} [L^{h(v_i)} : K^{h(v_i)}] \, . \tag{8.3}$$

The degree $[L^{h(v_i)} : K^{h(v_i)}]$ is called the **local degree** of the extension $(L, v_i)|(K, v)$. Actually, in the literature this name is mainly used for the degree $[L^{c(v_i)} : K^c]$ of the extension of the respective completions. But to work with the latter degree only makes sense in this connection if the valuations are of rank 1, that is, their value groups are archimedean.

By Lemma 7.7 we know that $[L^{h(v_i)} : K^{h(v_i)}] \ge (v_i L^{h(v_i)} : vK^{h(v_i)}) \cdot [L^{h(v_i)} v_i : K^{h(v_i)} v]$ for $1 \le i \le g$. Since the henselization is an immediate extension by Theorem 7.110, we have $v_i K^{h(v_i)} = vK$, $v_i L^{h(v_i)} = v_i L$, $K^{h(v_i)} v_i = Kv$ and $L^{h(v_i)} v_i = L v_i$. It follows that

$$[L^{h(v_i)} : K^{h(v_i)}] \ge (v_i L^{h(v_i)} : v_i K^{h(v_i)}) \cdot [L^{h(v_i)} v_i : K^{h(v_i)} v_i] = (v_i L : vK) \cdot [Lv_i : Kv] \, . \tag{8.4}$$

Together with equation (8.3), this proves:

Theorem 8.4 *Let (K, v) be a valued field and L a finite extension of K. Let $v_1, \ldots, v_g$ be the distinct extensions of v to L. Then we have the* **fundamental inequality**

$$[L : K] \ge \sum_{1 \le i \le g} (v_i L : vK) \cdot [Lv_i : Kv] \, . \tag{8.5}$$

Writing shortly $n = [L : K]$, $e_i = e(L|K, v_i) = (v_i L : vK)$ and $f_i = f(L|K, v_i) = [Lv_i : Kv]$, equation (8.5) can be expressed in the following mnemonic form:

$$n \ge \sum_{1 \le i \le g} e_i f_i \, .$$

From this theorem together with Corollary 7.70, we obtain:

Corollary 8.5 *Let $(L|K, v)$ be a finite normal extension. Let g be the number of extensions of v from K to L, and set $n = [L : K]$, $e = e(L|K, v)$ and $f = f(L|K, v)$. Then the fundamental inequality can be written as*

$$n \ \ge \ e \cdot f \cdot g \, . \tag{8.6}$$

If equality holds in (8.5), then we call it the **fundamental equality** and we will say that (K, v) is **defectless in** L; otherwise, we will call L a **defect extension of** (K, v). If $L|K$ is an arbitrary algebraic extension, then we will say that (K, v) is **defectless in** L if it is defectless in every finite normal subextension. If (K, v) is defectless in every finite extension, that is, if (K, v) is defectless in $\tilde{K}$, then we call (K, v) a **defectless field**.

Similarly, (K,v) is called a **separably defectless field** if it is defectless in every finite separable extension, and (K,v) is called an **inseparably defectless field** if it is defectless in every finite purely inseparable extension. If (K,v) is a defectless field, then it is separably defectless and inseparably defectless. Every perfect separably defectless field is a defectless field. If (K,v) is a defectless field, then it is separably defectless and inseparably defectless. At the first glimpse, the converse to this assertion may seem to be trivially true. But it isn't since it is not clear why a proper finite separable extension of an inseparably defectless field should again be an inseparably defectless field, even if this extension is defectless. But this converse is even true if we replace "inseparably defectless" by a weaker property, cf. [KU1], [KU2], Theorem 4.17.

In section 7.2 we have already introduced the notation "$(L|K,v)$ is defectless" for the case of $(K,v) \subset (L,v)$ being a defectless extension of valued K-vector spaces.

WARNING: Although we are using the same expression "defectless", the assertions "$(L|K,v)$ is defectless" and "(K,v) is defectless in L" are different. They coincide only in the case where the extension of v from K to L is unique; in particular, both notions coincide if (K,v) is henselian. But if there is a second extension of v from K to L, then $(L|K,v)$ will never be defectless.

Nevertheless, both notions are indispensable. While "$(L|K,v)$ is defectless" covers also the case of infinite and even transcendental extensions, the notion "(K,v) is defectless in L" covers the case of several extensions of the valuation.

Lemma 8.6 *An algebraic extension $(L|K,v)$ is defectless if and only if the extension of v from K to L is unique and every finite subextension $(E|K,v)$ satisfies the fundamental equality $n = e \cdot f$, that is, admits a valuation basis. In particular, a henselian field (K,v) is a defectless field if and only if every finite extension $(L|K,v)$ admits a valuation basis.*

Proof: From Lemma 7.12 we know that $(L|K,v)$ is defectless if and only if every finite subextension $(E|K,v)$ admits a valuation basis, which by Lemma 7.11 is the case if and only if it satisfies the fundamental equality $[E:K] = (vL:vK) \cdot [Kv:Lv]$. But in view of (8.5) this is only possible if there is only one extension of v from K to E. Since every algebraic extension is the union of finite extensions, this holds if and only if there is a unique extension of v from K to L. □

The property of being a defectless field is preserved under isomorphisms of valued fields. Indeed, every such isomorphism $\iota : (K,v) \to (\iota K, v\iota^{-1})$ can be extended to a field isomorphism $\tilde{\iota} : L \to \tilde{\iota}L$ for every given extension field L of K. Then, to every extension w of v from K to L there is a corresponding extension $w\tilde{\iota}^{-1}$ of $v\iota^{-1}$ from $\tilde{\iota}K$ to $\tilde{\iota}L$, so that $\tilde{\iota} : (L,w) \to (\tilde{\iota}L, w\tilde{\iota}^{-1})$ is an isomorphism of valued fields which extends ι. Since the situation is symmetrical, this correspondence is a bijection. By part c) of Lemma 7.23, ramification index and inertia degree of $(L|K,w)$ are equal to that of $(\tilde{\iota}L|\iota K, w\tilde{\iota}^{-1}))$ for every extension w; so (K,v) is defectless in L if and only if $(\iota K, v\iota^{-1})$ is defectless in $\tilde{\iota}L$. It follows that (K,v) is a defectless field if and only if $(\iota K, v\iota^{-1})$ is. Since the isomorphism also preserves properties like "separable" and "purely inseparable", we find that also the properties of being a separably defectless field and of being an inseparably defectless field are preserved.

The property of being a defectless extension is transitive:

Lemma 8.7 *Let (K,v) be an arbitrary valued field, $L|K$ a finite extension and $E|K$ a subextension of $L|K$. Let $v_1,\dots,v_g$ be all extensions of v from K to E. Then (K,v) is defectless in L if and only if (K,v) is defectless in E and (E,v_i) is defectless in L for $1\le i\le g$.*

Proof: Every extension of v from K to E is the extension of precisely one valuation v_i of E. So the distinct extensions of v from K to E may be denoted by v_{ij} with $1\le i\le g$, $1\le i\le g_i$, such that $v_{ij}|_E = v_i$. By virtue of the multiplicativity of ramification index and inertia degree (Lemma 7.8),

$$\sum_{i,j}(v_{ij}L:vK)\cdot[Lv_{ij}:Kv] = \sum_{1\le i\le g}(v_iE:vK)\cdot[Ev_i:Kv]\cdot\sum_{1\le j\le g_i}(v_{ij}L:v_iE)\cdot[Lv_{ij}:Ev_i]\ .$$

Now apply the fundamental inequality (8.5) to E over (K,v) and to L over (E,v_i) for every i. In view of $[L:K]=[L:E]\cdot[E:K]$, this shows that the fundamental equality $[L:K]=\sum_{i,j}(v_{ij}L:vK)\cdot[Lv_{ij}:Kv]$ will hold if and only if $\sum_{1\le i\le g}(v_iE:vK)\cdot[Ev_i:Kv]=[E:K]$, and $\sum_{1\le j\le g_i}(v_{ij}L:v_iE)\cdot[Lv_{ij}:Ev_i]=[L:E]$ for $1\le i\le g$. □

With the help of this lemma, we can prove:

Lemma 8.8 *a) Every finite valued field extension of a defectless field is again a defectless field. More precisely, if $v_1,\dots,v_g$ are all extensions of the valuation v from K to a finite extension L, then (K,v) is a defectless field if and only if (K,v) is defectless in L and (L,v_i) are defectless fields for all $i=1,\dots,g$. The same holds for "separably defectless" if $L|K$ is separable, and for "inseparably defectless" if $L|K$ is purely inseparable.*

b) Assume that $(L|K,v)$ is a finite extension of valued fields. If $L|K$ is normal or admits only one extension of v from K to L, then (K,v) is a defectless field if and only if (L,v) is a defectless field and (K,v) is defectless in L. The same holds for "separably defectless" if $L|K$ is separable, and for "inseparably defectless" if $L|K$ is purely inseparable.

c) (K,v) is a defectless field if and only if it is defectless in every finite normal extension. Similarly, (K,v) is a separably defectless field if and only if it is defectless in every finite Galois extension.

Proof: Let (K,v) be a valued field, $L|K$ a finite field extension and $v_1,\dots,v_g$ all extensions of v from K to L. If $L'|L$ is a finite extension such that (L,v_i) is not defectless in L', then by the foregoing lemma, (K,v) is not defectless in L'. Hence, if (K,v) is a defectless field, then (L,v_i) must be a defectless field, for every i, and moreover (K,v) must be defectless in L. Conversely, if (K,v) is a defectless field, then it is defectless in every finite extension L' of L (since it is also a finite extension of K), and again by the foregoing lemma, (L,v_i) will then be defectless in L', for every i.

If in addition $L|K$ is separable, then every finite separable extension L' of L is a finite separable extension of K, and every finite separable extension L' of K containing L is a finite separable extension of L. The same holds for "purely inseparable" in the place of "separable ". Hence, the above arguments also work for "separably defectless" in the place of "defectless" if $L|K$ is separable, and for "inseparably defectless" in the place of "defectless" if $L|K$ is purely inseparable. This proves part a).

If there is only one extension of v from K to L, then the assertion of part b) follows trivially from part a). If there is more than one extension and if $L|K$ is normal, then every other extension is of the form $v\iota$ for $\iota\in\mathrm{Gal}\,L|K$, and ι^{-1} is an isomorphism of (L,v) onto

$(L, v\iota)$. Hence (L, v) is a defectless field if and only if $(L, v\iota)$ is. Thus, part b) follows from part a).

Every finite extension $L|K$ is contained in a finite normal extension, namely the normal hull of $L|K$, and if $L|K$ is separable, then also the normal hull is a separable and thus a Galois extension of K. In view of the foregoing lemma, this proves part c). □

Contrary to the property "henselian field", the property "defectless field" is in general not inherited by infinite algebraic extensions (cf. Theorem 8.41 and Theorem 8.45). But there is an important class of algebraic extensions, called tame extensions, which always preserve this property (cf. Corollary 12.16).

In view of equation (8.3) we find that equality holds in (8.5) if and only if equality holds in (8.4) for $1 \leq i \leq g$. So if all henselizations $K^{h(v_i)}$ are defectless fields, then L is defectless over (K, v). But this formulation has an obvious disadvantage: the number of the henselizations in question depends on the extension $L|K$. But from Lemma 8.2 we know that all henselizations of (K, v) are isomorphic over K, so if one of them is a defectless field, then all henselizations are defectless fields.

If $L|K$ is separable, then all extensions $\iota_i L|K$ and thus also all extensions $\iota_i L.K^h|K^h$ are separable. Consequently, if $(K^{h(v)}, v)$ is separably defectless, then the fundamental equality holds for every finite separable extension L of K. If $L|K$ is purely inseparable, then $\iota_i L = L$ for all i, and the extensions $L|K$ and $L.K^h|K^h$ are purely inseparable. Consequently, if $(K^{h(v)}, v)$ is inseparably defectless, then the fundamental equality holds for every finite purely inseparable extension L of K.

For the converse, assume that L' is a finite extension of K^h. Then by Lemma 6.37 there is some finite extension L of K such that $L.K^h \supset L'$; if $L'|K^h$ is separable resp. purely inseparable, then $L|K$ may be chosen to be separable resp. purely inseparable. If (K, v) is defectless in L, then it follows that $(K^{h(v)}, v)$ is defectless in $L.K^h$. By Lemma 8.7 it then follows that $(K^{h(v)}, v)$ is also defectless in L'. Consequently, if (K, v) is defectless (resp. separably defectless, resp. inseparably defectless), then so is $(K^{h(v)}, v)$. We have proved:

Theorem 8.9 *The valued field (K, v) is defectless if and only if its henselization $(K, v)^h$ is defectless. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

8.2 The Lemma of Ostrowski and the defect

We shall now investigate what ramification theory can say about the fundamental inequality. So let (K, v) be a valued field and $L|K$ a finite normal extension. Let p denote the characteristic exponent of $\overline{K}$. We use the notation of (7.22).

Part e) of Theorem 7.84 has shown that the number $g(L|K, v)$ of all extensions of v from K to L is equal to [Z : K]. Theorem 7.88 has shown that $[\overline{L} : \overline{K}]_{\rm sep} = [\overline{\rm T} : \overline{\rm Z}] = [{\rm T} : {\rm Z}]$, and that $[\overline{L} : \overline{K}]_{\rm insep} = [\overline{L} : \overline{V}]$. Hence, the inertia degree $f(L|K, v) = [\overline{L} : \overline{K}]$ may be written as $p^\nu[\overline{\rm T} : \overline{\rm Z}] = p^\nu[{\rm T} : {\rm Z}]$ with $\nu \geq 0$ an integer such that $p^\nu = [\overline{L} : \overline{K}]_{\rm insep} = [\overline{L} : \overline{V}]$. Similarly, we know from Theorem 7.94 that $|\,(vL/vK)_{p'}\,|$ is equal to $(v{\rm V} : v{\rm T}) = [{\rm V} : {\rm T}]$, and that $(vL/vK)_p \cong vL/v{\rm V}$. Hence, the ramification index $e(L|K, v) = (vL : vK)$ can be written as $p^\mu(v{\rm V} : v{\rm T}) = p^\mu[{\rm V} : {\rm T}]$ with $\mu \geq 0$ an integer such that $p^\mu = |\,(vL/vK)_p\,| = |\,vL/v{\rm V}\,|$. Finally, we know from Theorem 7.91 that $[L : {\rm V}] = p^\lambda$ with $\lambda \geq 0$ an integer. We find that $[L : K] = [L : {\rm V}][{\rm V} : {\rm T}][{\rm T} : {\rm Z}][{\rm Z} : K] = p^{\lambda-\mu-\nu}efg$. From the fundamental inequality (8.5)

we know that $[L : \mathrm{V}] \geq [\overline{L} : \overline{V}](vL : v\mathrm{V}) = p^{\mu+\nu}$. Hence, $\lambda - \mu - \nu$ is an integer ≥ 0. We have proved the important

Lemma 8.10 (Lemma of Ostrowski for normal extensions)
Let $(L|K, v)$ be a finite normal extension of valued fields and p be the characteristic exponent of $\overline{K}$. Then there exists a natural number $d(L|K, v)$, which is a power of p, such that

$$[L : K] = d(L|K, v) \cdot e(L|K, v) \cdot f(L|K, v) \cdot g(L|K, v) \ . \tag{8.7}$$

In particular, $d(L|K, v)$, $e(L|K, v)$, $f(L|K, v)$ and $g(L|K, v)$ are divisors of $[L : K]$.

The above equation can be expressed in the following mnemonic form:

$$n = d \cdot e \cdot f \cdot g \ .$$

If (K, v) is henselian, then we have

$$n = d \cdot e \cdot f \ .$$

Observe that by virtue of part c) of Lemma 8.8, (K, v) is a defectless field if and only if $d(L|K, v) = 1$ for every finite normal extension $L|K$. The integer $d(L|K, v)$ is called the **defect** (or **ramification deficiency**) **of** (K, v) **in** L (or **of** $(L|K, v)$). In the literature, the name "ramification deficiency" is even more usual than "defect", and it is also common to denote by it the integer $\delta \geq 0$ if $d(L|K) = p^{\delta}$. (In this version, the multiplicativity of the defect that we will state below turns into additivity). However, we suggest our use of the defect because it makes formulas easier and allows a generalization to valued vector spaces.

The equality $n = d \cdot e \cdot f$ for finite normal extensions of henselian fields may also be read as $d = n/ef$. In this form, it shows us how to generalize the definition of the defect to finite not necessarily normal extensions of henselian fields. Indeed, if $L|K$ is a finite extension and (K, v) is henselian or more generally, there is only one extension of the valuation v from K to L, then we set

$$d(L|K, v) := \frac{[L : K]}{(vL : vK) \cdot \left[\overline{L} : \overline{K}\right]} \ .$$

If $L|K$ is normal, then in view of (8.7) and the fact that $g(L|K, v) = 1$ if the extension of v is unique, this coincides with the earlier definition of the defect. From the Lemma of Ostrowski for normal extensions, we deduce that the quotient is a natural number and a power of the characteristic exponent of the residue field.

Lemma 8.11 (Lemma of Ostrowski for henselian fields)
Let $(L|K, v)$ be a finite extension of valued fields and p the characteristic exponent of $\overline{K}$. Assume (K, v) to be henselian. Then there exists a natural number $d(L|K, v)$, which is a power of p, such that

$$[L : K] = d(L|K, v) \cdot e(L|K, v) \cdot f(L|K, v) \ . \tag{8.8}$$

In particular, $d(L|K, v)$, $e(L|K, v)$ and $f(L|K, v)$ are divisors of $[L : K]$.

Proof: Let N be the normal hull of L over K. Then $N|K$ and $N|L$ are finite normal extensions. Hence by Lemma 8.10, $d(N|K,v) = [N:K]e(N|K,v)^{-1}f(N|K,v)^{-1}$ and $d(N|L,v) = [N:L]e(N|L,v)^{-1}f(N|L,v)^{-1}$ are powers of p. Dividing one by the other, we find that

$$\begin{aligned} d(N|K,v)d(N|L,v)^{-1} &= [N:K][N:L]^{-1}\, e(N|L,v)e(N|K,v)^{-1}\, f(N|L,v)f(N|K,v)^{-1} \\ &= [L:K]e(L|K,v)^{-1}f(L|K,v)^{-1} \;=\; d(L|K,v)\;. \end{aligned}$$

On the other hand, $d(L|K,v) \geq 1$ since $[L:K] \geq e(L|K,v)f(L|K,v)$ by virtue of Lemma 7.7. This proves that $d(L|K,v)$ is a natural number and a power of p. □

Corollary 8.12 *Let (K,v) be a henselian field and p the characteristic exponent of $\overline{K}$. If $(L|K,v)$ is an immediate algebraic extension, then its degree is a power of p. Similarly, if $\mathbf{A}$ is an immediate algebraic approximation type over (K,v), then its degree is a power of p.*

Proof: If $(L|K,v)$ is a finite immediate extension, then the assertion follows directly from the foregoing lemma. If $L|K$ is infinite algebraic, then every finite subextension is also immediate, and the assertion thus holds when we understand the degree to be a supernatural number.

If $\mathbf{A}$ is an immediate algebraic approximation type over (K,v) then by Theorem 7.122 there is an immediate extension of (K,v) of degree equal to $\deg \mathbf{A}$. Hence by what we have already shown, $\deg \mathbf{A}$ is a power of p. □

Now let us make an important observation. Dividing (8.7) by $[\mathrm{Z}:K] = g(L|K,v)$, we obtain that

$$[L:\mathrm{Z}] = d(L|K,v)\cdot e(L|K,v)\cdot f(L|K,v) = d(L|K,v)\cdot e(L|\mathrm{Z},v)\cdot f(L|\mathrm{Z},v)\;.$$

This shows that $d(L|K,v) = d(L|\mathrm{Z},v)$. Similarly, dividing the last equation by $[\mathrm{T}:\mathrm{Z}] = f(\mathrm{T}|\mathrm{Z},v)$, we obtain

$$[L:\mathrm{T}] = d(L|K,v)\cdot e(L|\mathrm{Z},v)\cdot f(L|\mathrm{T},v) = d(L|K,v)\cdot e(L|\mathrm{T},v)\cdot f(L|\mathrm{T},v)\;.$$

This shows that $d(L|K,v) = d(L|\mathrm{T},v)$. Dividing further by $[\mathrm{V}:\mathrm{T}] = e(\mathrm{V}|\mathrm{T},v)$, we obtain

$$[L:\mathrm{V}] = d(L|K,v)\cdot e(L|\mathrm{V},v)\cdot f(L|\mathrm{T},v) = d(L|K,v)\cdot e(L|\mathrm{V},v)\cdot f(L|\mathrm{V},v)\;.$$

This shows that $d(L|K,v) = d(L|\mathrm{V},v)$. We have proved:

Lemma 8.13 *Let $(L|K,v)$ be a finite normal extension with decomposition field Z, inertia field T and ramification field V. Then*

$$d(L|K,v) = d(L|\mathrm{Z},v) = d(L|\mathrm{T},v) = d(L|\mathrm{V},v)\;.$$

Using the first equation, we prove:

Lemma 8.14 *Let $(L|K,v)$ be a finite normal subextension of $(\tilde{K}|K,v)$. Further, let (K^h,v) be the henselization of (K,v) and (L^h,v) the henselization of (L,v) in $(\tilde{K},v)$. Then*

$$d(L|K,v) = d(L^h|K^h,v)\;.$$

In particular, (K,v) is defectless in L if and only if $(L^h|K^h,v)$ is defectless.

Proof: By Theorem 7.109, $\mathrm{Z} = L \cap K^h$, and $L \mid L \cap K^h$ is linearly disjoint from $K^h \mid L \cap K^h$. Thus, $[L : \mathrm{Z}] = [L.K^h : K^h]$. From Lemma 7.108 we know that $L.K^h = L^h$. So we have $[L : \mathrm{Z}] = [L^h : K^h]$. Further, $(L^h|L, v)$ and $(K^h|K, v)$ are immediate by Theorem 7.110. That is, $vK^h = vK = v\mathrm{Z}$, $vL.K^h = vL$, $\overline{K^h} = \overline{K} = \overline{\mathrm{Z}}$ and $\overline{L^h} = \overline{L}$. Consequently, $e(L^h|K^h, v) = e(L|\mathrm{Z}, v)$ and $f(L^h|K^h, v) = f(L|\mathrm{Z}, v)$. Applying (8.7) to the extensions $(L.K^h|L, v)$ and $(L|K, v)$, we now find that also $d(L^h|K^h, v)$ must equal $d(L|\mathrm{Z}, v)$. In view of the foregoing lemma, we conclude that $d(L^h|K^h, v) = d(L|K, v)$. □

The last assertion holds even if we drop the condition that $L|K$ be finite. To show this, one just needs to observe that if $E|K$ is a normal subextension of $L|K$, then $E^h|K^h$ is a normal subextension of $L^h|K^h$ since $E^h = E.K^h$ by Lemma 7.108, and that every finite normal subextension of $L^h|K^h$ is contained in $E^h|K^h$ for $E|K$ a suitable finite normal subextension of $L|K$. The details are left to the reader.

Corollary 8.15 *Let $(L|K, v)$ be an arbitrary normal subextension of $(\tilde{K}|K, v)$. Further, let (K^h, v) be the henselization of (K, v) and (L^h, v) the henselization of (L, v) in $(\tilde{K}, v)$. Then (K, v) is defectless in L if and only if $(L^h|K^h, v)$ is defectless.*

From this corollary together with Lemma 7.10, we can deduce the **transitivity of defectless extensions** also for the case where the extension of the valuation is not unique:

Corollary 8.16 *Let $(L|K, v)$ be an arbitrary normal extension and $E|K$ a normal subextension of $L|K$. If (K, v) is defectless in E and (E, v) is defectless in L, then (K, v) is defectless in L. Consequently, if (K, v) is defectless in E and (E, v) is a defectless field, then (K, v) is a defectless field.*

Lemma 8.14 tells us that we can measure the defect of a finite normal extension by shifting the extension up to the henselizations. If we are working within an algebraically closed field with a once and for all fixed valuation, then this observation gives us the possibility to define the defect for all finite extensions by just measuring it at the extension of the respective henselizations. We can do this even for all algebraic extensions $(L|K, v)$ for which $L^h|K^h$ is finite. Following this idea, we fix a valuation v of $\tilde{K}$ and define: An extension $L|K$ will be called **h-finite** if the extension $(L^h|K^h, v)$ of the respective henselizations is finite. If this is the case, then we define the **henselian defect** of $(L|K, v)$ to be

$$d_h(L|K, v) := d(L^h|K^h, v) = \frac{[L^h : K^h]}{(vL : vK) \cdot \left[\overline{L} : \overline{K}\right]} .$$

Note that the second equation holds since $(L^h|L, v)$ and $(K^h|K, v)$ are immediate extensions. Our definition coincides with the earlier definitions of the defect, wherever it was defined:

Lemma 8.17 *If $(L|K, v)$ is a finite normal extension, then $d_h(L|K, v) = d(L|K, v)$. If $(L|K, v)$ is an h-finite extension such that the extension of v from K to L is unique, then it is finite and again, $d_h(L|K, v) = d(L|K, v)$.*

Further, (K, v) is a defectless field if and only if $d_h(L|K, v) = 1$ for every finite extension $(L|K, v)$.

Proof: The first assertion is a reformulation of Lemma 8.14. Now assume that $(L|K, v)$ is an h-finite extension such that the extension of v from K to L is unique. Then $L|K$

is algebraic, and by Theorem 7.109, it is linearly disjoint from $K^h|K$ (in $(\tilde{K}, v)$. By Lemma 7.108, $L^h = L.K^h$. By assumption, $L.K^h|K^h$ is finite, hence also $L|K$ must be finite, and $[L^h : K^h] = [L : K]$. The latter shows that $d_h(L|K, v) = d(L|K, v)$.

If (K, v) is a defectless field, then by Theorem 8.9, also $(K, v)^h$ is a defectless field, showing that for every finite extension $(L|K, v)$ we must have that $d_h(L|K, v) = d(L^h|K^h, v) = 1$. Conversely, if $d_h(L|K, v) = 1$ for all finite extensions $(L|K, v)$, then in particular, $d(L|K, v) = d_h(L|K, v) = 1$ for all finite normal extensions $(L|K, v)$. So by definition, (K, v) is a defectless field. □

Corollary 8.18 *Let (K, v) be a defectless field and $(L|K, v)$ an immediate extension such that the extension of v from K to L is unique. Then $L = K$.*

Proof: If the assertion were not true, then $(L|K, v)$ would admit a nontrivial finite immediate subextension $(L_0|K, v)$. Also for this subextension, the extension of v would be unique, and the preceding lemma would give that $d_h(L_0|K, v) = d(L_0|K, v) > 1$. But this contradicts our assumption that (K, v) be a defectless field. □

In view of Lemma 8.17 and since the henselian defect is the main notion of a defect in this book, we will frequently omit the word "henselian" and only speak of the **defect**.

In the same way, we generalize the notion of "defectless extension". We have to pay attention since there is already this notion for arbitrary extensions. If $(L|K, v)$ is an h-finite extension, then we call it **h-defectless** if the extension $(L^h|K^h, v)$ is defectless, that is, if $d_h(L|K, v) = 1$. Lemma 8.17 tells us that a finite normal extension $(L|K, v)$ is h-defectless if and only if it is defectless in the already defined sense. But this is not necessarily true if $L|K$ is not normal and there are more than one extension of v from K to L. Then it may happen that for one extension the henselian defect is 1 while for another extension it is > 1. In this case, the henselian defect also depends on the chosen extension of v from K to L. If $L|K$ is normal, then the henselian defect does not depend on this extension.

From the multiplicativity of extension degree, ramification index and inertia degree we obtain the **multiplicativity of the defect**:

Lemma 8.19 *If $(L|K, v)$ is an h-finite extension and $E|K$ is a subextension of $L|K$, then also $(L|E, v)$ and $(E|K, v)$ are h-finite, and*

$$d_h(L|K, v) = d_h(L|E, v) \cdot d_h(E|K, v) \ .$$

Consequently, $(L|K, v)$ is h-defectless if and only if $(L|E, v)$ and $(E|K, v)$ are h-defectless.

The following lemma provides important examples for h-defectless extensions.

Lemma 8.20 *Let $(L|K, v)$ be a normal extension and $K \subset K_0 \subset K_1 \subset (L|K, v)^r$ such that $(K_1|K_0, v)$ is finite. Then $(K_1|K_0, v)$ is h-defectless.*

Proof: Let $L_0 \subset L$ be the normal hull of $K_0|K_1$. By Lemma 7.82 and Lemma 7.95, $\mathrm{V}_0 := (L_0|K_0, v)^r = (L|K, v)^r \cap L_0 \supset K_1$. From Lemma 8.13 and Lemma 8.17 we infer that $d_h(L_0|K_0, v) = d_h(L_0|\mathrm{V}_0, v)$. Now the multiplicativity of the defect shows that $d_h(\mathrm{V}_0|K_0, v) = 1$ and $d_h(K_1|K_0, v) = 1$. □

We will also need the following easy observation. Suppose that $(L|K, v)$ is an extension of degree p a prime. If it is not immediate, then $e(L|K, v) > 1$ or $f(L|K, v) > 1$.

Assume that $L|K$ is normal or that (K,v) is henselian. Then by the Lemma of Ostrowski (Lemma 8.10 resp. Lemma 8.11), $e(L|K,v) \cdot f(L|K,v)$ divides $[L:K] = p$. It follows that $e(L|K,v) = p$ or $f(L|K,v) = p$. And in both cases, $g(L|K,v) = 1$. We have proved:

Lemma 8.21 *Let $(L|K,v)$ be an extension of degree p a prime. Assume that (K,v) is henselian or that $L|K$ is normal. If $(L|K,v)$ is not immediate, then it is defectless, and the extension of v from K to L is unique.*

Exercise 8.1 *Show that Lemma 8.20 also holds for h-finite extensions $(K_1|K_0,v)$.*

8.3 Defectless fields

By the Lemma of Ostrowski, the defect $d(L|K,v)$ is a power of the characteristic exponent p of the residue field $\overline{K}$. If the residue characteristic of (K,v) is 0, that is, if $p = 1$, then $d(L|K,v) = 1$ for every finite normal extension $L|K$. Hence:

Theorem 8.22 *Every valued field of residue characteristic zero is a defectless field.*

If (K,v) is a valued field such that vK is divisible and $\overline{K}$ is algebraically closed, then by virtue of Lemma 7.59, the extension $(\tilde{K},w)|(K,v)$ is immediate for every extension w of v from K to $\tilde{K}$. The same then holds for every henselization of $(K^{h(w)},w)$ in the place of (K,v). So if the residue characteristic of (K,v) and thus also that of $(K^{h(w)},w)$ is zero, then the foregoing theorem shows that the immediate extension $(\tilde{K},w)|(K^{h(w)},w)$ must be trivial. We have proved:

Corollary 8.23 *If (K,v) is a valued field of residue characteristic zero such that its value group vK is divisible and its residue field $\overline{K}$ is algebraically closed, then every henselization of (K,v) is algebraically closed. If in addition (K,v) is henselian, then K is itself algebraically closed.*

The last assertion of this corollary can be generalized as follows:

Lemma 8.24 *Let (K,v) be a henselian field of residue characteristic zero. If $(L|K,v)$ is a normal extension with $vL = vK$, then $\mathrm{Gal}\, L|K \cong \mathrm{Gal}\, \overline{L}|\overline{K}$. In particular, if vK is divisible, then $\mathrm{Gal}\, K \cong \mathrm{Gal}\, \overline{K}$.*

Proof: Since v admits a unique extension from K to L, it follows by Theorem 7.84 that $(L|K,v)^d = K$. On the other hand, $\mathrm{char}\, \overline{K} = 0$ implies that $\overline{L}|\overline{K}$ is separable and that $(L|K,v)^r = L$ (cf. Theorem 7.91). Further, $vL = vK$ implies that $(L|K,v)^i = L$ (cf. Theorem 7.94). Now Theorem 7.88 shows that $\mathrm{Gal}\, L|K = \mathrm{Gal}\,(L|K,v)^i \,|\, \mathrm{Gal}\,(L|K,v)^d \cong \mathrm{Gal}\, \overline{L}|\overline{K}$.

The second assertion follows from the first since if vK is divisible, then $vK = v\tilde{K}$ in view of Corollary 7.9. □

We shall now prove a theorem that provides examples of henselian defectless fields, independently of their residue characteristic. Recall the definition of (K,v)-vector spaces that we have given in Section 7.2. If $(L|K,v)$ is an extension of valued fields, then (L,v) is a (K,v)-vector space.

Lemma 8.25 *Let (K,v) be a spherically complete field. Then every finite dimensional (K,v)-vector space is spherically complete and admits a valuation basis (over the zero subspace).*

Proof: By virtue of Lemma 4.22, there is a K-subvector space $V_{\mathcal{B}}$ of V such that $(V_{\mathcal{B}},v)$ admits a valuation basis $\mathcal{B}$ (over the zero subspace), and $(V_{\mathcal{B}},v) \subset (V,v)$ is immediate. By the definition of a (K,v)-vector space, for every $b \in \mathcal{B}$ the 1-dimensional subvectorspace (Kb,v) of $(V_{\mathcal{B}},v)$ is isomorphic to (K,v) (as a valued K-vector space). Since (K,v) is spherically complete and this property is preserved under isomorphisms (already under those of ultrametric spaces), it follows that every (Kb,v) is spherically complete. Further, the subspaces (Kb,v), $b \in \mathcal{B}$ are valuation independent. Hence by Lemma 3.46, their sum $(V_{\mathcal{B}},v)$ is spherically complete. (Alternatively, this is proved by showing that the ultrametric space $(V_{\mathcal{B}},v)$ is isomorphic to the product of the ultrametric spaces (Kb,v), cf. Lemma 4.12, and then applying Lemma 2.10.) From Lemma 2.16 it now follows that $(V_{\mathcal{B}},v) \subset (V,v)$ can not be a proper immediate extension of valued vector spaces. Hence $V_{\mathcal{B}} = V$, showing that (V,v) is spherically complete and admits a valuation basis over the zero subspace. □

We apply this lemma to a finite valued field extension (L,v) of a spherically complete field (K,v). (In this case, we can take $V_{\mathcal{B}}$ to be the K-subvector space of L which is generated by a standard valuation independent set $\mathcal{B} = \{z_i u_j, \mid 1 \le i \le e(L|K,v)\,,\ 1 \le j \le f(L|K,v)\}$.) We obtain that (L,v) is spherically complete and that $(L|K,v)$ is defectless. Moreover, from Lemma 8.6 we infer that the extension of v from K to L is unique. This proves:

Theorem 8.26 *Every spherically complete field is a henselian defectless field. In particular, power series fields are henselian defectless fields. Every finite extension of a spherically complete field is again spherically complete.*

In view of Theorem 7.112 and Lemma 7.4, we obtain the following special cases:

Corollary 8.27 *For every prime p, the valued fields $(\mathbb{Q}_p, v_p)$ and $(\mathbb{F}_p((t)), v_t)$ are henselian defectless fields.*

From Theorem 7.124 and Theorem 8.26, we obtain

Corollary 8.28 *Every maximal field is henselian defectless.*

Recall that a valued field is **algebraically maximal** if it admits no proper immediate algebraic extensions. Similarly, we call it **separable-algebraically maximal** if it admits no proper immediate separable algebraic extensions. Since the henselization is an immediate separable algebraic extension (Theorem 7.110), we conclude:

Lemma 8.29 *Every separable-algebraically maximal field is henselian.*

The converse is not true in general (see Theorem 8.39 and Theorem 8.41 below). But if a field is henselian defectless, then it satisfies $n = e \cdot f$ for every finite extension, showing that every finite immediate extension must be trivial. Since every proper algebraic immediate extension admits a proper finite immediate subextension, this yields that every henselian defectless field is algebraically maximal. Adding the assertion of Theorem 8.22, we now obtain:

Corollary 8.30 *Every henselian defectless field is algebraically maximal, and every algebraically maximal field is henselian. Hence for valued fields of residue characteristic zero, these three notions are equivalent.*

Let us give a very important application of this corollary together with Theorem 7.107, Theorem 7.112 and the results of Section 7.10.

Theorem 8.31 *The maximal immediate extension of a valued field (K,v) of residue characteristic zero is unique up to valuation preserving isomorphism over K.*

Proof: Let (L_1,v_1) and (L_2,v_2) be two maximal immediate extensions of (K,v). By the usual argument using Zorn's Lemma it is shown that there exists a maximal subextension $(L_1|L,v_1)$ of $(L_1|K,v_1)$ with the property that there is a valuation preserving embedding of (L,v_1) in (L_2,v_2) over K. We can identify (L,v_1) with its image in L_2 and denote by (L,v) the so-obtained common subfield of (L_1,v_1) and (L_2,v_2). By Theorem 7.112, the maximal fields (L_1,v_1) and (L_2,v_2) are henselian. Hence by Theorem 7.107, there are henselizations of (L,v) in (L_1,v_1) and (L_2,v_2) which are isomorphic over (L,v). This contradicts our choice of (L,v) if the henselizations are proper extensions of (L,v). We thus find that (L,v) must be henselian. Since (L,v) has residue characteristic 0 like (K,v), we can infer from Corollary 8.30 that (L,v) is algebraically maximal. Hence by Theorem 7.131, every nontrivial immediate approximation type over (L,v) is transcendental.

Now suppose that there is some $x \in L_1 \setminus L$. Since $(L_1|L,v_1)$ is immediate, Corollary 2.28 shows that the approximation type $\operatorname{appr}(x,K)$ is nontrivial and immediate. By what we have shown before, it is transcendental. Since (L_2,v_2) is a maximal valued field, it is spherically complete by Theorem 7.124. Hence, $\operatorname{appr}(x,K)$ is realized in (L_2,v_2) (cf. Lemma 2.42), that is, there is some $y \in L_2$ such that $\operatorname{appr}(x,K) = \operatorname{appr}(y,K)$. Now Theorem 7.120 shows that $(L(x),v_1)$ and $(L(y),v_2)$ are isomorphic over K. But this again contradicts our choice of (L,v). Consequently, $L_1 = L$, and (L,v) is a maximal valued field. Thus, the immediate extension $(L_2|L,v_2)$ is trivial, and we have that $L_1 = L = L_2$. This proves that (L_1,v_1) and (L_2,v_2) are isomorphic over (K,v). □

Besides the henselian fields of residue characteristic zero, there is a second important class of defectless fields. Later, when we have treated the composition of valuations, then we will combine both classes to obtain further important examples of defectless fields, namely the formally p-adic and the finitely ramified fields.

Theorem 8.32 *Let (K,v) be a valued field with value group $vK \cong \mathbb{Z}$. Then (K,v) is a separably defectless field. If in addition* $\operatorname{char} K = 0$*, then (K,v) is a defectless field.*

Proof: Since the henselization is an immediate extension (Theorem 7.110), its value group is the same as that of (K,v). In view of Theorem 8.9, it thus suffices to prove our assertion in the case of (K,v) being henselian.

Let $(L|K,v)$ be a finite normal extension. We wish to show that it is defectless. Let V denote the ramification field of $(L|K,v)$. By Lemma 8.13, $d(L|K,v) = d(L|\mathrm{V},v)$. By Lemma 7.92, the extension $L|\mathrm{V}$ is a tower of normal extensions of degree p. In view of the multiplicativity of the defect, it suffices to show that every of these is defectless. Let $\mathrm{V} \subset K_0 \subset K_1 \subset L$ such that $[K_1 : K_0] = p$. Then $K_0|K$ is a finite extension, hence by the fundamental inequality, $(vK_0 : vK)$ is finite. This implies that vK_0 is again isomorphic to

$\mathbb{Z}$. Since we assume (K, v) to be henselian, the same holds for (K_0, v). Corollary 7.115 now shows that the separable extension $(K_1|K_0, v)$ is not immediate. Hence by Lemma 8.21, it is defectless. It follows that $(L|K, v)$ is defectless. We have proved that (K, v) is a separably defectless field.

Our second assertion follows from the fact that every perfect separably defectless field is a defectless field. □

8.4 Examples for nontrivial defect

In this section, we shall give examples for extensions with defect > 1. There is one basic example which is quick at hand. We just need a preliminary observation.

Lemma 8.33 *Let (K, v) be a countable nontrivially valued field with archimedean value group vK. Then its completion is an extension of transcendence degree $2^{\aleph_0}$. In particular, the extensions $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ and $\mathbb{Q}_p|\mathbb{Q}$ have transcendence degree $2^{\aleph_0}$.*

Proof: For an infinite field L, the cardinality of its algebraic closure is equal to that of L. (Indeed, the elements in the algebraic closure of L can be well-ordered by use of a well-ordering of all polynomials over L. These in turn are represented by their coefficients and thus correspond to finite tuples of elements of L. But if L is infinite, then the set of all finite tuples of elements of L has the same cardinality as L.) Consequently, every transcendence basis of L over its prime field k has the same cardinality as L. If K is a subfield of L, then every transcendence basis of L over K can be extended to a transcendence basis $\mathcal{T}$ of L over k by adjoining a transcendence basis $\mathcal{T}'$ of K over its prime field k. If K has smaller cardinality than L, then also $\mathcal{T}'$ has smaller cardinality than L, and so the cardinality of $\mathcal{T}$ must be equal to that of L. We have shown:

If L is an extension of K of cardinality $|L| > |K|$, then the transcendence degree of $L|K$ must be equal to $|L|$.

Hence it suffices to show that the cardinality of the completion K^c is $2^{\aleph_0}$.

We pick some element a of positive value in K (e.g., we may take $a = t$ in $\mathbb{F}_p(t)$ and $a = p$ in $\mathbb{Q}_p$). Since vK is assumed to be archimedean, the set $\{iva = va^i \mid i \in \mathbb{N}\}$ is cofinal in vK, and it is order isomorphic to $\mathbb{N}$. We use the fact that there are $2^{\aleph_0}$ many subsets of $\mathbb{N}$. With every such subset S we associate a completion nest as follows. We set $a_1 := a$ if $1 \in S$, and $a_1 = 0$ otherwise. If a_i is already defined, then we set $a_{i+1} := a_i + a^{i+1}$ if $i+1 \in S$, and $a_{i+1} := a_i$ otherwise. Now the balls $B_{iva}(a_i)$, $i \in \mathbb{N}$, form a completion nest. This completion nest determines uniquely a completion type (cf. Lemma 2.39). Every two distinct subsets S, S' of $\mathbb{N}$ will give two distinct completion types. Indeed, if i is such that $i-1 \in S$ and $i-1 \notin S'$, then $v(a_i - a'_i) < iva$, which yields that $B_{iva}(a_i) \cap B_{iva}(a'_i) = \emptyset$. Consequently, we obtain $2^{\aleph_0}$ many completion types over (K, v). Since the completion types are in one to one correspondence with the elements of the completion, we conclude that the cardinality of K^c is at least $2^{\aleph_0}$.

On the other hand, the number of approximation types over (K, v) is (by definition) bounded by the number of maps from vK into K. That is, there are at most $|K|^{|vK|} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ approximation types over (K, v). We conclude that $|K^c| = 2^{\aleph_0}$. □

Remark 8.34 It is quite easy to construct explicitly at least one element s in $\mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$ (and that is all we need for our first example). Let $s = \sum_{i=1}^{\infty} c_i t^{\nu_i}$ with $0 \neq c_i \in \mathbb{F}_p$

and $(\nu_i)_{i\in\mathbb{N}}$ an increasing sequence of natural numbers such that $\lim_i \nu_{i+1}/\nu_i = \infty$. We claim that s is transcendental over $\mathbb{F}_p(t)$. Assume the opposite. Then (after multiplying the minimal polynomial of s over $\mathbb{F}_p(t)$ by a common denominator of its coefficients), s satisfies a relation

$$g(s) = d_n(t)s^n + \ldots + d_1(t)s + c_0(t) = 0$$

of minimal degree n with coefficients $d_i(t) \in \mathbb{F}_p[t]$ and $d_n(t) \neq 0$. Let $m := \max_i \deg d_i$. For every j, the element $g(\sum_{i=1}^{j} c_i t^{\nu_i})$ is a polynomial in t of degree $\leq n\nu_j + m$. It must be nonzero since the above relation is of minimal degree and thus does not admit a zero in $\mathbb{F}_p(t)$. On the other hand, the Taylor expansion shows that with $z := -\sum_{i=j+1}^{\infty} c_i t^{\nu_i}$ and g_i the i-th formal derivative of g,

$$g\left(\sum_{i=1}^{j} c_i t^{\nu_i}\right) = g(s+z) = g_1(s)z + \ldots + g_n(s)z^n \in t^{\nu_{j+1}}\mathbb{F}_p[[t]]$$

since $g(s) = 0$. By hypothesis, we can choose j so large that $\nu_j \geq m$ and $\nu_{j+1}/\nu_j > n+1$. Then $\nu_{j+1} > (n+1)\nu_j \geq n\nu_j + m \geq \deg g(\sum_{i=1}^{j} c_i t^{\nu_i})$. This is a contradiction since a nonzero polynomial of degree $< \nu_{j+1}$ can not lie in $t^{\nu_{j+1}}\mathbb{F}_p[[t]]$.

Now we are able to present our first example, which is due to F. K. Schmidt.

Example 8.35 We consider $\mathbb{F}_p((t))$ with its canonical valuation $v = v_t$. By the foregoing lemma, we can choose some element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. Since $(\mathbb{F}_p((t))|\mathbb{F}_p(t), v)$ is an immediate extension, the same holds for $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t), v)$ and thus also for $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p), v)$. The latter extension is purely inseparable of degree p (since s, t are algebraically independent over $\mathbb{F}_p$, the extension $\mathbb{F}_p(s)|\mathbb{F}_p(s^p)$ is linearly disjoint from $\mathbb{F}_p(t,s^p)|\mathbb{F}_p(s^p)$). Hence, Corollary 7.72 shows that there is only one extension of the valuation v from $\mathbb{F}_p(t,s^p)$ to $\mathbb{F}_p(t,s)$. So we have $e = f = g = 1$ for this extension and consequently, its defect is p. ♢

Remark 8.36 This example is the easiest one used in commutative algebra to show that the integral closure of a noetherian ring of dimension 1 in a finite extension of its quotient field need not be finitely generated.

In some sense, the field $\mathbb{F}_p(t,s^p)$ is the smallest possible admitting a defect extension. We will see later that a field of transcendence degree 1 over its prime field is defectless under every valuation (this follows from Theorem 12.39).

We can use Theorem 7.107 to derive from Example 8.35 an example of a defect extension of henselian fields.

Example 8.37 We consider $\mathbb{F}_p((t))$ with its canonical valuation $v = v_t$ and choose an element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. In Example 8.35 we have shown that the extension $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p), v)$ is immediate. By Theorem 7.107, there is a henselization $(\mathbb{F}_p(t,s), v)^h$ of $(\mathbb{F}_p(t,s), v)$ in $\mathbb{F}_p((t))$ and a henselization $(\mathbb{F}_p(t,s^p), v)^h$ of $(\mathbb{F}_p(t,s^p), v)$ in $(\mathbb{F}_p(t,s), v)^h$. We find that $(\mathbb{F}_p(t,s), v)^h|(\mathbb{F}_p(t,s^p), v)^h$ is again a purely inseparable extension of degree p. Indeed, $\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p)$ is linearly disjoint from the separable extension $\mathbb{F}_p(t,s^p)^h|\mathbb{F}_p(t,s^p)$, and by virtue of Corollary 7.108, $\mathbb{F}_p(t,s)^h = \mathbb{F}_p(t,s).\mathbb{F}_p(t,s^p)^h$. Also for this extension, we have that $e = f = g = 1$ and again, the defect is p. Note that by Theorem 8.32, a proper immediate extension over a field like $(\mathbb{F}_p(t,s^p), v)^h$ can only be purely inseparable. ♢

The next example is easily found by considering the purely inseparable extension $\tilde{K}|K^{\mathrm{sep}}$. In comparison with the last example, the involved fields are "much bigger", for instance, they do not have value group $\mathbb{Z}$ anymore.

Example 8.38 Let K be a field which is not perfect. Then the extension $\tilde{K}|K^{\rm sep}$ is nontrivial. For every nontrivial valuation v on $\tilde{K}$, we have by Lemma 7.59 that $v\tilde{K}$ and $vK^{\rm sep}$ are both equal to the divisible hull of vK, and that $\overline{\tilde{K}}$ and $\overline{K^{\rm sep}}$ are both equal to the algebraic closure of $\overline{K}$. Consequently, $(\tilde{K}|K^{\rm sep}, v)$ is an immediate extension. Since the extension of v from $K^{\rm sep}$ to $\tilde{K}$ is unique (cf. Corollary 7.72), we find that the defect of every subextension is equal to its degree. Recall that $\tilde{K}$ admits a nontrivial valuation as soon as it is not the algebraic closure of a finite field (cf. Corollary 7.39).

We can refine this example as follows. Let $p > 0$ be the characteristic of the field K. In order that every purely inseparable extension of the valued field (K, v) be immediate, it suffices that vK be p-divisible and $\overline{K}$ be perfect. This follows from Lemma 7.62. But we know from Corollary 7.61 that these conditions are already satisfied for every Artin-Schreier closed field K. Hence, *the perfect hull of every Artin-Schreier closed valued field is an immediate extension.* ♢

Note that the separable-algebraically closed field $K^{\rm sep}$ is henselian for every valuation. Hence, the first part of our example shows:

Theorem 8.39 *There are henselian valued fields of positive characteristic which admit proper purely inseparable immediate extensions. Hence, the property "henselian" does not imply the property "algebraically maximal".*

Until now, we have only presented purely inseparable defect extensions. But the second part of the last example can give an idea of how to produce a separable defect extension by interchanging the role of purely inseparable extensions and Artin-Schreier extensions.

Example 8.40 Let (K, v) be a valued field of characteristic $p > 0$ whose value group is not p-divisible. For example, we can choose (K, v) to be $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{F}_p((t)), v_t)$. Let $c \in K$ such that $vc < 0$ is not divisible by p. Let a be a root of the Artin-Schreier polynomial $X^p - X - c$. Then it follows from Lemma 7.54 that $va = vc/p$, and it is shown as in the proof of Lemma 7.55 that $[K(a) : K] = p = (vK(a) : vK)$. The fundamental inequality shows that $\overline{K(a)} = \overline{K}$ and that the extension of v from K to $K(a)$ is unique. Also the further extension to $K(a)^{1/p^\infty} = K^{1/p^\infty}(a)$ is unique. It follows that the extension of v from K^{1/p^∞} to $K^{1/p^\infty}(a)$ is unique. On the other hand, $[K^{1/p^\infty}(a) : K^{1/p^\infty}] = p$ since the separable extension $K(a)|K$ is linearly disjoint from $K^{1/p^\infty}|K$. By Lemma 7.62, $vK^{1/p^\infty}(a)$ is the p-divisible hull of $vK(a) = vK + \mathbb{Z}va$. Since $pva \in vK$, this is the same as the p-divisible hull of vK, which in turn is equal to vK^{1/p^∞}. Again by Lemma 7.62, the residue field of $K^{1/p^\infty}(a)$ is the perfect hull of $\overline{K(a)} = \overline{K}$. Hence it is equal to the residue field of K^{1/p^∞}. It follows that the extension $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$ is immediate and that its defect is p like its degree.

Similarly, one can start with a valued field (K, v) of characteristic $p > 0$ whose residue field is not perfect. In this case, the Artin-Schreier extension $K(a)|K$ is constructed as in the proof of Lemma 7.56. We leave the details to the reader. ♢

In this example, we can always choose (K, v) to be henselian (since passing to the henselization does not change value group and residue field). Then all constructed extensions of (K, v) are also henselian, since they are algebraic extensions. Hence, our example shows:

Theorem 8.41 *There are henselian valued fields of positive characteristic which admit immediate Artin-Schreier extensions. Hence, "henselian" does not imply "separable-algebraically maximal". There are spherically complete fields of positive characteristic admitting an infinite purely inseparable extension which is not even separable-algebraically maximal.*

If the perfect hull of a given valued field (K,v) is not an immediate extension, then vK is not p-divisible or $\overline{K}$ is not perfect, and we can apply the procedure of our above example. This shows:

Theorem 8.42 *If the perfect hull of a given valued field of positive characteristic is not an immediate extension, then it admits an immediate Artin-Schreier extension.*

In a slightly different form, the above example was already given by A. Ostrowski in [OST3], Section 57. If one takes $(K,v) = (\mathbb{F}_p(t), v_t)$, then one can choose $c = 1/t$. Ostrowski works with the polynomial $X^p - tX - 1$ in the place of the Artin-Schreier polynomial $X^p - X - 1/t$. Over an extension of degree $p-1$, it also can be transformed into an Artin-Schreier polynomial. Indeed, if we take b to be an element which satisfies $b^{p-1} = t$, then replacing X by bX and dividing by b^p will transform $X^p - tX - 1$ into the polynomial $X^p - X - 1/b^p$. Now we replace X by $X + 1/b$. Since we are working in characteristic p, this transforms $X^p - X - 1/b^p$ into $X^p - X - 1/b$ (this sort of transformation will play a crucial role in later chapters of this book). Now we see that the Artin-Schreier polynomial $X^p - X - 1/b$ plays the same role as $X^p - X - 1/t$. Indeed, $vb = \frac{1}{p-1}vt$ and it follows that $(v\mathbb{F}_p(b) : v\mathbb{F}_p(t)) = p-1 = [\mathbb{F}_p(b) : \mathbb{F}_p(t)]$, so that $v\mathbb{F}_p(b) = \mathbb{Z}\frac{vt}{p-1}$. In this value group, vb is not divisible by p.

The interplay of Artin-Schreier extensions and radical extensions that we have used in the last examples can also be transferred to the mixed characteristic case. Before we continue, we need a lemma which shows that there is some **quasi-additivity** in the mixed characteristic case.

Lemma 8.43 *Let (K,v) be a valued field of characteristic 0 and residue characteristic $p > 0$. Further, let $c_1, \ldots, c_n$ be elements in K of value $\geq -\frac{vp}{p}$. Then*

$$(c_1 + \ldots + c_n)^p \;\equiv\; c_1^p + \ldots + c_n^p \pmod{\mathcal{O}_\mathbf{K}} .$$

Proof: Every product of p many c_i's has value $\geq -vp$. In view of the fact that every binomial coefficient $\binom{p}{i}$ is divisible by p for $1 \leq i \leq p-1$, we find that $(c_1+c_2)^p \equiv c_1^p + c_2^p \pmod{\mathcal{O}_\mathbf{K}}$. Now the assertion follows by induction on n. □

Example 8.44 Consider $\mathbb{Q}_p$ with its p-adic valuation $v = v_p$. We construct an algebraic extension (K,v) with a p-divisible value group as follows. By induction, we choose elements a_i in the algebraic closure of $\mathbb{Q}_p$ such that $a_1^p = 1/p$ and $a_{i+1}^p = a_i$. Then $va_1 = -\frac{vp}{p}$ and $va_i = -\frac{vp}{p^i}$ for every i. Hence, the field $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ must have p-divisible value group under the unique extension of v from $\mathbb{Q}_p$ to K (recall that $(\mathbb{Q}_p, v)$ is henselian by Theorem 7.112). Note that $a_1, \ldots, a_i \in \mathbb{Q}_p(a_{i+1})$ for every i. Since $(v\mathbb{Q}_p(a_{i+1}) : v\mathbb{Q}_p(a_i)) = p$, the fundamental inequality shows that $\overline{\mathbb{Q}_p(a_{i+1})} = \overline{\mathbb{Q}_p(a_i)}$ for every i. Hence, $\overline{K} = \overline{\mathbb{Q}_p} = \mathbb{F}_p$.

Now we let a be a root of the Artin-Schreier polynomial $X^p - X - 1/p$. It follows from Lemma 7.54 that $va = -\frac{vp}{p}$. Consequently, $(v\mathbb{Q}_p(a) : v\mathbb{Q}_p) = p = [\mathbb{Q}_p(a) : \mathbb{Q}_p]$ and $\overline{\mathbb{Q}_p(a)} = \overline{\mathbb{Q}_p}$. We define $b_i := a - a_1 - \ldots - a_i$. By construction, $va_i \geq -\frac{vp}{p}$ for all i. It follows that also $vb_i \geq -\frac{vp}{p}$ for all i. With the help of the foregoing lemma, and bearing in mind that $a_{i+1}^p = a_i$ and $a_1^p = 1/p$, we compute

$$\begin{aligned} 0 &= a^p - a - \frac{1}{p} = (b_i + a_1 + \ldots + a_i)^p - (b_i + a_1 + \ldots + a_i) - 1/p \\ &\equiv b_i^p - b_i + a_1^p + \ldots + a_i^p - a_1 - \ldots - a_i - 1/p = b_i^p - b_i - a_i \pmod{\mathcal{O}_{\mathbf{K}}} . \end{aligned}$$

Since $va_i < 0$, Lemma 7.54 shows that $vb_i = \frac{1}{p}va_i = va_{i+1}$. Hence, $(v\mathbb{Q}_p(a, a_i) : v\mathbb{Q}_p(a_i)) = p = [\mathbb{Q}_p(a, a_i) : \mathbb{Q}_p(a_i)]$ and $\overline{\mathbb{Q}_p(a, a_i)} = \overline{\mathbb{Q}_p(a_i)} = \mathbb{F}_p$ for every i. If $\mathbb{Q}_p(a)|\mathbb{Q}_p$ were not linearly disjoint from $K|\mathbb{Q}_p$, then there would exist some i such that $\mathbb{Q}_p(a)|\mathbb{Q}_p$ would not be linearly disjoint from $\mathbb{Q}_p(a_i)|\mathbb{Q}_p$. But we have just shown that this is not the case. Similarly, if $vK(a)$ would contain an element that does not lie in the p-divisible hull of $\mathbb{Z} = v\mathbb{Q}_p$, or if $\overline{K(a)}$ would be a proper extension of $\mathbb{F}_p$, then the same would already hold for $\mathbb{Q}_p(a, a_i)$ for some i. But we have shown that this is not the case. Hence, $(K(a)|K, v)$ is a proper immediate Artin-Schreier extension of degree p. ♢

Recall that $(\mathbb{Q}_p, v_p)$ is spherically complete (cf. Lemma 7.4). Hence, our example shows:

Theorem 8.45 *There are henselian valued fields of characteristic 0 with positive residue characteristic which admit immediate Artin-Schreier extensions. There are spherically complete fields admitting an infinite separable-algebraic extension which is not even separable-algebraically maximal.*

From the last example, we can derive a special case which was given by A. Ostrowski in [OST3], Section 39.

Example 8.46 In the last example, we take $p = 2$. Then $(K(\sqrt{3})|K, v)$ is an immediate extension of degree 2. Indeed, this is nothing but the Artin-Schreier extension that we have constructed. If one substitutes $Y = 1 - 2X$ in the minimal polynomial $Y^2 - 3$ of $\sqrt{3}$ and then divides by 4, one obtains the Artin-Schreier polynomial $X^p - X - 1/2$.

This is Ostrowski's original example. A slightly different version was presented by P. Ribenboim (cf. Exemple 2 of Chapter G, p. 246): The extension $(K(\sqrt{-1})|K, v)$ is immediate. Indeed, the minimal polynomial $Y^2 + 1$ corresponds to the Artin-Schreier polynomial $X^p - X + 1/2$ which does the same job as $X^p - X - 1/2$. ♢

Let us come back to Example 8.44 to determine the approximation type and the distance of a over (K, v).

Example 8.47 Let the notation be as in Example 8.44. Since $(K(a)|K, v)$ is immediate, part d) of Lemma 7.1 shows that $\mathrm{appr}\,(a, K)$ is an immediate approximation type. Set $c_i := a_1 + \ldots + a_{i-1}$ for $i > 1$. We showed that the value of $b_{i-1} = a - c_i$ is $va_i = -(vp)/p^i$. Hence, $\mathrm{appr}\,(a, K)_{va_i} = B_{va_i}(c_i)$. If we are able to show that the values $-(vp)/p^i$ are cofinal in $\Lambda(a, K)$, then we know that $\mathrm{appr}\,(a, K)$ is uniquely determined by these balls (cf. Lemma 2.39).

Assume that there is $c \in K$ such that $v(a - c) > -(vp)/p^i$ for all i. Then $v(c - c_i) = -(vp)/p^i$ for all i. On the other hand, there is some i such that $c \in \mathbb{Q}_p(a_1, \ldots, a_{i-1}) =$

$\mathbb{Q}_p(a_{i-1})$. But this contradicts the fact that $v(c-a_1-\ldots-a_{i-1}) = v(c-c_i) = -(vp)/p^i \notin \mathbb{Q}_p(a_{i-1})$. This proves that the values $-(vp)/p^i$ are indeed cofinal in $\Lambda(a,K)$. Since vK is a subgroup of the rationals, the least upper bound of the values $-(vp)/p^i$ in vK is the element 0. Hence, $\operatorname{dist}(a,K) = 0$. It follows that $(K(a)|K,v)$ is immediate without (K,v) being dense in $(K(a),v)$. ♢

As in the equal characteristic case, we can interchange the role of radical extensions and Artin-Schreier extensions:

Example 8.48 We proceed as in Example 8.44, with the only difference that we define a_1 to be a root of the Artin-Schreier polynomial $X^p - X - 1/p$ and a_{i+1} to be a root of the Artin-Schreier polynomial $X^p - X + a_i$, and that we choose a such that $a^p = 1/p$. Note that also in this case, $a_1, \ldots, a_i \in \mathbb{Q}_p(a_{i+1})$ for every i, because $a_i = a_{i+1}^p - a_{i+1}$ for every i. By use of Lemma 7.54 and induction on i, we again deduce that $va_1 = -\frac{vp}{p}$ and that $va_i = -\frac{vp}{p^i}$ for every i. As before, we define $b_i := a - a_1 - \ldots - a_i$. Using Lemma 8.43 and bearing in mind that $a_{i+1}^p = a_{i+1} - a_i$ and $a_1^p = a_1 + 1/p$, we compute

$$\begin{aligned} 0 &= a^p - \frac{1}{p} = (b_i + a_1 + \ldots + a_i)^p - 1/p \\ &\equiv b_i^p + a_1^p + \ldots + a_i^p - 1/p = b_i^p + a_i \pmod{\mathcal{O}_{\mathbf{K}}}. \end{aligned}$$

It follows that $v(b_i^p + a_i) \geq 0 > va_i$. Consequently, $vb_i^p = va_i$, that is, $vb_i = \frac{1}{p}va_i = va_{i+1}$. As before, we set $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$. Now the same argument as in Example 8.44 shows that $(K(a)|K,v)$ is an immediate extension. For the approximation type $\operatorname{appr}(a,K)$, one can take over literally the arguments of our last example. In particular, the values $-(vt)/p^i$, $i \in \mathbb{N}$, are cofinal in $\Lambda(a,K)$ and again, we have that $\operatorname{dist}(a,K) = 0$. ♢

Our last examples can be transferred to the equal characteristic case:

Example 8.49 In our last examples, replace $\mathbb{Q}_p$ by $\mathbb{F}_p((t))$ and the element p by the element t. Everything transfers literally, except that quasi-additivity becomes additivity. We obtain:
1) There is an infinite purely inseparable extension (K,v) of $(\mathbb{F}_p((t)), v_t)$ admitting an immediate Artin-Schreier extension $(K(a)|K,v)$ such that $a^p - a = 1/t$ and $\operatorname{dist}(a,K) = 0$. Here, K is obtained from $\mathbb{F}_p((t))$ by adjoining the elements t^{1/p^i} for all $i \in \mathbb{N}$. We will show later that K is the perfect hull of $\mathbb{F}_p((t))$, cf. Corollary 8.67 and Corollary 8.68. This will show that our example is identical with the example given by S. K. Khanduja in [KH1].
2) There is an infinite separable extension (K,v) of $(\mathbb{F}_p((t)), v_t)$ admitting an immediate purely inseparable extension $(K(a)|K,v)$ of degree p such that $a^p = 1/t$ and $\operatorname{dist}(a,K) = 0$. Here, K is obtained from $\mathbb{F}_p((t))$ by adjoining the elements a_1 such that $a_1^p - a_1 = 1/t$ and a_i such that $a_{i+1}^p - a_{i+1} = -a_i$ for all $i \in \mathbb{N}$. That is, $K|\mathbb{F}_p((t))$ is an infinite tower of Artin-Schreier extensions. ♢

From Corollary 7.130 we know that if (K,v) is henselian and $(K(a)|K,v)$ is a proper separable immediate extension, then $\operatorname{dist}(a,K) < \infty$. This argument does not hold if $K(a)|K$ is purely inseparable. Hence, it is worthwile to note:

Theorem 8.50 *There exists a henselian field (K,v) admitting an immediate purely inseparable extension $(K(a)|K,v)$ of degree p such that* $\operatorname{dist}(a,K) = 0$.

This fact will lay the basis for our considerations in the next section.

We can develop Example 8.49 a bit further in order to treat complete fields.

Example 8.51 Take the immediate extensions $(K(a)|K, v)$ of Example 8.49 and consider the completion $(K, v)^c = (K^c, v)$ of (K, v). Then by Lemma 7.21, $(K^c(a), v) = (K(a).K^c, v)$ is the completion of $(K(a), v)$ for every extension of the valuation v from (K^c, v) to $K(a).K^c$. Consequently, the extension $(K^c(a)|K(a), v)$ and thus also the extension $(K^c(a)|K(a), v)$ is immediate. It follows that $(K^c(a)|K^c, v)$ is immediate. On the other hand, this extension is nontrivial. Indeed, since $\operatorname{dist}(a, K) \neq \infty$ we have that $a \notin K^c$. $\diamondsuit$

This example proves:

Theorem 8.52 *There are complete fields of rank 1 which admit immediate separable algebraic and immediate purely inseparable extensions. Consequently, not every complete field of rank 1 is spherically complete.*

As examples for immediate extensions which do not lie in the completion, we have constructed extensions generated by elements of distance 0. But the distance has to be handled with much caution. The same element can have a quite different distance if we blow up our ground field. So "distance 0" is not implicit in the construction principle that we have used.

Example 8.53 We let Γ be the lexicographic product $\mathbb{Z}\amalg\mathbb{Z}$ and replace $\mathbb{F}_p((t))$ by $\mathbb{F}_p((\Gamma))$ in Example 8.49. Let v be its canonical valuation. Let the elements $s, t \in \mathbb{F}_p((\Gamma))$ be chosen such that $vs = (0, 1)$ and $vt = (1, 0)$. Then $\mathbb{Z}vs$ is a convex subgroup of $v\mathbb{F}_p((\Gamma)) = \Gamma$. Again taking a to be a root of the Artin-Schreier polynomial $X^p - X - 1/t$, we find that everything works as before, except for the determination of the distance. As before, the values $-(vt)/p^i$, $i \in \mathbb{N}$, are cofinal in $\Lambda(a, K)$. But now, 0 is not the least value of K which is bigger than $\Lambda(a, K)$. In fact, we have that $\mathbb{Z}vs > \Lambda(a, K)$, and there is no least upper bound for $\Lambda(a, K)$. That is, $\operatorname{dist}(a, K)$ can not be an element of vK, and since there are elements of vK which are smaller than 0 bigger than $\Lambda(a, K)$, we obtain that $\operatorname{dist}(a, K) < 0$. $\diamondsuit$

For the conclusion of this section, we shall give an example which is due to F. Delon (cf. [DEL1], Exemple 1.51). It is based on Example 6.52.

Example 8.54 We consider $\mathbb{F}_p((t))$ with its valuation $v = v_t$. According to Lemma 8.33 we can choose elements $x, y \in \mathbb{F}_p((t))$ which are algebraically independent over $\mathbb{F}_p(t)$. As in Example 6.52, we define $L := \mathbb{F}_p(t, x, y)$, $s := x^p + ty^p$ and $K := \mathbb{F}_p(t, s)$. The elements t, s are algebraically independent over $\mathbb{F}_p$. Consequently, the p-degree of K is p^2. We know from Example 6.52 that K is relatively algebraically closed in L. We define F to be the relative algebraic closure of K in $\mathbb{F}_p((t))$. We claim that $F|K$ is separable. Assume that this were not true, and choose some $a \in K \setminus K^p$ such that $a^{1/p} \in F$. From Example 6.52 we know that $L(a^{1/p})$ contains $K^{1/p}$. Then in particular, it contains $t^{1/p}$. Since $vt^{1/p} \notin \mathbb{Z}vt = v\mathbb{F}_p((t))$, we find that $t^{1/p} \notin \mathbb{F}_p((t))$. But this is a contradiction since $L(a^{1/p}) \subset \mathbb{F}_p((t))$. Hence indeed, $F|K$ is separable algebraic, and from Lemma 6.32 it follows that the p-degree of F is p^2 like that of K. That is, $[F(t^{1/p}, s^{1/p}) : F] = p^2$. On the other hand, we have that $L.F(t^{1/p}, s^{1/p}) = F.L(t^{1/p}, s^{1/p}) = F.L(t^{1/p}) \subset \mathbb{F}_p((t))(t^{1/p})$.

Now $(v\mathbb{F}_p((t))(t^{1/p}) : v\mathbb{F}_p((t))) = p = [\mathbb{F}_p((t))(t^{1/p}) : \mathbb{F}_p((t))]$ and thus $\overline{\mathbb{F}_p((t))(t^{1/p})} = \overline{\mathbb{F}_p((t))}$. It follows that $e(F(t^{1/p}, s^{1/p})|F, v) = p$ and $f(F(t^{1/p}, s^{1/p})|F, v) = 1$ and thus, $d(F(t^{1/p}, s^{1/p})|F, v) = p$.

Like $(\mathbb{F}_p(t), v)$, also (F, v) is dense in $(\mathbb{F}_p((t)), v)$ and has value group $\mathbb{Z}$. Hence by Corollary 7.129, $(\mathbb{F}_p((t)), v)$ is the unique maximal immediate extension of (F, v) (up to valuation preserving isomorphism over F). If (F, v) would admit a proper immediate algebraic extension (F', v), then a maximal immediate extension of (F', v) would also be a maximal immediate extension of (F, v) and would thus be isomorphic over F to $\mathbb{F}_p((t))$. But we have chosen F to be relatively algebraically closed in $\mathbb{F}_p((t))$. This proves that (F, v) must be algebraically maximal. The defect of the extension $(F(t^{1/p}, s^{1/p})|F, v)$ comes from the fact that $(F(s^{1/p}), v)$ is not algebraically maximal. Note that the maximal immediate extension $(\mathbb{F}_p((t)), v)$ of (F, v) is not a separable extension since it is not linearly disjoint from $K^{1/p}|K$. ♢

This example proves:

Theorem 8.55 *There are algebraically maximal fields which are not inseparably defectless. Hence, "algebraically maximal" does not imply "defectless". There are algebraically maximal fields admitting a finite purely inseparable extension which is not an algebraically maximal field.*

Exercise 8.2 *Prove Lemma 8.33, replacing the condition that vK be archimedean by the condition that the cofinality of vK be ω. Try to generalize the result to other cofinalities and other cardinalities.*

Exercise 8.3 *Show that (K, v) has residue characteristic 0 if and only if every valued field extension of (K, v) is a defectless field.*

8.5 Transforming purely inseparable into separable defect extensions

We have seen in the last section that there exist proper purely inseparable immediate extensions which do not lie in the completion (cf. Theorem 8.50). We wish to show in this section that such extensions can be transformed into immediate separable extensions. Let us start with a "simple" observation.

Lemma 8.56 *Let p be any prime, (K, v) a henselian field and $(K(a)|K, v)$ an extension of degree p. Assume that* appr(a, K) *is an immediate approximation type. Then $(K(a)|K, v)$ is immediate and p is the characteristic exponent of $\overline{K}$.*

Proof: Suppose that p is not the characteristic exponent of $\overline{K}$. Then by the Lemma of Ostrowski (Lemma 8.11), the extension $(K(a)|K, v)$ is defectless. Since it is of degree p, we know that $a \notin K$. From Lemma 7.14 we infer that appr(a, K) is not immediate. This contradiction to our assumption shows that p is indeed the characteristic exponent of $\overline{K}$.

From Corollary 7.126 we know that the degree of appr(a, K) is smaller or equal to $[K(a) : K]$. But Corollary 8.12 tells us that this degree is a power of p. Suppose that it is 1. By Lemma 7.116 this means that appr(a, K) is trivial. But then $a \in K$ (cf. Lemma 2.27), which is a contradiction. Consequently, the degree of appr(a, K) is p. In view of Lemma 7.119, we conclude that $(K(a)|K, v)$ is immediate. □

Now suppose that $(K(a)|K,v)$ is an arbitrary extension such that $\operatorname{dist}(a,K) < \infty$. Let $f \in K[X]$ be the minimal polynomial of a over K. Then by the Continuity of Roots (Lemma 7.49 or Corollary 7.51), there is some value $\alpha \in vK$ with the following property: If $g \in K[X]$ such that $v(f-g) > \alpha$ with respect to the Gau valuation on $K(X)$, then g admits a root $a' \in \tilde{K}$ such that $v(a-a') > \Lambda(a,K)$. It then follows from Lemma 2.36 that $\operatorname{appr}(a,K) = \operatorname{appr}(a',K)$.

If we start with an immediate extension $(K(a)|K,v)$, then by part d) of Lemma 7.1, $\operatorname{appr}(a,K) = \operatorname{appr}(a',K)$ is an immediate approximation type. This will in general not tell too much about the extension $(K(a')|K,v)$. For instance, if we take g of higher degree than f, then $K(a')|K$ may be of higher degree than $K(a)|K$, and it can not be expected that $(K(a')|K,v)$ be immediate. But even if $[K(a'):K] = [K(a):K]$, it still lacks an argument for $(K(a')|K,v)$ to be immediate. If we know that $\deg \mathbf{A} = [K(a):K] = [K(a'):K]$, then this will follow from Lemma 7.119. So we have to make sure that $\deg \mathbf{A} = [K(a):K]$. For this, we employ the foregoing lemma. It tells us that if we start with a henselian field (K,v) and an immediate extension $(K(a)|K,v)$ of prime degree p, then we obtain that $\deg \mathbf{A} = p = [K(a):K]$. Let us treat the special case where a is purely inseparable of degree p over K, where $p > 0$ is the characteristic of K. We want to derive an Artin-Schreier extension. We choose some $c \in K$ such that $(p-1)vc > \alpha$, where α is as above. Then we let $g(X) := X^p - c^{p-1}X - a^p$. Since $f(X) := X^p - a^p$ is the minimal polynomial of a over K, $v(f-g) = vc^{p-1} > \alpha$ yields that there is a root $a' \in \tilde{K}$ of g such that $\operatorname{appr}(a,K) = \operatorname{appr}(a',K)$. Consequently, $(K(a')|K,v)$ is immediate by our above arguments (since (K,v) is assumed to be henselian, we have a unique extension of v). This extension is an Artin-Schreier extension since a'/c is a root of the Artin-Schreier polynomial $X^p - X - (a/c)^p$. (The element a' may not itself be an Artin-Schreier root. For example, if $va' > 0$, then Example 9.4 in the next chapter will show that an Artin-Schreier polynomial with root a' must split over every henselian field. If $va' = 0$, then an extension by an Artin-Schreier root a' will always be tame and in particular defectless (cf. Chapter 12.2). We have proved:

Lemma 8.57 *Let (K,v) be a henselian field of characteristic $p > 0$ and $(K(a)|K,v)$ a purely inseparable immediate extension of degree p such that $\operatorname{dist}(a,K) < \infty$. Then there exists an immediate Artin-Schreier extension $(K(a')|K,v)$ such that $\operatorname{appr}(a',K) = \operatorname{appr}(a,K)$. (But it may not be possible to choose a' as an Artin-Schreier root.)*

In the following, assume that (K,v) is a henselian Artin-Schreier closed field. Then from Example 8.38 we know that the perfect hull of (K,v) is an immediate extension. Since K admits no Artin-Schreier extensions at all, the above lemma shows that there exist no purely inseparable immediate extensions $(K(a)|K,v)$ such that $\operatorname{dist}(a,K) < \infty$. That is, every purely inseparable immediate extension of (K,v) of degree p lies in the completion of (K,v). It follows that $(K^{1/p},v)$ lies in $(K,v)^c$. Now we observe that the Frobenius sends the extension $K^{1/p^2}|K^{1/p}$ onto the extension $K^{1/p}|K$. It is valuation preserving since $va < vb \Leftrightarrow va^p < vb^p$. Thus, since $(K^{1/p},v)$ lies in $(K,v)^c$, we find that $(K^{1/p^2},v)$ lies in $(K^{1/p},v)^c = (K,v)^c$ (cf. Lemma 7.23). By induction, one shows that $(K^{1/p^m},v)$ lies in $(K,v)^c$ for every $m \in \mathbb{N}$. Consequently, $(K^{1/p^\infty},v)$ lies in $(K,v)^c$. This proves:

Theorem 8.58 *Let (K,v) be a henselian Artin-Schreier closed field. Then it is dense in its perfect hull (endowed with the unique extension of v). In particular, a separable-algebraically closed field (K,v) is dense in its algebraic closure.*

The first assertion can also be proven in the following way. We represent the extension $K^{1/p^\infty}|K$ as an (infinite) tower of purely inseparable extensions $K_{\nu+1}|K_\nu$ ($\nu < \nu_0$ where ν_0 is some ordinal). Then we only have to show that $(K_{\nu+1}, v)$ lies in $(K_\nu, v)^c$ for every $\nu < \nu_0$. Since (K_ν, v) is henselian being an algebraic extension of (K, v), it just remains to show that K_ν is Artin-Schreier closed. This is the content of the next lemma:

Lemma 8.59 *Let K be an Artin-Schreier closed field of characteristic $p > 0$. Then also every purely inseparable extension of K is Artin-Schreier closed.*

Proof: If $\operatorname{char} K = 0$ then every purely inseparable extension is trivial and there is nothing to show. So let $\operatorname{char} K = p > 0$. Assume L to be a purely inseparable extension of the Artin-Schreier closed field K. Let $a \in L$ and $b \in \tilde{L}$ be a root of $X^p - X - a$. Choose a minimal integer $m \geq 0$ such that $a^{p^m} \in K$. Then $(b^{p^m})^p - b^{p^m} = (b^p - b)^{p^m} = a^{p^m}$. Since K is Artin-Schreier closed by assumption, it follows that $b^{p^m} \in K$. The field $K(b)$ contains $a = b^p - b$ and thus, $[K(b) : K] \geq [K(a) : K] = p^m$. On the other hand, $p^m \geq [K(b) : K]$ since $b^{p^m} \in K$. Consequently, $[K(b) : K] = [K(a) : K]$, showing that $b \in K(a) \subset L$. □

At this point, let us cite a result which can also be proved by means of the methods of this section (cf. Corollary 4.20 of [KU1] or [KU2]):

Lemma 8.60 *Let (K, v) be a henselian field of characteristic $p > 0$. Then (K, v) is a separably defectless field if and only if $(K, v)^c$ is a defectless field, and vice versa.*

8.6 Inseparably defectless fields

In this section, we shall give a characterization of inseparably defectless fields. Recall that a valued field (K, v) is called **inseparably defectless** if every purely inseparable extension of (K, v) is defectless. Recall also that every purely inseparable algebraic extension admits a unique extension of the valuation, cf. Corollary 7.72. Every defectless field and in particular every trivially valued field is inseparably defectless. Note that a valued field can be **inseparably maximal**, that is, it does not admit proper immediate purely inseparable extensions, without being inseparably defectless. The field (F, v) of Example 8.54 is of this kind.

Let us observe that for an inseparably defectless field (K, v), every immediate extension is separable. Indeed, it follows from Lemma 7.15 that every immediate extension of (K, v) is linearly disjoint from the defectless extension $(K^{1/p^\infty}|K, v)$. In the literature, one can find the expression **excellent** for those fields for which all immediate extensions are separable (cf. [DEL1], Définition 1.41). But there are also other properties of certain valuation rings for which this expression is used.

By definition, (K, v) is an inseparably defectless field if and only if the extension $(K^{1/p^\infty}|K, v)$ is defectless. For this to hold, it is sufficient that the subextension $(K^{1/p}|K, v)$ is defectless:

Lemma 8.61 *(K, v) is an inseparably defectless field if and only if $(K^{1/p}|K, v)$ is a defectless extension, and this holds if and only if $(K|K^p, v)$ is a defectless extension.*

Proof: The first implication "⇒" follows from the fact that every subextension of a defectless extension is again defectless. Now assume that $(K^{1/p}|K, v)$ is a defectless extension. The Frobenius sends the extension $(K^{1/p^2}|K^{1/p}, v)$ onto the extension $(K^{1/p}|K, v)$

and is valuation preserving. Consequently, also the former extension is defectless (cf. Lemma 7.23). By induction, we find that $(K^{1/p^m}|K^{1/p^{m-1}}, v)$ is defectless for every $m \geq 1$. Hence by the transitivity of defectless extensions (Lemma 7.10), also $(K^{1/p^m}|K, v)$ is defectless. Since every finite subextension of $K^{1/p^\infty}|K$ is already contained in K^{1/p^m} for some m, it follows that $(K^{1/p^\infty}|K, v)$ is defectless.

The second equivalence is proved again by use of the Frobenius. □

For a valued field of finite p-degree, we can give several properties which are equivalent to "inseparably defectless". The following theorem is due to F. Delon [DEL1]:

Theorem 8.62 *Let K be a field of characteristic $p > 0$ and finite p-degree $[K : K^p]$. Then for the valued field (K, v), the property of being inseparably defectless is equivalent to each of the following properties:*

a) $[K : K^p] = (vK : pvK)[\overline{K} : \overline{K}^p]$, *i.e., $(K|K^p, v)$ is a defectless extension*

b) $(K^{1/p}|K, v)$ *is a defectless extension*

c) *every immediate extension of (K, v) is separable*

d) *there is a separable maximal immediate extension of (K, v).*

Proof: We have already shown at the beginning of this section that for an inseparably defectless field, every immediate extension is separable. Since every valued field admits a maximal immediate extension by Theorem 5.7, it follows that property c) implies property d).

By the foregoing lemma, (K, v) is an inseparably defectless field if and only if it has property b). Further, Properties a) and b) are equivalent since a) just says that that the extension $(K|K^p, v)$ is defectless and since the Frobenius sends the extension $(K^{1/p}|K, v)$ onto the extension $(K|K^p, v)$.

It now suffices to show that property d) implies property a). Let (L, v) be a separable maximal immediate extension of (K, v). According to Corollary 6.42, the separability implies that $[L : L^p] \geq [K : K^p]$. On the other hand, we have that $vL = vK$ and $\overline{L} = \overline{K}$. By our choice of (L, v), it is a maximal field. Theorem 7.124 thus shows that (L, v) is spherically complete, and Theorem 8.26 tells us that (L, v) is defectless. Hence, the extension $(L^{1/p}|L, v)$ is defectless, and by the foregoing lemma we conclude that also $(L|L^p, v)$ is defectless. Since $(vL : pvL) = (vK : pvK)$ and $[\overline{L} : \overline{L}^p] = [\overline{K} : \overline{K}^p]$ are finite, it follows that $[L : L^p]$ is finite and equal to $(vL : pvL)[\overline{L} : \overline{L}^p]$. Consequently, using also the fundamental inequality for $(K|K^p, v)$, we obtain that

$$[L : L^p] = (vL : pvL)[\overline{L} : \overline{L}^p] = (vK : pvK)[\overline{K} : \overline{K}^p] \leq [K : K^p] \leq [L : L^p] \ .$$

Thus, equality holds everywhere, showing that a) holds. □

From the proof, we also obtain:

Corollary 8.63 *A given maximal immediate extension of a valued field (K, v) has the same p-degree as K if and only if (K, v) is an inseparably defectless field.*

If a p-basis of the field K is at the same time a valuation basis of the extension $(K|K^p, v)$, then we will call it a **valuation p-basis**. Since by Lemma 7.11 every finite defectless extension admits a valuation basis and every valuation basis is a basis, we obtain from the foregoing theorem:

Corollary 8.64 *A valued field of finite p-degree is inseparably defectless if and only if it admits a valuation p-basis.*

The following very useful lemma was also proved by F. Delon ([DEL1], Proposition 1.44):

Lemma 8.65 *For valued fields of finite p-degree, the property of being inseparably defectless is inherited by every finite extension.*

Proof: By Lemma 6.32, the p-degree of a field does not change under finite extensions. On the other hand, if $(L|K,v)$ is finite, then also $vL|vK$ and $\overline{L}|\overline{K}$ are finite by virtue of Lemma 7.7. Hence, also the p-degree of $\overline{L}$ is equal to that of $\overline{K}$. The same can be shown for abelian groups: $(vL:pvL)=(vK:pvK)$ (the details are left to the reader). It follows that $[L:L^p]=[K:K^p]=(vK:pvK)[\overline{K}:\overline{K}^p]=(vL:pvL)[\overline{L}:\overline{L}^p]$, which by Theorem 8.62 means that (L,v) is inseparably defectless. □

Theorem 8.66 *Let (K,v) be an inseparably defectless field. Assume that the valued field extension $(F|K,v)$ admits a finite standard valuation transcendence basis $\mathcal{T}$ such that $F=K(\mathcal{T})$. Then also (F,v) and $(F,v)^h$ are inseparably defectless fields.*

Proof: In view of Theorem 8.9, it suffices to prove that (F,v) is an inseparably defectless field. We write $\mathcal{T}=\{x_i\,,\,y_j\mid i\in I\,,\,j\in J\}\subset F$ such that the values $vx_i\,,\ i\in I$, are rationally independent over vK, and that the residues $\overline{y}_i\,,\ j\in J$, are algebraically independent over $\overline{K}$. Every finite purely inseparable extension L of $K(\mathcal{T})$ is contained in an extension $E=K'(\mathcal{T}^{1/p^m})=K'(t^{1/p^m}\mid t\in\mathcal{T})$ for a suitable $m\in\mathbb{N}$ and some finite purely inseparable extension K' of K. Since $K'|K$ is algebraic, we know from Corollary 7.9 that vK'/vK is a torsion group and $\overline{K'}|\overline{K}$ is algebraic. Consequently, the values $vx_i^{1/p^m}=\frac{vx_i}{p^m}$, $i\in I$, are still rationally independent over vK', and the residues $\overline{y^{1/p^m}}_i=\overline{y}_i^{1/p^m}$, $j\in J$, are still algebraically independent over $\overline{K'}$. This proves that $\mathcal{T}^{1/p^m}$ is a standard valuation transcendence basis of $(E|K',v)$. Now Lemma 7.31 shows that

$$vE=vK'\oplus\mathbb{Z}vx_1^{1/p^m}\oplus\ldots\oplus\mathbb{Z}vx_r^{1/p^m}=vK'\oplus\mathbb{Z}\frac{vx_1}{p^m}\oplus\ldots\oplus\mathbb{Z}\frac{vx_r}{p^m}$$

and that

$$\overline{E}=\overline{K'}\left(\overline{y_1^{1/p^m}},\ldots,\overline{y_s^{1/p^m}}\right)=\overline{K'}(\overline{y_1}^{1/p^m},\ldots,\overline{y_s}^{1/p^m})\ ,$$

whence

$$\begin{aligned}[E:K(\mathcal{T})] &= [K'(\mathcal{T}^{1/p^m}):K'(\mathcal{T})]\cdot[K'(\mathcal{T}):K(\mathcal{T})]=p^{m(r+s)}\cdot[K':K]\\ &= p^{mr}\cdot p^{ms}\cdot(vK':vK)\cdot[\overline{K'}:\overline{K}]=p^{mr}\cdot(vK':vK)\cdot p^{ms}\cdot[\overline{K'}:\overline{K}]\\ &= (vE:vK(\mathcal{T}))\cdot\left[\overline{E}:\overline{K(\mathcal{T})}\right]\end{aligned}$$

since $(K'|K,v)$ is defectless by hypothesis. This equation shows that $(E|K(\mathcal{T}),v)$ and thus also its subextension $(L|K(\mathcal{T}),v)$ is defectless. □

Every valuation on the finite field $\mathbb{F}_p$ is trivial and thus defectless. Hence we can apply the foregoing theorem to $(\mathbb{F}_p(t),v_t)$ to obtain that $(\mathbb{F}_p(t),v_t)$ and $(\mathbb{F}_p(t),v_t)^h$ are inseparably defectless fields. Let us note:

Corollary 8.67 *Let (K,v) be one of the fields $(\mathbb{F}_p(t),v_t)$, $(\mathbb{F}_p(t),v_t)^h$ or $(\mathbb{F}_p((t)),v_t)$. Then (K,v) is an inseparably defectless field with p-degree p and valuation p-basis $1,t,\ldots,t^{p-1}$. In particular,*

$$\mathbb{F}_p((t))^{1/p^\infty} = \mathbb{F}_p((t))(t^{1/p^i} \mid i \in \mathbb{N}) \ .$$

Proof: It follows from the foregoing theorem that $(\mathbb{F}_p(t),v_t)$ and $(\mathbb{F}_p(t),v_t)^h$ are inseparably defectless fields. Corollary 8.27 tells us that $(\mathbb{F}_p((t)),v_t)$ is a defectless and hence also inseparably defectless field. From Corollary 8.63 we know that the p-degrees of $\mathbb{F}_p(t)$ and $\mathbb{F}_p((t))$ are equal. Since $\mathbb{F}_p$ is perfect and $\mathbb{F}_p(t)$ is a function field in one variable over $\mathbb{F}_p$, its p-degree is p. Since $\mathbb{F}_p(t)^h$ is a separable extension of $\mathbb{F}_p(t)$, Lemma 6.32 shows that its p-degree is again p. Since the values $v1, vt, \ldots, (p-1)vt$ belong to distinct cosets modulo $vK^p = pvK$, it follows that $1,t,\ldots,t^{p-1}$ is a valuation p-basis for all three fields.

By definition, $K^{1/p^\infty} = \bigcup_{i\in\mathbb{N}} K^{1/p^i}$ for every field K of characteristic p. Since the p-degree of $\mathbb{F}_p((t))$ is p, we know that $\mathbb{F}_p((t))^{1/p} = \mathbb{F}_p((t))(t^{1/p})$ and by induction, $\mathbb{F}_p((t))^{1/p^i} = \mathbb{F}_p((t))(t^{1/p^i})$. It follows that $\mathbb{F}_p((t))^{1/p^\infty} = \bigcup_{i\in\mathbb{N}} \mathbb{F}_p((t))(t^{1/p^i}) = \mathbb{F}_p((t))(t^{1/p^i} \mid i \in \mathbb{N})$. □

Now we see that the field K of part 1) of Example 8.49 is indeed equal to $\mathbb{F}_p((t))^{1/p^\infty}$. We conclude:

Corollary 8.68 *For every element $a \in \widetilde{\mathbb{F}_p(t)}$ such that $a^p - a = 1/t$, the Artin-Schreier extension of $\mathbb{F}_p((t))^{1/p^\infty}$ generated by a is immediate with* $\operatorname{dist}(a, \mathbb{F}_p((t))^{1/p^\infty}) = 0$.

We leave it to the reader to prove the assertions of the last two corollaries for the case that $\mathbb{F}_p$ is replaced by any perfect field k of characteristic p (the valuation v_t being trivial on k).

For the conclusion of this section, let us consider an example of an immediate Artin-Schreier extension which is *not* a defect extension.

Example 8.69 We consider the Artin-Schreier polynomial $X^p - X - t$ over the valued field $(\mathbb{F}_p(t),v_t)$ and over its perfect hull. Extend v_t to the algebraic closure of $\mathbb{F}_p(t)$. Let a be a root of $X^p - X - t$. Then $a \notin \mathbb{F}_p(t)^{1/p^\infty}$. To show this, we observe that $\mathbb{F}_p(t) = \mathbb{F}_p(1/t)$. So we set $s = 1/t$ and consider $\mathbb{F}_p(t)$ with the s-adic valuation. Then we can infer from the foregoing corollary that the root a of the polynomial $X^p - X - 1/s$ generates a proper immediate extension of $(\mathbb{F}_p((s))^{1/p^\infty}, v_s)$. In particular, a can not lie in $\mathbb{F}_p(s)^{1/p^\infty} = \mathbb{F}_p(t)^{1/p^\infty}$. So $X^p - X - t$ is irreducible over $\mathbb{F}_p(t)$ and $\mathbb{F}_p(t)^{1/p^\infty}$. So all roots of $X^p - X - t$ are conjugate to a over these fields. But they are all of the form $a_j = a + j$ with $j = 0, 1, \ldots, p-1$. It is impossible that $v_t a < 0$ since then $v_t(a^p - a - t) = v_t a^p < 0$ in contradiction to $a^p - a - t = 0$. Hence, $v_t a_j \geq 0$, and all elements of $\mathbb{F}_p$ appear as the residue of some root a_j. Without loss of generality, we can choose our enumeration of the roots such that $\overline{a} = 0$. Let σ_j denote the automorphism in $\operatorname{Gal}\mathbb{F}_p(t)$ which sends a to a_j. Then if $j \neq \ell$, then $v_t\sigma_j(a-j) = v_t(\sigma_j a - j) = v_t(a + j - j) = v_t a > 0 = v_t(a + \ell - j) = v_t(\sigma_\ell a - j) = v_t\sigma_\ell(a-j)$. This proves that v_t admits p distinct extensions from $\mathbb{F}_p(t)$ to $\mathbb{F}_p(t,a)$ and from $\mathbb{F}_p(t)^{1/p^\infty}$ to $\mathbb{F}_p(t)^{1/p^\infty}(a)$. That is, $(\mathbb{F}_p(t,a),v_t)$ lies in the henselization of $(\mathbb{F}_p(t),v_t)$ in $(\widetilde{\mathbb{F}_p(t)},v_t)$ and in particular, $(\mathbb{F}_p(t,a)|\mathbb{F}_p(t),v_t)$ is an immediate extension. The same holds for the extension $(\mathbb{F}_p(t)^{1/p^\infty}(a)|\mathbb{F}_p(t)^{1/p^\infty},v_t)$.

We define $b_i := a + t + t^p + \ldots + t^{p^i}$. Since $v_t a > 0$, we obtain that also $v_t b_i > 0$ for every i. We compute

$$\begin{aligned} 0 &= a^p - a - t = (b_i - t - t^p - \ldots - t^{p^i})^p - b_i + t + t^p + \ldots + t^{p^i} - t \\ &= b_i^p - b_i - t^{p^i} \ . \end{aligned}$$

Since $v_t b_i > 0$, we have that $v_t b_i^p > v_t b_i$ and consequently, $v_t b_i = v_t t^{p^i} = p^i v_t t$. Since the values $p^i v_t t$ are cofinal in $\mathbb{Z}v_t t = v_t \mathbb{F}_p(t)$, we find that appr $(a, \mathbb{F}_p(t))$ is a completion type. ♢

Trivially, every perfect field is inseparably defectless. Hence, our example shows that there are perfect and thus also inseparably defectless fields which are not henselian. See also Example 9.4 for a discussion of Artin-Schreier polynomials of the above type over henselian fields.

8.7 Valuation disjoint extensions

Let $(\Omega|K, v)$ be an extension of valued fields and $L|K$ and $F|K$ two subextensions of $\Omega|K$. We say that $(L|K, v)$ is **valuation disjoint from** $(F|K, v)$ **(in** (Ω, v)**)** if every standard valuation independent set $\mathcal{B}$ of $(L|K, v)$ is also a standard valuation independent set of $(L.F|F, v)$. Let us observe that it is possible that a standard valuation independent set $\mathcal{B}$ of $(L|K, v)$ remains valuation independent over (F, v) without remaining a standard valuation independent set of $(L.F|F, v)$:

Example 8.70 Let k be a non-perfect field of characteristic p and $c \in k \setminus k^p$. Let t be transcendental over k and consider the valued field $(k(t), v_t)$ with the t-adic valuation. Extend v_t to $\widetilde{k(t)}$ and choose $a, b \in \widetilde{k(t)}$ such that $a^p = ct$ and $b^p = t$. Define $K := k(t)$, $L := K(a)$ and $F := K(b)$. Then the standard valuation basis $\{1, a, \ldots, a^{p-1}\}$ of $(K(a)|K, v_t)$ will remain valuation independent over $(K(b), v_t)$ since $a^i/b^i = c^{i/p}$ and $\{1, c^{1/p}, \ldots, c^{p-1/p}\}$ is valuation independent over $(K(b), v_t)$. But $\{1, a, \ldots, a^{p-1}\}$ is not a standard valuation basis of $(K(a, b)|K(b), v_t)$ since the values of its elements are all in the same coset modulo $vK(b)$ without being equal.

With this example, we can observe also another phenomenon which will play a role later in this section. We have that $vK(a) = vK(b) = \mathbb{Z}\frac{vt}{p}$, hence $\overline{K(a)} = \overline{K(b)} = \overline{K}$ by the fundamental inequality. Consequently, $\overline{K(a).K(b)} = \overline{K(a, b)} = \overline{K}(\bar{c}^{1/p}) \neq \overline{K} = \overline{K(a)}.\overline{K(b)}$. To obtain a similar phenomenon for the value groups, it suffices to change the definition of a slightly, requiring now that $a^p = c + t$. Then we have that $\overline{K(a)} = \overline{K(b)} = \overline{K}(\bar{c}^{1/p})$, hence $vK(a) = vK(b) = vK$ by the fundamental inequality. Then $K(a, b) = K(t^{1/p}, c^{1/p})$ and consequently, $vK(a, b) = \mathbb{Z}\frac{vt}{p} \neq \mathbb{Z}vt = vK(a) + vK(b)$. ♢

We have chosen the strong condition using *standard* valuation independent sets since then, valuation disjoint extensions can be characterized by means of their value groups and residue fields. We need some preparation.

Let G and G' be two subgroups of some group $\mathcal{G}$, and H be a common subgroup of G and G'. The elements $\alpha_1, \ldots, \alpha_n \in G$ are said to be **H-independent** if they belong to distinct cosets modulo H. We will say that the group extension $G|H$ is **disjoint from** the group extension $G'|H$ **(in** $\mathcal{G}$**)** if for every $n \in \mathbb{N}$ and every choice of H-independent elements $\alpha_1, \ldots, \alpha_n \in G$, these elements will also be G'-independent. Contrary to the field case, this is already the case if every two H-independent elements α, α' remain G'-independent. Observe that α, α' are H-independent if and only if $\alpha - \alpha' \notin H$. So if α, α' do not remain G'-independent, then $\alpha - \alpha' \in G \cap G'$, showing that $H \neq G \cap G'$. Conversely, if there exists $\alpha \in G \cap G'$, $\alpha \notin H$, then α and 0 are H-independent, but not G'-independent. We have proved:

$G|H$ is disjoint from $G'|H$ if and only if $G \cap G' = H$.
Hence, the notion "disjoint from" is symmetrical, like "linearly disjoint from" in the field case.

Now valuation disjoint extensions can be characterized as follows:

Lemma 8.71 *Let $(\Omega|K, v)$ be an extension of valued fields and $L|K$ and $F|K$ two subextensions of $\Omega|K$. Then $(L|K, v)$ is valuation disjoint from $(F|K, v)$ in (Ω, v) if and only if*
1) $vL|vK$ is disjoint from $vF|vK$ in $v\Omega$, and
2) $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$ in $\overline{\Omega}$.
Consequently, valuation disjointness is symmetrical: if $(L|K, v)$ is valuation disjoint from $(F|K, v)$, then $(F|K, v)$ is valuation disjoint from $(L|K, v)$.

Proof: From the characterization of valuation independence given in Lemma 7.16 we infer that $(L|K, v)$ is valuation disjoint from $(F|K, v)$ if the following two conditions are satisfied:
1) if $b, b' \in L$ have values which belong to distinct cosets modulo vK, then these values also belong to distinct cosets modulo vF, and
2) if $b_1, \ldots, b_n \in \mathcal{O}_{\mathbf{L}}$ have residues which are $\overline{K}$-linearly independent, then these residues are also $\overline{F}$-linearly independent.
Each of these conditions is equivalent to the corresponding condition in the assertion of our lemma.

For the converse, assume that condition 1) or 2) is not satisfied. If 1) is not satisfied, then there are two elements $b, b' \in L$ whose values belong to distinct cosets modulo vK but to the same coset modulo vF. From the former it follows that $\{b, b'\}$ is a standard valuation independent set in $(L|K, v)$, and from the latter it follows that $\{b, b'\}$ is not a standard valuation independent set in $(L.F|F, v)$. If 2) is not satisfied, then there are elements $b_1, \ldots, b_n \in \mathcal{O}_{\mathbf{L}}$ such that their residues are $\overline{K}$-linearly independent but not $\overline{F}$-linearly independent. Now $\{b_1, \ldots, b_n\}$ is a standard valuation independent set in $(L|K, v)$, but not in $(L.F|F, v)$. This shows that $(L|K, v)$ is not valuation disjoint from $(F|K, v)$. □

The following is a partial analogue to Lemma 6.12; we leave its straightforward proof to the reader:

Lemma 8.72 *Let $(\Omega|K, v)$ be an extension of valued fields, and let $L|K$ and $F \supset E \supset K$ be subextensions of $\Omega|K$. If $(L|K, v)$ is valuation disjoint from $(F|K, v)$ in (Ω, v), then it is also valuation disjoint from $(E|K, v)$. Conversely, if $(L|K, v)$ is valuation disjoint from $(E|K, v)$ and $(L.E|E, v)$ is valuation disjoint from $(F|E, v)$, then $(L|K, v)$ is valuation disjoint from $(F|K, v)$.*

An extension $(F|K, v)$ will be called **valuation separable** if it is valuation disjoint from $(K^{1/p^\infty}|K, v)$. Note that this does not depend on the embedding of (F, v) and $(K^{1/p^\infty}, v)$ in a common valued extension field, since K^{1/p^∞} admits a unique embedding in F^{1/p^∞} and the extension of v from F to F^{1/p^∞} is unique. From Lemma 8.71 we infer that $(F|K, v)$ is valuation separable if and only if $vF \cap vK^{1/p^\infty} = vK$ and $\overline{F}|\overline{K}$ is linearly disjoint from $\overline{K^{1/p^\infty}}|\overline{K}$. But $\overline{K^{1/p^\infty}} = \overline{K}^{1/p^\infty}$, so the latter is equivalent to the condition that $\overline{F}|\overline{K}$ is separable. Similarly, vK^{1/p^∞} is the p-divisible hull of vK, so the former is equivalent to the condition that every torsion element in vF/vK has order prime to p. We have proved:

Lemma 8.73 *An extension $(F|K,v)$ is valuation separable if and only if*
1) the torsion subgroup of vF/vK is a p'-group,
2) $\overline{F}|\overline{K}$ is separable.

An extension $(F|K,v)$ will be called **valuation regular** if it is valuation disjoint from $(\tilde{K}|K,v)$ in $(\tilde{F},\tilde{v})$ for some extension $\tilde{v}$ of the valuation v from F to $\tilde{F}$. As it was done for the notion "valuation separable", one deduces the following characterization for "valuation regular", using that $v\tilde{K}$ is the divisible hull of K and that $\widetilde{\overline{K}} = \overline{\tilde{K}}$:

Lemma 8.74 *An extension $(F|K,v)$ is valuation regular if and only if*
1) vF/vK is torsionfree,
2) $\overline{F}|\overline{K}$ is regular.
Consequently, $(F|K,v)$ is valuation regular if and only if it is valuation disjoint from $(\tilde{K}|K,v)$ for every *extension $\tilde{v}$ of the valuation v from F to $\tilde{F}$.*

By this lemma, every extension of an algebraically closed valued field is valuation regular. For the scope of this book, the most important examples of valuation regular extensions are the valued field extensions which are generated by algebraically valuation independent sets. Indeed, it follows from Lemma 7.31 that they satisfy the conditions of the above lemma. So we obtain:

Lemma 8.75 *Let $(L|K,v)$ be an extension of valued fields containing a standard algebraically valuation independent set $\mathcal{T}$. Then $(K(\mathcal{T})|K,v)$ is a valuation regular extension.*

Lemma 8.76 *Assume that $(F|K,v)$ is a valuation regular subextension of a valued field extension $(\Omega|K,v)$. If $\mathcal{T}$ is a standard algebraically valuation independent set in $(\Omega|F,v)$, then also $(F(\mathcal{T})|K(\mathcal{T}),v)$ is a valuation regular extension.*

Proof: We write $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\} \subset F$ such that the values $vx_i\,,\ i \in I$, are rationally independent over vF, and that the residues $\overline{y}_i\,,\ j \in J$, are algebraically independent over $\overline{F}$. Since $(F|K,v)$ is assumed to be valuation regular we know from Lemma 8.74 that vF/vK is torsion free and that $\overline{F}|\overline{K}$ is regular. The former implies that also $vF \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$ is torsion free modulo $vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$. Similarly, the latter implies by Corollary 6.51 that also the extension $\overline{F}(\overline{y}_j \mid j \in J)\,|\,\overline{K}(\overline{y}_j \mid j \in J)$ is regular. Again by Lemma 8.74, we conclude that $(F(\mathcal{T})|K(\mathcal{T}),v)$ is a valuation regular extension. □

Valuation disjoint extensions are particularly interesting in combination with defectless extensions:

Lemma 8.77 *Let $(\Omega|K,v)$ be a valued field extension. Further, let $(L|K,v)$ be a defectless algebraic and $(F|K,v)$ an arbitrary subextension of $(\Omega|K,v)$. Assume that $(L|K,v)$ is valuation disjoint from $(F|K,v)$. Then $L|K$ is linearly disjoint from $F|K$, the extension of v from F to $L.F$ is unique and $(L.F|F,v)$ is defectless. Furthermore,*

$$v(L.F) = vL + vF \quad \textit{and} \quad \overline{L.F} = \overline{L}.\overline{F}\,. \tag{8.9}$$

Proof: We prove our assertion first under the assumption that $L|K$ is finite. The main point of the proof is to determine $v(L.F)$ and $\overline{L.F}$. Certainly, $v(L.F)$ contains both vL and vF, and $\overline{L.F}$ contains both $\overline{L}$ and $\overline{F}$. Since $(L|K,v)$ is assumed to be valuation

disjoint from $(F|K,v)$, we have that $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$ and that every set of vK-independent elements of vL is also vF-independent. It follows that

$$(v(L.F) : vF) \geq (vL + vF : vF) = (vL : vK)$$
$$[\overline{L.F} : \overline{F}] \geq [\overline{L}.\overline{F} : \overline{F}] = [\overline{L} : \overline{K}] \ .$$

By assumption, $(L|K,v)$ is defectless, i.e. $[L:K] = [\overline{L} : \overline{K}] \cdot (vL : vK)$. Consequently,

$$\begin{aligned} [L.F : F] &\geq (v(L.F) : vF) \cdot [\overline{L.F} : \overline{F}] \\ &\geq (vL : vK) \cdot [\overline{L} : \overline{K}] = [L:K] \geq [L.F : F] \ , \end{aligned}$$

showing that in all of these inequalities, "=" holds everywhere. The equality $[L.F : F] = [L : K]$ shows that $L|K$ is linearly disjoint from $F|K$. The equality $[L.F : F] = (v(L.F) : vF) \cdot [\overline{L.F} : \overline{F}]$ shows that the extension $(L.F|F,v)$ is defectless. It follows from the fundamental inequality (8.5) (cf. Theorem 8.4) that there exists only one extension of the valuation v from F to $L.F$. Further, we find that (8.9) holds.

Now assume that $L|K$ is infinite. By what we have already shown, it follows that every finite subextension is linearly disjoint from $F|K$, hence $L|K$ is itself linearly disjoint from $F|K$. Since every finite subextension of $L.F|F$ is contained in some subextension of the form $L'.F|F$ where $L'|K$ is a finite subextension of $L|K$, we also obtain that $(L.F|F,v)$ is defectless. Finally, for every element $\alpha \in v(L.F)$ there is already a finite extension $L'|K$ such that $\alpha \in v(L'.F) = vL' + vF$, showing that $\alpha \in vL + vF$ and consequently, $v(L.F) = vL + vF$. A similar argument works for the residue fields. □

The condition that $(L|K,v)$ be defectless is indispensable:

Example 8.78 In Example 8.49 2), replace $\mathbb{F}_p$ by a field k of characteristic $p > 0$ which is not perfect. Everything works the same, and we obtain an immediate purely inseparable extension $(K(a)|K,v)$ of degree p such that $a^p = 1/t$ and $\mathrm{dist}\,(a,K) = 0$. Now let $c \in k \setminus k^p$ and b be such that $b^p = c + 1/t$. Then the extension $(K(b)|K,v_t)$ is immediate like $(K(a)|K,v_t)$ (observe that a and b have the same approximation type over (K,v_t) by virtue of Lemma 2.36 since $v_t(a-b) = 0 \geq \Lambda(a,K)$). Consequently, $(K(a)|K,v_t)$ is valuation disjoint from $(K(b)|K,v_t)$ in $(K(a,b),v_t)$. But $\overline{K(a).K(b)} = \overline{K(a,b)} = k(c^{1/p}) \neq k = \overline{K(a)}.\overline{K(b)}$. There are similar but slightly more complicated examples which show the same phenomenon at the value groups. ♢

Since every finite extension $(L|K,v)$ is defectless if (K,v) is a henselian defectless field, we obtain directly from the foregoing lemma:

Corollary 8.79 *Let (K,v) be a henselian defectless field and $(F|K,v)$ a valuation regular extension. Then there exists a unique extension of v from F to $\tilde{K}.F$. With this extension, $(\tilde{K}.F|F,v)$ is defectless, and for every algebraic extension $L|K$ we have that $v(L.F) = vL + vF$ and $\overline{L.F} = \overline{L}.\overline{F}$.*

Assume that $(F|K,v)$ is valuation separable. If a given finite purely inseparable extension $(L|K,v)$ is defectless, then $L|K$ is linearly disjoint from $F|K$ by the foregoing lemma. Hence if (K,v) is inseparably defectless, then $F|K$ is linearly disjoint from every purely inseparable extension of K, that is, $F|K$ is separable.

Now assume that $(F|K,v)$ is valuation regular. If a given finite extension $(L|K,v)$ is defectless, then again, $L|K$ is linearly disjoint from $F|K$ by the foregoing lemma. If (K,v) is henselian defectless, then every finite extension $(L|K,v)$ is defectless, which shows that $F|K$ is linearly disjoint from every finite extension of K, i.e., $F|K$ is regular. Let us note:

Corollary 8.80 *Every valuation separable extension of an inseparably defectless field is separable, and every valuation regular extension of a henselian defectless field is regular.*

In view of Lemma 8.73 and Lemma 8.74, we find:

Corollary 8.81 *Every immediate extension is valuation regular. Consequently, every immediate extension of an inseparably defectless field is separable, and every immediate extension of a henselian defectless field is regular.*

As a further corollary to Lemma 8.77, we obtain the assertion which is lacking in Lemma 8.72, if compared to Lemma 6.12.

Lemma 8.82 *Let $(\Omega|K,v)$ be an extension of valued fields and $L|K$ and $E \supset F \supset K$ subextensions of $\Omega|K$. If $(L|K,v)$ is defectless and valuation disjoint from $(F|K,v)$ in (Ω,v), then $(L.E|E,v)$ is valuation disjoint from $(F|E,v)$.*

Proof: Since $(L|K,v)$ is assumed to be valuation disjoint from $(F|K,v)$, it is also valuation disjoint from $(F|K,v)$ by virtue of Lemma 8.72. Since $(L|K,v)$ is assumed to be defectless, Lemma 8.77 thus shows that $v(L.E) = vL + vE$ and $\overline{L.E} = \overline{L}.\overline{E}$. By assumption, we have that $vL \cap vF = vK$, hence $v(L.E) \cap vF = (vL + vE) \cap vF = vE$. Also by assumption, $\overline{L}$ and $\overline{F}$ are $\overline{K}$-linearly, hence by Lemma 6.12, $\overline{L.E} = \overline{L}.\overline{E}$ and $\overline{F}$ are $\overline{E}$-linearly disjoint. This proves that $(L.E|E,v)$ is valuation disjoint from $(F|E,v)$. □

One of the assertions of Lemma 8.77 can be generalized in the following way:

Lemma 8.83 *Let $(F|K,v)$ be an arbitrary valued field extension, and extend v from F to $\tilde{F}$. Further, let $(L|K,v)$ be an h-finite subextension of $(\tilde{F}|K,v)$. If $(L|K,v)$ is valuation disjoint from $(F|K,v)$ in $(\tilde{F},v)$, then $d_h(L|K,v) \geq d_h(L.F|F,v)$.*

Proof: Since $vL|vK$ is disjoint from $vF|vK$ and $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$, we find that

$$\begin{aligned} (v(L.F) : vF) &\geq (vL + vF \,:\, vF) = (vL : vK) \\ [\overline{L.F} : \overline{F}] &\geq [\overline{L}.\overline{F} : \overline{F}] = [\overline{L} : \overline{K}] \,. \end{aligned}$$

Now our assertion follows from the definition of the henselian defect and the fact that $[(L.F)^h : F^h] = [L^h.F^h : K^h.F^h] \leq [L^h : K^h]$. □

Corollary 8.84 *Let (K,v) be a defectless field and $(F|K,v)$ a valuation regular extension. Then (F,v) is defectless in $\tilde{K}.F$. If $(\tilde{K}.F,v)$ is a defectless field, then also (F,v) is a defectless field.*

Proof: Let $L|K$ be a normal extension. Since (K,v) is assumed to be a defectless field, we know that $d_h(L|K,v) = 1$. Since $(F|K,v)$ is assumed to be a valuation regular extension, it follows from the foregoing lemma that $d_h(L.F|F,v) = 1$. Since every normal subextension of $\tilde{K}.F|F$ is contained in an extension $L.F|F$ for $L|K$ a normal extension, it follows by Lemma 8.14 that (F,v) is defectless in $\tilde{K}.F$. If in addition $(\tilde{K}.F,v)$ is a defectless field, then it follows from the transitivity of defectless extensions (Lemma 8.16) that (F,v) is a defectless field. □

For an important special case, we can improve Lemma 8.83 so as to obtain equality of the defects.

Lemma 8.85 *Let $(F'|K,v)$ be a normal extension and $K \subset F \subset (F'|K,v)^r$. Let $(L|K,v)$ be an arbitrary finite subextension of $(F'|K,v)$. Then*

$$d_h(L|K,v) = d_h(L.F|F,v) \; .$$

Proof: Let $L' \subset F'$ be the normal hull of $L|K$, and let $\mathrm{V} := (F'|K,v)^r$ and $\mathrm{V}' := (L'|K,v)^r$. By Lemma 7.95, $\mathrm{V}' = \mathrm{V} \cap L'$. Hence, $L'|\mathrm{V}'$ is linearly disjoint from $\mathrm{V}|\mathrm{V}'$. By Lemma 7.82, $(L'|L,v)^r = L.\mathrm{V}'$. Since $d_h(L.\mathrm{V}'|L,v) = 1$ by virtue of Lemma 8.20, we find that $d_h(L.\mathrm{V}'|K,v) = d_h(L|K,v)$. By assumption, $F \subset \mathrm{V}$, showing that $E := F.\mathrm{V}' \subset \mathrm{V}$. Hence by Lemma 8.20, $d_h(E|F,v) = 1$. On the other hand, Lemma 7.82 shows that $(F'|L.F,v)^r = L.\mathrm{V} \supset L.E$. Hence again by Lemma 8.20, $d_h(L.E|L.F,v) = 1$. We find that

$$\begin{aligned} d_h(L.F|F,v) &= d_h(L.E|L.F,v) \cdot d_h(L.F|F,v) = d_h(L.E|F,v) \\ &= d_h(L.E|E,v) \cdot d_h(E|F,v) = d_h(L.E|E,v) \; . \end{aligned}$$

We see that it suffices to prove our lemma with L replaced by $L.\mathrm{V}$ and F replaced by $F.\mathrm{V}'$ and K replaced by V'. After this replacement, we can assume that $(L'|L,v)$ is a finite normal extension with ramification field K and that $L'|K$ is linearly disjoint from $\mathrm{V}|K$. The latter implies that $[L.F : F] = [L : K]$, and it follows from Lemma 6.12 that $L'.F|F$ is linearly disjoint from $\mathrm{V}|F$, that is, $L'.F \cap \mathrm{V} = F$. Hence F is equal to $(L'.F|F,v)^r$ by virtue of Lemma 7.95. Since $L.F \subset L'.F$, we can infer from Theorem 7.91 that $v(L.F)/vF$ is a p-group (where p is the characteristic exponent of $\overline{K}$), and that $\overline{L.F}|\overline{F}$ is purely inseparable. On the other hand, Lemma 7.82 shows that $(F'|L,v)^r = L.\mathrm{V}$. Since $L.F \subset L.\mathrm{V}$, we can thus conclude from Theorem 7.94 and Theorem 7.88 that $v(L.F)/vL$ is a p'-group and that $\overline{L.F}|\overline{L}$ is separable. This yields that $v(L.F)/(vL + vF)$ is at the same time a p-group and a p'-group, i.e., $v(L.F) = (vL + vF)$. Similarly, we obtain that $\overline{L.F}|\overline{L}.\overline{F}$ is at the same time a purely inseparable and a separable extension, i.e., $\overline{L.F} = \overline{L}.\overline{F}$.

Furthermore, Theorem 7.91 shows that vL/vK is a p-group and that $\overline{L}|\overline{K}$ is purely inseparable. On the other hand, Theorem 7.94 and Theorem 7.88 show that vF/vK is a p'-group and $\overline{F}|\overline{K}$ is separable. Consequently, vL/vK is disjoint from vF/vK and $(vL+vF : vF) = (vL : vK)$. Similarly, $\overline{L}|\overline{K}$ is linearly disjoint from $\overline{F}|\overline{K}$ and $[\overline{L}.\overline{F} : \overline{F}] = [\overline{L} : \overline{K}]$.

Since the ramification field of $(L'|K,v)$ is K, Theorem 7.84 shows that the extension of v from K to L' and thus also to L is unique. The same holds for the extension of v from F to $L.F$ because F is ramification field of $(L'.F|F,v)$. In view of Lemma 8.17, we can now compute:

$$\begin{aligned} d_h(L|K,v) &= \frac{[L:K]}{(vL:vK)[\overline{L}:\overline{K}]} = \frac{[L.F:F]}{(vL+vF:vF)[\overline{L}.\overline{F}:\overline{F}]} \\ &= \frac{[L.F:F]}{(v(L.F):vF)[\overline{L.F}:\overline{F}]} = d_h(L.F|F,v) \; . \end{aligned}$$

□

From this lemma, we derive:

Theorem 8.86 *Let $(F'|K,v)$ be a normal extension and $K \subset F \subset (F'|K,v)^r$. Then (F,v) is a defectless field if and only if (K,v) is.*

Proof: From Lemma 7.95 we know that $(F'|K,v)^r = (\tilde{K}|K,v)^r \cap F'$. Hence we can assume from the start that $F' = \tilde{K}$. Suppose that (F,v) is a defectless field, and let $(L|K,v)$ be an arbitrary finite extension. Then by the foregoing lemma, $d_h(L|K,v) = d_h(L.F|F,v) = 1$. This shows that (K,v) is a defectless field.

For the converse, suppose that (K,v) is a defectless field, and let $(E|F,v)$ be a finite extension. Then there is a finite extension $(L|K,v)$ such that $E \subset L.F$. We have that $d_h(L.F|F,v) = d_h(L|K,v) = 1$. By the multiplicativity of the defect, it follows that also $d_h(E|F,v) = 1$. □

Lemma 8.87 *Let $(K'|K,v)$ be a valuation regular extension. Further, let $(L'|K',v)$ and $(L|K,v)$ be normal extensions such that $L \subset L'$. Then $(L \cap K' \,|\, K,v)$ and also the extensions $(L \cap (L'|K',v)^d, v) \supset (L \,|\, L \cap K', v)^d$ and $(L \cap (L'|K',v)^i, v) \supset (L \,|\, L \cap K', v)^i$ and $(L \cap (L'|K',v)^r, v) \supset (L \,|\, L \cap K', v)^r$ are immediate. If (K,v) is a defectless field, then equalities hold:*

$$\begin{array}{l} \mathrm{res}_L(G^d(L'|K',v)) = G^d(L \,|\, L \cap K', v) \quad \textit{and} \quad (L \cap (L'|K',v)^d, v) = (L \,|\, L \cap K', v)^d \\ \mathrm{res}_L(G^i(L'|K',v)) = G^i(L \,|\, L \cap K', v) \quad \textit{and} \quad (L \cap (L'|K',v)^i, v) = (L \,|\, L \cap K', v)^i \\ \mathrm{res}_L(G^r(L'|K',v)) = G^r(L \,|\, L \cap K', v) \quad \textit{and} \quad (L \cap (L'|K',v)^r, v) = (L \,|\, L \cap K', v)^r \ . \end{array}$$

Proof: The field $L_1 := L \cap (L'|K',v)^d$ is contained in the field $(L'|K',v)^d$ which by Lemma 7.87 is an immediate extension of (K',v). Hence, $vL_1 \subset vK'$ and $\overline{L}_1 \subset \overline{K'}$. But vL_1/vK is a torsion group and $\overline{L}_1|\overline{K}$ is algebraic. Since $(K'|K,v)$ is assumed to be valuation regular, this implies that $vL_1 = vK$ and $\overline{L}_1 = \overline{K}$, which yields our first assertion for the decomposition fields and that $(L \cap K' \,|\, K,v)$ is immediate.

The field $L_2 := L \cap (L'|K',v)^i$ is contained in the field $L_2' := (L'|K',v)^i$. By Theorem 7.88, the latter has the same value group as (K',v), so as before it follows that $vL_2 = vK$. Since $(K'|K,v)$ is valuation regular, $\overline{K'}|\overline{K}$ is separable. By Theorem 7.88, also $\overline{L_2'}|\overline{K'}$ is separable. Hence, $\overline{L}_2|\overline{K}$ is an algebraic subextension of the separable extension $\overline{L_2'}|\overline{K}$. Since by Theorem 7.88, $\overline{L}_2$ is a purely inseparable extension of the residue field of $(L \,|\, L \cap K', v)^i$, it must be equal to the latter. We have proved that $(L \cap (L'|K',v)^i, v) \supset (L \,|\, L \cap K', v)^i$ is immediate.

The field $L_3 := L \cap (L'|K',v)^r$ is contained in the field $L_3' := (L'|K',v)^r$. The equality of the residue fields is shown as in the case of the inertia fields. Since $(K'|K,v)$ is valuation regular, vK'/vK is torsion free. By Theorem 7.94, vL_3'/vK' is a p'-group. Hence also the torsion subgroup of vL_3'/vK is a p'-group. It follows that its subgroup vL_3/vK is a p'-group. But by Theorem 7.91, $vL_3/v(L \,|\, L \cap K', v)^r$ is a p-group, which consequently must be trivial. We have proved that $(L \cap (L'|K',v)^r, v) \supset (L \,|\, L \cap K', v)^r$ is immediate.

Now assume that (K,v) is a defectless field. Then by Theorem 8.86, also $(L \,|\, L \cap K', v)^d$, $(L \,|\, L \cap K', v)^i$ and $(L \,|\, L \cap K', v)^r$ are defectless fields. On the other hand, the extensions $(L \cap (L'|K',v)^d, v) \supset (L \,|\, L \cap K', v)^d$ and $(L \cap (L'|K',v)^i, v) \supset (L \,|\, L \cap K', v)^i$ and also $(L \cap (L'|K',v)^r, v) \supset (L \,|\, L \cap K', v)^r$ admit a unique extension of the valuation since they all lie between L and $(L \,|\, L \cap K', v)^d$ (cf. Theorem 7.84). Since they are immediate, it follows by Corollary 8.18 that they must be trivial. This proves our last assertion.

Finally, let us show that $(L \,|\, L \cap K', v)^d = (L|K,v)^d$. By Lemma 7.82, $(L \,|\, L \cap K', v)^d \supset (L|K,v)^d$. By what we have shown already, it is immediate. On the other hand, it admits a unique extension of the valuation. Hence, equality must hold. □

With $L = K^{\mathrm{sep}}$ and $L' = K'^{\mathrm{sep}}$, we obtain from this lemma:

Theorem 8.88 *Let (K,v) be a defectless field and $(K'|K,v)$ a valuation regular extension. Then the relative algebraic closure of (K,v) in $(K',v)^h$ is precisely $(K,v)^h$.*

To illustrate the use of these results, we apply them to extensions generated by not necessarily finite standard algebraically valuation independent sets.

Lemma 8.89 *Let $(\Omega|K,v)$ be an arbitrary valued field extension and $\mathcal{T}$ a (not necessarily finite) standard algebraically valuation independent set in this extension. Suppose that $(L|K,v)$ is a finite subextension of $(\Omega|K,v)$ such that the extension of v from K to L is unique. Then*

$$d_h(L(\mathcal{T})|K(\mathcal{T}),v) = d_h(L|K,v)\ . \tag{8.10}$$

Proof: By Theorem 7.31 we see that also the extension of v from $K(\mathcal{T})$ to $L(\mathcal{T})$ is unique (since the valuation on $L(\mathcal{T})$ is uniquely determined by the valuation on L and the values of the elements of $\mathcal{T}$). From that theorem, we also deduce that $(vL(\mathcal{T}) : vK(\mathcal{T})) = (vL : vK)$ and $[\overline{L(\mathcal{T})} : \overline{K(\mathcal{T})}] = [\overline{L} : \overline{K}]$. Since the elements of $\mathcal{T}$ are algebraically independent over K, the extension $K(\mathcal{T})|K$ is regular and thus, $[L(\mathcal{T}) : K(\mathcal{T})] = [L : K]$. In view of Lemma 8.17, we can thus compute:

$$d_h(L(\mathcal{T})|K(\mathcal{T}),v) = \frac{[L(\mathcal{T}) : K(\mathcal{T})]}{(vL(\mathcal{T}) : vK(\mathcal{T}))\cdot[\overline{L(\mathcal{T})} : \overline{K(\mathcal{T})}]} = \frac{[L:K]}{(vL:vK)\cdot[\overline{L}:\overline{K}]} = d_h(L|K,v)\ .$$

□

By use of this lemma, we can improve Lemma 8.87 for this special situation:

Lemma 8.90 *Let $(L'|K',v)$ and $(L|K,v)$ be normal extensions such that $L \subset L'$. Further, suppose that $\mathcal{T}$ is a (not necessarily finite) standard algebraically valuation independent set in the extension $(K'|K,v)$ such that $K' = K(\mathcal{T})$. Then*

$$\begin{array}{lll} \mathrm{res}_L(G^d(L'|K(\mathcal{T}),v)) = G^d(L|K,v) & \text{and} & (L\cap(L'|K(\mathcal{T}),v)^d,v) = (L|K,v)^d \\ \mathrm{res}_L(G^i(L'|K(\mathcal{T}),v)) = G^i(L|K,v) & \text{and} & (L\cap(L'|K(\mathcal{T}),v)^i,v) = (L|K,v)^i \\ \mathrm{res}_L(G^r(L'|K(\mathcal{T}),v)) = G^r(L|K,v) & \text{and} & (L\cap(L'|K(\mathcal{T}),v)^r,v) = (L|K,v)^r\ . \end{array}$$

Proof: First note that $L \cap K(\mathcal{T}) = K$ since $K(\mathcal{T})|K$ is regular (see the proof of the last lemma). Now if one of the asserted equalities would not hold, then one of the immediate extension described in Lemma 8.87 would be nontrivial and would thus contain a finite nontrivial immediate subextension, which we will call $(E_2|E_1,v)$. Since all these extensions lie between L and $(L|K,v)^d$, they admit unique extensions of the valuation v. Thus, $d_h(E_2|E_1,v) > 1$ and by the last lemma, also $d_h(E_2(\mathcal{T})|E_1(\mathcal{T}),v) > 1$. But $(E_2(\mathcal{T})|E_1(\mathcal{T})$ is a subextension of $(L'|K(\mathcal{T}),v)^r$, and Lemma 8.20 shows that it must be h-defectless. This contradiction shows that all asserted equalities hold. □

Corollary 8.91 *Let $(K(\mathcal{T})|K,v)$ be as before. Then the relative algebraic closure of (K,v) in $(K(\mathcal{T}),v)^h$ is precisely $(K,v)^h$.*

We have already seen that $(K(\mathcal{T})|K,v)$ is valuation regular (Lemma 8.75) and that $K(\mathcal{T})|K$ is regular (see the last proofs). Since $vK^h = vK$, $vK(\mathcal{T})^h = vK(\mathcal{T})$, $\overline{K^h} = \overline{K}$ and $\overline{K(\mathcal{T})^h} = \overline{K(\mathcal{T})}$, we immediately see from Lemma 8.74 that also the extension $(K(\mathcal{T})^h|K^h,v)$ is valuation regular. The preceding lemma shows that K^h is relatively algebraically closed in $K(\mathcal{T})^h$. On the other hand, $K^h(\mathcal{T})|K^h$ is regular and $K(\mathcal{T})^h|K^h(\mathcal{T})$ is separable algebraic, hence $K(\mathcal{T})^h|K^h$ is separable. By Lemma 6.48 we find that $K(\mathcal{T})^h|K^h$ is regular. Let us summarize:

Corollary 8.92 *Let $(K(\mathcal{T})|K,v)$ be as before. Then the extensions $(K(\mathcal{T})|K,v)$ and $(K(\mathcal{T})^h|K^h,v)$ are valuation regular and regular.*

Let us give a criterion for $(K(\mathcal{T}),v)$ to be a defectless field, which is of interest if $\mathcal{T}$ is infinite.

Lemma 8.93 *Let $(\Omega|K,v)$ be an arbitrary valued field extension and $\mathcal{T}$ a (not necessarily finite) standard algebraically valuation independent set in this extension. If for every $n \in \mathbb{N}$ and every choice of elements $t_1,\ldots,t_n \in \mathcal{T}$ the field $(K(\{t_1,\ldots,t_n\}),v)$ is defectless, then also $(K(\mathcal{T}),v)$ is defectless. The same holds with "separably defectless" or "inseparably defectless" in the place of "defectless".*

Proof: Let $L|K(\mathcal{T})$ be a finite extension. Then there are elements $t_1,\ldots,t_n \in \mathcal{T}$ and a finite extension $L'|K(t_1,\ldots,t_n)$ such that $L = L'.K(\mathcal{T})$ and that $L'|K(t_1,\ldots,t_n)$ is separable resp. purely inseparable if $L|K(\mathcal{T})$ is. Observe that $\mathcal{T}' := \mathcal{T} \setminus \{t_1,\ldots,t_n\}$ is a standard valuation transcendence basis of $(K(\mathcal{T})|K(t_1,\ldots,t_n),v)$. Hence by Lemma 8.75, this extension is valuation regular. From Lemma 8.83 we infer that $(L|K(\mathcal{T}),v)$ is defectless if $(L'|K(t_1,\ldots,t_n),v)$ is. This implies our assertion. □

We conclude that Theorem 8.66 also holds for infinite standard valuation transcendence bases $\mathcal{T}$.

Corollary 8.94 *Let (K,v) be an inseparably defectless field. Assume that the valued field extension $(F|K,v)$ admits a (not necessarily finite) standard valuation transcendence basis $\mathcal{T}$ such that $F = K(\mathcal{T})$. Then also (F,v) and $(F,v)^h$ are inseparably defectless fields.*

Chapter 9

Hensel's Lemma

9.1 Equivalent forms of Hensel's Lemma

For a valued field $\mathbf{K} = (K, v)$, the property of being henselian is equivalent to a variety of criteria for polynomials $f \in \mathcal{O}_{\mathbf{K}}[X]$ to admit a zero in $\mathcal{O}_{\mathbf{K}}$. Recall that Lemma 7.45 has shown that the residue map induces a bijection between the integral roots of f in $\tilde{K}$ and the roots of $\overline{f}$ in $\overline{\tilde{K}}$ (counted with multiplicity). In view of this, the question arises under which conditions a root of $\overline{f}$ lying in $\overline{K}$ is the residue of some root of f which lies in K. If $b \in K$ is such that $\overline{b}$ is a root of $\overline{f}$, then b could be thought of as an **approximative root**. Hensel's original lemma showed how an approximative can be **refined to a root of** f. This refinement procedure works in every spherically complete field (see Theorem 7.112 and Exercise 9.3). However, Hensel's Lemma turned out to be one of a lot of properties of a valued field which are equivalent to the property of being henselian, the proof not anymore involving a refinement procedure. Note that the condition $f \in \mathcal{O}_{\mathbf{K}}[X]$ implies that also the derivative f' and all higher derivatives of f are polynomials in $\mathcal{O}_{\mathbf{K}}[X]$ (cf. Lemma 6.60). Since the residue map is a ring homomorphism, we have $\overline{f'} = \overline{f}\,'$ (and the same holds for the higher derivatives).

Theorem 9.1 *For a valued field $\mathbf{K} = (K, v)$, the property of being henselian is equivalent to each of the following properties:*

1) *If $f = 1 + X + X^2 g(X)$ with $g(X) \in \mathcal{M}_{\mathbf{K}}[X]$, then f admits a root in $\mathcal{O}_{\mathbf{K}}$ whose residue is -1.*

2) *Every monic polynomial $f = X^n + c_{n-1}X^{n-1} + \ldots + c_1 X + c_0 \in \mathcal{O}_{\mathbf{K}}[X]$ with $\overline{c}_{n-1} \neq 0$ and $\overline{c}_{n-2} = \ldots = \overline{c}_0 = 0$ admits a linear factor $X + c$ in $\mathcal{O}_{\mathbf{K}}[X]$ such that $\overline{c} = \overline{c}_{n-1}$.*

3) *For every monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ the following holds: if $\overline{f}$ has a simple root $\overline{b} \in \overline{K}$, then f admits a root $a \in \mathcal{O}_{\mathbf{K}}$ such that $\overline{a} = \overline{b}$.*

4) ("Hensel's Lemma")
For every monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ the following holds: if $b \in \mathcal{O}_{\mathbf{K}}$ satisfies $vf(b) > 0$ and $vf'(b) = 0$, then f admits a root $a \in \mathcal{O}_{\mathbf{K}}$ such that $\overline{a} = \overline{b}$.

5) ("Newtons Lemma")
For every monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ the following holds: if $b \in \mathcal{O}_{\mathbf{K}}$ satisfies $vf(b) > 2vf'(b)$, then f admits a root $a \in \mathcal{O}_{\mathbf{K}}$ such that $v(a - b) > vf'(b)$.

6) (strong "Hensel's Lemma")

For arbitrary polynomials $f \in \mathcal{O}_K[X]$ the following holds: if there is a factorization $\overline{f} = \overline{g}\overline{h}$ such that $\overline{g}$ is relatively prime to $\overline{h}$ in the polynomial ring $\overline{K}[X]$, then there exist polynomials $g, h \in \mathcal{O}_K[X]$ reducing to $\overline{g}, \overline{h}$ respectively and such that $f = gh$ and $\deg g = \deg \overline{g}$.

7) ("Krasners Lemma")
Assume v to be extended to the separable closure K^{sep} of K. Then for every element $a \in K^{\text{sep}}$ the following holds: if $b \in K^{\text{sep}} \setminus K$ satisfies

$$v(a-b) > \max\{v(a-\sigma a) \mid \sigma \in \operatorname{Gal} K \wedge \sigma a \neq a\} ,$$

then $a \in K(b)$.

3′), 4′), 5′) *The same as 3), 4), 5) respectively, but without the hypothesis that f be monic.*

2″), 3″), 4″), 5″) *The same as 2), 3), 4), 5) respectively, but with the additional hypothesis that f be separable over K.*

Proof: During this proof, let $f = c_n X^n + c_{n-1}X^{n-1} + \ldots + c_0$ always be a polynomial in $K[X]$ with roots $a_1, \ldots, a_n \in \tilde{K}$, so that

$$f = c_n \prod_{i=1}^{n} (X - a_i) .$$

• We show that 6) holds in every henselian field (K, v). This field admits a unique extension of v to $\tilde{K}$ which we will again denote by v. Hence, $v\sigma = v$ for all $\sigma \in \operatorname{Gal} K$. Now let f, $\overline{g}$ and $\overline{h}$ satisfy the hypothesis of 6). Observe that the condition that $\overline{g}$ be prime to $\overline{h}$ implies that $\overline{g} \neq 0 \neq \overline{h}$, so $\overline{f} = \overline{g}\overline{h} \neq 0$. We define

$$\tilde{g} := \prod_{i \in J} (X - a_i) \in \tilde{K}[X] \quad \text{with} \quad J = \{i \mid 1 \leq i \leq n \wedge va_i \geq 0 \wedge \overline{g}(\overline{a_i}) = 0\} .$$

Since (K, v) is assumed to be henselian, from (7.15) we know that every $\sigma \in \operatorname{Gal} K$ induces an automorphism $\overline{\sigma} \in \overline{K}$ (the reduction of σ) via $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. For all $\sigma \in \operatorname{Gal} K$ and all $i \in J$ we have that $v\sigma a_i = va_i \geq 0$ and that $\overline{\sigma}\,\overline{a_i}$ is again a root of $\overline{g}$, which yields that σa_i is again a root of $\tilde{g}$. So the set of roots of $\tilde{g}$ is closed under the application of all $\sigma \in \operatorname{Gal} K$. Since $\overline{g}$ is assumed to be prime to $\overline{h}$, our construction yields that $\tilde{g}$ and $\tilde{h} = c_n \prod_{i \notin J}(X - a_i) \in \tilde{K}[X]$ have no common root in $\tilde{K}$. Since $f = \tilde{g}\tilde{h} \in K[X]$, part b) of Lemma 6.1 now shows that $\tilde{g}, \tilde{h} \in K[X]$.

We know from Lemma 7.45 that for every root $\overline{a}$ of $\overline{g}$ of multiplicity m, there are at least m roots (counted with multiplicity) of f whose residue is equal to $\overline{a}$. From our choice of $\tilde{g}$ it thus follows that $\overline{g}$ divides $\overline{\tilde{g}} = \prod_{i \in J}(X - \overline{a_i})$. On the other hand, $f = \tilde{g}\tilde{h}$ yields that $\overline{\tilde{g}}$ divides $\overline{f} = \overline{g}\overline{h}$. By assumption, $\overline{g}$ and $\overline{h}$ are prime to each other, so by part a) of Lemma 6.1 they have no common root in $\widetilde{\overline{K}}$. Since by our choice of $\tilde{g}$, every root of $\overline{\tilde{g}}$ is also a root of $\overline{g}$, this shows that $\overline{\tilde{g}}$ must divide $\overline{g}$. This proves that $\deg \tilde{g} = \deg \overline{g}$ and that there is $c \in \mathcal{O}_{\mathbf{K}}^{\times}$ such that $g := c\tilde{g} \in \mathcal{O}_{\mathbf{K}}[X]$ reduces to $\overline{g}$. We find that $f = gh$ with $h = c^{-1}\tilde{h} \in \mathcal{O}_{\mathbf{K}}[X]$ by virtue of Gau' Lemma.

• If 6) holds, then also 1) holds: If $f = 1 + X + X^2 g(X)$ with $g(X) \in \mathcal{M}_{\mathbf{K}}[X]$, then the reduction of f is just $\overline{f} = 1 + X =: \overline{g}$, so the factor g obtained by 6) is linear, and the corresponding root of f has residue -1.

• If 1) holds, then also 5′) and thus also 5) and 5″) hold: Let f, b satisfy the assumptions of 5). We may assume that $f(b) \neq 0$ because otherwise we are done. Then it follows from the assumption $vf(b) > 2vf'(b)$ that also $f'(b) \neq 0$. We employ a substitution used by P. Ribenboim in [RIB14]. From the Taylor Expansion (6.7) we infer the equality

$$f(X+Y) = f(X) + f'(X)Y + \tilde{g}(X,Y)Y^2$$

with $\tilde{g}(X,Y) \in \mathcal{O}_{\mathbf{K}}[X,Y]$. We set $X = b$ and $Y = f(b)f'(b)^{-1}Z$ with Z a new variable. We obtain

$$f(b + f(b)f'(b)^{-1}Z) = f(b) + f(b)Z + \frac{f(b)^2}{f'(b)^2}Z^2\,\tilde{g}(b, f(b)f'(b)^{-1}Z)\ .$$

We have $\tilde{g}(b, f(b)f'(b)^{-1}Z) \in \mathcal{O}_{\mathbf{K}}[Z]$. Since $f(b)/f'(b)^2$ is an element of $\mathcal{M}_{\mathbf{K}}$ by hypothesis,

$$g(Z) := f(b)f'(b)^{-2}\tilde{g}(b, f(b)f'(b)^{-1}Z) \in \mathcal{M}_{\mathbf{K}}[Z]\ .$$

We obtain

$$h(Z) := \frac{f(b + f(b)f'(b)^{-1}Z)}{f(b)} = 1 + Z + Z^2 g(Z)\ .$$

By 1), h admits a root $a' \in \mathcal{O}_{\mathbf{K}}$ such that $\overline{a'} = -1$ and thus, $va' = 0$. Consequently, $a := b + f(b)f'(b)^{-1}a'$ is a root of f, and it satisfies $v(a-b) = v(f(b)f'(b)^{-1}a') > vf'(b)$. Furthermore, $vf'(b) \geq 0$ since $f' \in \mathcal{O}_{\mathbf{K}}[X]$ and $b \in \mathcal{O}_{\mathbf{K}}$.

• Hensel's Lemma in the versions 4), 4′), 4″) are special cases of Newton's Lemma 5), 5′), 5″) respectively, because $\overline{a} = \overline{b}$ follows from $v(a-b) > 0$.

• Property 4) and property 3) are equivalent since their hypotheses are, and similarly for 4′) and 3′) as well as for 4″) and 3″). Indeed, for $b \in \mathcal{O}_{\mathbf{K}}$, $\overline{b}$ is a simple root of $\overline{f}$ if and only if $\overline{f}(\overline{b}) = 0$ and $\overline{f}\,'(\overline{b}) \neq 0$. Since $\overline{f}(\overline{b}) = \overline{f(b)}$ and $\overline{f}\,'(\overline{b}) = \overline{f'(b)}$, this in turn is equivalent to $vf(b) > 0$ and $vf'(b) = 0$.

• If property 3) or 3′) holds, then also 2) holds, and if property 3″) holds, then also 2″) holds. For, if $f = X^n + c_{n-1}X^{n-1} + \ldots + c_1X + c_0 \in \mathcal{O}_{\mathbf{K}}[X]$ with $\overline{c}_{n-1} \neq 0$ and $\overline{c}_{n-2} = \ldots = \overline{c}_0 = 0$, then $\overline{f} = X^n + \overline{c}_{n-1}X^{n-1}$ admits $-\overline{c}_{n-1}$ as a simple root.

• If property 2) or 2″) holds, then (K,v) is henselian: Assume that (K,v) admits more than one extension of v to $\tilde{K}$. Then there is already a finite Galois extension $L|K$ admitting more than one extension of v from K to L. Hence, $\mathrm{Z} := (L|K,v)^d$ is a proper extension of K. Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ be the distinct embeddings of Z into L. From Theorem 7.84 we know that $v\sigma_i \neq v$ on Z for all $i \neq 1$. From Lemma 7.75 we infer the existence of some $a \in L$ such that $va = 0$ and $v\sigma_i a > 0$ for all $i \neq 1$. Consequently, $a \notin K$ is the only root of value 0 of its minimal polynomial over K, and all other roots have value > 0. This minimal polynomial is thus of the form as in the hypothesis of 2), but it is irreducible. We have shown that 2) can not hold if (K,v) is not henselian.

• If (K,v) is henselian, then Krasner's Lemma holds: Assume that $a, b \in K^{\mathrm{sep}}$ satisfy the hypothesis of Krasner's Lemma. We will show that a is fixed by every automorphism $\tau \in \mathrm{Gal}\,K(b)$; since $K(b)$ is the fixed field of $\mathrm{Gal}\,K(b)$ in K^{sep} and $a \in K^{\mathrm{sep}}$, this will yield that $a \in K(b)$. Note that $\tau b = b$ because of $\tau \in \mathrm{Gal}\,K(b)$. Since (K,v) is henselian, the same holds for $(K(b),v)$ by Lemma 7.102, hence we have

$$v(b-\tau a) = v\tau(b-a) = v(b-a) = v(a-b) > \max\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\,(K) \wedge \sigma a \neq a\}\ .$$

It follows that

$$v(a-\tau a) \geq \min\{v(a-b), v(b-\tau a)\} > \max\{v(a-\sigma a) \mid \sigma \in \operatorname{Gal} K \wedge \sigma a \neq a\}$$

which yields that $\tau a = a$.

• Krasner's Lemma implies property 2): Assume that f satisfies the assumption of 2). Then in view of Lemma 7.45, there is precisely one root of f in $\tilde{K}$ with residue $-c_{n-1}$, say a, and all other roots have residue zero. Observe that f is separable because $c_{n-1} \neq 0$ and not both n and $n-1$ can be divisible by the characteristic. Hence, $a \in K^{\mathrm{sep}}$. Since all conjugates of a over K are among these roots, we find that $v(a-\sigma a) = va = 0$ for all $\sigma \in \operatorname{Gal} K$ such that $\sigma a \neq a$. On the other hand, $v(a + c_{n-1}) > 0$, hence by Krasner's Lemma, $a \in K(-c_{n-1}) = K$. Now the desired linear factor is $X + c$ with $c = -a$. □

Remark 9.2 The proof that 6) holds in a henselian field is due to Rayner [RAY]. The proof that 2) implies "henselian" is due to Nagata [NAG]. Ribenboim's substitution which we have employed for the proof of 1)⇒5) is a refinement of a substitution that was used by Iversen in [IVER] to prove the equivalence of Hensel's Lemma with Newton's Lemma. This trick was apparently rediscovered by L. v. d. Dries in his thesis [VDD1]. It works as follows: Suppose that f and b satisfy the hypothesis of Newton's Lemma. Set $g(X) := f(b + f'(b)X)/f'(b)^2$. Then $vg(0) = vf(b) - 2vf'(b) > 0$. Further, $g'(0) = 1$ by virtue of the chain rule, and thus, $vg'(0) = 0$. Then by Hensel's Lemma, there is $\tilde{a} \in \mathcal{M}_{\mathbf{K}}$ such that $g(\tilde{a}) = 0$. Now $a := b + f'(b)\tilde{a}$ is a root of f which satisfies $v(a-b) > vf'(b)$.

We have found the name "**Hensel's Lemma**" for property 4) as well as for property 6) or slightly different versions of them. We prefer to use it for property 4). Originally, Hensel's Lemma is the lemma which states that every complete valued field of rank 1 satisfies property 4) resp. property 6) (cf. Theorem 9.7 below).

See P. Ribenboim [RIB14] for more equivalent forms of Hensel's Lemma, alternative proofs and a different, probably historically more adequate naming of the various forms.

Remark 9.3 In Hensel's Lemma, the root a satisfying $\overline{a} = \overline{b}$ is uniquely determined. This follows from Lemma 7.45 and the hypothesis that $\overline{b}$ be a simple root of $\overline{f}$. The same holds for Newton's Lemma:
Let $f \in \mathcal{O}_{\mathbf{K}}[X]$ and $b \in \mathcal{O}_{\mathbf{K}}$ such that $vf(b) > 2vf'(b)$. Then there is a unique root $a \in \tilde{K}$ of f such that $v(a-b) > vf'(b)$, and if $a \neq b$, then $v(a-b) = vf(b) - vf'(b) > vf'(b)$.
This is seen as follows. From the Taylor expansion (cf. Lemma 6.60), we know that $0 = f(a) = f(b) + f'(b)(a-b) + \tilde{h}(b,a)(a-b)^2$ with $\tilde{h}(b,a) \in \mathcal{O}_{\mathbf{K}}$. Hence, $f(b) = -f'(b)(a-b) - \tilde{h}(b,a)(a-b)^2$. Now assume that $a \neq b$, which means that $v(a-b) < \infty$. From our hypothesis that $v(a-b) > vf'(b)$, it then follows that $vf'(b)(a-b) < v\tilde{h}(b,a)(a-b)^2$. Consequently, $vf(b) = vf'(b)(a-b)$, that is, $v(a-b) = vf(b) - vf'(b)$.

Suppose that also $a' \in \tilde{K}$ is a root of f with $v(a'-b) > vf'(b)$. Then $v(a'-a) \geq \min\{v(a'-b), v(a-b)\} > vf'(b)$. By the Taylor expansion, $0 = f(a') - f(a) = f'(a)(a'-a) + \tilde{h}(a,a')(a'-a)^2$ with $\tilde{h}(a,a') \in \mathcal{O}_{\mathbf{K}}$. From $v(a-b) > vf'(b)$, it also follows that $vf'(a) = vf'(b)$. Hence if $a' \neq a$, then we would have $vf'(a)(a'-a) = vf'(b)(a'-a) < v\tilde{h}(a,a')(a'-a)^2$, and it would follow that $v0 = vf'(a)(a'-a) < \infty$, a contradiction. Hence $a' = a$, as asserted.

Example 9.4 Consider the Artin-Schreier polynomial $f = X^p - X - c$ over the henselian field (K, v). If $vc > 0$, then $\overline{f} = X^p - X$. This polynomial admits 0 and 1 as simple roots, hence by Hensel's Lemma, f admits roots in K with residue 0 and 1. If $\operatorname{char} \overline{K} = p$, then $X^p - X$ admits precisely the elements of the prime field $\mathbb{F}_p$ as its roots, that is, it splits completely over $\mathbb{F}_p$. In this case, Hensel's Lemma shows that f splits completely over K, the residues of its roots being precisely the elements of $\mathbb{F}_p$.

If $vc = 0$ then $\overline{f} = X^p - X - \overline{c}$ with $\overline{c} \neq 0$. If this polynomial admits a root ϑ in $\overline{K}$, then it splits completely over $\overline{K}$, the roots being $\vartheta + i$, $i \in \mathbb{F}_p$. In this case again by Hensel's Lemma, f splits completely over K, the residues of its roots being just $\vartheta + i$, $i \in \mathbb{F}_p$. On the other hand, if $\overline{f}$ does not admit a root in $\overline{K}$ then f does not admit a root in K.

By the preceding discussion, we see that an irreducible Artin-Schreier polynomial $f = X^p - X - c$ over a henselian field (K, v) with Artin-Schreier closed residue field $\overline{K}$ must satisfy $vc < 0$. The case $vc < 0$ has already appeared in Examples 8.40, 8.44, 8.47 and 8.49 as well as in Lemma 7.54, Lemma 7.55 and Lemma 7.56. ♢

Recall that $\wp(X)$ denotes the Artin-Schreier polynomial $X^p - X$. Further, we set $\wp(S) := \{\wp(a) \mid a \in S\}$ for every $S \subset K$. From the considerations of our example, we conclude:

Corollary 9.5 *If (K, v) is henselian, then $\mathcal{M}_{\mathbf{K}} = \wp(\mathcal{M}_{\mathbf{K}}) \subset \wp(K)$. If in addition $\overline{K}$ is Artin-Schreier closed, then $\mathcal{O}_{\mathbf{K}} = \wp(\mathcal{O}_{\mathbf{K}}) \subset \wp(K)$.*

Although we know already from Theorem 7.112 as well as from Theorem 8.26 that every spherically complete valued field is henselian, we will now give alternative proofs on the basis of Theorem 9.1. Let $\mathbf{K} = (K, v)$ be spherically complete.

First proof: Following an idea of S. Prie-Crampe, we will employ the Ultrametric Fixed Point Theorem to prove that (K, v) satisfies property 1) of Theorem 9.1. So let $f = 1 + X + X^2 g(X)$ with $g(X) \in \mathcal{M}_{\mathbf{K}}[X]$. For every $b \in K$, set

$$\Xi b \ := \ b - f(b) \ = \ -1 - b^2 g(b) \ . \tag{9.1}$$

By assumption, all coefficients of g have value > 0, hence $vb^2 g(b) > 0$ for all $b \in \mathcal{O}_{\mathbf{K}}$. It follows that Ξ sends $\mathcal{O}_{\mathbf{K}}$ into $-1 + \mathcal{M}_{\mathbf{K}}$. Moreover, we can show that Ξ is contractive on $\mathcal{O}_{\mathbf{K}}$. For $b, c \in \mathcal{O}_{\mathbf{K}}$, we have

$$\Xi b - \Xi c = c^2 g(c) - b^2 g(b) \ .$$

Applying Lemma 6.60 with $R = \mathcal{M}_{\mathbf{K}}$ to the polynomial $X^2 g(X) \in \mathcal{M}_{\mathbf{K}}[X]$, we find that there exists $h(X, Z) \in \mathcal{M}_{\mathbf{K}}[X, Z]$ such that

$$Z^2 g(Z) - X^2 g(X) = (Z - X) h(X, Z) \ .$$

Evaluating with $Z = b$ and $X = c$, we find $h(c, b) \in \mathcal{M}_{\mathbf{K}}$, whence $v(\Xi b - \Xi c) = v(c^2 g(c) - b^2 g(b)) > v(b - c)$. This proves Ξ to be contractive on $\mathcal{O}_{\mathbf{K}}$. In view of Lemma 5.4 and our assumption that (K, v) be spherically complete, we may infer from the Ultrametric Fixed Point Theorem 2.11 that Ξ has a fixed point $a \in \mathcal{O}_{\mathbf{K}}$. It satisfies $a = \Xi a = a - f(a)$ which is the same as $f(a) = 0$. Hence a is a root of f and it satisfies $\overline{a} = -1$ since it lies in $-1 + \mathcal{M}_{\mathbf{K}}$. □

Second proof: The idea is to employ Theorem 3.36 with $c = 0$ to prove that $\mathbf{K} = (K, v)$ satisfies property 4) of Theorem 9.1. So we have to determine $\mathcal{S}$ and Ψ. Assume that $b \in \mathcal{O}_{\mathbf{K}}$ satisfies $vf(b) > 0$ and $vf'(b) = 0$. We set $\mathcal{S} := b + \mathcal{M}_{\mathbf{K}}$. Then $\overline{b'} = \overline{b}$ for every $b' \in \mathcal{S}$. In particular, it follows that $\overline{f}(\overline{b'}) = \overline{f}(\overline{b}) = 0$ and $\overline{f'}(\overline{b'}) = \overline{f'}(\overline{b}) \neq 0$, hence $vf(b') > 0$ and $vf'(b') = 0$ as well as $v(f'(b') - f'(b)) > 0$ for every $b' \in \mathcal{S}$.

From the Taylor expansion (cf. Lemma 6.60) we infer the existence of some $\tilde{h}(X, Z) \in \mathcal{O}_{\mathbf{K}}[X, Z]$ such that

$$f(Z) - f(X) = f'(X)(Z - X) + (Z - X)^2 \tilde{h}(X, Z) \ .$$

Let $b', b'' \in \mathcal{S}$. Then $v(b'-b'') > 0$ and thus $v(b'-b'')^2 > v(b'-b'')$. Further, $v\tilde{h}(b'', b') \geq 0$ since $b', b'' \in \mathcal{O}_{\mathbf{K}}$ and $\tilde{h}$ has coefficients in $\mathcal{O}_{\mathbf{K}}$. Consequently,

$$\begin{aligned} v\left(f(b') - f(b'') - f'(b)(b'-b'')\right) &= v\left(f'(b'') - f'(b) \ + \ (b'-b'')\tilde{h}(b'', b')\right) + v(b'-b'') \\ &> v(b'-b'') \end{aligned}$$

since $v(f'(b'') - f'(b)) > 0$ and $v(b'-b'')\tilde{h}(b'', b') > 0$. It follows that

$$v(f(b') - f(b'')) = vf'(b)(b'-b'') = v(b'-b'') < v\left(f(b') - f(b'') - f'(b)(b'-b'')\right) \ ,$$

which proves that

$$\Psi(X) = f'(b)X$$

is a valuation differential for f on $\mathcal{S}$. Given $b' \in \mathcal{S}$ such that $f(b') \neq 0$, then

$$b'' := b' - \frac{f(b')}{f'(b)} \tag{9.2}$$

is an element of $\mathcal{S}$ because of $vf(b') > 0$ and $vf'(b) = 0$, and it satisfies

$$v(f(b') - f'(b)(b'-b'')) = \infty > vf(b') \ .$$

This proves that Ψ satisfies condition (3.10) of Theorem 3.36. From this theorem, we conclude that $0 \in f(b + \mathcal{M}_{\mathbf{K}})$, that is, there exists $a \in \mathcal{O}_{\mathbf{K}}$ such that $f(a) = 0$ and $\overline{a} = \overline{b}$. □

Remark 9.6 The element b'' in (9.2) may be viewed as a **refinement** of the approximative root b'. Observe that the denominator $f'(b)$ does not depend on b'. One can also use the refinement $b'' := b' - f(b')/f'(b')$. Newton's Lemma for spherically complete fields of rank 1 is commonly proved by an iteration of the latter refinement (see also Exercise 9.3). The proof would equally work with an iteration of (9.2). (Analogously, for the usual Newton method of calculus it is known that one can work with a fixed denominator — although convergence may not be as fast). Note that one may even replace $f'(b)$ by any element c such that $v(f'(b) - c) > vf'(b)$. This is why the refinement (9.1) works with denominator 1.

Historically, the first examples for henselian fields have been the complete fields of rank 1. Recall that "rank 1" just says that the value group is archimedean. If the value group is not isomorphic to $\mathbb{Z}$, then a complete field may not be spherically complete (cf. Example 8.51). So we have to adapt the above proof for this case. We will see later that the rank 1 condition is indispensable (cf. Corollary 10.9).

Theorem 9.7 *Every complete field of rank 1 is henselian.*

Proof: Let (K, v) be a complete field of rank 1, and let the notation be as in the first of the above two proofs. Since $h(X, Z) \in \mathcal{M}_{\mathbf{K}}[X, Z]$, we may write $h(X, Z) = c \cdot \tilde{h}(X, Z)$ with $c \in \mathcal{M}_{\mathbf{K}}$ and $\tilde{h}(X, Z) \in \mathcal{O}_{\mathbf{K}}[X, Z]$ (for instance, c may be taken to be the coefficient of smallest value in h). Then we have $\tilde{h}(b, a) \in \mathcal{O}_{\mathbf{K}}$ and $v(\Xi a - \Xi b) = v(ag(a) - bg(b)) \geq vc + v(a-b)$. By induction on n we find that $v(\Xi^{n+1}a - \Xi^n a) \geq nvc + v(\Xi a - a) \geq nvc$. Since vK is archimedean, for every $\alpha \in vK$ there is some $n \in \mathbb{N}$ such that $nvc > \alpha$. Now we can use Lemma 2.12 to infer the existence of the desired fixed point. □

For the conclusion of this section, we apply Theorem 9.1 to show:

Lemma 9.8 *The completion of a henselian field is again henselian.*

Proof: Let $\mathbf{L} = (L, v)$ be the completion of $\mathbf{K} = (K, v)$ and $f \in \mathcal{O}_{\mathbf{L}}[X]$ a monic polynomial. We extend v to the Gauß valuation on $L[X]$. Assume that $\overline{f}$ has a simple root $\overline{b} \in \overline{L}$. If $g \in \mathcal{O}_{\mathbf{K}}[X]$ such that $v(f-g) > 0$, then $\overline{g} = \overline{f}$, so that also $\overline{g}$ admits $\overline{b}$ as a simple root (we can choose $b \in K$). Since $\mathbf{K}$ is assumed to be henselian, we can infer from assertion 3) of Theorem 9.1 that there is a root $a_g \in \mathcal{O}_{\mathbf{K}}$ of g such that $\overline{a}_g = \overline{b}$.

Now extend the valuation v to the algebraic closure of L and choose the (unique) root $a \in \tilde{L}$ of f which under this extension satisfies $\overline{a} = \overline{b}$. Given an arbitrary $\alpha \in v\tilde{L}$ we wish to show that there is some g of the above form such that $v(a-a_g) > \alpha$. Since $v\tilde{L}$ lies in the divisible hull of vL by virtue of Corollary 7.9, we can enlarge α such that $\alpha \in vL = vK$. Now from the Continuity of Roots (Theorem 7.49) we know that for a sufficiently large $\beta \in vL$, we will have that $v(a-a_g) > \alpha$ if $v(f-g) > \beta$. Since the coefficients of f lie in the completion of $\mathbf{K}$, we can always find such a polynomial $g \in \mathcal{O}_{\mathbf{K}}[X]$. This proves that a lies in $\mathbf{K}^c = \mathbf{L}$. By condition 3) of Theorem 9.1, this shows that $\mathbf{L}$ is henselian. □

Exercise 9.1 *Show the equivalence of 1) of Theorem 9.1 with a weaker version of 2) where $c_{n-1} = 1$ by means of a suitable substitution. What happens to the zeros under this substitution? Try to deduce "henselian" from this weaker version of 2); what is the problem?*

Exercise 9.2 *Show that $\mathbf{K}$ is henselian if and only if it satisfies the following property: If $f \in \mathcal{O}_{\mathbf{K}}[X]$ is irreducible over K and $\overline{f}$ is not constant, then $\overline{f} = \overline{c}\varphi^m$ where $c \in \mathcal{O}_{\mathbf{K}}$ and $\varphi \in \overline{K}[X]$ is an irreducible polynomial such that* $\deg f = m \deg \varphi$.

Exercise 9.3 *Use the ultrametric fixed point theorem to show directly that every spherically complete field satisfies Hensel's Lemma. (Hint: define $\Xi b := b - f(b)/f'(b)$.) Try the same with Newton's Lemma.*

Exercise 9.4 *(F. Pop)*
Deduce Fermat's Last Theorem from Hensel's Lemma. (Hint: see page 385.)

9.2 The diameter of the conjugates

Let (K, v) be a valued field and let v be extended to $\tilde{K}$. Further, let $a \in \tilde{K} \setminus K^{1/p^\infty}$. Following J. Ax [AX2], we call the value

$$\operatorname{diam}(a, K) := \min\{v(a-\sigma a) \mid \sigma \in \operatorname{Gal} K \wedge \sigma a \neq a\}$$

the **diameter of the conjugates of** a (**over** K). We know the counterpart

$$\operatorname{kras}(a, K) := \max\{v(a-\sigma a) \mid \sigma \in \operatorname{Gal} K \wedge \sigma a \neq a\}$$

of $\operatorname{diam}(a, K)$ already from Krasner's Lemma, which is property 7) of Theorem 9.1. Let us also define $\operatorname{diam}(a, K) := \infty$ and $\operatorname{kras}(a, K) := \infty$ for $a \in K^{1/p^\infty}$.

Krasner's Lemma tells us that if (K, v) is henselian and if we have $b \in K^{\mathrm{sep}}$ such that $v(a-b) > \operatorname{kras}(a, K)$, then $a \in K(b)$. In particular, we see that if (K, v) is henselian and if $a \in K^{\mathrm{sep}} \setminus K$, then there is no $b \in K$ such that $v(a-b) > \operatorname{kras}(a, K)$. But this result can be improved:

Theorem 9.9 *Let (K,v) be a valued field, and let v be extended to $\tilde{K}$. Then (K,v) is henselian if and only if it satisfies the following condition:*
For every $a \in \tilde{K}$,

$$\operatorname{diam}(a,K) \geq v(a-b)$$

for every $b \in K$.
In this condition, "$\tilde{K}$" can be replaced by "$K^{\rm sep}$".

Proof: Assume (K,v) to be henselian, $a \in \tilde{K}$ and $b \in K$. Then for every $\sigma \in \operatorname{Gal} K$, we have that $v(\sigma a - b) = v\sigma(a-b) = v(a-b)$. Hence, $v(a-\sigma a) \geq \min\{v(a-b), v(\sigma a - b)\} = v(a-b)$. This proves that $\operatorname{diam}(a,K) \geq v(a-b)$.

For the converse, assume that (K,v) satisfies the condition which is described in our assertion. We wish to show that (K,v) has property 3″) of Theorem 9.1. So let be given a monic separable polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ such that $\overline{f}$ admits a simple root $\bar{b}$ with $b \in \mathcal{O}_{\mathbf{K}}$. Hence, there is a root $a \in K^{\rm sep}$ such that $\overline{a} = \bar{b}$. All conjugates of a over K are also roots of f (since the minimal polynomial of a over K must divide f). Hence, $\overline{\sigma a} \neq \overline{a}$ for all $\sigma \in \operatorname{Gal} K$ such that $\sigma a \neq a$. (Note that $v\sigma a \geq 0$ for all σ since f is monic and has integral coefficients.) We conclude that $v(a-\sigma a) = 0$ for all σ such that $\sigma a \neq a$. On the other hand, $\overline{a} = \bar{b}$ yields that $v(a-b) > 0$. Hence by assumption, $\operatorname{diam}(a,K)$ can not be equal to 0. Consequently, there can not exist $\sigma \in \operatorname{Gal} K$ such that $\sigma a \neq a$. But a is separable over K, being a root of a separable polynomial. This proves that $a \in K$. □

Corollary 9.10 *If (K,v) is a henselian field and $a \in \tilde{K}$, then*

$$\operatorname{diam}(a,K) \geq \operatorname{dist}(a,K)\ .$$

The question arises whether $\operatorname{dist}(a,K)$ is always equal to $\operatorname{diam}(a,K)$. The following lemma, due to Ax [AX2], seems to lend credibility to our presumption:

Lemma 9.11 *Let $(K(a)|K,v)$ be an extension of degree not divisible by the residue characteristic of K. Then $v(a-b) \geq \operatorname{diam}(a,K)$ for $b := [K(a):K]^{-1}\operatorname{Tr}_{K(a)|K}(a)$.*

Proof: We know that $v[K(a):K] = 0$ since the residue characteristic does not divide $[K(a):K]$. We compute $v(a - [K(a):K]^{-1}\operatorname{Tr}_{K(a)|K}(a)) = v([K(a):K]\,a - \operatorname{Tr}_{K(a)|K}(a)) = v\sum a - \sigma a \geq \min v(a-\sigma a) = \operatorname{diam}(a,K)$. □

Corollary 9.12 *Let (K,v) be a valued field of residue characteristic 0. Then for all $a \in \tilde{K}$,*

$$\operatorname{diam}(a,K) \leq \operatorname{dist}(a,K)\ ,$$

and if (K,v) is henselian, then

$$\operatorname{diam}(a,K) = \operatorname{dist}(a,K)\ .$$

Unfortunately, the situation is not as nice for extensions of degree divisible by the residue characteristic. Here, Ax has shown:
If (K,v) is a henselian field with archimedean value group vK, and if $a \in \tilde{K}$, then

$$\operatorname{diam}(a,K) = \operatorname{dist}(a,K^{1/p^\infty})\ . \tag{9.3}$$

But this is not anymore true if the value group is non-archimedean, as we have shown in Example 8.53. If the characteristic of K is $p > 0$, then it may also happen that $\operatorname{diam}(a, K) > \operatorname{dist}(a, K)$. At present, it remains to draw the reader's attention to an interesting error in Ax' paper [AX2]. This error was found by S. K. Khanduja, who gave a counterexample in [KH1]. If we have the equality (9.3), does it follow that there is some $b \in K^{1/p^\infty}$ such that $v(a-b) = \operatorname{diam}(a, K)$? After having read Section 8.4, the reader should be able to answer "No!" (this is what we hope). Indeed, if $\operatorname{diam}(a, K) = \operatorname{dist}(a, K^{1/p^\infty})$ is the distance of a proper immediate extension of $(K^{1/p^\infty}, v)$, then we know that this distance can not be assumed. So let us reconsider Example 8.49. Before we continue, let us note:

Lemma 9.13 *Let K be a field of characteristic $p > 0$ and a a root of the Artin-Schreier polynomial $X^p - X - c$ with $c \in K$. If $a \notin K$, then* $\operatorname{kras}(a, K) = \operatorname{diam}(a, K) = 0$.

Proof: According to Lemma 6.2, the roots of $X^p - X - c$ are $a, a+1, \ldots, a+p-1$. If $a \notin K$, then $X^p - X - c$ is irreducible over K, and the conjugates of a are precisely these roots. Hence, $a - \sigma a = i$ for some $i \in \{0, \ldots, p-1\}$. If $a \neq \sigma a$, then consequently, $v(a - \sigma a) = 0$. This proves our assertion. □

Example 9.14 Let $(K, v) = (\mathbb{F}_p((t)), v_t)$ and a a root of the Artin-Schreier polynomial $X^p - X - 1/t$. Then $\operatorname{dist}(a, K^{1/p^\infty}) = 0 = \operatorname{diam}(a, K^{1/p^\infty})$ by Corollary 8.68 and the foregoing lemma. But this distance is not assumed by any $b \in K^{1/p^\infty}$ since the extension $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$ is nontrivial and immediate. This is the example given by Khanduja.

To make the example a bit more drastical, one can replace $\mathbb{F}_p$ by $k := \tilde{\mathbb{F}}_p$. For $L := k((t))$, everything works as before, and we obtain an immediate Artin-Schreier extension $(L^{1/p^\infty}(a)|L^{1/p^\infty}, v)$. Now if there were some $b \in L^{1/p^\infty}$ such that $v(a-b) \geq 0$, then the polynomial $X^p - X - (1/t - b^p - b)$ would have integral coefficients. Since $\overline{L} = \tilde{\mathbb{F}}_p$ is algebraically closed, there would exist some $b_0 \in L^{1/p^\infty}$ such that $\overline{b_0}$ is a root of the reduction of $X^p - X - (1/t - b^p - b)$. That is, $v(1/t - b^p - b - b_0^p - b_0) > 0$, and the polynomial $X^p - X - (1/t - b^p - b - b_0^p - b_0)$ would have a root in L according to Example 9.4. This contradicts the fact that $L^{1/p^\infty}(a)|L^{1/p^\infty}$ is an Artin-Schreier extension. ♢

This example disproves Corollary 2 and case b) of Proposition 2′ of [AX2].

9.3 The multidimensional Hensel's Lemma

Let $\mathbf{K} = (K, v)$ be a valued field. Suppose that we are given a system of polynomials $\underline{f} = (f_1, \ldots, f_n)$ in the variables $\underline{X} = (X_1, \ldots, X_n)$, with coefficients in K. **In this section, indeces will not indicate formal derivatives but distinguish between arbitrary polynomials.**

Let i be fixed and $\underline{b'}, \underline{b''} \in K^n$. Plugging in $\underline{b'}$ for $\underline{Z}$ and $\underline{b''}$ for $\underline{X}$ in equation (6.10), we obtain that

$$f_i(\underline{b'}) - f_i(\underline{b''}) = \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(\underline{b''}) \cdot (b'_j - b''_j) + \sum_{j=1}^{n}\sum_{k=1}^{n} \tilde{h}_{jk}(\underline{b''}, \underline{b'}) \cdot (b'_j - b''_j) \cdot (b'_k - b''_k)\,. \qquad (9.4)$$

Let us now assume that $f_i \in \mathcal{O}_{\mathbf{K}}[\underline{X}]$ for all i. Hence $f'_i \in \mathcal{O}_{\mathbf{K}}[\underline{X}]$ for all i, and $\tilde{h}_{jk}(\underline{X}, \underline{Z}) \in \mathcal{O}_{\mathbf{K}}[\underline{X}, \underline{Z}]$ for all j, k. Assume further that $\underline{b} \in \mathcal{O}_{\mathbf{K}}^n$ and define $\mathcal{S} := \underline{b} + \mathcal{M}_{\mathbf{K}}^n \subset \mathcal{O}_{\mathbf{K}}^n$. For every $\underline{b'}, \underline{b''} \in \mathcal{S}$, we have $\underline{b'} - \underline{b''} \in \mathcal{M}_{\mathbf{K}}^n$ and $\tilde{h}_{jk}(\underline{b''}, \underline{b'}) \in \mathcal{O}_{\mathbf{K}}$.

As in Section 3.6, we will work with the group valuation $\underline{v}$ of the additive group K^n which is defined by $\underline{v}\,\underline{b} = \min_i vb_i$. We have $\underline{v}\,\underline{b} = vb_{i_0}$ if and only if $\underline{b} \in (b_{i_0}\mathcal{O}_{\mathbf{K}})^n$. Now let i_0 be such that $\underline{v}(\underline{b'} - \underline{b''}) = v(b'_{i_0} - b''_{i_0})$. Then from (9.4) and the above assumptions, we deduce:

$$f_i(\underline{b'}) - f_i(\underline{b''}) - \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(\underline{b''}) \cdot (b'_j - b''_j) \in (b'_{i_0} - b''_{i_0})\mathcal{M}_{\mathbf{K}} \ .$$

For $\underline{c} \in K^n$, the **Jacobian matrix of** $\underline{f}$ is the matrix of partial derivatives

$$M_{\underline{f}}(\underline{c}) := \left(\frac{\partial f_i}{\partial X_j}(\underline{c}) \right)_{i,\,j} \ .$$

With its help, we can now write

$$\underline{f}(\underline{b'}) - \underline{f}(\underline{b''}) - M_{\underline{f}}(\underline{b''})(\underline{b'} - \underline{b''}) \in ((b'_{i_0} - b''_{i_0})\mathcal{M}_{\mathbf{K}})^n \ .$$

If $\underline{b'} \neq \underline{b''}$, then this means that

$$\underline{v}\left(\underline{f}(\underline{b'}) - \underline{f}(\underline{b''}) - M_{\underline{f}}(\underline{b''})(\underline{b'} - \underline{b''})\right) > \underline{v}(\underline{b'} - \underline{b''}) \ . \tag{9.5}$$

Let $J_{\underline{f}}(\underline{c})$ denote the determinant of $M_{\underline{f}}(\underline{c})$. Note that $J_{\underline{f}}(\underline{c})$ is obtained from $J_{\underline{f}}(\underline{X})$ by plugging in $\underline{c}$ for $\underline{X}$. On the other hand, $J_{\underline{f}}(\underline{X})$ is a polynomial in $X_1, \ldots, X_n$, and since we have assumed that $f_i \in \mathcal{O}_{\mathbf{K}}[\underline{X}]$, this polynomial also has coefficients in $\mathcal{O}_{\mathbf{K}}$. For every $\underline{b'} \in \mathcal{S}$, it follows that $\overline{J_{\underline{f}}(\underline{b'})} = \overline{J_{\underline{f}}(\underline{b})}$, because $\overline{b'_i} = \overline{b_i}$ for all i.

From now on, let us assume that $vJ_{\underline{f}}(\underline{b}) = 0$. Consequently, $vJ_{\underline{f}}(\underline{b'}) = 0$ for every $\underline{b'} \in \mathcal{S}$. Since $M_{\underline{f}}(\underline{b'}) \in \mathcal{O}_{\mathbf{K}}^{n\times n}$ and $J_{\underline{f}}(\underline{b'})$ is a unit in $\mathcal{O}_{\mathbf{K}}$, we find that $M_{\underline{f}}(\underline{b'})$ is invertible and $M_{\underline{f}}(\underline{b'})^{-1} \in \mathcal{O}_{\mathbf{K}}^{n\times n}$.

We leave it to the reader to prove the following:

$$\begin{aligned} M \in \mathcal{O}_{\mathbf{K}}^{n\times n} &\implies \forall \underline{c} \in K^n : \ \underline{v}(M\underline{c}) \geq \underline{v}\,\underline{c} \ , \\ M \in \mathcal{M}_{\mathbf{K}}^{n\times n} &\implies \forall \underline{c} \in K^n \setminus \{\underline{0}\} : \ \underline{v}(M\underline{c}) > \underline{v}\,\underline{c} \ . \end{aligned}$$

It follows that

$$M, M^{-1} \in \mathcal{O}_{\mathbf{K}}^{n\times n} \implies \forall \underline{c} \in K^n : \ \underline{v}(M\underline{c}) = \underline{v}\,\underline{c} \ .$$

Let $\underline{b'} \in \mathcal{S}$. As the partial derivatives $\partial f_i/\partial X_j$ have coefficients in $\mathcal{O}_{\mathbf{K}}$, we have that $\overline{\partial f_i/\partial X_j(\underline{b'})} = \overline{\partial f_i/\partial X_j(\underline{b})}$. Consequently,

$$M_{\underline{f}}(\underline{b'}) - M_{\underline{f}}(\underline{b}) \in \mathcal{M}_{\mathbf{K}}^{n\times n} \ .$$

Hence for $\underline{b'}$ and $\underline{b''}$ as above, with $\underline{b'} \neq \underline{b''}$, we have that

$$\underline{v}\left((M_{\underline{f}}(\underline{b'}) - M_{\underline{f}}(\underline{b}))(\underline{b'} - \underline{b''})\right) > \underline{v}(\underline{b'} - \underline{b''}) \ ,$$

and inequality (9.5) yields the inequality

$$\underline{v}\left(\underline{f}(\underline{b'}) - \underline{f}(\underline{b''}) - M_{\underline{f}}(\underline{b})(\underline{b'} - \underline{b''})\right) > \underline{v}(\underline{b'} - \underline{b''}) \ .$$

It follows that

$$\underline{v}\left(\underline{f}(\underline{b'}) - \underline{f}(\underline{b''})\right) = \underline{v}\, M_{\underline{f}}(\underline{b})(\underline{b'} - \underline{b''}) = \underline{v}(\underline{b'} - \underline{b''}) < \underline{v}\left(\underline{f}(\underline{b'}) - \underline{f}(\underline{b''}) - M_{\underline{f}}(\underline{b})(\underline{b'} - \underline{b''})\right) \ .$$

This proves that the map represented by the matrix $M_{\underline{f}}(\underline{b})$ is a valuation differential for $\underline{f}$ on $\mathcal{S}$. Moreover, we see from this inequality that $\underline{f}$ is injective on $\mathcal{S}$. Indeed, $\underline{v}(\underline{f}(\underline{b'})-\underline{f}(\underline{b''})) = \infty$ is impossible for $\underline{b'} \neq \underline{b''}$.

Assume in addition that $\underline{v}\,\underline{f}(\underline{b}) > 0$. Then for every $b' \in \mathcal{S}$, $\overline{f_i(\underline{b'})} = \overline{f_i(\underline{b})} = 0$, hence $\underline{v}\,\underline{f}(\underline{b'}) > 0$. For every $b' \in \mathcal{S}$,

$$\underline{b''} := \underline{b'} - M_{\underline{f}}(\underline{b})^{-1}\underline{f}(\underline{b'})$$

is an element of $\mathcal{S}$ because $\underline{v}\,\underline{f}(\underline{b'}) > 0$ implies $\underline{v}(M_{\underline{f}}(\underline{b})^{-1}\underline{f}(\underline{b'})) > 0$. For this $\underline{b''}$,

$$\underline{v}\left(\underline{f}(\underline{b'}) - M_{\underline{f}}(\underline{b})(\underline{b'} - \underline{b''})\right) = \infty > \underline{v}\,\underline{f}(\underline{b'})\ .$$

This proves that $\Psi = M_{\underline{f}}(\underline{b})$ satisfies condition (3.10) of Theorem 3.36. From this theorem, we conclude that $\underline{0} \in \mathcal{S} = \underline{f}(\underline{b} + \mathcal{M}_{\mathbf{K}}^n)$. By the injectivity of $\underline{f}$ on $\mathcal{S}$, it follows that there is a unique solution of $\underline{f}(\underline{X}) = 0$ on $\mathcal{S}$. We have proved:

Theorem 9.15 *Let $\mathbf{K} = (K, v)$ be a spherically complete valued field. Then $\mathbf{K}$ satisfies the multidimensional Hensel's Lemma:*
Let $\underline{f} = (f_1, \ldots, f_n)$ be a system of polynomials in the variables $\underline{X} = (X_1, \ldots, X_n)$ and with coefficients in $\mathcal{O}_{\mathbf{K}}$. Assume that $\underline{b} \in \mathcal{O}_{\mathbf{K}}^n$ such that $vJ_{\underline{f}}(\underline{b}) = 0$ and $\underline{v}\underline{f}(\underline{b}) > 0$. Then there exists a unique $\underline{a} \in \mathcal{O}_{\mathbf{K}}^n$ such that $\underline{f}(\underline{a}) = 0$ and $\underline{v}(\underline{a} - \underline{b}) > 0$.

We are now going to show that the multidimensional Hensel's Lemma holds in every henselian field. Beforehand, we need the following lemma. See [LANG3], Chapter X, §7, Proposition 8 for its proof, which uses the theory of derivations.

Lemma 9.16 *Let $x_1, \ldots, x_n$ be elements in an arbitrary extension field of the field K. Suppose that there are n polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_n]$ such that*
1) $f_i(x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, n$,
2) $J_{\underline{f}}(x_1, \ldots, x_n) \neq 0$.
Then the elements $x_1, \ldots, x_n$ are separable algebraic over K.

Theorem 9.17 *For a valued field (K, v), the following properties are equivalent:*
1) *(K, v) is henselian.*
2) *(K, v) satisfies the multidimensional Hensel's Lemma.*
3) *(K, v) satisfies the multidimensional Newton's Lemma:*
Let $\underline{f} = (f_1, \ldots, f_n)$ be a system of polynomials in the variables $\underline{X} = (X_1, \ldots, X_n)$ and with coefficients in $\mathcal{O}_{\mathbf{K}}$. Assume that $\underline{b} \in \mathcal{O}_{\mathbf{K}}^n$ such that $\underline{v}\,\underline{f}(\underline{b}) > 2vJ_{\underline{f}}(\underline{b})$. Then there exists a unique $\underline{a} \in \mathcal{O}_{\mathbf{K}}^n$ such that $\underline{f}(\underline{a}) = 0$ and $\underline{v}(\underline{a} - \underline{b}) > vJ_{\underline{f}}(\underline{b})$.

Proof: **1)⇒2)**: Let (K, v) be henselian. Take (L, v) to be a maximal immediate extension of $(\tilde{K}, v)$. Then by virtue of Theorem 7.124, (L, v) is spherically complete. In view of Corollary 7.60, we find that L must be algebraically closed. By Theorem 9.15, (L, v) satisfies the multidimensional Hensel's Lemma. Now assume that the hypothesis of the multidimensional Hensel's Lemma is satisfied by $\underline{b} \in \mathcal{O}_{\mathbf{K}}^n$. It follows that there is a unique $\underline{a} \in \mathcal{O}_{\mathbf{L}}^n$ such that $\underline{f}(\underline{a}) = 0$ and $\underline{v}(\underline{a} - \underline{b}) > 0$. From the latter, it follows that $vJ_{\underline{f}}(\underline{a}) = vJ_{\underline{f}}(\underline{b}) = 0$ and in particular, $J_{\underline{f}}(\underline{a}) \neq 0$. Now Lemma 9.16 shows that the elements $a_1, \ldots, a_n$ are separable algebraic over K. On the other hand, for every

$\sigma \in \operatorname{Gal} K$, the element $\sigma\underline{a} = (\sigma a_1, \ldots, \sigma a_n)$ satisfies $\underline{f}(\sigma\underline{a}) = \sigma\underline{f}(\underline{a}) = 0$ and $\underline{v}(\sigma\underline{a} - \underline{b}) = \min_i v(\sigma a_i - b_i) = \min_i v\sigma(a_i - b_i) = \min_i v(a_i - b_i) = \underline{v}\underline{a} > 0$ (note that $v\sigma = v$ because (K, v) is henselian). By the uniqueness of $\underline{a}$, it follows that $\sigma\underline{a} = \underline{a}$ for every $\sigma \in \operatorname{Gal} K$, that is, $\underline{a} \in K^n$, as required.

2)⇒3): Assume that $\underline{b} \in \mathcal{O}_{\mathbf{K}}^n$ satisfies the hypothesis of the multidimensional Newton's Lemma. We employ a multidimensional version of the **trick of Iversen**. We introduce the new variables $\underline{Y} = (Y_1, \ldots, Y_n)$ and set

$$\underline{g}(\underline{Y}) := J_{\underline{f}}(\underline{b})^{-2} \cdot \underline{f}\left(\underline{b} + J_{\underline{f}}(\underline{b})\underline{Y}\right) .$$

Then $\underline{g}(\underline{0}) = J_{\underline{f}}(\underline{b})^{-2}\underline{f}(\underline{b})$ so that

$$\underline{v}\,\underline{g}(\underline{0}) = \underline{v}J_{\underline{f}}(\underline{b})^{-2}\underline{f}(\underline{b}) = \underline{v}\underline{f}(\underline{b}) - 2vJ_{\underline{f}}(\underline{b}) > 0 .$$

Using the chain rule of multidimensional calculus, we find that

$$M_{\underline{g}}(\underline{0}) = J_{\underline{f}}(\underline{b})^{-2}M_{\underline{f}}(\underline{b})J_{\underline{f}}(\underline{b}) \left(\frac{\partial Y_i}{\partial Y_j}(\underline{0})\right)_{i,\,j} = J_{\underline{f}}(\underline{b})^{-1}M_{\underline{f}}(\underline{b}) ,$$

since $\left(\partial Y_i/\partial Y_j\,(\underline{0})\right)_{i,j}$ is the $n \times n$-unit matrix. Hence, $J_{\underline{g}}(\underline{0}) = J_{\underline{f}}(\underline{b})^{-1}J_{\underline{f}}(\underline{b}) = 1$ and

$$vJ_{\underline{g}}(\underline{0}) = v1 = 0 .$$

Hence by the multidimensional Hensel's Lemma, there is $\tilde{\underline{a}} \in \mathcal{M}_{\mathbf{K}}^n$ such that $\underline{g}(\tilde{\underline{a}}) = \underline{0}$. Now

$$\underline{a} \;:=\; \underline{b} + J_{\underline{f}}(\underline{b})\,\tilde{\underline{a}} \;\in\; \underline{b} + J_{\underline{f}}(\underline{b})\mathcal{M}_{\mathbf{K}}^n$$

satisfies $\underline{f}(\underline{a}) = \underline{0}$ and $\underline{v}(\underline{a} - \underline{b}) > vJ_{\underline{f}}(\underline{b})$.

3)⇒1): If $n = 1$, then $J_{\underline{f}}(\underline{b}) = M_{\underline{f}}(\underline{b}) = f_1'(b_1)$, and the assertion is precisely the assertion of the onedimensional Newton's Lemma. Hence Theorem 9.1 shows that the multidimensional Newton's Lemma implies that (K, v) is henselian. □

Using the multidimensional Newton's Lemma, one can prove the multidimensional **Implicit Function Theorem**:

Theorem 9.18 *Let (K, v) be a henselian field, and let $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ with $m < n$. Set*

$$J(\underline{X}) := \begin{pmatrix} \frac{\partial f_1}{\partial X_{n-m+1}}(\underline{X}) & \cdots & \frac{\partial f_1}{\partial X_n}(\underline{X}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial X_{n-m+1}}(\underline{X}) & \cdots & \frac{\partial f_m}{\partial X_n}(\underline{X}) \end{pmatrix} .$$

Assume that $f_1, \ldots, f_m$ admit a common zero $\underline{a} = (a_1, \ldots, a_n) \in K^n$ and that the determinant $J(\underline{a})$ is nonzero. Then there are $\alpha, \beta \in vK$ such that for all $(a_1', \ldots, a_{n-m}') \in K^{n-m}$ with $v(a_i - a_i') > \alpha$, $1 \le i \le n - m$, there exists a unique $(b_1, \ldots, b_m) \in K^m$ such that $(a_1', \ldots, a_{n-m}', b_1, \ldots, b_m)$ is a common zero of $f_1, \ldots, f_m$ and $v(a_{n-m+i} - b_i) > \beta$, $1 \le i \le m$.

Proof: After multiplying all f_i with an element of K of suitably large value, we can assume that all f_i have coefficients in $\mathcal{O}_{\mathbf{K}}$. Then also $J(\underline{X})$ is a polynomial in $\underline{X}$ with coefficients in $\mathcal{O}_{\mathbf{K}}$. By assumption, $\underline{f}(\underline{a}) = 0$. Hence, $v(a_i - a_i') > 2vJ(\underline{a})$, $1 \leq i \leq n-m$, will imply that for $1 \leq i \leq m$

$$vf_i(a_1', \ldots, a_{n-m}', a_{n-m+1}, \ldots, a_n) > 2vJ(\underline{a}) = 2vJ(a_1', \ldots, a_{n-m}', a_{n-m+1}, \ldots, a_n) .$$

Hence by the multidimensional Newton's Lemma (property **3)** of Theorem 9.17), there is a unique common zero $(b_1, \ldots, b_m) \in K^m$ of the polynomials $f_i(a_1', \ldots, a_{n-m}', X_1, \ldots, X_m)$, $1 \leq i \leq m$, such that $v(a_{n-m+i} - b_i) > vJ(\underline{a})$, $1 \leq i \leq m$. With $\alpha = 2vJ(\underline{a})$ and $\beta = vJ(\underline{a})$, this proves our assertion. □

Exercise 9.5 *Let (K, v) be a valued field of rank 1.*

a) Let $0 < \alpha \in vK$ and prove that (K, v) is henselian if and only if it satisfies the following condition

Every monic polynomial $f = X^n + c_{n-1}X^{n-1} + \ldots + c_1X + c_0 \in K[X]$ with $vc_{n-1} = 0$ and $vc_i > \alpha$, $0 \leq i \leq n-2$, admits a linear factor $X + c$ in $\mathcal{O}_{\mathbf{K}}[X]$ such that $\overline{c} = \overline{c}_{n-1}$.

(Hint: use the Approximation Theorem). b) Prove that (K, v) is henselian if and only if it satisfies the Implicit Function Theorem.

9.4 n-th powers in henselian fields

Let $\mathbf{K} = (K, v)$ be a henselian field, $a \in K$ and n a natural number. We ask whether a is an n-th power in K, that is, whether there exists $b \in K$ such that $a = b^n$. A first observation is the following. If $a = b^n$ with $b \in K$, then $va = nvb \in nvK$, and if $a \in \mathcal{O}_{\mathbf{K}}^{\times}$, then $\overline{a} = \overline{b}^n \in (\overline{K}^{\times})^n$. A partial converse of the latter implication is given by the next lemma.

Before we continue, let us note the following. If k is an arbitrary field and the natural number n is not divisible by the characteristic of k, then a polynomial $X^n - a$ with $0 \neq a \in k$ has no multiple roots. Indeed, a multiple root would also be a root of the derivative nX^{n-1} of $X^n - a$. But since $n \neq 0$ in k, the only root of nX^{n-1} is 0, which is not a root of $X^n - a$.

Lemma 9.19 *Let (K, v) be a henselian field, $a \in \mathcal{O}_{\mathbf{K}}^{\times}$ and n a natural number not divisible by the residue characteristic* $\operatorname{char}\overline{K}$. *If $\overline{a} \in (\overline{K}^{\times})^n$, then a is an n-th power in K.*

Proof: Consider the polynomial $X^n - a \in \mathcal{O}_{\mathbf{K}}[X]$. Its reduction is $X^n - \overline{a}$. By hypothesis, this polynomial has a root in $\overline{K}$. Since n is not divisible by the characteristic of $\overline{K}$ and $\overline{a} \neq 0$, this root is a simple root. Hence by Hensel's Lemma in the version 3) of Theorem 9.1, $X^n - a$ admits a root in K, and a is thus an n-th power in K. □

Since a 1-unit has residue 1 and the polynomial $X^n - 1$ always admits 1 as a root, we obtain the following corollary:

Corollary 9.20 *Let (K, v) be a henselian field and n a natural number not divisible by the residue characteristic* $\operatorname{char}\overline{K}$. *Then every 1-unit in (K, v) is an n-th power.*

In the case of $a \in \mathcal{O}_{\mathbf{K}}^{\times}$, what can be said if n is divisible by the residue characteristic? In this case one can employ Newton's Lemma. The derivative of $X^n - a$ is nX^{n-1}, hence an approximative root b will satisfy the hypothesis of Newton's Lemma if $v(b^n - a) > 2vn + 2(n-1)vb$. But then, $vb^n = va = 0$ and thus, $vb = 0$. So in fact our condition is

$v(b^n - a) > 2vn$. (Note that the expression "vn" denotes the value of the element $n \cdot 1$, where 1 is the multiplicative unit of K.) Now Newton's Lemma shows:

If a is an n-th power up to a summand of value $> 2vn$, then a is an n-th power.

Let us again consider the important special case of 1-units. If a is a 1-unit, then the value $v(a-1)$ is called the **level of the 1-unit** a. We have proved:

Lemma 9.21 *Let (K, v) be a henselian field and n a natural number. If a is a 1-unit of level $> 2vn$, then a is an n-th power in K.*

This lemma will be applied to improve the assertion of Lemma 9.24 for an important class of valued fields of mixed characteristic. See Lemma 9.24 in the next section.

What can we say about elements of value $\neq 0$? We have seen that $va \in nvK$ if a is an n-th power in K. So we can answer: If there is some $c \in K$ such that $vac^n = 0$ and if ac^n satisfies one of the above criteria for being an n-th power, then a is an n-th power. In particular, we have:

Corollary 9.22 *Let (K, v) be a valued field and n a natural number. If K is closed under n-th roots, then vK is n-divisible, and $\overline{K}$ is closed under n-th roots. The converse holds if (K, v) is henselian and n is not divisible by the residue characteristic* $\operatorname{char} \overline{K}$.

(For the first assertion, cf. Lemma 7.55 and Lemma 7.56).

But this answer is not satisfactory if we want to dicide for every single element whether it is an n-th power. There is an easy answer to this problem if (K, v) is equipped with a multiplicative coefficient map $\operatorname{co} : K \to \overline{K}$. In this case, we can prove the following generalization of Lemma 9.19:

Lemma 9.23 *Let (K, v) be a henselian field and n a natural number not divisible by the residue characteristic* $\operatorname{char} \overline{K}$. *Assume that (K, v) admits a multiplicative coefficient map. Then $a \in K$ is an n-th power in K if and only if $va \in nvK$ and $\operatorname{co} a$ is an n-th power in $\overline{K}$.*

Proof: Since co is assumed to be multiplicative, $\operatorname{co} a$ is an n-th power in $\overline{K}$ if $a \in K$ is an n-th power in K. The latter also implies that $va \in nvK$. Now assume that $va \in nvK$ and $\operatorname{co} a$ is an n-th power in $\overline{K}$. We choose $c \in K$ such that $va = nvc$. Then $\operatorname{co} ac^{-n} = (\operatorname{co} a)(\operatorname{co} c)^{-n}$ is again an n-th power in $\overline{K}$. Since $vac^{-n} = 0$, we have that $\overline{ac^{-n}} = \operatorname{co} ac^{-n}$. By Lemma 9.19 it follows that ac^{-n} is an n-th power in K and thus, the same holds for a. □

9.5 Relatively algebraically closed subfields of henselian fields

From the already given examples of henselian fields, we obtain an abundance of other examples by an application of Corollary 7.106 or of the following lemma, which gives a new proof for that corollary. In this proof, we make significant use of the fact that a field is henselian if and only if it satisfies Hensel's Lemma restricted to separable polynomials. The latter is property 4″) of Theorem 9.1.

Lemma 9.24 *Assume (L,v) to be henselian and K to be relatively separable-algebraically closed in L. Then (K,v) is henselian too. Further, $\overline{K}$ is relatively separable-algebraically closed in $\overline{L}$. If in addition $\overline{K}=\overline{L}$, then the torsion subgroup of vL/vK is a p-group, with p the characteristic exponent of $\overline{K}$.*

Proof: By Theorem 9.1, a valued field is henselian if and only if it satisfies Hensel's Lemma in the version 4″). If (L,v) is henselian and $f\in K[X]$ is separable over K and satisfies the assumptions of 4″), then there exists a root $a\in L$ according to 4″). The minimal polynomial of a over K divides f and is thus separable over K. Since K is assumed to be relatively separable-algebraically closed in L, it follows that $a\in K$. Hence also (K,v) has property 4″), that is, (K,v) is henselian. (Note that a proof using the uniqueness of extensions which serves as the definition for the property "henselian" is not at all as obvious as this proof which uses Hensel's Lemma. For the proof of the fact that every algebraic extension of a henselian field is henselian it is just the opposite case.)

Now let $\overline{a}\in\overline{L}$ be separable-algebraic over $\overline{K}$ and let $\overline{g}(X)\in\overline{K}[X]$ be its irreducible polynomial over $\overline{K}$. We choose a monic polynomial $g(X)\in K[X]$ whose reduction modulo v is $\overline{g}(X)$. Since $\overline{a}$ is assumed to be separable-algebraic over $\overline{K}$ we know that $\overline{a}$ is a simple root of $\overline{g}$. Hence by Hensel's Lemma, there is a root $a\in K$ of g whose residue is $\overline{a}$, showing that $\overline{a}\in\overline{K}$. This proves that $\overline{K}$ is relatively separable-algebraically closed in $\overline{L}$.

Assume that $\overline{K}=\overline{L}$ and let $\alpha\in vL$ and $n\in\mathbb{N}$ not divisible by p such that $n\alpha\in vK$. Choose $a\in L$ and $b\in K$ such that $va=\alpha$ and $vb=n\alpha$. Then $v(a^n/b)=0$. Since we have $\overline{K}=\overline{L}$, there exists some $c\in K$ satisfying $vc=0$ and $\overline{c}=\overline{a^n/b}$, hence $\overline{a^n/bc}=1$. So a^n/bc is a 1-unit, and since n is not divisible by p, Corollary 9.20 (applied to (L,v)) shows that it admits an n-th root $d\in L$. We have that $(a/d)^n=bc\in K$. Since the polynomial X^n-bc has no multiple roots, it is separable and thus, a/d is separable algebraic over K. Since K was assumed to be relatively separable-algebraically closed in L, we find that $a/d\in K$. Since $vc=0$, we have the equality $nv(a/d)=vbc=vb=n\alpha$, which shows that $\alpha=v(a/d)\in vK$. □

Concerning the embedding of a residue field in a valued field, we can derive the following assertion:

Lemma 9.25 *Let (K,v) be a henselian field of equal characteristic. Let k be the prime field of K and $\overline{K}$. If $k'|k$ is a separably generated subextension of $\overline{K}|k$, then there is a field embedding of k' in K such that $\overline{\iota\zeta}=\zeta$ for all $\zeta\in\overline{K}$. If (K,v) is of residue characteristic 0, then $\overline{K}$ itself admits such an embedding.*

Proof: Let $\mathcal{T}$ be a transcendence basis of $k'|k$ such that $k'|k(\mathcal{T}$ is a separable algebraic extension. For every $t\in\mathcal{T}$, choose an element $a_t\in K$ such that $\overline{a_t}=t$. Then by Lemma 7.26, the elements a_t, $t\in\mathcal{T}$, are algebraically valuation independent and thus also algebraically independent in K (over k). Hence, the assignment $t\mapsto a_t$ induces a field embedding ι of $k(\mathcal{T})$ in K over k. In view of Lemma 7.31, it satisfies that $\overline{\iota\zeta}=\zeta$ for all $\zeta\in k(\mathcal{T})$. Now we take K_0 to be the relative algebraic closure of $\iota k(\mathcal{T})$ in K. By Lemma 9.24, its residue field contains the relative separable-algebraic closure of $k(\mathcal{T})$ in $\overline{K}$. Thus, it also contains k' because $k'|k(\mathcal{T})$ is separable algebraic. But K_0 is an algebraic extension of the trivially valued field $\iota k(\mathcal{T}),v)$ and is thus also trivially valued. Consequently, the residue map restricted to K_0 is an isomorphism. Its inverse, restricted to k', is the required embedding of k' in K.

The second assertion follows from what we have just proved and the fact that every extension of fields of characteristic 0 is separably generated. □

As a further corollary to Theorem 9.7 and Lemma 9.24, we have something like a converse to Theorem 7.114. Recall that the completion of a valued field is an immediate extension. So if a valued field has rank 1, then the same is true for its completion.

Corollary 9.26 *The relative separable-algebraic closure of a valued field of rank 1 in its completion is henselian.*

A valued field (K, v) is called **finitely ramified** if it admits a prime element π (i.e., $v\pi$ is the least positive element in vK), and there is a prime p and a natural number e such that $vp = ev\pi$. Such a finitely ramified field has residue characteristic p; indeed, $vp > 0$ shows that $p = 0$ in $\overline{K}$. On the other hand, $vp = ev\pi < \infty$ shows that the characteristic of K is $\neq p$ and thus, it must be equal to 0. This shows that every finitely ramified field is of mixed characteristic.

An important subclass of the class of finitely ramified fields is that of formally $\wp$-adic fields. We call a finitely ramified field a **formally $\wp$-adic field** if its residue field is a finite field. If in addition, this residue field is equal to $\mathbb{F}_p$ and p is a prime element of the field, then we call it a **formally p-adic field**.

In Lemma 9.24, we have shown that if K is relatively algebraically closed in a henselian field (L, v) such that $\overline{K} = \overline{L}$, then the order of every torsion element in vL/vK is a power of the characteristic exponent of $\overline{K}$. This result can be improved for extensions of finitely ramified fields.

Lemma 9.27 *Let (L, v) be a henselian finitely ramified field and K relatively algebraically closed in L. If $\overline{L}|\overline{K}$ is separable algebraic and if (L, v) and (K, v) have a common prime element π, then (K, v) is a henselian field, $\overline{L} = \overline{K}$ and vL/vK is torsion free.*

Proof: As in the proof of Lemma 9.24 it is shown that (K, v) is henselian and that $\overline{L} = \overline{K}$.

Let p be the residue characteristic of (L, v). It suffices to show the following: If $a \in L$ such that $pva \in vK$, then $va \in vK$. Without loss of generality, we can assume that $va \geq 0$; otherwise, we replace va by $-va$ and a by a^{-1}. Let $b = a^p\tilde{c}$, where $\tilde{c} \in K$ is chosen such that $v\tilde{c} = -va^p$, that is, $vb = 0$. We claim that $mv\pi \in \Lambda(b, K)$ for every positive integer m. Assume the contrary, that is, that there exists a minimal $m_0 \geq 0$ such that $(m_0 + 1)v\pi \notin \Lambda(b, K)$. In view of $m_0v\pi \in \Lambda(b, K)$, let $c_0 \in K$ such that $v(b - c_0) \geq m_0v\pi$. Since $vL = \mathbb{Z}v\pi$, the inequality $v(b - c_0) > m_0v\pi$ would imply that $(m_0 + 1)v\pi \in \Lambda(b, K)$, a contradiction. Hence $v((b - c_0)\pi^{-m_0}) = 0$. Since $\overline{L} = \overline{K}$ by assumption, there is some $c \in K$ such that $v((b - c_0)\pi^{-m_0} - c) > 0$. That is, $v(b - c_0 - \pi^{m_0}c) > m_0v\pi$. As before, it follows that $(m_0 + 1)v\pi \in \Lambda(b, K)$, a contradiction. Hence, $mv\pi \in \Lambda(b, K)$ for every positive integer m.

By what we have shown, we can choose $c \in K$ such that $v(b - c) > 2ev\pi = 2vp$. Then $vc = vb = 0$, and $1 + \frac{b-c}{c}$ is a 1-unit of level $> 2vp$. Hence, in the henselian field (L, v) there is some unit $\tilde{a}$ such that $\tilde{a}^p = 1 + \frac{b-c}{c}$. Consequently, $b = c\tilde{a}^p$, that is, $(a/\tilde{a})^p = c/\tilde{c} \in K$. Since K was assumed to be relatively algebraically closed in L, we find that $a/\tilde{a} \in K$. This shows that $va = v(a/\tilde{a}) \in vK$. □

9.6 Henselian places

Remark 9.28 Apart from the paper "Allgemeine Bewertungstheorie" of W. Krull [KRU7], the paper "The theory of real places" by S. Lang [LANG2] was the first attempt to study systematically the relation between formally real fields and their places. Since then, this approach has been very fruitful (the list of papers at the beginning of Section 5.4 is certainly not exhaustive). For an interesting historical survey, see the notes on the literature of §5 in T. Y. Lam [LAM1]. For further reading, we refer the reader to §7 and §8 in A. Prestel [PR1], §2 and §3 of Chapter III in S. Prie-Crampe [PC1], and §5 of [LAM1].

The choice of the material in the present section and the last part of Section 5.4 is inspired by the paper [RIB15] of P. Ribenboim, in which the results concerning euclidean fields, the level and the Pythagoras number were proved for generalized power series fields.

In this section, we will give examples for properties that are transferred through henselian places between valued fields and their residue fields. In particular, we will consider properties that have been introduced in Section 5.4, and complement them in the case of henselian places.

Recall that if P is a real place of the field K, then K is formally real by virtue of Lemma 5.14, and Corollary 5.18 shows that v_P is compatible with some ordering of K. Recall further that a field K is called **euclidean** if it is formally real and a or $-a$ is a square for every $a \in K$. In view of the condition that K be formally real, the latter condition is equivalent to the condition that *either* a *or* $-a$ is a square whenever $0 \neq a \in K$. The standard example for a euclidean field is $\mathbb{R}$. We leave it to the reader to prove that every relatively algebraically closed subfield of a euclidean field is again euclidean.

Lemma 9.29 *Let P be a real place of the field K. If K is euclidean, then v_PK is 2-divisible and KP is euclidean. The converse holds if (K,P) is henselian.*

Proof: The first assertion was already stated for convex valuations in Section 5.4. Cf. also Corollary 9.22. Now assume that (K,P) is henselian, v_PK is 2-divisible and KP is euclidean. Since KP is formally real, it is of characteristic 0. Let $a \in K$. Since v_PK is 2-divisible, there is some $b \in K$ such that $v_Pab^2 = 0$. Since KP is euclidean, either $(ab^2)P$ or $-(ab^2)P$ is a square in KP. Since 2 is not divisible by the residue characteristic of (K,P), it follows from Lemma 9.19 that ab^2 or $-ab^2$ and thus also a or $-a$ is a square in K. Since K is also formally real by Lemma 5.14, we have thus proved that it is euclidean. □

Let us observe that a euclidean field admits precisely one ordering. Indeed, since a euclidean field is formally real, it admits at least one ordering. On the other hand, every positive cone contains all squares. So it contains precisely all squares since it can not contain $a \neq 0$ and $-a$ at the same time. Hence, the only possible positive cone on K is the set of all squares of K.

A field K is called **real closed** if K is formally real but no proper algebraic extension of K is formally real. E. Artin and O. Schreier have proved that K is real closed if and only if $K(\sqrt{-1}) = \tilde{K}$, and this is the case if and only if the absolute Galois group of K is finite (in this case, it is $\mathbb{Z}/2\mathbb{Z}$); cf. [AR–SCHR]. We wish to characterize real closed by their convex places.

Lemma 9.30 *Let K be a real closed field with a real place P. Then (K,P) is henselian, v_PK is divisible and KP is real closed.*

Proof: Since P is assumed to be a real place, KP is formally real. Since the henselization $(K,P)^h$ is an immediate extension of (K,P), its residue field is still KP. Hence also K^h

is formally real. By our assumption on K we must have that $K^h = K$, that is, (K, P) is henselian.

Assume that k is a formally real algebraic extension of KP. Further, let $\widetilde{v_P K}$ denote the divisible hull of $v_P K$. Then by Theorem 7.57 there exists an algebraic extension $(L|K, P)$ such that $LP = k$ and $v_P L = \widetilde{v_P K}$. But then, also L is formally real and by assumption on K we must have that $L = K$. Hence, $v_P K = \widetilde{vK}$ is divisible, and $KP = k$ shows that KP is real closed. □

Corollary 9.31 *Every real closed field is euclidean. Consequently, every real closed field admits precisely one ordering, the positive cone of it being the set of all squares.*

Proof: Let K be a real closed field and P the natural place with respect to some ordering of K. Then KP is a subfield of $\mathbb{R}$. Let k be the relative algebraic closure of KP in $\mathbb{R}$. Then k is euclidean. In particular, k is formally real, so the foregoing lemma shows that $k = KP$. Again by the foregoing lemma, (K, P) is henselian and $v_P K$ is 2-divisible. Now it follows from Lemma 9.29 that K is euclidean. □

Theorem 9.32 *Let P be a henselian place of K. If* $\operatorname{char} KP \neq 2$*, then the level of K is equal to the level of $\overline{K}$.*

Proof: We have already shown in Section 5.4 that the level of KP is smaller or equal to the level of K. It remains to show that the level of K is smaller or equal to the level of KP. If the latter is ∞, then there is nothing to show. Now assume that $-1 = \sum_{i=1}^{n} \zeta_i^2$ with $\zeta_i \in KP$. We choose elements $a_i \in K$ such that $a_i P = \zeta_i$. Then $(a_1 P)^2 = -1 - \sum_{i=2}^{n} (a_i P)^2$. Since (K, P) is assumed to be henselian, we can infer from Lemma 9.19 that $-1 - \sum_{i=2}^{n} a_i^2$ is a square, say b_1^2 with $b_1 \in K$. Then $-1 = b_1^2 + a_2^2 + \ldots + a_n^2$, showing that the level of K is at most n. This proves that the level of K is smaller or equal to the level of KP. □

Since a field is formally real if and only if its level is ∞, we obtain: *If (K, P) is henselian and KP is of characteristic $\neq 2$ and not formally real, then K is not formally real.* This suggests that the residue field of a henselian place of a formally real field is again formally real. The only disturbing point is the condition on the characteristic. To overcome it, we prove a more informative theorem which is due to M. Knebusch and M. Wright [KN–WR].

Theorem 9.33 *A henselian valuation v of a formally real field K is compatible with every ordering of K.*

Proof: We consider the polynomial $X^2 + X + a$ for every $a \in \mathcal{M}_v$. Its reduction $X^2 + X$ has two distinct roots (independently of the characteristic of $\overline{K}$!). Since (K, v) is assumed to be henselian, it follows that $X^2 + X + a$ admits a root b in K. We have that $0 = b^2 + b + a = (b + \frac{1}{2})^2 + a - \frac{1}{4}$, hence $\frac{1}{4} - a$ is a square in K. So $a \leq \frac{1}{4} < 1$ for every ordering of K and every $a \in \mathcal{M}_v$. By Lemma 5.15, this shows that v is compatible with every ordering of K. □

In view of Lemma 5.14 and Lemma 5.16, we conclude:

Corollary 9.34 *If (K, P) is henselian, then K is formally real if and only if KP is formally real.*

The following theorem was proved by A. Prestel [PR1], Theorem (8.6).

Theorem 9.35 *Let P be a real place of the field K. Then K is real closed if and only if*
1) $v_P K$ is divisible,
2) KP is real closed,
3) (K, v_P) is henselian.

Proof: Implication "$\Rightarrow$" is the assertion of Lemma 9.30. To prove "$\Leftarrow$", assume that 1), 2) and 3) hold. Since KP is formally real, its characteristic is 0. Hence by Theorem 8.22, (K, v_P) is defectless. By assumption 3), it is also henselian, and by assumption 1), every algebraic extension of $v_P K$ is trivial. So for every finite extension $L|K$, the unique extension of P from K to L satisfies $[L : K] = [LP : KP]$. That is, if $L|K$ is nontrivial, then $LP|KP$ is nontrivial and it follows from assumption 2) that LP is not formally real. Since (L, v_P) is again henselian (cf. Lemma 7.102), we obtain from the foregoing corollary that L is not formally real. We have thus proved that K is real closed. □

In view of our convention that the residue field of the natural valuation is a subfield of $\mathbb{R}$, we can deduce from this theorem the following assertion:
Let P be the natural place of the ordered field $(K, <)$. Then K is real closed if and only if 1) $v_P K$ is divisible, 2) KP is relatively algebraically closed in $\mathbb{R}$, 3) (K, P) is henselian.
For the proof of implication "$\Leftarrow$" of the above theorem, we can also employ the characterization of real closed fields by their Galois groups. Let us assume that (K, P) is a henselian field with divisible value group $v_P K$. If we also know that KP has characteristic 0, then we can infer from Lemma 8.24 that $\mathrm{Gal}\, K \cong \mathrm{Gal}\, KP$. Hence $\mathrm{Gal}\, K \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\mathrm{Gal}\, KP \cong \mathbb{Z}/2\mathbb{Z}$, showing that K is real closed if and only if KP is real closed.

Using the characterization by Galois groups, we are able to classify the valuations of a real closed field. The following theorem is due to M. Knebusch and M. Wright [KN–WR].

Theorem 9.36 *Let v be a valuation of the real closed field K. Then $\overline{K}$ is real closed or algebraically closed. $\overline{K}$ is real closed if and only if (K, v) is henselian, and this in turn holds if and only if v is compatible with the unique ordering of K.*

Proof: Since K is assumed to be real closed, we have that $[\tilde{K} : K] = 2$. Hence if (K, v) is not henselian, that is, if there is more than one extension of v from K of $\tilde{K}$, then by the fundamental inequality (8.5), there are precisely two extensions v_1, v_2, and both are immediate. Then $\overline{K} = \tilde{K} v_1$ is algebraically closed (cf. Lemma 7.59). On the other hand, if (K, v) is henselian, then we know from Theorem 9.33 that v is compatible with the unique ordering of K. If the latter holds, then the place associated with v is a real place by Lemma 5.16, and it follows from Theorem 9.35 that $\overline{K}$ is real closed. If the latter holds, then $[\tilde{K} : K] = 2 = [\tilde{\overline{K}} : \overline{K}]$, and the fundamental inequality shows that v admits a unique extension from K to $\tilde{K}$, i.e. (K, v) is henselian. □

Now we consider the Pythagoras number.

Theorem 9.37 *Let P be a real henselian place of the field K. Then the Pythagoras number of K is equal to that of KP.*

Proof: We have already shown in Section 5.4 that the Pythagoras number of KP is smaller or equal to that of K. It remains to show that the Pythagoras number of K is smaller or equal to that of KP. Let n be the Pythagoras number of KP and $a_1, \ldots, a_m \in K$ with $m > n$. We have to show that $\sum_{i=1}^{m} a_i^2$ is the sum of at most n squares in K. Without

loss of generality we may assume that $v_P a_1 \leq v_P a_i$ for all i. It suffices to show that $s := 1 + \sum_{i=2}^{m} (a_i/a_1)^2$ is the sum of at most n squares in K. By assumption on a_1, we have that $v_P(a_i/a_1) \geq 0$ for all i, and $sP \neq 0$ is thus a sum of squares in KP. So it is the sum of at most n squares in KP. It follows as in the proof of Theorem 9.32, where we replace -1 by s, that also s is a sum of at most n squares in K. □

Finally, let us mention the following analogue to Theorem 9.35. Its proof is left to the reader.

Theorem 9.38 *Let P be a place of K and assume that* $\operatorname{char} KP = 0$. *Then K is algebraically closed if and only if*

1) $v_P K$ is divisible,

2) KP is algebraically closed,

3) (K, v_P) is henselian.

For $\operatorname{char} KP \neq 0$, *the assertion remains valid if condition 3) is replaced by*

3′) (K, v_P) is algebraically maximal.

Exercise 9.6 *Suppose that it was already shown that a subfield of $\mathbb{R}$ is real closed if and only if its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Deduce the same for every formally real field, using Theorem 9.35 and Lemma 8.24.*

9.7 Puiseux series fields

In this section, we shall consider a special class of valued fields which can be represented as the union over a countable ascending chain of power series fields. Let k be an arbitrary field and t transcendental over k. We choose elements $t_n \in \widetilde{K(t)}$ for every $n \in \mathbb{N}$ such that

1) $t_1 = t$,

2) $t_n^n = t$,

3) $t_n^\ell = t_m$ whenever $\ell m = n$.

Now we consider the power series fields $K_n := k((t_n))$ for $n \in \mathbb{N}$. Every two fields K_n and K_m are contained in the common extension field K_{nm}. Hence, the union $K := \bigcup_{n\in\mathbb{N}} K_n$ is again a field, and it is endowed with a valuation which extends the t_n-adic valuation of every K_n. (This follows from Lemma 13.7 and the fact that valued fields can be axiomatized by universal existential axioms). Such a field is called a **Puiseux series field over** k. We will denote its canonical valuation by v_t. We leave it to the reader to prove that up to isomorphism, the Puiseux series field over k does not depend on the particular choice of the t_n for $n > 1$ if k contains all roots of unity.

The field K can be viewed as the union over an ascending chain of fields K_{n_i}. Indeed, there exists a sequence n_i, $i \in \mathbb{N}$, of natural numbers such that

1) n_i divides n_j whenever $i < j$,

2) for every $m \in \mathbb{N}$ there is some $i \in \mathbb{N}$ such that m divides n_i.

For the fields K_{n_i} this means that

1) $K_{n_i} \subset K_{n_j}$ whenever $i < j$,

2) for every $m \in \mathbb{N}$ there is some $i \in \mathbb{N}$ such that $K_m \subset K_{n_i}$.

Consequently, the fields K_{n_i} form an ascending chain of fields with union K.

For the proof of the following theorem, we make significant use of the fact that a valued field is henselian if and only if it satisfies Hensel's Lemma (cf. Theorem 9.1).

Theorem 9.39 *Every Puiseux series field K over k is henselian with respect to its canonical valuation v_t. Its residue field is k and its value group is isomorphic to $\mathbb{Q}$ and thus divisible. Further, K is an algebraic extension of $k((t))$.*

Proof: In order to show that (K, v_t) is henselian, it suffices to prove that it has property 4) of Theorem 9.1, that is, it satisfies Hensel's Lemma. Let $f \in K[X]$ and $b \in K$ satisfy the assumptions of Hensel's Lemma. Since K is the union over the ascending chain of fields K_{n_i}, there must be some $i \in \mathbb{N}$ such that K_{n_i} contains the finitely many coefficients of f and the element b. But $K_{n_i} = k((t_{n_i}))$ is a power series field and thus henselian with respect to its t_{n_i}-adic valuation (cf. Corollary 8.27). Again by Theorem 9.1, K_{n_i} satisfies Hensel's Lemma. Thus, the required root a of f can already be found in the subfield K_{n_i} of K. This proves that (K, v_t) is henselian.

The value group of every K_n is $\mathbb{Z}vt_{n_i} = \mathbb{Z}\frac{vt}{n_i}$, so the union over all K_n has value group $\bigcup_{n\in\mathbb{N}} \mathbb{Z}\frac{vt}{n} = \mathbb{Q}vt$. This is divisible and isomorphic to $\mathbb{Q}$. The residue field of every K_n is k, hence also the residue field of the union of all K_n is k.

For every $n \in \mathbb{N}$, the element t_n is algebraic over $k((t))$. Hence it follows from Corollary 7.22 that $K_n = k((t_n)) = k((t))(t_n)$ is algebraic over $k((t))$. Consequently, also the union of all K_n is algebraic over $k((t))$. □

In view of Theorem 8.22 and Corollary 8.23, we conclude from the foregoing theorem:

Corollary 9.40 *Let k be a field of characteristic 0 and K a Puiseux series field over k with canonical valuation v_t. Then (K, v_t) is a henselian defectless field. Further, K is the algebraic closure of $k((t))$ if and only if k is algebraically closed.*

The assertion of this corollary does not hold if k has positive characteristic. This fact will not appear too odd if one reconsiders the principle of Example 8.49. This example has shown that even algebraic extensions can induce approximation types $\mathrm{appr}\,(a, K)$ for which the possible values $v(a - c)$ with $c \in K$ run through all value groups vK_{n_i}, or to be more precise, where every value $-vt_{p^j}$, $j \in \mathbb{N}$, appears as the value $v(a - c)$ for some $c \in K$.

Example 9.41 In Example 8.49 resp. Example 9.14, we replace $\mathbb{F}_p$ by an arbitrary field k of characteristic $p > 0$. Again, we let a be a root of the polynomial $X^p - X - 1/t$. Everything works the same, and we again obtain that $(k((t))^{1/p^\infty}(a)|k((t))^{1/p^\infty}, v_t)$ is nontrivial and immediate and that the distance of a is 0. A straightforward modification of the procedure of Example 8.47 shows that $k((t))^{1/p^\infty}$ can be replaced by every Puiseux series field K over k and still, $(K(a)|K, v_t)$ will be immediate with $\mathrm{dist}\,(a, K) = 0$. Note that this also shows that even the completion of K is not algebraically closed. ♢

Our example proves:

Theorem 9.42 *Let k be a field of characteristic $p > 0$ and K be a Puiseux series field over k with canonical valuation v_t. Then (K, v_t) is not defectless. In particular, K is not algebraically closed, even if k is algebraically closed.*

We can further conclude that the union over an ascending chain of power series field of positive characteristic may not be defectless and thus not spherically complete. Consequently, such a union can have quite different properties than the fields in the chain.

On the other hand, if k is of characteristic 0, then the Puiseux series fields over k are well behaved, and they are nice examples for fields having henselian places with divisible value group. For instance, Theorem 9.39 in combination with Theorem 9.35 and Theorem 9.36 shows:

Theorem 9.43 *Let k be a field of characteristic 0 and K a Puiseux series field over k. Then K is real closed if and only if k is.*

One can always construct henselian defectless fields having residue field k and divisible value group, even if k has positive characteristic. By Corollary 8.28, it suffices to take a maximal immediate extension of the field $(k(t_n \mid n \in \mathbb{N}), v_t)$. But this field is "very large": it has uncountable transcendence degree over $k((t))$.

The p-adic counterpart of Theorem 9.43 was proved by G. Cherlin in [CH1]. We have to postpone it to the next chapter since we have to employ the theory of the composition of valuations.

Chapter 10

Composition of valuations and places

If P_1 is a place on the field K and P_2 is a place on its residue field KP_1, then the **composition** P_1P_2 of P_1 and P_2 is a place on K with residue field $KP_1P_2 = (KP_1)P_2$. If $v = v_{P_1}$ and $w = v_{P_2}$ are the valuations associated with P_1 and P_2 respectively, then the valuation associated with P_1P_2 is denoted by $v \circ w$ and called the **composition of** v **and** w. The value group $v \circ wK$ is isomorphic $vK \amalg wKP_1$ (we will frequently identify both groups). The valuation v is said to be a **coarsening of** $v \circ w$. Here, w may be trivial, i.e., we view v as a coarsening of itself. For the basic properties of composite valuations and places, we refer the reader to [ZA–SA2]. Here, let us state some special properties that we will need later. When we talk of "unique extensions" of places, then we mean up to equivalence. With this use, a valued field (K, P) is henselian if and only if P admits a unique extension to every algebraic extension field of K.

Lemma 10.1 *Let P be a place on K with $P = P_1P_2$. If $L|K$ is an algebraic extension and Q is an extension of P to L, then there is a unique extension Q_1 of P_1 to L and a unique extension Q_2 of P_2 to LQ_1 such that $Q = Q_1Q_2$. If $L|K$ is finite, then*

$$(v_QL : v_PK) = (v_{Q_1}L : v_{P_1}K)(v_{Q_2}LQ_1 : v_{P_2}KP_1) .$$

Lemma 10.2 *Let P be a place on K with $P = P_1P_2$. Further, let $L|K$ be a finite extension and $Q_{11}, \ldots, Q_{1n}$ all extensions of P_1 to L. Then (K, P) is defectless in L if and only if (K, P_1) is defectless in L and (KP_1, P_2) is defectless in every LQ_{1i}, $1 \leq i \leq n$.*

Theorem 10.3 *Let P be a place on K with $P = P_1P_2$. Then:*

a) (K, P) is henselian if and only if (K, P_1) and (KP_1, P_2) are henselian.

b) (K, P) is a defectless field if and only if (K, P_1) and (KP_1, P_2) are defectless fields.

Proof: a): Assume that (K, P) is not henselian, and that $L|K$ is a finite extension such that P admits two extensions Q, Q' to L. If also P_1 admits two extensions to L, then (K, P_1) is not henselian. Otherwise, if P_1 has a unique extension Q_1 to L, then we have $Q = Q_1Q_2$ and $Q' = Q_1Q_2'$ with two extensions $Q_2 \neq Q_2'$ of P_2 to LQ_1, showing that (KP_1, P_2) is not henselian.

Assume that (K, P_1) is not henselian, and that $L|K$ is a finite extension such that P_1 admits two extensions $Q_1 \neq Q_1'$ to L. Let Q_2 be an extension of P_2 to LQ_1 and Q_2' an extension of P_2 to LQ_1'. Then $Q = Q_1Q_2$ and $Q' = Q_1'Q_2'$ are two distinct extensions of P to L, showing that (K, P) is not henselian.

Now assume that (KP_1, P_2) is not henselian, and that $L_1|KP_1$ is a finite extension such that P_2 admits two extensions $Q_2 \neq Q_2'$ to L_1. By Theorem 7.57 there exists an algebraic extension (L, Q_1) of (K, P_1) such that $LQ_1 = L_1$. Now $Q = Q_1Q_2$ and $Q' = Q_1Q_2'$ are two distinct extensions of P to L, showing again that (K, P) is not henselian.

b): This follows from the foregoing lemmata; if there is a finite extension $L_1|KP_1$ such that (KP_1, P_2) is not defectless in L_1, then one uses again Theorem 7.57 to obtain an extension (L, Q_1) of (K, P_1) such that $LQ_1 = L_1$. □

Lemma 10.4 *Let P be a place on K with $P = P_1P_2$. Further, let $L|K$ be a finite extension such that P admits a unique extension Q to L. Then also the extension Q_1 of P_1 to L is unique, and the extension Q_2 of P_2 to LQ_1 is unique. Further,*

$$d(L|K, Q) = d(L|K, Q_1) \cdot d(LQ_1|KP_1, Q_2) \ .$$

Proof: The proof of the first assertion is straightforward. For the defects, we compute:

$$\frac{[L:K]}{(v_QL : v_PK)[LQ : KP]} = \frac{[L:K]}{(v_{Q_1}L : v_{P_1}K)(v_{Q_2}LQ_1 : v_{P_2}KP_1)[LQ_1Q_2 : KP_1P_2]}$$

$$= \frac{[L:K]}{(v_{Q_1}L : v_{P_1}K)[LQ_1 : KP_1]} \cdot \frac{[LQ_1 : KP_1]}{(v_{Q_2}LQ_1 : v_{P_2}KP_1)[LQ_1Q_2 : KP_1P_2]} \ .$$

□

Lemma 10.5 *Let $(L|K, P)$ be an arbitrary extension and $P = P_1P_2$. Then $(L|K, P)$ is immediate if and only if $v_{P_1}L = v_{P_1}K$ and $(LP_1|KP_1, P_2)$ is immediate.*

Theorem 10.6 *Let P be a place on K with $P = P_1P_2$. Then (K, P) is algebraically maximal if and only if (K, P_1) and (KP_1, P_2) are.*

For the proof of the next theorem, one uses that by Theorem 7.57 for every algebraic extension $L_1|KP_1$ there exists a *separable* algebraic extension (L, Q_1) of (K, P_1) such that $LQ_1 = L_1$, provided that Q_1 is nontrivial on K.

Theorem 10.7 *Let P be a place on K with $P = P_1P_2$. Assume that P_1 is nontrivial on K. Then:*

a) (K, P) is a separably defectless field if and only if (K, P_1) is a separably defectless field and (KP_1, P_2) is a defectless field.

b) (K, P) is separable-algebraically maximal if and only if (K, P_1) is separable-algebraically maximal and (KP_1, P_2) is algebraically maximal.

Lemma 10.8 *The following are equivalent for a valued field (K, v):*

a) (K, v) is complete,

b) (K, w) is complete for every coarsening of v,

c) (K, w) is complete for some nontrivial coarsening of v.

Take a nontrivially valued field (k, v) which is neither complete nor henselian. Since the power series field $(k((t)), v_t)$ is complete, it follows by the foregoing lemma that $(k((t)), v_t \circ v)$ is complete, although (k, v) is not complete. By Theorem 10.3, $(k((t)), v_t \circ v)$ is not henselian. (Note that by Theorem 9.7, every complete field of rank 1 is henselian, but that the constructed field is of rank at least two.) We note:

Corollary 10.9 *There are complete valued fields which are not henselian.*

10.1 The Hensel-Rychlik Property

We will now add two further properties to the list of equivalent forms of Hensel's Lemma. For an arbitrary polynomial f of degree n with roots $a_1, \ldots, a_n$ and leading coefficient c_n, we define the **discriminant of** f to be

$$\operatorname{discr} f \;:=\; (-1)^{\frac{n(n-1)}{2}} c_n^{2n-2} \prod_{1\le i<j\le n} (a_i - a_j)^2 \;=\; c_n^{2n-2} \prod_{i\neq j} (a_i - a_j)\,. \tag{10.1}$$

We will also need the following notion. Two valuations v and w of a field K are called **independent** if they do not admit a nontrivial common coarsening, that is, if there is no nontrivial valuation ring $\mathcal{O}$ of K which contains both $\mathcal{O}_v$ and $\mathcal{O}_w$.

Theorem 10.10 *For a valued field $\mathbf{K} = (K, v)$, the property of being henselian is equivalent to each of the following properties:*

1) (strong "Hensel-Rychlik")
For every polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ the following holds: if some $b \in K$ satisfies $vf(b) > v \operatorname{discr} f$, then f admits a root in K.

2) (weak "Hensel-Rychlik")
For every monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ the following holds: if some $b \in K$ satisfies $vf(b) > v \operatorname{discr} f$, then f admits a root in K.

Proof: Every henselian field (K, v) satisfies property **1)**:
The following proof is due to E. Ehrhardt, a student of F. Lorenz (cf. [LOR2], pp. 112–113).

Assume that f satisfies the assumption of **1)**, and let c denote its leading coefficient. Let $n = \deg f$ and $a_1, \ldots, a_n$ be the roots of f in $\tilde{K}$, enumerated such that $va_1 \le va_2 \le \ldots \le va_n$. Choose $m \le n$ such that $a_{m+1}, \ldots, a_n$ are precisely the integral roots of f. Since f has integral coefficients, it follows from part b) of Lemma 7.42 that $vc + va_1 + \ldots + va_m \ge 0$. Hence, $vc + va_1 + \ldots + va_i \ge vc + va_1 + \ldots + va_m \ge 0$ for every i. We write

$$\operatorname{discr} f = \prod_{i=2}^{n} d_i \quad \text{with} \;\; d_i := c^2 \prod_{j=1}^{i-1} (a_i - a_j)^2\,.$$

From our enumeration of the roots it follows that $v(a_i - a_j) \ge va_j$ for every $j \le i$. Hence for every i,

$$vd_i \;=\; 2vc + \sum_{j=1}^{i-1} 2v(a_i - a_j) \;\ge\; 2(vc + \sum_{j=1}^{i-1} va_j) \;\ge\; 0\,. \tag{10.2}$$

Further,

$$vc + \sum_{\substack{j=1\\ j\neq k}}^{i-1} v(a_i - a_j) \;\geq\; vc + \sum_{\substack{j=1\\ j\neq k}}^{i-1} va_j \;\geq\; vc + \sum_{j=1}^{m} va_j \;\geq\; 0 \tag{10.3}$$

for every i, because $a_1, \ldots, a_m$ are precisely the roots of negative value by assumption.

Now suppose that there are two indeces $k < \ell$ such that $v(b-a_k) = v(b-a_\ell) \geq v(b-a_i)$ for every i. Note that $v(a_\ell - a_i) \geq v(b - a_i)$ for every i. From (10.3), we obtain that

$$vd_\ell \;=\; v(c\prod_{\substack{j=1\\ j\neq k}}^{\ell-1}(a_\ell - a_j)) + v(c(a_\ell - a_k)^2 \prod_{\substack{j=1\\ j\neq k}}^{\ell-1}(a_\ell - a_j)) \;\geq\; v(c(a_\ell - a_k)^2 \prod_{\substack{j=1\\ j\neq k}}^{\ell-1}(a_\ell - a_j))$$

and for $i > \ell$,

$$vd_i \;=\; v(c\prod_{j=1}^{i-1}(a_i - a_j)) + v(c\prod_{\substack{j=1\\ j\neq \ell}}^{i-1}(a_i - a_j)) + v(a_i - a_\ell) \;\geq\; v(a_i - a_\ell) \;=\; v(a_\ell - a_i)\;.$$

With these inequalities and with $vd_i \geq 0$ for $i < \ell$, which we know from (10.2), we deduce:

$$\begin{aligned} v\,\mathrm{discr}\, f \;&\geq\; v\prod_{i\geq \ell} d_i \;\geq\; v(c(a_\ell - a_k)^2 \prod_{\substack{j=1\\ j\neq k,\ell}}^{n}(a_\ell - a_j)) \\ &\geq\; v(c(b-a_k)(b-a_\ell)\prod_{\substack{j=1\\ j\neq k,\ell}}^{n}(b-a_j)) \;=\; vf(b)\;. \end{aligned}$$

But this contradicts our assumption that $vf(b) > v\mathrm{discr}\, f$. Hence, there is an index k such that $v(a_k - b) > v(a_i - b)$ for all $i \neq k$. All conjugates of a_k over K are among the a_i. Since $v\,\mathrm{discr}\, f < \infty$, we have that $a_i \neq a_j$ for $i \neq j$. This shows that a_k is separable over K. Since Krasner's Lemma holds in every henselian field by Theorem 9.1, we find that $a_k \in K(b) = K$, which shows that f admits a root in K.

The implication **1)⇒2)** is trivial. Now assume that (K, v) is not henselian. We wish to show that **2)** does not hold. Let $L|K$ be a finite Galois extension which admits at least two extensions of v from K to L. Let $v_1, \ldots, v_n$ be all extensions of v to L. If w_i is a coarsening of v_i then let $\overline{w}_i$ denote the valuation induced by v_i on Lw_i (such that $v_i = w_i \circ \overline{w}_i$), let w denote the restriction of w_i to K and $\overline{w}$ the restriction of $\overline{w}_i$ to Kw. We are going to prove that there is some i and a coarsening w_i of v_i such that all extensions of $\overline{w}$ from Kw to Lw_i are independent.

If already $v_1, \ldots, v_n$ are independent, then there is nothing to show. Otherwise, we have a nonempty set $\mathfrak{O}$ of all nontrivial valuation rings of L which contain at least two valuation rings $\mathcal{O}_{v_i}$ and $\mathcal{O}_{v_j}$. Observe that the intersection over every descending chain of rings $\mathcal{O}_\nu$, $\nu < \lambda$, ($\mathcal{O}_\nu \supset \mathcal{O}_\mu$ for $\nu \leq \mu$) is again a member of $\mathfrak{O}$. Indeed, it is a valuation ring since every intersection over a descending chain of valuation rings is a valuation ring, and it satisfies the above condition since there are only finitely many valuation rings $\mathcal{O}_{v_i}$. By Zorn's Lemma, it follows that $\mathfrak{O}$ has minimal elements. Let $\mathcal{O}$ be such a minimal element in $\mathfrak{O}$. After a suitable renumbering, we can assume that precisely $\mathcal{O}_{v_1}, \ldots, \mathcal{O}_{v_m}$ are the valuation rings contained in $\mathcal{O}$ ($1 < m \leq n$). Let w_1 be the common coarsening of $v_1, \ldots, v_m$ such that $\mathcal{O} = \mathcal{O}_{w_1}$, and let $\overline{w}_1, \ldots, \overline{w}_m$ be the valuations induced by $v_1, \ldots, v_m$

on Lw_1. Further, let w be the restriction of w_1 to K and let $\overline{w}$ denote the valuation induced by v on Kw. Then $\overline{w}_1, \ldots, \overline{w}_m$ are all extensions of $\overline{w}$ from Kw to Lw_1. Indeed, if $\overline{w}'$ is an extension of $\overline{w}$ from Kw to Lw_1, then $v' = w_1 \circ \overline{w}'$ is an extension of v from K to L which satisfies $\mathcal{O}_{v'} \subset \mathcal{O}$ and thus, $v' = v_i$ for some $i \leq m$. By the minimality of $\mathcal{O}$, there is no valuation ring properly contained in $\mathcal{O}$ which contains more than one of these rings. Consequently, there is no proper valuation ring of Lw_1 which contains more than one $\mathcal{O}_{\overline{w}_i}$. This shows that the extensions $\overline{w}_1, \ldots, \overline{w}_m$ are independent, and our claim is proved.

Note that by Lemma 7.76, the extension $Lw_1|Kw$ is normal. Let Z denote the decomposition field of $(Lw_1|Kw, \overline{w}_1)$. Since $m \geq 2$, Z is a proper extension of Kw. Since it is a separable extension, we can choose a primitive element $\overline{a}$ such that $\mathrm{Z} = Kw(\overline{a})$. W.l.o.g. we can choose $\overline{a}$ such that $\overline{w}_i\overline{a} \geq 0$ for all i. Let $\overline{f} \in \mathcal{O}_{(Kw,\overline{w})}[X]$ be the minimal polynomial of $\overline{a}$ over Kw. Set $\alpha := \overline{w}\operatorname{discr}\overline{f}$. Further, let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_m$ be the distinct embeddings of Z into L. Since the valuations $\overline{w}_1, \ldots, \overline{w}_m$ are independent on Lw_1, it follows that also the extensions of $\overline{w}$ from Kw to Z are independent. From Theorem 7.84 we can thus infer that $\overline{w}_1\sigma_i \neq \overline{w}_1$ on Z for all $i \neq 1$. From the Strong Approximation Theorem (cf. [ZA-SA2], §10, Theorem 18′) we infer the existence of some $\overline{b}_1 \in \mathrm{Z}$ such that such that $\overline{w}_1(\overline{a} - \overline{b}_1) > \alpha$ and $\overline{w}_1\sigma_i\overline{b}_1 > \alpha$ for all $i \neq 1$. For $\overline{b} := \mathrm{Tr}_{\mathrm{Z}|Kw_1}(\overline{b}_1)$ we find:

$$\overline{w}_1(\overline{a} - \overline{b}) = \overline{w}_1(\overline{a} - \overline{b}_1 - \sum_{i>1} \sigma_i\overline{b}_1 \geq \min\{\overline{w}_1(\overline{a} - \overline{b}_1), \overline{b}_1)\sigma_i\overline{b}_1 \mid i = 2, \ldots, m\} > \alpha \ .$$

Since $\overline{w}_1\overline{b} \geq 0$ and thus also $\overline{w}_1(\sigma_i\overline{a} - \overline{b}) \geq 0$ for all i, we find that

$$\overline{w}_1\overline{f}(\overline{b}) = \overline{w}_1(\overline{a} - \overline{b}) + \sum_{i>1} \overline{w}_1(\sigma_i\overline{a} - \overline{b}) > \alpha = \overline{w}\operatorname{discr}\overline{f} \ .$$

Now choose a monic polynomial $f \in \mathcal{O}_{\mathbf{K}}[X]$ whose w-reduction fw is $\overline{f}$. Further, choose $b \in K$ such that $bw = \overline{b}$. It follows that $vf(b) > v\operatorname{discr} f$. But f is irreducible over K since $\overline{f}$ is irreducible over Kw. This proves that **2)** does not hold. □

Note that in contrast to Newton's Lemma, the Hensel-Rychlik property gives a lower bound for the value of $vf(b)$ which does not depend on the approximative root b. Property **2)** is the version of the "Hensel-Rychlik" property used by Ax and Kochen in their important paper [AX–KOC1].

10.2 Formally $\wp$-adic and finitely ramified fields

We have given the definition of finitely ramified fields and formally $\wp$-adic fields in Section 8.3. Now let us state some important properties of these fields.

Lemma 10.11 *Every immediate extension of a finitely ramified field is again a finitely ramified field. Every immediate extension of a formally $\wp$-adic (resp. formally p-adic) field is again a formally $\wp$-adic (resp. formally p-adic) field.*

Proof: If (K, v) is a finitely ramified field and $(L|K, v)$ is an immediate extension, then because of $vL = vK$, the prime element of (K, v) will also be a prime element of (L, v), showing that also (L, v) is a finitely ramified field. If in addition, $\overline{K}$ is finite, then so is $\overline{L}$, and (L, v) is a formally $\wp$-adic field (resp. a formally p-adic field if $\overline{L} = \overline{K} = \mathbb{F}_p$. □

Lemma 10.12 *Let (K,v) be a valued field of mixed characteristic and P the place associated with v. Then P admits a decomposition $P = P_1P_2P_3$ where:*

1) P_1 is a (possibly trivial) place such that $\operatorname{char} KP_1 = 0$,

2) (KP_1, P_2) is a valued field of mixed characteristic and rank 1,

3) $\operatorname{char} KP_1P_2 > 0$ *(and P_3 is possibly trivial).*

If in addition (K,v) is finitely ramified, then P_3 is trivial and $v_{P_2}KP_1 \cong \mathbb{Z}$.

Proof: Let Γ be the least convex subgroup of vK containing vp and Δ the largest convex subgroup of vK not containing vp. Let P_1 be the coarsening of P having value group vK/Γ. On its residue field KP_1, the place induces a place $\overline{P}$ with value group Γ. Let P_2 be the coarsening of $\overline{P}$ with value group Γ/Δ and let P_3 be the place induced by $\overline{P}$ on KP_1P_2. Then $P = P_1P_2P_3$.

By our choice of P_1, we have that $v_{P_1}p = 0$, that is, the characteristic of KP_1 is not p. Since the characteristic of its residue field $KP = KP_1P_2P_3$ is p, it follows that the characteristic of KP_1 must be 0.

By our choice of P_2, we have that $v_{P_2}p > 0$, that is, $\operatorname{char} KP_1P_2 = p$. Since Γ/Δ is archimedean, P_2 is of rank 1.

If (K,v) is finitely ramified, then the least positive element in vK is archimedean equivalent to vp, that is, there are no elements which are archimedean smaller than vp. This yields $\Delta = \{0\}$, showing that P_3 is trivial. If π is the prime element of (K,v), then $\Gamma = \mathbb{Z}v\pi$, hence $v_{P_2}KP_1 = \Gamma \cong \mathbb{Z}$. □

Theorem 10.13 *Every finitely ramified field is a defectless field.*

Proof: Let (K,v) be a finitely ramified field. If P is the place associated with v, then by the foregoing lemma, it admits a decomposition $P = P_1P_2$ (where P_1 may be trivial) such that $\operatorname{char}(KP_1) = 0$ and P_2 is a place on KP_1 with value group $\cong \mathbb{Z}$ and residue characteristic $p > 0$. By Theorem 8.22, (K, P_1) is a defectless field. Theorem 8.32 shows that also (KP_1, P_2) is a defectless field. Now it follows by Theorem 10.3 that (K,v) is a defectless field. □

A formally $\wp$-adic field (K,v) is called a **$\wp$-adically closed field** if no proper algebraic extension has at the same time the same prime element and the same residue field as (K,v). Using Theorem 10.13, it is easy to prove a characterization of $\wp$-adically closed fields which is similar to that given for real closed fields in Theorem 9.35.

Theorem 10.14 *A formally $\wp$-adic field (K,v) is $\wp$-adically closed if and only if vK is a $\mathbb{Z}$-group and (K,v) is henselian. Every formally $\wp$-adic field (K,v) admits an algebraic extension which is $\wp$-adically closed and has the same prime element and residue field as (K,v).*

Proof: Assume that (K,v) is a formally $\wp$-adic field. Suppose first that vK is a $\mathbb{Z}$-group and (K,v) is henselian, and let $L|K$ be a proper finite extension. Since (K,v) is henselian and defectless by Theorem 10.13, we obtain that $(vL : vK) > 1$ or $[\overline{L} : \overline{K}] > 1$. In the first case, the least positive element α in vK is not anymore the least positive element in vL. This is seen as follows. Let Δ be the convex hull of $\mathbb{Z}\alpha$ in vL. Since $vK/\mathbb{Z}\alpha$ is divisible and $vL|vK$ is finite, it follows that also vL/Δ is divisible and that $1 < (vL : vK) = (\Delta : \mathbb{Z}\alpha) < \infty$. This yields that α is not anymore the least positive

element of Δ and of vL. In the second case where $[\overline{L} : \overline{K}] > 1$, we obtain that (K, v) and (L, v) do not have the same residue field. We have proved that (K, v) is $\wp$-adically closed.

Now suppose that (K, v) is not henselian. Then the henselization is a proper immediate extension, thus having the same prime element and the same residue field as (K, v). So (K, v) is not $\wp$-adically closed. Suppose that vK is not a $\mathbb{Z}$-group. By Lemma 3.34, there is a proper algebraic extension $\Gamma | vK$ such that Γ is a $\mathbb{Z}$-group having the same least positive element as vK. By Theorem 7.57, there is a proper algebraic extension $(L|K, v)$ such that $vL = \Gamma$ and $\overline{L} = \overline{K}$. This again shows that (K, v) is not $\wp$-adically closed.

The last argument also shows how to find an algebraic extension of (K, v) which is $\wp$-adically closed: If (K, v) is not henselian, replace (K, v) by its henselization. Then, if vK is not a $\mathbb{Z}$-group, choose (L, v) as above. □

Corollary 10.15 *Let K be a relatively algebraically closed subfield of L and assume that (L, v) is a $\wp$-adically closed field. If both (K, v) and (L, v) have a common prime element, then also (K, v) is a $\wp$-adically closed field, and $\overline{K} = \overline{L}$.*

Proof: Since $\overline{L}$ is a finite field, also $\overline{K}$ is a finite field and the extension $\overline{L}|\overline{K}$ is separable algebraic. Hence, Lemma 9.27 shows that $\overline{L} = \overline{K}$ and that vL/vK is torsion free. From the latter, it follows that also vK is a $\mathbb{Z}$-group. Hence by the foregoing theorem, (K, v) is $\wp$-adically closed. □

Now we are able to prove the $\wp$-adic counterpart of Theorem 9.43:

Theorem 10.16 *Let (k, v) be a valued field and K a Puiseux series field over k with canonical valuation v_t. Then $(K, v_t \circ v$ is a $\wp$-adically closed field if and only if (k, v) is.*

Proof: The residue field of (K, v_t) is k. Hence, $v_t \circ vK \cong v_t K \amalg vk$. So if π is a prime element in (k, v), then it is also a prime element in $(K, v_t \circ v)$. Conversely, if $(K, v_t \circ v)$ has a prime element π, then (k, v) has a prime element of the same value (for instance, πv_t). It follows that $(K, v_t \circ v)$ is formally $\wp$-adic if and only if (k, v) is.

Since $v_t K \cong \mathbb{Q}$ by Theorem 9.39, we find that $v_t \circ vK$ is a $\mathbb{Z}$-group if and only if vk is. Again by Theorem 9.39, (K, v_t) is henselian. Hence, part a) of Theorem 10.3 shows that $(K, v_t \circ v)$ is henselian if and only if (k, v) is. Now our assertion follows from the foregoing theorem. □

Remark 10.17 It can be shown that if a field K admits a valuation v such that (K, v) is $\wp$-adically closed, then this valuation is uniquely determined. Thus, let us call K $\wp$-adically closed if it admits such a valuation. Then the theorem can also be stated without mentioning the valuations: *Let K be a Puiseux series field over k. Then K is $\wp$-adically closed if and only if k is.*

10.3 Rolle fields

In this section, we shall give a generalization of a theorem on Rolle fields that P. Ribenboim [RIB15] has proved for fields of generalized power series. It follows the spirit of Section 9.6. See also F. Delon [DEL3] for further information on Rolle fields and Rolle rings. It should be noted that the following results admit a straitforward generalization to Rolle rings.

An ordered field $(K, <)$ is called a **Rolle field** if it satisfies the following **Rolle property**:

for every polynomial $f \in K[X]$ and every pair $a < b$ of elements of K such that $f(a) = f(b) = 0$, there exists an element $c \in K$ satisfying $a < c < b$ and $f'(c) = 0$

where f' denotes the formal derivative of f. The following lemma is an easy consequence of this definition:

Lemma 10.18 *Assume $(L, <)$ to be a Rolle field and $(K, <)$ is a subfield, relatively algebraically closed in L (the ordering being the restriction of the ordering of L). Then (K, v) is also a Rolle field.*

Proof: Given a polynomial $f \in K[X]$ and elements $a < b$ of K such that $f(a) = f(b) = 0$. Since (L, v) is a Rolle field, there exists an element $c \in L$ satisfying $a < c < b$ and $f'(c) = 0$. But like f, its derivative f' is also a polynomial with coefficients in K; since K is relatively algebraically closed in L, it follows that $c \in K$. □

Theorem 10.19 *Assume that P is a henselian place of the Rolle field $(L, <)$. Then LP admits an ordering with respect to which it is a Rolle field.*

Proof: Since L admits an ordering, it has characteristic 0. Consider the relative algebraic closure k of $\mathbb{Q}$ in L. Since k admits an ordering (the restriction of the ordering of L), it cannot have a nontrivial henselian p-adic valuation. Thus P is trivial on k which yields that the characteristic of LP is 0.

Since LP has characteristic 0 and (L, P) is henselian, there exists an embedding ι of LP into L such that $\forall x \in LP : (\iota x)P = x$ (i.e. ιLP is a field of representatives for the residue field LP). If ιLP would admit a proper algebraic extension L' within L then $L'P$ would be a proper extension of LP within LP which is absurd. Hence ιLP is relatively algebraically closed in L. Equipped with the restriction of the ordering of L it is thus a Rolle field by virtue of the above lemma. The same holds for LP with respect to the ordering induced by the isomorphism ι. □

In the following we will discuss the converse of Theorem 10.19. We need the characterization for Rolle fields which was given by R. Brown, T. C. Craven and M. J. Pelling in [BRW–CR–PE]. An abelian group is called **odd-divisible** if every element is divisible by every odd number.

Theorem 10.20 *$(K, <)$ is a Rolle field if and only if it admits a henselian valuation v whose residue field Kv is real closed and whose value group vK is odd-divisible.*

Note that it is not necessary to require that the valuation v be the natural valuation of $(K, <)$ (i.e. that the valuation ring be convex). Indeed, Brown, Craven and Pelling have also shown in [BCP] that if $(K, <)$ is a Rolle field, then also $(K, <')$ for every other ordering $<'$ of K.

The following is a generalization of one direction of the preceding theorem and gives a converse of Theorem 10.19.

Theorem 10.21 *Assume that the field L admits a henselian valuation v whose value group vL is odd-divisible. If Lv admits an ordering such that $(Lv, <)$ is a Rolle field, then L admits an ordering (inducing $<$ via v) and is a Rolle field (with respect to every ordering).*

Proof: Assume the hypothesis of the theorem. Then by virtue of the Theorem of Brown, Craven and Pelling, the Rolle field $(Lv, <)$ admits a valuation w whose residue field $(Lv)w$ is real closed and whose value group $w(Lv)$ is odd-divisible. Consider the composite $v \circ w$ on L. Its residue field $L(v \circ w) = (Lv)w$ is real closed. Its value group $v \circ wL$ is odd-divisible; this is seen as follows. $v \circ wL$ has a convex subgroup Γ isomorphic to $w(Lv)$ and hence odd-divisible. The quotiont of $v \circ wL$ by Γ is isomorphic to vL and hence also odd-divisible by hypothesis. It follows that the group $v \circ wL$ itself is odd-divisible. Again using the Theorem of Brown, Craven and Pelling, we conclude that L is a Rolle field for every ordering. Let us show that there exists at least one ordering. As L admits a place $P = P_v$ onto the ordered field $(Lv, <)$, it admits an ordering $<'$ which induces $<$ on Lv via this place, that is, the positive cone of $<$ on Lv is the image of the positive cone of $<'$ under the place P. □

Chapter 11

Additive polynomials

11.1 Definition and basic properties

A polynomial $f \in K[X]$ is called **additive** if $f(a+b) = f(a) + f(b)$ for all elements a, b in every extension field L of K, that is, if the map induced by f on L is an endomorphism of the additive group $(L, +)$. In our definition, the field K does not appear; indeed, it can be shown that a polynomial which is additive on K, is also additive on every extension field.

Lemma 11.1 *Let $f \in K[X]$.*
a) If f is additive, then the set A of its roots in $\tilde{K}$ is a subgroup of the additive group of $\tilde{K}$. Conversely, if A is a subgroup of the additive group of $\tilde{K}$ and f has no multiple roots, then f is additive.
b) If f is additive, then f is of the form

$$\sum_{i=0}^{m} c_i X^{p^i}$$

where p is the characteristic exponent of K.

Proof: a): If f is additive and a, b are roots of f, then $f(a+b) = f(a) + f(b) = 0$; this shows that the set of roots of f form a subgroup A of $(\tilde{K}, +)$. To prove the converse, assume the latter and that f has no multiple roots. Consider the following polynomial in two variables:

$$g(X, Y) = f(X+Y) - f(X) - f(Y) \; .$$

Since $A + A = A$, we have that g induces the zero function on $A \times A$. We shall prove that g also induces the zero function on $K \times K$, which shows that f induces an endomorphism of $(K, +)$.

Observe first that for $c \in K$, the polynomials $g(c, Y)$ and $g(X, c)$ are of lower degree than f. This follows from the Taylor expansion (6.7). By our assumption that f has no multiple roots, it follows that $|A| = \deg f$ and that every polynomial of degree $< \deg f$ vanishing on A must be identically 0. Assume that there exists $c \in K$ such that $g(c, Y)$ is not identically 0. Then there must be $a \in A$ such that $g(c, a) \neq 0$. Consequently, $g(X, a)$ is not identically 0, contradicting the fact that it vanishes on A. This contradiction shows that $g(c, Y)$ is identically 0 for all $c \in K$. That is, g induces the zero map on $K \times K$.

If we replace K by any extension field L, then still A is a subgroup of the additive group of $\tilde{L}$, and our proof shows that f induces also an additive map on $(L,+)$. We have thus proved that f is additive.

b): Let us consider again $g(X,Y)$. Since we may replace K by an arbitrary extension field L, we may assume w.l.o.g. that K is not finite. We have shown that $g(c,Y)$ vanishes identically for every $c \in K$. That means that the polynomial $g(X,Y) \in K(Y)[X]$ has infinitely many zeros in $K(Y)$. Hence, it must be identically 0.

Write $f = c_n X^n + \ldots + c_0$. Then $g(X,Y)$ is the sum of the forms $c_i(X+Y)^i - c_i X^i - c_i Y^i$ of degree i, $1 \le i \le \deg f$. Since g is identically 0, the same must be true for every of these forms and thus for all $(X+Y)^i - X^i - Y^i$ for which $c_i \neq 0$. But $(X+Y)^i - X^i - Y^i \equiv 0$ can only hold if i is a power of the characteristic of K. □

Part b) of this lemma shows that over a field of characteristic 0, every additive polynomial is of the form cX with $c \in K$. Now assume that $\operatorname{char} K = p > 0$. Then not only the monomials aX are additive, but also the monomial X^p. Indeed, as a map on K, it is equal to the Frobenius endomorphism φ. As a map, the monomial X^{p^2} is equal to the composition of φ with itself, written as φ^2, and by induction, we can replace 2 by every integer n. On the other hand, the monomial X induces the identity map, which we may write as φ^0. Note that addition and composition of additive maps on $(K,+)$ give again additive maps (in particular, addition of additive polynomials gives additive polynomials). It remains to interprete the coefficients of additive polynomials as maps. This is easily done by viewing K as a K-vector space: the map $c\cdot$ induced by $c \in K$ is given by multiplication $a \mapsto ca$, and it is an automorphism of $(K,+)$ if $c \neq 0$. So cX^{p^n} as a map is the composition of φ^n with $c\cdot$. We will write this composition as $c\varphi^n$. Adding these monomials generates new additive maps of the form $\sum_{i=0}^m c_i \varphi^i$, and addition of such maps gives again additive maps of this form. Composition of such additive maps generates again additive maps, and the reader may compute that they may again be written in the above form. In this way, we are naturally led to considering the ring $K[\varphi]$ of all polynomials in φ over K, where multiplication is given by composition. The correspondence that we have worked out, now reads as

$$\sum_{i=0}^m c_i X^{p^i} \quad \longleftrightarrow \quad \sum_{i=0}^m c_i \varphi^i \in K[\varphi] \tag{11.1}$$

which means that both expressions describe the same additive map on K. For instance, the additive **Artin-Schreier polynomial** $\wp(X) = X^p - X$ corresponds to $\varphi - 1$. Through the above correspondence, the ring $K[\varphi]$ may be considered as the **ring of additive polynomials over** K. Note that this ring is not commutative; in fact, we have

$$\varphi c = c^p \varphi \quad \text{for all } c \in K\ .$$

Let us state some basic properties of the ring $K[\varphi]$. We need the following definitions. (For deeper information about the following notions, see the comprehensive book "Free rings and their relations" by P. M. Cohn ([COHN1], [COHN2]). Let R be a ring with 1. Equipped with a function $\deg : R \to \mathbb{N}$, the ring R is called **left euclidean** if for all elements $s, s' \in R$, $s \neq 0$, there exist $q, r \in R$ such that

$$s' = qs + r \quad \text{with } r = 0 \text{ or } \deg r < \deg s\ ,$$

and it is called **right euclidean** if the same holds with "$s' = sq + r$" in the place of "$s' = qs + r$". Usually, the function deg is extended to 0 by setting $\deg 0 = -\infty$. For

example, polynomial rings over fields equipped with the usual degree function are both-sided euclidean rings. Further, an integral domain R is called a **left principal ideal domain** if every left ideal in R is principal (and analogously for "right" in the place of "left"). We leave it to the reader to show that every left (resp. right) euclidean integral domain is a left (resp. right) principal ideal domain. Finally, an integral domain R is called a **left Ore domain** if

$$Rr \cap Rs \neq \{0\} \quad \text{for all } r, s \in R \setminus \{0\} ,$$

and it is called a **right Ore domain** if $rR \cap sR \neq \{0\}$ for all $r, s \in R \setminus \{0\}$. Every left (resp. right) Ore domain can be embedded into a skew field (cf. [COHN], §0.8, Corollary 8.7). The reader may prove that a left (resp. right) principal ideal domain is a left (resp. right) Ore domain.

The ring $K[\varphi]$ may be equipped with a degree function which satisfies $\deg 0 = -\infty$ and $\deg \sum_{i=0}^{m} c_i \varphi^i = m$ if $c_m \neq 0$. This degree function is a homomorphism of the multiplicative monoid of $K[\varphi]$ onto $\mathbb{Z}$ since it satisfies $\deg rs = \deg r + \deg s$. In particular, this shows that $K[\varphi]$ is an integral domain. The following theorem was proved by O. Ore [ORE]:

Theorem 11.2 *The ring $K[\varphi]$ is a left euclidean integral domain and thus also a left principal ideal domain and a left Ore domain. It is right euclidean if and only if K is perfect; if K is not perfect, then $K[\varphi]$ is not even right Ore.*

Proof: Let $s = \sum_{i=0}^{m} c_i \varphi^i$ and $s' = \sum_{i=0}^{n} d_i \varphi^i$. If $\deg s' < \deg s$, then set $q = 0$ and $r = s'$. Now assume that $\deg s' = n \geq m = \deg s$. Then

$$\deg(s' - d_n c_m^{-p^{n-m}} \varphi^{n-m}\, s) \leq n - 1 < \deg s' .$$

Now let $q \in R$ such that $\deg(s' - qs)$ is minimal. Then $\deg(s' - qs) < \deg s$. Otherwise, we could apply the above to $s' - qs$ in the place of s', finding some $q' \in R$ such that $\deg(s' - (q + q')s) = \deg(s' - qs - q's) < \deg(s' - qs)$ contradicting the minimality of q. Setting $r = s' - qs$, we obtain $s' = qs + r$ with $\deg r < \deg s$. We have proved that $K[\varphi]$ is left euclidean. If K is perfect, hence $K = K^{p^m}$, then we also have

$$\deg(s' - s\,(c_m^{-1} d_n)^{1/p^m} \varphi^{n-m}) \leq n - 1 < \deg s' ,$$

and in the same way as above one deduces that $K[\varphi]$ is right euclidean.

Now assume that K is not perfect and choose some element $c \in K$ not admitting a p-th root in K. Then $K^p \cap cK^p = \{0\}$ and

$$\varphi K[\varphi] \,\cap\, c\varphi K[\varphi] = \{0\}$$

since every nonzero additive polynomial in the set $\varphi K[\varphi]$ has coefficients in K^p whereas every nonzero additive polynomial in $c\varphi K[\varphi]$ has coefficients in cK^p. □

Example 11.3 The ring $\mathbb{F}_p[\varphi]$ is a both-sided euclidean integral domain, and every field of characteristic p is a left $\mathbb{F}_p[\varphi]$-module, where the action of φ on K is just the application of the Frobenius endomorphism. K is perfect if and only if every element of K is divisible by the ring element φ. But this does not imply that K is a divisible $\mathbb{F}_p[\varphi]$-module. For instance, if K admits Artin-Schreier extensions, that is, if $K \neq \wp(K) = (\varphi - 1)K$, then

there are elements in K which are not divisible by $\varphi - 1$. On the other hand, K is a divisible $\mathbb{F}_p[\varphi]$-module if K is algebraically closed.

Observe that K is not torsion free as an $\mathbb{F}_p[\varphi]$-module. Indeed, K contains $\mathbb{F}_p$ which satisfies

$$(\varphi - 1)\mathbb{F}_p = \{0\} .$$

If k is any subfield of K, then K is also a $k[\varphi]$-module. Again, K is a divisible $K[\varphi]$-module if K is algebraically closed. ♢

Remark 11.4 Let us state some further properties of the ring $K[\varphi]$ which follow from Theorem 11.2. More generally, let R be any left principal ideal domain. Then R is a left free ideal ring (fir), and it is thus a semifir, i.e. every finitely generated left or right ideal is free of unique rank (note that this property is left-right symmetrical, cf. [COHN2], Chapter 1, Theorem 1.1). Consequently, every finitely generated submodule of a (left or right) free R-module is again free, cf. [COHN2], Chapter 1, Theorem 1.1. On the other hand, every finitely generated torsion free (left or right) R-module is embeddable in a (finitely generated) free R-module if and only if R is right Ore, cf. [COHN2], Chapter 0, Corollary 9.5 and [GENT], Proposition 4.1. Being a semifir, R is right Ore if and only if it is a right Bezout ring. But if R is not right Ore, then it contains free right ideals of arbitrary finite or countable rank, and R is thus not right noetherian, cf. [COHN2], Chapter 0, Proposition 8.9 and Corollary 8.10. From all this we conclude that all finitely generated torsion free (left or right) $K[\varphi]$-modules are free if and only if K is perfect, that is, $K[\varphi]$ is euclidean on both sides. Every projective (left or right) R-module is free, cf. [COHN2], Chapter 1, Theorem 4.1. A right R-module is flat if and only if it is torsion free, and a left R-module M is flat if and only if every finitely generated submodule of M is free, cf. [COHN2], Chapter 1, Corollary 4.7 and Proposition 4.5. In view of the above, the latter is the case if and only if every finitely generated submodule of M is embeddable in a free R-module. Further, a left R-module M is flat if and only if for every $n \in \mathbb{N}$ and all right linearly independent elements $r_1, \ldots, r_n \in R$,

$$\forall x_1, \ldots, x_n \in M : \ \sum r_i x_i = 0 \ \Rightarrow \ \forall i : \ x_i = 0 ,$$

cf. [COHN1], Chapter 1, Lemma 4.3. As a semifir, R is a coherent ring. Finally, since R is left Ore, it can be embedded in a skew field of left fractions, cf. [COHN2], Chapter 0, Corollary 8.7.

Exercise 11.1 *a) Let K be a finite field. Give an example of a polynomial $f \in K[X]$ which is not additive but induces an additive map on $(K, +)$.*
b) Show that the second assertion in part a) of Lemma 11.1 fails if we drop the condition that f has no multiple roots. Replace this condition by a suitable condition on the multiplicity of the roots.
b) Deduce Lemma 11.1 from the theorem of Artin as cited in [LANG3], VIII, §11, Theorem 18.

Exercise 11.2 *Describe the relation of the degree functions on $K[X]$ and $K[\varphi]$ via the correspondence (11.1), giving thereby a proof of $\deg rs = \deg r + \deg s$. Show that it also satisfies $\deg r + s \leq \max\{\deg r, \deg s\}$ with equality holding if $\deg r \neq \deg s$. Can it be transformed into a valuation?*

11.2 Frobenius-closed bases of function fields

If $F|K$ is an extension of fields of characteristic $p > 0$, then a K-basis B of $F|K$ will be called **Frobenius-closed** if $B^p \subset B$, where $B^p = \{b^p \mid b \in B\} = \varphi B$. We leave it to the reader to show that a Frobenius-closed basis necessarily contains the element 1. We will show the existence of Frobenius-closed bases for algebraic function fields of transcendence degree 1 over a perfect field of characteristic $p > 0$, provided that K is relatively algebraically closed in F. To get acquainted with this task, we first prove the following:

Lemma 11.5 *If F is an algebraic function field of transcendence degree 1 over an algebraically closed field K of arbitrary characteristic and q is an arbitrary natural number > 1, then there exists a basis of $F|K$ which is closed under q-th powers.*

If $F = K(x)$ is a rational function field, then our lemma follows from the **Partial Fraction Decomposition**: Every element $f \in F$ has a unique representation

$$f = c + \sum_{n>0} c_n x^n + \sum_{a \in K} \sum_{n>0} c_{a,n} \frac{1}{(x-a)^n}$$

where only finitely many of the coefficients $c, c_n, c_{a,n} \in K$ are nonzero. If we put

$$t_a = \frac{1}{x-a}\ , \quad t_\infty = x$$

then it follows that the elements

$$1,\ t_a^n \ \text{ with } \ a \in K \cup \{\infty\},\ n \in \mathbb{N}$$

form a K-basis of F; this basis has the property that **every** power of a basis element is again a basis element.

For general function fields the Partial Fraction Decomposition remains true in a modified form (according to Hasse) that we shall describe now. At this point, we need the Riemann-Roch Theorem. In order to apply it, we have to introduce some notation. In what follows, we always assume that K is relatively algebraically closed in F. A **divisor** of $F|K$ is an element of the (multiplicatively written) free abelian group generated by all places of $F|K$. (The places themselves are called **prime divisors**.) A divisor may thus be written in the form

$$A = \prod_P P^{v_P A}$$

where the product is taken over all places of $F|K$ and the $v_P A$ are integers, only finitely many of them nonzero. Recall that we identify equivalent places. The **degree of a nontrivial place** P of $F|K$, denoted by $\deg P$, is defined to be the degree $[FP : K]$ (which is finite since $F|K$ is an algebraic function field in one variable, cf. Corollary 7.32). Accordingly, the **degree of a divisor** A, denoted by $\deg A$, is defined to be the integer $\sum_P v_P A \cdot \deg P$. The set

$$L(A) := \{f \in F \mid v_P f \geq -v_P A \text{ for all places } P \text{ of } F|K\}$$

is a K-vector space since $v_P K = \{0\}$, hence $\forall c \in K :\ v_P(cf) = v_P f$ for every place P of $F|K$. We write $\dim A$ for $\dim_K A$.

Theorem 11.6 (Riemann-Roch)
Let $F|K$ be an algebraic function field in one variable with K relatively algebraically closed in F. There exists a smallest nonnegative integer g, called the genus of $F|K$, such that

$$\dim A \geq \deg A - g + 1$$

for all divisors A of $F|K$. Furthermore,

$$\dim A = \deg A - g + 1$$

whenever $\deg A > 2g - 2$.

For the proof, see [DEU].

Let P_∞ be a fixed place of $F|K$ and R^∞ the ring of all $f \in F$ which satisfy $v_P f \geq 0$ for every $P \neq P_\infty$. The following is an application of the Riemann-Roch Theorem:

Corollary 11.7 *For every $P \neq P_\infty$ there exists an element $t_P \in F$ such that*

$$\begin{aligned} v_P t_P &= -1 \\ v_Q t_P &\geq 0 \quad \text{for } Q \neq P, P_\infty \, . \end{aligned}$$

Proof: If we choose $n \in \mathbb{N}$ as large as to satisfy $n \dim P_\infty > 2g-2$, then by the Riemann-Roch Theorem,

$$\dim(P + nP_\infty) = \dim P + n \dim P_\infty - g + 1 > n \dim P_\infty - g + 1 = \dim nP_\infty \, .$$

Hence there is an element $t_P \in L(P + nP_\infty) \setminus L(nP_\infty)$. This element has the required properties. □

We return to the proof of our lemma, assuming that K is algebraically closed. Hence, K is the residue field of every place P of $F|K$ (that is, $\deg P = 1$). Every t_P of the foregoing corollary is the inverse of a uniformizing parameter for P. Every $f \in F$ can be expanded P-adically with respect to such a uniformizing parameter, and the principal part appearing in this expansion has the form

$$h_P(f) = \sum_{n>0} c_{P,n} t_P^n \, ,$$

where only finitely many of the coefficients $c_{P,n} \in K$ are nonzero, namely $n \leq -v_P f$. By construction, t_P has only a single pole $\neq P_\infty$ and this pole is P; the same holds for $h_P(f)$ (if $h_P(f) \neq 0$). Consequently,

$$h = f - \sum_{P \neq P_\infty} h_P(f)$$

has no pole other than P_∞ and is thus an element of R^∞. We have shown that f has a unique representation

$$f = h + \sum_{P \neq P_\infty} \sum_{n>0} c_{P,n} t_P^n$$

with coefficients $c_{P,n} \in K$ and an element $h \in R^\infty$. This shows that the elements

$$t_P^n \;\text{ with }\; P \neq P_\infty,\ n \in \mathbb{N}$$

form a K-basis of F modulo R^∞ which has the property that every power of a basis element is again a basis element.

Now it remains to show that R^∞ admits a basis which is closed under q-th powers. An integer $n \in \mathbb{N}$ is called **pole number** of P_∞ if there exists $t_n \in R^\infty$ such that $v_{P_\infty} t_n = -n$. Let $H_\infty \subseteq \mathbb{N}$ be the set of all pole numbers. Fixing a t_n for every $n \in H_\infty$, we get a K-basis

$$1,\ t_n \;\text{ with }\; n \in H_\infty$$

of R^∞. To get a basis which is closed under q-th powers, we have to carry out our choice as follows:

Observe that H_∞ is closed under addition; in particular

$$qH_\infty \subset H_\infty \ .$$

For every $m \in H_\infty \setminus qH_\infty$ we choose an arbitrary element $t_m \in R^\infty$ with $v_{P_\infty} t_m = -m$. Every $n \in H_\infty$ can uniquely be written as

$$n = q^\nu m \ \text{ where } \ \nu \geq 0 \ \text{ and } \ m \in H_\infty \setminus qH_\infty \ .$$

Accordingly we put

$$t_n = t_m^{q^\nu}$$

which implies

$$v_{P_\infty} t_n = q^\nu \cdot v_{P_\infty} t_m = -q^\nu m = -n \ .$$

This construction produces a K-basis

$$1, \ t_m^{q^\nu} \ \text{ with } \ m \in H_\infty \setminus qH_\infty, \ \nu \geq 0$$

of R^∞, which is closed under q-th powers. This concludes the proof of our lemma.

For the generalization of this lemma to perfect ground fields of characteristic $p > 0$ we have to choose $q = p$:

Lemma 11.8 *Let F be an algebraic function field of transcendence degree 1 over a perfect field K of characteristic $p > 0$. If K is relatively algebraically closed in F, then there exists a Frobenius-closed basis for $F|K$.*

Proof: If K is not algebraically closed, we have to modify the proof of the previous lemma since not every place P of K has degree 1. (Such a modification is also necessary for the Partial Fraction Decomposition in $K(x)$ if K is not algebraically closed.) The modification reads as follows:

For every place P of $F|K$, let

$$d_P = \deg P = [FP : K]$$

be the degree of P. For every $P \neq P_\infty$ we choose elements $u_{P,i} \in R^\infty$, $1 \leq i \leq d_P$, such that their residues $u_{P,1}P, ..., u_{P,d_P}P$ form a K-basis of FP. We note that for every $\nu \geq 0$, the p^ν-th powers $u_{P,i}^{p^\nu}$ of these elements have the same property: their P-residues also form a K-basis of FP since K is perfect.
We write every $n \in \mathbb{N}$ in the form

$$n = p^\nu m \ \text{ with } \ m \in \mathbb{N}, \ (p, m) = 1, \ \nu \geq 0$$

and observe that the elements

$$u_{P,i}^{p^\nu} t_P^n \ \text{ with } \ P \neq P_\infty, \ n \in \mathbb{N}, \ 1 \leq i \leq d_P$$

form a Frobenius-closed K-basis of F modulo R^∞.

It remains to construct a Frobenius-closed K-basis of R^∞. This is done as follows: We consider the K-vector spaces

$$L_n = L(nP_\infty) = \{x \in F \mid v_{P_\infty} x \geq -n \text{ and } v_P(x) \geq 0 \text{ for } P \neq P_\infty\} \ .$$

By our assumption that K is relatively algebraically closed in F, we have $L_0 = K$. Further,

$$R^\infty = \bigcup_{n\in\mathbb{N}} L_n \ .$$

For

$$d_{\infty,n} := \dim L_n/L_{n-1}$$

we have $d_{\infty,n} \geq 0$ and

$$d_{\infty,n} \leq [FP_\infty : K] =: d_\infty \ .$$

(Note that by the Riemann-Roch Theorem, equality holds for large enough n; cf. the proof of the above corollary.) Now for $n = 1, 2, ...$ we shall choose succesively basis elements $t_{n,i} \in L_n$ modulo L_{n-1}. Then the elements

$$1,\ t_{n,i} \ \text{ with } \ n \in \mathbb{N},\ 1 \leq i \leq d_{\infty,n}$$

form a K-basis of R^∞. To obtain that this basis is Frobenius-closed, we organize our choice as follows:

If $n = pm$, the p-th powers $t_{m,i}^p \in L_n$ are linearly independent modulo $L_{p(m-1)}$ and even modulo $L_{pm-1} = L_{n-1}$. This fact follows from our hypothesis that K is perfect: the existence of nonzero elements $c_i \in K$ with $\sum c_i t_{m,i}^p \in L_{pm-1}$, i.e. $v_{P_\infty} \sum c_i t_{m,i}^p > -pm$, would yield $v_{P_\infty} \sum c_i^{1/p} t_{m,i} > -m$, hence $\geq -m+1$, showing that $\sum c_i^{1/p} t_{m,i} \in L_{m-1}$, which is a contradiction. In our choice of the elements $t_{n,i}$ we are thus free to take all the elements $t_{m,i}^p$ and to extend this set to a basis of L_n modulo L_{n-1} by arbitrary further elements, if necessary (for n large enough, the elements $t_{m,i}^p$ will already form such a basis). This procedure guarantees that the p-th power of every basis element $t_{m,i}$ is again a basis element, namely equal to $t_{pm,j}$ for suitable j. Hence a basis constructed in this way will be Frobenius-closed. □

Let $F|K$ be an arbitrary extension of fields of characteristic $p > 0$. Both F and K are $K[\varphi]$-modules, and so is the quotient module F/K. Suppose that F/K is a free $K[\varphi]$-module. Then it admits a $K[\varphi]$-basis. Let $B_0 \subset F$ be a set of representatives for such a $K[\varphi]$-basis of F/K. It follows that

$$B \ = \ \bigcup_{n=0}^{\infty} B_0^{p^n} \ \cup \ \{1\} \ = \ \bigcup_{n=0}^{\infty} \varphi^n B_0 \ \cup \ \{1\}$$

is a set of generators of the K-vector space F. By our construction of B, every K-linear combination of elements of $B \setminus \{1\}$ may be viewed as a $K[\varphi]$-linear combination of elements of B_0. This shows that the elements of B are K-linearly independent, and B is thus a Frobenius-closed basis of $F|K$. Note that B_0 is the basis of a free $K[\varphi]$-submodule M of F which satisfies $F = M \oplus K$.

The converse to this procedure would mean to extract a $K[\varphi]$-basis B_0 from a Frobenius-closed K-basis B. But B_0 can only be found if for every element $b \in B \setminus \{1\}$ there is some element b_0 which is not a p-th power in F and such that $b = b_0^{p^n}$ for some $n \in \mathbb{N}$. This will hold if no element of $F \setminus K$ has a p^n-th root for every $n \in \mathbb{N}$.

Lemma 11.9 *If $F|K$ is an algebraic function field (of arbitrary transcendence degree), and if K is relatively algebraically closed in F, then no element of $F \setminus K$ has a p^n-th root for every $n \in \mathbb{N}$.*

Proof: Let $f \in F \setminus K$. Since K is relatively algebraically closed in F, we know that f is transcendental over K. So we may choose a transcendence basis $\mathcal{T}$ of $F|K$ containing f. According to Lemma 7.53, we may choose a K-rational valuation v on the rational function field $K(\mathcal{T})$ such that the values of all elements in $\mathcal{T}$ are rationally independent, which yields that $vK(\mathcal{T}) = \bigoplus_{t\in\mathcal{T}} \mathbb{Z}vt$. In particular, vf is not divisible by p in $vK(\mathcal{T})$. Since $F|K(\mathcal{T})$ is finite, the same is true for $(vF : vK(\mathcal{T}))$ by Lemma 7.7. This yields that there is some $n \in \mathbb{N}$ such that vf is not divisible by p^n in vF. Hence, f does not admit a p^n-th root in F. □

This lemma shows that if $F|K$ is an algebraic function field with K relatively algebraically closed in F, admitting a Frobenius-closed basis B and if we let B_0 be the set of all elements in B which do not admit a p-th root in F, then we obtain $B = \bigcup_{n=0}^{\infty} B_0^{p^n} \cup \{1\}$. Since the elements of B are K-linearly independent, the elements of B_0 are $K[\varphi]$-linearly independent over K. Moreover, B_0 is a set of generators of the $K[\varphi]$-module F over K. Hence, the set B_0/K is a $K[\varphi]$-basis of F/K. We have thus proved:

Lemma 11.10 *Let $F|K$ be an algebraic function field (of arbitrary transcendence degree), and K relatively algebraically closed in F. Then F admits a Frobenius-closed K-basis if and only if F/K is a free $K[\varphi]$-module.*

Lemma 11.8 now reads as follows:

Theorem 11.11 *If F is an algebraic function field of transcendence degree 1 over a perfect field K of characteristic $p > 0$ and if K is relatively algebraically closed in F, then F/K is a free $K[\varphi]$-module.*

Open Problem 11.1 Does Theorem 11.11 also hold for transcendence degree > 1?

Open Problem 11.2 Does Theorem 11.11 also hold if the assumption that K be perfect is replaced by the assumption that $F|K$ be separable? Note that if K is not perfect, then there exist places P of $F|K$ such that $FP|K$ is not separable, even if $F|K$ is separable. In this case, the construction of the proof of Lemma 11.8 breaks down and it cannot be expected that there is a Frobenius-closed K-basis of F which is as "natural" as the ones produced by that construction.

Exercise 11.3 *Show that F/K cannot be a free $K[\varphi]$-module if K is not relatively algebraically closed in F. Does there exist an algebraic field extension which admits a Frobenius-closed basis? Prove a suitable version of Lemma 11.10 which does not use the assumption that K be relatively algebraically closed in F.*

11.3 Polynomials as maps on valued fields

We intend to study the action of additive polynomials on valued fields of positive characteristic. As a preparation, we will give some general results for arbitrary polynomials in this section. Given a valued field (L, v) of arbitrary characteristic and a polynomial $f \in L[X]$, we will view f as a map on the underlying valued abelian group of (L, v). The coefficient map co will be used in the sense of valued abelian groups, it need not be multiplicative.

Recall that all components of the valued abelian group (L, v) are isomorphic to the additive group of the residue field $\overline{L}$.

We write $f(X) = \sum_{1\leq i\leq n} c_i X^i \in L[X]$. We can as well view f as a map from $\tilde{L}$ into $\tilde{L}$, with an arbitrary extension of v to $\tilde{L}$. If $b \in \tilde{L}$ is such that all monomials $c_k b^k$ have different values and if the minimal value is assumed by $c_i b^i$, then

$$vf(b) = vc_i + i \cdot vb \quad \text{and} \quad \operatorname{co} f(b) = (\operatorname{co} c_i)(\operatorname{co} b)^i \,,$$

hence in this case, the bone of b determines the bone of $f(b)$ uniquely. If on the other hand, $c_i b^i$ and $c_j b^j$ are both of minimal value for some $i \neq j$, then

$$vb = (vc_j - vc_i)/(i-j) \in \widetilde{vL} \,,$$

where $\widetilde{vL} = v\tilde{L}$ is the divisible hull of vL. From these equations, we obtain finitely many values $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ in $\widetilde{vL}$ for vb for which not all monomials have different values. We will call them **critical values for** f. All values of possible exceptional bones for f must be critical values. We set $\mathfrak{I}_0 := \{\alpha \in \widetilde{vL} \mid \alpha < \alpha_1\}$, $\mathfrak{I}_m := \{\alpha \in \widetilde{vL} \mid \alpha > \alpha_m\}$ and $\mathfrak{I}_\ell := \{\alpha \in \widetilde{vL} \mid \alpha_\ell < \alpha < \alpha_{\ell+1}\}$ for $0 < \ell < m$. These sets are convex subsets of $\widetilde{vL}$, and we will call them **monomial value sets for** f. The reason for this name is the fact that for vb in a fixed monomial value set, there is a unique monomial of least value whose index is also fixed, as we will show now.

If $va < vb$ and if $c_i a^i$ and $c_j b^j$ are unique monomials of minimal value in $f(a)$ and $f(b)$ respectively, then $i \neq j$ will imply that the interval $[va, vb]$ in $\widetilde{vL}$ contains some α_ℓ (cf. the proof of Lemma 7.117). Consequently, for every monomial value set $\mathfrak{I}_\ell$ for f, there is a fixed i such that $c_i b^i$ is the unique monomial of minimal value for all b with $vb \in \mathfrak{I}_\ell$. We will call $c_i x^i$ the **minimal value monomial for** $\mathfrak{I}_\ell$. Observe that

$$v(f(b) - c_i b^i) > vf(b) \quad \text{whenever } vb \in \mathfrak{I}_\ell \,. \tag{11.2}$$

Note that if $vc_i + i\alpha = vc_j + j\alpha$ with $i < j$, then $vc_i + i\beta > vc_j + j\beta$ whenever $\beta < \alpha$, and $vc_i + i\beta < vc_j + j\beta$ whenever $\beta > \alpha$. If i is the minimal and j the maximal index such that $vc_i + i\alpha_\ell = vc_j + j\alpha_\ell$, then it follows that $c_j b^j$ is the unique monomial of minimal value for all b with $vb \in \mathfrak{I}_{\ell-1}$ and that $c_i b^i$ is the unique monomial of minimal value for all b with $vb \in \mathfrak{I}_\ell$. This shows that for every fixed i, the monomial $c_i x^i$ can be a minimal value monomial for at most one monomial value set. Since there are at most $n+1$ monomials in f, it follows that there are at most $n+1$ monomial value sets and at most n critical values α_ℓ. Consequently, there are at most n exceptional values for f.

The map $\alpha \mapsto vc_i + i\alpha$ on $\widetilde{vL}$ is monotone, and if $i \neq 0$, then it is strictly monotone (that is, $\alpha < \beta$ implies that $vc_i + i\alpha < vc_i + i\beta$). Define a map mvm_f on $\widetilde{vL}$ as follows: If $\alpha \in \mathfrak{I}_\ell$ or $\alpha = \alpha_\ell$ and if $c_i x^i$ is the minimal value monomial for $\mathfrak{I}_\ell$, then we set $\operatorname{mvm}_f(\alpha) := vc_i + i\alpha$. If $\alpha = \alpha_\ell$ and if $c_j x^j$ is the minimal value monomial for $\mathfrak{I}_{\ell-1}$, then $vc_i + i\alpha = vc_j + j\alpha$, as we have seen above. By the monotonicity of the maps $\alpha \mapsto vc_i + i\alpha$ for every i it thus follows that the map mvm_f is monotone. It is strictly monotone if and only if the constant c_0 never appears as a minimal value monomial (which for instance is the case if $c_0 = 0$). If it appears, then it must appear as the minimal value monomial for $\mathfrak{I}_m$. But if $c_0 \neq 0$, then c_0 will always be the minimal value monomial for $\mathfrak{I}_m$ because for large enough α we will have $vc_i + i\alpha > vc_0$ for all $i \geq 1$. We note:

The map mvm_f *on* $\widetilde{vL}$ *is monotone. It is strictly monotone if and only if* $c_0 = 0$.

The equality $vf(b) = \mathrm{mvm}_f(vb)$ holds if vb is not equal to any of the critical values α_ℓ. But note that we always have the inequality

$$vf(b) \geq \mathrm{mvm}_f(vb) \ .$$

Let us discuss the case where $vb = \alpha_\ell$ for some ℓ. In this case, there are more than one monomial of minimal value, say $c_{i_\nu} b^{i_\nu}$, $0 \leq \nu \leq n'$ (with $n' \leq n$), all other monomials having higher values. Then

$$\mathrm{co}\, f(b) = \sum_\nu \mathrm{co}\, c_{i_\nu} (\mathrm{co}\, b)^{i_\nu}$$

unless the sum is zero; if it is not, then again, the bone of b determines the bone of $f(b)$ uniquely and moreover, we have the equality $vf(b) = \mathrm{mvm}_f(vb)$. But there are only finitely many elements $\zeta \in \overline{L}$ which are roots of the polynomial $\sum_\nu \mathrm{co}\, c_{i_\nu} X^{i_\nu} \in \overline{L}[X]$. Hence, there are only finitely many exceptional bones of the form (α_ℓ, ζ) for f. Since exceptional bones for f can only have value α_ℓ for some $\ell \in \{1, \ldots, m\}$, we conclude that there are only finitely many exceptional bones for f. That is, f is finitely exceptional on $(\tilde{L}, v)$. Moreover, if $(vb, \mathrm{co}\, b)$ is not of the form (α_ℓ, ζ) as above, then for every a with $va \geq vb$, we find that

$$vf(b) = \mathrm{mvm}_f(vb) \leq \mathrm{mvm}_f(va) \leq vf(a) \ .$$

This shows that f is almost value compatible on $(\tilde{L}, v)$. Note that for such an element b, the inequality $va > vb$ will imply the inequality $vf(a) > vf(b)$, provided that $c_0 = 0$. If neither a nor b have bones of the above form and if $c_0 = 0$, then

$$va < vb \ \Leftrightarrow \ vf(a) < vf(b) \ .$$

Let us restrict the map f to the field L. Every exceptional value for f on (L, v) is also an exceptional value for f on $(\tilde{L}, v)$. The properties "finitely exceptional" and "almost value-compatible" are inherited by restrictions. So we have proved:

Lemma 11.12 *Assume that (L, v) is a valued field and that f is a polynomial in one variable over L, viewed as a map of L into itself. Then the following assertions hold:*
1) the set $\mathrm{Exv}_f(L, v)$ of exceptional values for f has at most n elements,
2) f is finitely exceptional,
3) f is almost value-compatible.

Now assume that $a \neq 0$ is a zero of f. If the valuation v is nontrivial on L, then the set $\{b \in L \mid vb > va\}$ is infinite. So we can choose some b in this set such that $f(a+b) \neq 0$. Because of $vb > va$, the elements a and $a+b$ have equal bones. But $f(a) = 0$ and $f(a+b) \neq 0$ do not have equal bones, so we find that the bone of a is exceptional. Let us note:

If the valuation v is nontrivial on L, then the bones of all roots $\neq 0$ of f in L are exceptional for f.

11.4 Valued fields as valued $K[\varphi]$-modules

We have seen already that every field of characteristic $p > 0$ may be viewed as a left module over a ring of additive polynomials. If the field is valued, then by our general definition,

it will be a valued module over this ring. We also have seen that the module may not be torsion free. So it cannot be expected that the valuation is compatible with the module structure in any "classical" sense.

Example 11.13 The power series field $\mathbb{F}_p((t))$ and the fields $\mathbb{F}_p(t)$ and $\mathbb{F}_p(t)^h$ are valued $\mathbb{F}_p[\varphi]$- and $\mathbb{F}_p(t)[\varphi]$-modules. None of them is torsion free since they all contain $\mathbb{F}_p$. (One could also form the ring $\mathbb{F}_p[t,\varphi]$ and view these fields as valued $\mathbb{F}_p[t,\varphi]$-modules, but it seems that we already have enough open problems ...) ♢

It was indeed this example that led to our general definition of a valued module. The idea to view $\mathbb{F}_p((t))$ as a module admitting multiplication by t and application of φ is due to L. v. d. Dries. We will discuss later the open problems in connection with the structure and model theory of $\mathbb{F}_p((t))$, and we will see why it makes sense to view $\mathbb{F}_p((t))$ as a valued module over the rings $\mathbb{F}_p[\varphi]$ and $\mathbb{F}_p(t)[\varphi]$.

In the following, let $(L|K,v)$ be an extension of valued fields of characteristic $p>0$. We wish to consider the valued $K[\varphi]$-module (L,v). Since every element of $K[\varphi]$ is a polynomial, we can make use of the results of the last section. Lemma 11.12 tells us:

Theorem 11.14 *Let $(L|K,v)$ be an extension of valued fields of characteristic $p>0$. Then (L,v) is a finitely exceptional almost value-compatible left $K[\varphi]$-module.*

It seems that this structure is an important aspect of the structure theory for valued fields in positive characteristic. As we will see, it gives us information about the properties of the power series field $(\mathbb{F}_p((t)),v_t)$ that are particularly encoded in this structure and that can not be deduced from Hensel's Lemma or the fact that it is a defectless field. To our knowledge, the structure of such valued $K[\varphi]$-modules has not yet been studied. We have made a start by treating the theory of their immediate extensions in Section 4.8 in analogy to that of valued fields. Moreover, we have developed the theory of valued modules and the results of the last three sections of Chapter 3 having in mind the valued $K[\varphi]$-modules.

Let us see what Chapter 3 can tell us in the present situation. From assertion 3) of Lemma 11.12 and Corollary 3.41, we can deduce:

Theorem 11.15 *Every additive polynomial on a valued field of characteristic $p>0$ is a spherically continuous map.*

From this theorem together with Corollary 3.40 we obtain:

Corollary 11.16 *Let f be an additive polynomial on the valued field (L,v) of characteristic $p>0$. If (G,v) is a spherically complete subgroup of (L,v), then also $(f(G),v)$ is a spherically complete subgroup of (L,v). If in addition $(f(G),v)\subset(L,v)$ is immediate, then $f(G)=L$.*

In the special case of a perfect field L, this corollary together with the approach of the last section provides a handy criterion for an additive polynomial to be surjective on a spherically complete field (L,v), or even for a an element to lie in the subgroup $f(L)$. Indeed, if L is perfect, then all monomials in f are surjective. So the only values of interest are precisely the critical values α_ℓ for f that we have introduced in the last section. A condition like (3.16) of Corollary 3.40 has only to be checked if the value $v(f(b)-c)$ is a critical value. We note:

Corollary 11.17 *Let f be an additive polynomial on the perfect spherically complete field (L,v) of characteristic $p>0$. Assume that for every critical value $\alpha_\ell \in vL$ of f and every $c \in L$ of value $vc = \alpha_\ell$ there is some $b \in L$ such that $v(f(b)-c) > vc$. Then f is surjective on L. In particular, if no critical value of f lies in vL, then f is surjective on L.*

Example 11.18 Let (L,v) be a maximal immediate extension of $(\mathbb{F}_p((t))^{1/p^\infty}, v_t)$ and consider the additive polynomial $f(X) = X^p - tX$. Its only critical value is $\frac{1}{p-1} \notin \frac{1}{p^\infty}\mathbb{Z} = vL$. Hence, f is surjective on L. The same holds if we replace t by t^n with $n \in \mathbb{Z}$ not divisible by $p-1$. ♢

Even more interesting for our further studies is the application of Corollary 3.48:

Theorem 11.19 *Let $f_1, \ldots, f_n$ be additive polynomials on the valued field (L,v) of characteristic $p>0$. Further, let $G_1, \ldots, G_n$ be spherically complete subgroups of (L,v). Assume that for every $a \in f_1(G_1) + \ldots + f_n(G_n)$,*

$$\exists b_i \in G_i : \ vf_i(b_i) \geq va \ (1 \leq i \leq n) \quad \textit{and} \quad v(f_1(b_1) + \ldots + f_n(b_n) - a) > va \ . \tag{11.3}$$

Then $f_1(G_1) + \ldots + f_n(G_n)$ is a spherically complete and nice subgroup of (L,v). If (11.3) holds for every $a \in L$, then

$$L = f_1(G_1) + \ldots + f_n(G_n) \ .$$

More generally, if (H,v) is a v-convex subgroup of (L,v) and if (11.3) holds for every $a \in L$ with $va < vH$, then

$$L = f_1(G_1) + \ldots + f_n(G_n) + H \ .$$

Recall that every v-convex subgroup of a spherically complete field is spherically complete. Consequently, this theorem gives quite a lot of information on spherically complete fields of characteristic $p>0$. We wish to illustrate this by applying it to the power series field $(\mathbb{F}_p((t)), v_t)$.

11.5 Additive decompositions of power series fields

From Corollary 8.67 we know that $(\mathbb{F}_p((t)), v_t)$ has valuation p-basis $1, t, \ldots, t^{p-1}$. So let us apply Theorem 11.19 to an arbitrary spherically complete field (L,v) having this valuation p-basis. We choose the following additive polynomials f_i and subgroups G_i of L:

$f_i = t^i X$ and $G_i = L$ for $1 \leq i \leq p-1$,

$f_p = \wp(X) = X^p - X$ and $G_p = L$,

$H = \mathcal{O}_{\mathbf{L}}$.

Now let $a \in L$ such that $va < 0$. Then we write

$$a = tc_1^p + t^2 c_2^p + \ldots + t^{p-1} c_{p-1}^p + c_p^p \ ,$$

using the hypothesis that $1, t, \ldots, t^{p-1}$ is a p-basis of L. Since this is even a valuation p-basis, we know that va is equal to the minimum of the values of the summands. Now we $b_i = c_i$ for $1 \leq i \leq p$. Then

$$v(f_1(b_1) + \ldots + f_p(b_p) - a) = v(tc_1^p + t^2 c_2^p + \ldots + t^{p-1} c_{p-1}^p + c_p^p - c_p - a) = vc_p \ .$$

If $vc_p < 0$, then $vc_p > vc_p^p \geq va$. Otherwise, $vc_p \geq 0 > va$. This shows that condition (11.3) is satisfied for all $a \in L$ such that $va < 0$. Hence, an application of Theorem 11.19 gives:

Lemma 11.20 *Let (L, v) be a spherically complete field (L, v) having valuation p-basis $1, t, \ldots, t^{p-1}$. Then*

$$L = \wp(L) + \mathcal{O}_{\mathbf{L}} + tL^p + \ldots + t^{p-1}L^p \ . \tag{11.4}$$

But note that Theorem 11.19 does not require that (L, v) be spherically complete. Even if it is not, it may have interesting valued subgroups which are spherically complete. For example, every subgroup with finite value set is spherically complete. More generally, if (G, v) is a valued subgroup such that vG admits no infinite ascending sequences, then (G, v) is spherically complete (cf. Section 2.2). Let us choose a group complement $\mathbf{A}$ for the valuation ring $\mathcal{O}_{\mathbf{L}}$ in the abelian group L (cf. our use of such group complements in Section 5.5, page 144. We leave it to the reader to prove that $v\mathbf{A}$ admits no infinite ascending sequences if and only if $vL = \{0\}$ or $vL \cong \mathbb{Z}$. In the latter case, we have that $v\mathbf{A} = \{n \in \mathbb{Z} \mid n < 0\}$, and $(\mathbf{A}, v)$ is spherically complete. Given $a \in L$ with $va < 0$, we choose c_i as before. Now we can choose $c_i' \in \mathbf{A}$ such that $c_i - c_i' \in \mathcal{O}_{\mathbf{L}}$. With $b_i = c_i'$, we obtain:

$$f_1(b_1) + \ldots + f_p(b_p) - a \equiv tc_1^p + t^2c_2^p + \ldots + t^{p-1}c_{p-1}^p + c_p^p - c_p - a = c_p \pmod{\mathcal{O}_{\mathbf{L}}} \ ,$$

which again shows that condition (11.3) is satisfied for all $a \in L$ such that $va < 0$. From Theorem 11.19, we obtain:

$$L = \wp(\mathbf{A}) + \mathcal{O}_{\mathbf{L}} + t\mathbf{A}^p + \ldots + t^{p-1}\mathbf{A}^p \ .$$

In particular:

Lemma 11.21 *The decomposition (11.4) also holds for all valued fields (L, v) with value group $vL \cong \mathbb{Z}$.*

Corollary 11.22 *The decomposition (11.4) holds if (L, v) is one of the fields $(\mathbb{F}_p(t), v_t)$, $(\mathbb{F}_p(t), v_t)^h$ or $(\mathbb{F}_p((t)), v_t)$ (or any intermediate field).*

If the field (L, v) is henselian (which it is if it is spherically complete), then by Corollary 9.5, we find that $\mathcal{M}_{\mathbf{L}} \subset \wp(L)$. Hence in this case, (11.4) is equivalent to

$$L = \wp(L) + (\mathcal{O}_{\mathbf{L}}^{\times} \cup \{0\}) + tL^p + \ldots + t^{p-1}L^p \ . \tag{11.5}$$

If the residue field $\overline{L}$ is Artin-Schreier closed, then Corollary 9.5 tells us that $\mathcal{O}_{\mathbf{L}} \subset \wp(L)$, showing that (11.5) in turn is equivalent to

$$L = \wp(L) + tL^p + \ldots + t^{p-1}L^p \tag{11.6}$$

It was this decomposition in the case of an Artin-Schreier closed residue field that was noticed by L. v. d. Dries and led us to the consideration of such decompositions.

More generally, one can deduce decompositions like the following. Assume again that (L, v) is a spherically complete field of p-degree p. Then the extension $(L|L^{p^n}, v)$ is of

degree p^n and admits a valuation basis t_i, $1 \le i \le p^n$. Let $f_i \in \mathcal{O}_{\mathbf{L}}$, $1 \le i \le p^n$, be monic additive polynomials of degree p^n. Then the following decomposition holds:

$$L = t_1 f_1(L) + \ldots + t_{p^n} f_{p^n}(L) \ .$$

We do not know whether the decompositions deduced from Theorem 11.19 are stable under algebraic or finite extensions, but this seems unlikely. For instance, we have shown in Example 11.18 that the additive polynomial $X^p - tX$ is surjective on a maximal immediate extension (L, v) of the perfect hull of $(\mathbb{F}_p((t)), v)$. But if we adjoin a root s of the polynomial $X^{p-1} - t$, which gives a maximal immediate extension of the perfect hull of the power series field $\mathbb{F}_p((s))$, then $s^p = st$, and st is not in the range of $X^p - tX$ on $L(s)$, because $X^p - X - 1$ is irreducible over $\mathbb{F}_p$. It seems more likely that the niceness of subgroups as asserted by Theorem 11.19 is stable under finite extensions, perhaps under additional conditions.

We also do not know whether the decompositions deduced from Theorem 11.19 are the only properties of this type of the power series field $\mathbb{F}_p((t))$. For instance, we do not know whether all similar properties involving arbitrary polynomials are consequences of Theorem 11.19 and the fact that $(\mathbb{F}_p((t)), v)$ is henselian defectless. Model theoretically speaking, we do not know all elementary properties of $(\mathbb{F}_p((t))$. See Section 14.11 for further details. Let us only mention here that the structural information given by Theorem 11.19 is certainly of relevance for the examination of the elementary properties of $\mathbb{F}_p((t))$. In contrast to this open problem, we are able to describe the model theory of perfect valued fields (cf. Section 14.5). Probably it is more accessible because for perfect fields, the structure described by Theorem 11.19 is less complex. Note that if $L = L^p$ is perfect, then $t \in L$, yielding that $L = tL^p$ and that the above decompositions are trivially true.

11.6 G-modules and group complements

In this section, we introduce some notions that we will need in the next section. For $\sigma \in G$, **conjugation by** σ means the automorphism

$$G \ni \tau \ \mapsto \ \tau^\sigma := \sigma^{-1}\tau\sigma \ .$$

Note that

$$\tau^{\sigma\rho} = \rho^{-1}\sigma^{-1}\tau\,\sigma\,\rho = \rho^{-1}(\tau^\sigma)\rho = (\tau^\sigma)^\rho \quad \text{for all } \tau, \sigma, \rho \in G \ . \tag{11.7}$$

Further, we set $\tau^{-\sigma} := (\tau^{-1})^\sigma$ (which indeed is the inverse of τ^σ). As usual, we set $M^\sigma = \{m^\sigma \mid m \in M\}$ for every subset $M \subset G$. A subgroup N is normal in G if and only if $N^\sigma = N$ for all $\sigma \in G$. We always have $G^\sigma = G$. Hence, if H is a group complement of the normal subgroup N in G, that is,

$$HN = G \quad \text{and} \quad H \cap N = \{1\} \ , \tag{11.8}$$

then so is every conjugate H^σ for $\sigma \in G$. Uniqueness up to conjugation would mean that these are the only group complements of N in G.

We shall now introduce two notions that will play an important role in Section 11.7. A **right G-module** is an arbitrary group N together with a map μ from G into the group of automorphisms of N such that $\mu(\sigma\rho) = \mu(\rho) \circ \mu(\sigma)$. For example, to every $\sigma \in G$ we may associate the conjugation by σ; in view of (11.7), this turns G into a right G-module.

In this setting, a subgroup N of G is normal if and only if it is a G-submodule of G. A map ϕ from G into a G-module N is called a **twisted homomorphism** (or a **crossed homomorphism**) if it satisfies

$$\phi(\sigma\rho) = \phi(\sigma)^\rho \phi(h) \quad \text{for all } \sigma, \rho \in G\,. \tag{11.9}$$

As for a usual homomorphism, also the kernel of a twisted homomorphism is a subgroup of G, but it may not be normal in G.

Let us assume that H is a group complement of the normal subgroup N in G. It follows from (11.8) that every element $\sigma \in G$ admits a unique representation

$$\sigma = \sigma_H \sigma_N \quad \text{with } \sigma_H \in H\,,\ \sigma_N \in N \tag{11.10}$$

Note that H is a system of representatives for the left cosets of G modulo N. Since $N \lhd G$, we have $HN = NH$, and H is also a system of representatives for the right cosets of G.

Now assume in addition that N is abelian. Then the scalar multiplication of the G-module N given by conjugation reads as

$$\sigma^\rho = \rho_N^{-1}(\rho_H^{-1}\sigma\rho_H)\rho_N = \rho_H^{-1}\sigma\rho_H = \sigma^{\rho_H} \quad \text{for all } \sigma \in N\,,\ \rho \in G \tag{11.11}$$

since ρ_N and $\rho_H^{-1}\sigma\rho_H$ are elements of N. According to (11.10) and (11.11) we write

$$\sigma\rho = \sigma_H\sigma_N\rho_H\rho_N = \sigma_H\rho_H\rho_H^{-1}\sigma_N\rho_H\rho_N = \sigma_H\rho_H\sigma_N^\rho\rho_N\,.$$

Hence, the projection $\sigma \mapsto \sigma_H$ onto the first factor in (11.10) is the canonical epimorphism from G onto H with kernel N. The other projection $\sigma \mapsto \sigma_N$ is a twisted homomorphism from G onto N, satisfying

$$(\sigma\rho)_N = \sigma_N^\rho\rho_N \quad \text{for all } \sigma, \rho \in G\,;$$

it induces the identity on N, and its kernel is H.

11.7 Additively generated field extensions

In this section, let K be a field of characteristic $p > 0$. A field extension $L|K$ is called **p-elementary extension** if it is a finite Galois extension and its Galois group is an elementary-abelian p-group, that is, an abelian p-group in which every nonzero element has order p. In particular, $[L : K]$ is a power of p.

In this section, we will consider the following larger class of all extensions $L|K$ which satisfy the following condition:

$$\left.\begin{array}{l}\text{there exists a Galois extension } K'|K \text{ which is linearly disjoint from } L|K,\\ \text{such that } L.K'|K' \text{ is a } p\text{-elementary extension}\\ \text{and also } L.K'|K \text{ is a Galois extension.}\end{array}\right\} \tag{11.12}$$

From the linear disjointness it follows that $\operatorname{Gal} L.K'|L \cong \operatorname{Gal} K'|K$ and that $[L : K] = [L.K' : K']$ which yields that $[L : K] = p^n$ for some natural number n. For a further investigation of this situation, we will use the following notation. We set $L' := L.K'$ and define

$G := \operatorname{Gal} L'|K$,
$N := \operatorname{Gal} L'|K' \lhd G$,
$H := \operatorname{Gal} L'|L \cong \operatorname{Gal} K'|K \cong G/N$.

The group N is abelian of order p^n. Since $K'|K$ is assumed to be a Galois extension, N is a normal subgroup of G. That is, N is a right G-module with scalar multiplication given by conjugation:

$$(\sigma, \tau) \;\mapsto\; \sigma^\tau = \tau^{-1}\sigma\tau \quad \text{for all } \sigma \in N\,,\ \tau \in G\,.$$

Since $L.K' = L'$ and $L \cap K' = K$, we have that $H \cap N = 1$ and $G = HN$, that is, H is a group complement for N in G. As we have seen in the last section, every element $\sigma \in G$ admits a unique representation

$$\sigma = \sigma_H \sigma_N \quad \text{with } \ \sigma_H \in H\,,\ \sigma_N \in N\,. \tag{11.13}$$

Since N is abelian, the scalar multiplication of the G-module N reads as

$$\sigma^\tau = \tau_N^{-1}(\tau_H^{-1}\sigma\tau_H)\tau_N = \tau_H^{-1}\sigma\tau_H = \sigma^{\tau_H}\,. \tag{11.14}$$

The projection $\sigma \mapsto \sigma_N$ is a twisted homomorphism from G onto N, satisfying

$$(\sigma\tau)_N = \sigma_N^\tau \tau_N \quad \text{for all } \sigma, \tau \in G\,;$$

it induces the identity on N, and its kernel is H.

Lemma 11.23 *If $L|K$ satisfies condition (11.12) then w.l.o.g., the extensions $L'|K$ and $K'|K$ may assumed to be finite.*

Proof: Suppose that $K'|K$ is an arbitrary algebraic extension such that (11.12) holds. Let $N_0 := \{\tau \in G \mid \forall \sigma_N \in N :\ \tau^{-1}\sigma_N\tau = \sigma_N\}$ be the subgroup of all automorphisms in G whose action on N is trivial (i.e., N_0 is the centralizer of N in G). Since N is abelian, it is contained in N_0. Consequently, the fixed field K_0 of N_0 in L' is contained in K' (which is the fixed field of N in L' by definition of N). Since N is a normal subgroup of G, also its centralizer N_0 is a normal subgroup of G, showing that $K_0|K$ is a Galois extension. We set $H_0 := G/N_0$. By our choice of N_0, the action of G on N induces an action of H_0 on N which is given by $\rho^{-1}\sigma_N\rho = \tau^{-1}\sigma_N\tau$ for $\rho = \tau N_0 \in H_0$. Consequently, H_0 must be finite, being a group of automorphisms of the finite group N. This proves that $K_0|K$ is a finite Galois extension with Galois group H_0. Recall that it follows from (11.12) that also $L|K$ is finite.

We claim that $H \cap N_0$ is a normal subgroup of G. Let $\tau \in H \cap N_0$ and $\sigma \in G$; we want to show that $\tau^\sigma \in H \cap N_0$. Write $\sigma = \sigma_H\sigma_N$ according to (11.13). Then $\tau^\sigma = \sigma_N^{-1}(\sigma_H^{-1}\tau\sigma_H)\sigma_N$; since $\sigma_H \in H$ and $N_0 \lhd G$, we find that $\tau' := \sigma_H^{-1}\tau\sigma_H \in H \cap N_0$. In particular, τ' lies in the centralizer of N. In view of $\sigma_N \in N$ we obtain $\tau^\sigma = \sigma_N^{-1}\tau'\sigma_N = \sigma_N^{-1}\sigma_N\tau' = \tau' \in H \cap N_0$. We have proved that $H \cap N_0$ is a normal subgroup of G. With L_0 the fixed field of $H \cap N_0$ in L', we hence obtain a Galois extension $L_0|K$. Since $L_0 = L.K_0$, the extension $L_0|K$ is finite.

Finally, it remains to show that $\operatorname{Gal} L_0|K_0 \cong \operatorname{Gal} L'|K'$ which also yields that $L_0|K_0$ is p-elementary. Observe that $H.N_0 = G$ since it contains $HN = G$. Now we compute: $\operatorname{Gal} L_0|K_0 = \operatorname{Gal} L'|K_0 \,/\, \operatorname{Gal} L'|L_0 = N_0/(H \cap N_0) \cong H.N_0/H = G/H \cong N = \operatorname{Gal} L'|K'$. We have proved that condition (11.12) also holds with K_0, L_0 in the place of K', L'. $\square$

In view of this lemma, we will assume in the sequel that all field extensions are finite.

Like N, also the additive group $(L',+)$ is a right G-module, the scalar multiplication given by

$$(a,\tau) \;\mapsto\; a^\tau := \tau^{-1}a \quad \text{for all } a \in L' \,,\, \tau \in G \,.$$

Let us show:

Lemma 11.24 *There is an embedding $\phi :\; N \longrightarrow (L',+)$ of right G-modules.*

Proof: By [LANG3], VIII, §12, Theorem 20, the finite Galois extension $K'|K$ admits a normal basis. That is, there exists $b \in K'$ such that b^ρ, $\rho \in \operatorname{Gal} K'|K$, is a basis of K' over K. Since H is a set of representatives in G for $\operatorname{Gal} K'|K$, we may represent these conjugates as b^ρ, $\rho \in H$. Let $\psi :\; N \to (K,+)$ be any homomorphism of groups (there is always at least the trivial one), and set

$$\phi(\sigma_N) := \sum_{\rho\in H} \psi(\rho\sigma_N\rho^{-1})\, b^\rho \quad \text{for all } \sigma_N \in N \,. \tag{11.15}$$

Since ψ is a group homomorphism from N into $(L',+)$, the same is true for ϕ. Given $\tau \in G$, we write $\tau = \tau_H\tau_N$; then $b^{\rho\tau} = (b^{\rho\tau_H})^{\tau_N} = b^{\rho\tau_H}$ since $b^{\rho\tau_H} \in K'$ and $\tau_N \in N = \operatorname{Gal} L'|K'$. Observing also that $H = H\tau_H$, we may compute:

$$\begin{aligned}
\phi(\sigma_N)^\tau &= \sum_{\rho\in H} \psi(\rho\sigma_N\rho^{-1})\, b^{\rho\tau} \;=\; \sum_{\rho\in H} \psi(\rho\tau_H^{-1}\sigma_N(\rho\tau_H^{-1})^{-1})\, b^\rho \\
&= \sum_{\rho\in H} \psi(\rho\sigma_N^{\tau_H}\rho^{-1})\, b^\rho \;=\; \phi(\sigma_N^{\tau_H}) \;=\; \phi(\sigma_N^\tau)
\end{aligned}$$

which shows that ϕ is a homomorphism of right G-modules.

Now we have to choose ψ so well as to guarantee that ϕ becomes injective. If $\phi(\sigma_N) = 0$ then $\psi(\rho\sigma_N\rho^{-1}) = 0$ for all $\rho \in H$ since by our choice of b, the conjugates b^ρ, $\rho \in H$, are linearly independent over K. In particular, $\phi(\sigma_N) = 0$ implies $\psi(\sigma_N) = 0$. Hence, ϕ will be injective if we are able to choose ψ to be injective. This is done as follows. The elementary-abelian p-group N may be viewed as a finite-dimensional $\mathbb{F}_p$-vector space. If K is an infinite field (which by our general assumption has characteristic p), then it contains $\mathbb{F}_p$-vector spaces of arbitrary finite dimension; so there exists an embedding ψ of N into $(K,+)$. If K is a finite field, then all finite extensions of K are cyclic, their Galois groups being generated by a suitable power of the Frobenius φ (as we have mentioned in section 6.1); consequently, N must be cyclic. Since it is also elementary-abelian, N is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ which is the additive group of $\mathbb{F}_p \subset K$. Hence also in this case, N admits an embedding ψ into $(K,+)$. □

By composition with ϕ, the twisted homomorphism $\sigma \mapsto \sigma_N$ is turned into a map $\sigma \mapsto \phi(\sigma_N)$ from G into $(L',+)$. We shall write $\phi(\sigma)$ instead of $\phi(\sigma_N)$, thereby considering the G-module homomorphism $\phi :\; N \to (L',+)$ as being extended to $\phi :\; G \to (L',+)$. By construction, the latter has kernel H and is injective on N. Further, it satisfies $\phi(\sigma\tau) = \phi((\sigma\tau)_N) = \phi(\sigma_N^\tau\tau_N) = \phi(\sigma_N)^\tau + \phi(\tau_N) = \phi(\sigma)^\tau + \phi(\tau)$ showing that ϕ is a twisted homomorphism in the following sense:

$$\phi(\sigma\tau) = \phi(\sigma)^\tau + \phi(\tau) \quad \text{for all } \sigma,\tau \in G \,. \tag{11.16}$$

We claim that there exists an element $\vartheta \in L'$ such that

$$\vartheta^\tau = \vartheta + \phi(\tau) \quad \text{for all } \tau \in G\,. \tag{11.17}$$

Note that (11.17) determines ϑ up to addition of elements from K. It may be constructed as follows. We choose an element $a \in L'$ such that the trace $s := \mathrm{Tr}_{L'|K}(a) = \sum_{\sigma\in G} \sigma a = \sum_{\sigma\in G} a^\sigma$ is not zero (we have seen in the foregoing proof that such an element exists: we could choose a to be the generator of a normal basis of $L'|K$; the linear independence will then force the trace to be nonzero). We set

$$\vartheta := -\frac{1}{s}\sum_{\sigma\in G} \phi(\sigma)\, a^\sigma\,. \tag{11.18}$$

Given $\tau \in G$, we have $G\tau = G$ and

$$1 = \frac{1}{s}\sum_{\sigma\in G} a^\sigma = \frac{1}{s}\sum_{\sigma\in G} a^{\sigma\tau}$$

which we use to compute

$$\begin{aligned}
\vartheta^\tau &= -\frac{1}{s}\sum_{\sigma\in G} \phi(\sigma)^\tau a^{\sigma\tau} = -\frac{1}{s}\sum_{\sigma\in G} ((\phi(\sigma)^\tau + \phi(\tau))a^{\sigma\tau} - \phi(\tau)a^{\sigma\tau}) \\
&= -\frac{1}{s}\sum_{\sigma\in G} \phi(\sigma\tau)a^{\sigma\tau} + \phi(\tau)\frac{1}{s}\sum_{\sigma\in G} a^{\sigma\tau} \\
&= -\frac{1}{s}\sum_{\sigma\in G} \phi(\sigma)a^{\sigma} + \phi(\tau)\frac{1}{s}\sum_{\sigma\in G} a^{\sigma} = \vartheta + \phi(\tau)\,.
\end{aligned}$$

This proves that ϑ indeed satisfies (11.17).

Remark 11.25 The additive analogue of Hilbert's Satz 90 (cf. [LANG3], VIII, §10 or [JAC], chapter 1, section 15) says that $\mathrm{H}^1(G,(L',+)) = 0$. Since the twisted homomorphism $\phi : G \to (L',+)$ may be interpreted as a 1-cocycle, this implies that ϕ splits, which indicates the existence of ϑ. Replacing the twisted homomorphism ϕ by an arbitrary 1-cocycle in our above computation will in fact provide a proof of this additive analogue.

Since H is the kernel of ϕ, (11.17) yields that H is the group of all automorphisms of $L'|K$ which fix ϑ. Since on the other hand, by definition of $H = \mathrm{Gal}\, L'|L$ the fixed field of H in L' is L, we know from Galois theory that $L = K(\vartheta)$. Let us now compute the minimal polynomial f of ϑ over K. The group N may be viewed as a system of representatives for the left cosets of G modulo H. Consequently, the elements ϑ^τ, $\tau \in N$, are precisely all conjugates of ϑ over K. So

$$f(X) = \prod_{\tau\in N} (X - \vartheta^\tau) = \prod_{\tau\in N} (X - \vartheta - \phi(\tau)) = \mathcal{A}(X - \vartheta)\,,$$

where

$$\mathcal{A}(X) := \prod_{\tau\in N} (X - \phi(\tau))\,.$$

The roots of $\mathcal{A}$ form the additive group $\phi(N)$. Since we have chosen ϕ to be injective, we have $|\phi(N)| = \deg \mathcal{A}$. By Lemma 11.1 it follows that $\mathcal{A}$ is an additive polynomial. In particular,

$$f(X) = \mathcal{A}(X - \vartheta) = \mathcal{A}(X) - \mathcal{A}(\vartheta)\,.$$

Since $f(X) \in K[X]$, we have $\mathcal{A}(X) \in K[X]$ and $\mathcal{A}(\vartheta) \in K$. We have proved:

Lemma 11.26 *Let $L|K$ be an extension which satisfies condition (11.12). Then there exist an additive polynomial $\mathcal{A}(X) \in K[X]$ and an element $\vartheta \in L$ such that $L = K(\vartheta)$ and $\mathcal{A}(\vartheta) \in K$.*

As an example, let us treat an important special case. Let us assume that $L|K$ is a Galois extension of degree p. Then its Galois group is just $\mathbb{Z}/p\mathbb{Z}$, and the extension is thus p-elementary. In the above setting, we may then choose $K' = K$ which yields $L' = L$, $G = N = \mathbb{Z}/p\mathbb{Z}$ and $H = 1$. The embedding $\phi : N \longrightarrow (L', +)$ may be chosen "by hand" to be the most natural one: $N = \mathbb{Z}/p\mathbb{Z} = (\mathbb{F}_p, +) \subset (L', +)$. We obtain

$$\mathcal{A}(X) = \prod_{i \in \mathbb{F}_p} (X - i) = X^p - X$$

since the latter is the unique polynomial of degree which vanishes on all elements of $\mathbb{F}_p$. The extension $L|K$ is thus generated by the root ϑ of the polynomial $f(X) = X^p - X - \mathcal{A}(\vartheta)$ which is called an **Artin-Schreier polynomial**. The extension $L|K$ is called an **Artin-Schreier extension**, and ϑ is called an **Artin-Schreier root**. So we have shown:

Corollary 11.27 *Every Galois extension of degree p of a field of characteristic $p > 0$ is an Artin-Schreier extension.*

Inspired by this special case, we want to investigate whether we can get more information about the additive polynomial $\mathcal{A}$ if we strengthen the hypotheses. For instance, ϕ may be injective even if ψ is not. In our special case, $N = \mathbb{Z}/p\mathbb{Z}$ was an irreducible G-module, that is, it did not admit any proper nonzero G-submodule. But if N is an irreducible G-module, then every G-module homomorphism ϕ can only have kernel 0 or N, so if it does not vanish, then it is injective. For ϕ as defined in (11.15), we obtain $\phi \neq 0$ if $\psi \neq 0$. So it will suffice to take $\psi : N \to (\mathbb{F}_p, +)$ as a nonzero (additive) character; it exists since N is a nontrivial p-group. With this choice of ψ, we obtain

$$\phi(N) \subset \sum_{\rho \in H} \mathbb{F}_p \, b^\rho = \sum_{\rho \in H} \mathbb{F}_p \, \rho b \, .$$

Since the coefficients of the polynomial $\mathcal{A}$ are the elementary symmetric polynomials of the elements $\phi(\tau)$, $\tau \in N$, they lie in the ring $\mathbb{F}_p[\, \rho b \mid \rho \in H]$.

The condition that N be an irreducible G-module has turned out to be of certain importance. It is satisfied in the following special case:

Lemma 11.28 *Assume that $L|K$ is minimal with the property (11.12), that is, there is no proper nontrivial subextension with the same property. Then N is an irreducible G-module.*

Proof: Assume that M is a G-submodule of N, that is, M is a normal subgroup of G. Then HM is a subgroup of G containing H. In view of the unique representation (11.13), we have $HM = H$ if and only if $M = 1$ and $HM = G$ if and only if $M = N$. Note that the fixed field L_1' of M in L' is a Galois extension of K containing K'. Further, the fixed field L_1 of HM is contained in L, and it satisfies $L_1.K' = L_1'$ since $HM \cap N = M \cap N = M$. Consequently, also $L_1|K$ has property (11.12).

Suppose now that $L|K$ is minimal with the property (11.12). Then $L_1 = L$ or $L_1 = K$. Hence $HM = H$ or $HM = G$, that is, $M = 1$ or $M = N$ showing that the G-module N is irreducible. □

We may summarize our preceding discussion in the following

Lemma 11.29 *Let the situation be as in Lemma 11.26. Assume in addition that $L|K$ is minimal with the property (11.12). We may assume w.l.o.g. that the extension $K'|K$ is finite. For every $b \in K'$ generating a normal basis of $K'|K$, and for every nonzero additive character $\psi : N \to (\mathbb{F}_p, +)$, the G-module homomorphism ϕ defined in (11.15) is injective. Moreover, the coefficients of the correponding additive polynomial $\mathcal{A}(X)$ lie in the ring*

$$K \cap \mathbb{F}_p[\, \rho b \mid \rho \in H] \, .$$

Exercise 11.4 *Since we may take $\psi : N \to (\mathbb{F}_p, +)$, it will also suffice to choose $b \in K'$ such that the conjugates σb are linearly independent over $\mathbb{F}_p$. Does this yield more freedom in the choice of b?*

Exercise 11.5 *Give additional conditions under which the converse of Lemma 11.28 holds.*

Chapter 12

The tame and the wild valuation theory

12.1 Absolute ramification theory

Let (K,v) be a henselian field. The inertia field of the normal extension $(\tilde{K}|K,v)$ (or equivalently, of the normal extension $(K^{\rm sep}|K,v)$) will be called the **absolute inertia field** or just **inertia field of** (K,v) and denoted by $(K,v)^i$ or by (K^i,v^i). Similarly, the ramification field of $(\tilde{K}|K,v)$ will be called the **absolute ramification field** or just **ramification field of** (K,v) and denoted by $(K,v)^r$ or by (K^r,v^r). Observe that both fields are uniquely determined by the henselian field (K,v) since the extension of v from K to L is unique. In the following, we will always refer to the valuation v and its unique extension to $\tilde{K}$. So instead of writing e.g. (K^i,v^i), we will just write K^i.

Lemma 12.1 *Let (K,v) be a henselian field. Its absolute inertia field K^i and its absolute ramification field K^r are Galois extensions of K. The residue fields of both are equal to the separable-separable closure $\overline{K}^{\rm sep}$ of $\overline{K}$. The value group of the absolute inertia field is equal to vK, and that of the absolute ramification field is the p'-divisible hull of vK.*

Proof: Since (K,v) is henselian, the decomposition field of $(\tilde{K}|K,v)$ is equal to (K,v). Hence by Lemma 7.81, the inertia field $(K,v)^i = (\tilde{K}|K,v)^i$ and the ramification field $(K,v)^r = (\tilde{K}|K,v)^r$ are Galois extensions of (K,v). Theorem 7.88 shows that $vK^i = vK$ and that $\overline{K^i}$ is the relative separable-algebraic closure of $\overline{K}$ in $\tilde{\overline{K}}$. By Theorem 7.94, vK^r is the p'-divisible closure of vK in $v\tilde{K}$, and $\overline{K^r} = \overline{K^i}$. On the other hand, we know from Lemma 7.59 that $v\tilde{K}$ is the divisible hull of vK and that $\tilde{\overline{K}}$ is the algebraic closure of $\overline{K}$. It now follows that vK^r is the p'-divisible hull of vK and that $\overline{K^r} = \overline{K}^{\rm sep}$. □

From Lemma 7.82 we obtain:

Lemma 12.2 *Let $(L|K,v)$ be an algebraic extension of henselian fields. Then $L^i = L.K^i$ and $L^r = L.K^r$.*

As a consequence of Corollary 7.97, we have

Lemma 12.3 *Let (K,v) be a henselian field and $K \subset K_0 \subset K_1 \subset K^r$ such that $K_1|K_0$ is finite. Then $(K_1|K_0,v)$ is defectless. Further, $e(K_1|K_0,v)$ is not divisible by the characteristic of $\overline{K}$, and $\overline{K_1}|\overline{K_0}$ is a separable-algebraic extension.*

Proof: Since (K,v) is henselian, the same is true for (K_0,v), and we thus have $g(K_1|K_0,v)=1$. Hence, our assertion follows directly from Corollary 7.97 if $K_1|K_0$ is normal. If this is not the case, we choose N to be the normal hull of K_1 over K_0. Since $K^r|K_0$ is normal, N lies in K^r. Now Corollary 7.97 applied to the finite normal extension $(N|K_0,v)$ gives the corresponding assertion for this extension. From this, the assertion also follows for the subextension $(K_1|K_0,v)$ of $(N|K_0,v)$. □

By use of this lemma, we can add the following fact to the assertion of Lemma 12.2:

Lemma 12.4 *Let $(L|K,v)$ be an immediate extension of henselian fields. Then $L^i=L.K^i$ and $L^r=L.K^r$.*

Proof: Since $vK=vL$, we have by Theorem 7.88 that $vK^i=vK=vL=vL^i$. It also follows that the p'-divisible hull of vK is the same as that of vL. Hence in view of Theorem 7.94, $vK^r=vL^r$. Since $\overline{K}=\overline{L}$, the separable-algebraic closure of $\overline{K}$ is the same as that of $\overline{L}$. Hence in view of Theorem 7.88 and Theorem 7.94, $\overline{K^i}=\overline{L^i}$ and $\overline{K^r}=\overline{L^r}$. This shows that

$$(L^i|K^i,v) \quad \text{and} \quad (L^i|K^i,v)$$

are immediate extensions. Consequently, also

$$(L^i|L.K^i,v) \quad \text{and} \quad (L^i|L.K^i,v)$$

are immediate algebraic extensions. Now Lemma 12.3 shows that they must be trivial. □

The absolute inertia field and the absolute ramification field of (K,v) are universal in the sense that their intersection with an arbitrary normal extension of (K,v) produces the inertia and ramification field of that extension. Indeed, this follows from Lemma 7.89 and Lemma 7.95. So let us note:

Lemma 12.5 *Let $(L|K,v)$ be a normal extension of henselian fields. Then $(L|K,v)^i=(L\cap K^i,v)$ and $(L|K,v)^r=(L\cap K^r,v)$. In particular, the inertia field of every normal extension of (K,v) is contained in K^i, and the ramification field of every normal extension of (K,v) is contained in K^r.*

The following lemma shows that even if the extension $L|K$ is not normal, then the intersections have the main properties of the inertia and ramification field:

Lemma 12.6 *Let $(L|K,v)$ be an algebraic extension of henselian fields, and let p denote the characteristic exponent of $\overline{K}$. Then the following holds:*
a) $\overline{L\cap K^i}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$, and $v(L\cap K^i)=vK$. If $L|K$ is finite, then $[L\cap K^i:K]=[\overline{L}:\overline{K}]_{\rm sep}$.
b) $v(L\cap K^r)$ is the p'-divisible hull of vK in vL, and $\overline{L\cap K^r}=\overline{L\cap K^i}$. If $L|K$ is finite and $e(L|K,v)=p^\mu\cdot e'$ with e' prime to p, then $[L\cap K^r:L\cap K^i]=e'$.
Consequently, if $\overline{L}|\overline{K}$ is not purely inseparable, then $L|K$ is not linearly disjoint from $K^i|K$. Similarly, if $(vL:vK)$ is not a power of p, then $L|K$ is not linearly disjoint from $K^r|K$.

Proof: We prove our assertions for the case of a finite extension $L|K$. The deduction of the assertions for arbitrary algebraic extensions is left to the reader as a straightforward exercise. We set $\mathrm{V}_0:=L\cap K^r$ and $\mathrm{T}_0:=L\cap K^i\subset\mathrm{V}_0$. Let N be the normal hull of

$L|K$. Since (K,v) is henselian, we have $g=1$ for every extension of valued fields which are algebraic extensions of (K,v).

By Lemma 7.89, $\mathrm{T} := N \cap K^i$ is the inertia field of $(N|K,v)$ and of $(N|\mathrm{T}_0,v)$. Hence, $\overline{\mathrm{T}}|\overline{\mathrm{T}_0}$ is separable with $[\mathrm{T}:\mathrm{T}_0]=[\overline{\mathrm{T}}:\overline{\mathrm{T}_0}]$, and $\overline{N}|\overline{\mathrm{T}}$ is purely inseparable. This shows that

$$[\overline{L.\mathrm{T}}:\overline{\mathrm{T}_0}]_{\mathrm{sep}} = [\overline{\mathrm{T}}:\overline{\mathrm{T}_0}] = [\mathrm{T}:\mathrm{T}_0]\ . \tag{12.1}$$

By Lemma 6.14, $\mathrm{T}|\mathrm{T}_0$ is linearly disjoint from $L|\mathrm{T}_0$, hence we have $[\mathrm{T}:\mathrm{T}_0]=[L.\mathrm{T}:L]$. By Lemma 7.82, $L.\mathrm{T}$ is the inertia field of $(N|L,v)$. Thus, the extension $\overline{L.\mathrm{T}}|\overline{L}$ is separable, and $[L.\mathrm{T}:L]=[\overline{L.\mathrm{T}}:\overline{L}]$. Altogether, we find that

$$[\overline{L.\mathrm{T}}:\overline{\mathrm{T}_0}]_{\mathrm{sep}} = [\overline{L.\mathrm{T}}:\overline{L}]\ , \tag{12.2}$$

which proves that the extension $\overline{L}|\overline{\mathrm{T}_0}$ must be purely inseparable. Consequently, $\overline{\mathrm{T}_0}$ contains the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$. But $\overline{\mathrm{T}_0}|\overline{K}$ is separable, being a subextension of the separable extension $\overline{K^i}|\overline{K}$. This proves that $\overline{\mathrm{T}_0}=(\overline{L}|\overline{K})^{\mathrm{sep}}$. By Theorem 7.88, $v\mathrm{T}_0=vK$ and $[\mathrm{T}_0:K]=[\overline{\mathrm{T}_0}:\overline{K}]=[\overline{L}:\overline{K}]_{\mathrm{sep}}$.

Similarly, we treat the field V_0. By Lemma 7.95 and Lemma 7.82, $\mathrm{V} := N \cap K^r$ is the ramification field of $(N|K,v)$, of $(N|\mathrm{T}_0,v)$ and of $(N|\mathrm{V}_0,v)$. Hence by Theorem 7.94, $\overline{\mathrm{V}}|\overline{\mathrm{T}_0}$ is separable and $v\mathrm{V}/v\mathrm{V}_0$ is a p'-group. The former proves that $\overline{\mathrm{V}_0}=\overline{\mathrm{T}_0}$. Indeed, we have already seen that $\overline{\mathrm{T}_0}\subset\overline{\mathrm{V}_0}$ is the relative separable-algebraic closure of $\overline{K}$ in $\overline{L}$, hence the separable subextension $\overline{\mathrm{V}_0}|\overline{\mathrm{T}_0}$ of $\overline{\mathrm{V}}|\overline{\mathrm{T}_0}$ must be trivial.

By Theorem 7.94, we know that $\overline{\mathrm{V}}=\overline{\mathrm{T}}$. Since also $\overline{\mathrm{V}_0}=\overline{\mathrm{T}_0}$, we have that $[\overline{\mathrm{V}}:\overline{\mathrm{V}_0}]=[\overline{\mathrm{T}}:\overline{\mathrm{T}_0}]=[\mathrm{T}:\mathrm{T}_0]$. In view of Lemma 12.3, it follows that

$$[\mathrm{V}:\mathrm{V}_0]=(v\mathrm{V}:v\mathrm{V}_0)[\overline{\mathrm{V}}:\overline{\mathrm{V}_0}]=(v\mathrm{V}:v\mathrm{V}_0)[\mathrm{T}:\mathrm{T}_0]\ . \tag{12.3}$$

By Lemma 7.82, $L.\mathrm{V}$ is the ramification field of $(N|L.\mathrm{T},v)$. Hence by Theorem 7.94, $\overline{L.\mathrm{V}}=\overline{L.\mathrm{T}}$. Hence from (12.3) and (12.4), we obtain that

$$[\overline{L.\mathrm{V}}:\overline{L}]=[\overline{L.\mathrm{T}}:\overline{L}]=[\mathrm{T}:\mathrm{T}_0]\ .$$

Again in view of Lemma 12.3, we conclude that

$$[L.\mathrm{V}:L]=(v(L.\mathrm{V}):vL)[\overline{L.\mathrm{V}}:\overline{L}]=(v(L.\mathrm{V}):vL)[\mathrm{T}:\mathrm{T}_0]\ . \tag{12.4}$$

By Lemma 6.34, $\mathrm{V}|\mathrm{V}_0$ is linearly disjoint from $L|\mathrm{V}_0$, hence we have that $[\mathrm{V}:\mathrm{V}_0]=[L.\mathrm{V}:L]$. From this together with (12.3) and (12.4), we find that $(v\mathrm{V}:v\mathrm{V}_0)=(v(L.\mathrm{V}):vL)$. Consequently, $(v(L.\mathrm{V}):v\mathrm{V})=(vL:v\mathrm{V}_0)$. From Theorem 7.91 we know that $vN/v\mathrm{V}$ is a p-group. Hence also its subgroup $v(L.\mathrm{V})/v\mathrm{V}$ is a p-group, which yields that also $vL/v\mathrm{V}_0$ is a p-group. Consequently, $v\mathrm{V}_0$ contains the p'-divisible hull of vK in vL. But $v\mathrm{V}_0/vK$ is a p'-group, being a subgroup of the p'-group vK^r/vK. This proves that $v\mathrm{V}_0$ is the p'-divisible hull of vK in vL. So if $e(L|K,v)=p^\mu\cdot e'$ with e' prime to p, then $e'=(v\mathrm{V}_0:vK)$. By Corollary 7.97, $[\mathrm{V}_0:K]=(v\mathrm{V}_0:vK)\cdot[\overline{\mathrm{V}_0}:\overline{K}]$. As we have seen already, $\overline{\mathrm{V}_0}=\overline{\mathrm{T}_0}$ and $[\mathrm{T}_0:K]=[\overline{\mathrm{T}_0}:\overline{K}]$. Dividing by $[\mathrm{T}_0:K]$, we thus find that $[\mathrm{V}_0:\mathrm{T}_0]=(v\mathrm{V}_0:vK)=e'$.

□

In view of this lemma, one may define the inertia and ramification field of an arbitrary algebraic extension of a henselian field (K,v) to be the respective intersections with K^i and K^r. We leave it to the reader to work out the details of such a generalized ramification theory.

12.2 Tame and purely wild extensions

Lemma 12.3 motivates the following definition. An algebraic extension $(L|K, v)$ of henselian fields is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $e(E|K, v)$ is prime to the characteristic exponent of $\overline{K}$.

(TE2) The residue field extension $\overline{E}|\overline{K}$ is separable.

(TE3) The extension $(E|K, v)$ is defectless.

Remark 12.7 This notion of "tame extension" does not coincide with the notion of "tamely ramified extension" as defined in the book of O. Endler [END2], page 180. The latter definition requires (TE1) and (TE2), but not necessarily (TE3). Our tame extensions are the defectless tamely ramified extensions in the sense of Endler's book. In particular, in our terminology, proper immediate algebraic extensions of henselian fields are not called tame, which is justified in view of the difficulties that they will give us later in this book.

It follows directly from this definition that every subextension of a tame extension is again a tame extension. If the residue characteristic of (K, v) is 0, then (TE1) and (TE2) are trivially satisfied for every finite extension $(E|K, v)$. By Theorem 8.22, also (TE3) is satisfied. Hence, every finite extension of henselian fields of residue characteristic 0 is tame. The following theorem characterizes the tame extensions of an arbitrary henselian field (K, v):

Theorem 12.8 *Let (K, v) be a henselian field. Then the absolute ramification field (K^r, v) is a tame extension of (K, v), and every other tame extension of (K, v) is contained in $(K, v)^r$. Hence, $(K, v)^r$ is* the *maximal tame extension of (K, v). In particular, the compositum of every two tame extensions of (K, v) is again a tame extension of (K, v).*

Proof: Let $(L|K, v)$ be a subextension of $(K^r|K, v)$ and $(E|K, v)$ a finite subextension of $(L|K, v)$. Then an application of Lemma 12.3 with $K_0 = K$ and $K_1 = E$ shows that $(E|K, v)$ satisfies (TE1), (TE2) and (TE3). Hence, $(L|K, v)$ is a tame extension.

Now let $(L|K, v)$ be a tame extension. To show that $L \subset K^r$, it suffices to prove that every finite subextension $(E|K, v)$ of $(L|K, v)$ is a subextension of $(K^r|K, v)$. By assumption, $(E|K, v)$ satisfies (TE1), (TE2) and (TE3). By Lemma 12.6 and (TE1), (TE2), $[L \cap K^r : K] = e(L|K, v) \cdot f(L|K, v)$. By (TE3), the latter is equal to $[L : K]$. Hence $L = L \cap K^r \subset K^r$. □

An extension $(L|K, v)$ of henselian fields is called **unramified** if the extension $vL|vK$ is trivial.

Theorem 12.9 *Let (K, v) be a henselian field. Every tame unramified extension of (K, v) is contained in the absolute inertia field $(K, v)^i$ of (K, v). Hence, $(K, v)^i$ is* the *maximal tame unramified extension of (K, v). In particular, the compositum of every two tame unramified extensions of (K, v) is again a tame extension of (K, v).*

The proof is similar to that of the foregoing theorem, taking K^i in the place of K^r.

The following consequence of Lemma 8.85 is one of the most important properties of tame extensions. It serves to reduce the study of the defect to more suitable situations (cf. Chapter 3 of [KU1] and [KU2]).

Lemma 12.10 *Let (K,v) be henselian, $(F|K,v)$ a tame extension and $(L|K,v)$ a finite extension. Then*

$$d(L|K,v) = d(L.F|F,v) \ .$$

A (not necessarily algebraic) extension $(L|K,v)$ of a henselian field (K,v) is called **purely wild** if it satisfies the following conditions:

(PW1) vL/vK is a p-group, where p is the characteristic exponent of $\overline{K}$.
(In particular, vL/vK is a torsion group.)

(PW2) The residue field extension $\overline{E}|\overline{K}$ is purely inseparable.
(In particular, $\overline{E}|\overline{K}$ is algebraic.)

We see: Every subextension of a purely wild extension is purely wild. Every purely inseparable extension is purely wild.

Since $K^r|K$ is normal, $L|K$ is linearly disjoint from $K^r|K$ if and only if $L \cap K^r = K$. By Lemma 12.6, the latter holds if and only if $e(L|K,v)$ is a power of the characteristic exponent of $\overline{K}$ and $\overline{L}|\overline{K}$ is purely inseparable. This proves:

Lemma 12.11 *An algebraic extension $(L|K,v)$ is purely wild if and only if $L|K$ is linearly disjoint from $K^r|K$.*

Let us consider a very special sort of purely wild extensions. Note that in the case of $\operatorname{char} K = 0$, we have by convention that $K^{1/p^\infty} = K$ (which is the perfect hull of K), even if $p = \operatorname{char} \overline{K}$.

Lemma 12.12 *Let (K,v) be a henselian field with $\operatorname{char} \overline{K} = p > 0$. Suppose that (K,v) admits no purely wild Artin-Schreier extensions. Then its value group vK is p-divisible and its residue field $\overline{K}$ is perfect, and (K,v) is dense in its perfect hull $(K^{1/p^\infty},v)$.*

Proof: To find that vK is p-divisible, employ Lemma 7.55 and note that the Artin-Schreier extensions given there are defectless and thus purely wild. Similarly, it follows from Lemma 7.56 that $\overline{K}$ is perfect. Now the proof for Theorem 8.58 shows that (K,v) is dense in its perfect hull $(K^{1/p^\infty},v)$; just note that the immediate Artin-Schreier extensions provided by Lemma 8.57 are purely wild. □

Observe that the hypothesis that (K,v) be henselian is only required since we have defined purely wild extensions only over henselian fields. The reader may try a more general definition but note that we do not want the henselization to be called a purely wild extension. The way out is to require that the extension in question is linearly disjoint from the henselization.

12.3 Tame fields

Let (K,v) be a henselian valued field and denote by p the characteristic exponent of its residue field $\overline{K}$. Then (K,v) is said to be a **tame field** if $(\tilde{K}|K,v)$ is a tame extension, and to be a **separably tame field** if $(K^{\rm sep}|K,v)$ is a tame extension. On the other extreme, we will call (K,v) a **purely wild field** if $(\tilde{K}|K,v)$ is a purely wild extension. For example, for every henselian field (K,v), its ramification field $(K,v)^r$ is a purely wild field.

Lemma 12.13 *a) Let (K,v) be a henselian field. Then (K,v) is a tame field if and only if $K^r = \tilde{K}$. Similarly, (K,v) is a tame field if and only if $K^r = K^{\rm sep}$. Further, (K,v) is a purely wild field if and only if $K^r = K$.*

b) Every algebraic extension of a tame (resp. separably tame, resp. purely wild) field is again a tame (resp. separably tame, resp. purely wild) field.

Proof: a): If (K,v) is a tame field, then by definition and Theorem 12.8, $\tilde{K} \subset K^r$, showing that $\tilde{K} = K^r$. The converse also follows from Theorem 12.8. For "separably tame", the same proof works with $K^{\rm sep}$ in the place of $\tilde{K}$ because by definition, K^r is a subfield of $K^{\rm sep}$.

By definition and Lemma 12.11, (K,v) is a purely wild field if and only if $\tilde{K}|K$ is linearly disjoint from $K^r|K$. This in turn is equivalent to $K^r = K$.

b): Every algebraic extension $L|K$ satisfies $\tilde{L} = \tilde{K} = L.\tilde{K}$, $L^{\rm sep} = L.K^{\rm sep}$ and by Lemma 12.2 $L^r = L.K^r$. In view of this, assertion b) follows directly from a). □

If (K,v) is a henselian field of residue characteristic 0, then every algebraic extension $(L|K,v)$ is tame, as we have seen in the last section. So we note:

Lemma 12.14 *Every algebraic extension of henselian fields of residue characteristic 0 is a tame extension. Every henselian field of residue characteristic 0 is a tame field.*

From the definition and the fact that every tame extension is defectless and separable, we obtain:

Lemma 12.15 *Every tame field is henselian defectless and perfect.*

In general, infinite extensions of defectless fields need not be defectless fields (cf. Theorem 8.41 and Theorem 8.45). But from the foregoing lemma and Lemma 12.13, we can deduce:

Corollary 12.16 *Every algebraic extension of a tame field is a defectless field.*

In [PAN1], [PAN2], M. Pank proved the following important theorem (for a proof, see [KU–PA–ROQ]):

Theorem 12.17 *Let (K,v) be a henselian field with residue characteristic $p > 0$. There exist algebraic field complements W_s of $K^{\rm sep}$ over K, i.e. $K^r.W_s = K^{\rm sep}$ and W_s is linearly disjoint from $K^{\rm sep}$ over K. The perfect hull $W = W_s^{1/p^\infty}$ is an algebraic field complement of K^r over K, i.e. $K^r.W = \tilde{K}$ and W is linearly disjoint from K^r over K. The valued complements (W_s,v) can be characterized as the maximal separable algebraic purely wild extensions of (K,v), and the (W,v) are the maximal algebraic purely wild extensions of (K,v). Both are unique up to isomorphism over K if there does not exist any finite tame extension of (K,v) whose degree is divisible by p. Moreover, vW is the p-divisible hull of vK, and $\overline{W}$ is the perfect hull of $\overline{K}$.*

Using this theorem, we give some characterizations for tame fields. Recall that every algebraically maximal field is henselian (cf. Corollary 8.30).

Lemma 12.18 *The following assertions are equivalent:*
1) (K,v) is tame,
2) Every algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is algebraically maximal and closed under purely wild extensions by p-th roots,
4) (K,v) is algebraically maximal, vK is p-divisible and $\overline{K}$ is perfect.

Proof: Let (K,v) be a tame field, i.e., $K^r = \tilde{K}$. Then by Lemma 12.11, every algebraic purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Suppose that (K,v) has no algebraic purely wild extension. Then in particular, it has no purely wild extension by p-th roots. Since every immediate algebraic extension of a henselian field is purely wild by definition, we also obtain that (K,v) admits no proper immediate algebraic extension, i.e., (K,v) is algebraically maximal. This proves 2)⇒3).

Assume now that (K,v) is an algebraically maximal field closed under purely wild extensions by p-th roots. Let a be an arbitrary element of K. Assume that va is not divisible by p in vK; then the extension $K(b)|K$ generated by an element $b \in \tilde{K}$ with $b^p = a$ satisfies $(vK(b) : vK) = p = [K(b) : K]$ and thus admits a unique extension of v. With this extension it is purely wild, contrary to our assumption on (K,v). Assume that $va = 0$ and that $\overline{a}$ has no p-th root in $\overline{K}$; then the extension $K(b)|K$ generated as above satisfies $[\overline{K(b)} : \overline{K}] = p = [K(b) : K]$ and is again purely wild, contrary to our assumption. By this, we have shown that vK is p-divisible and $\overline{K}$ is perfect. This proves 3)⇒4).

Suppose that (K,v) is an algebraically maximal (and thus henselian) field such that vK is p-divisible and $\overline{K}$ is perfect. Choose a maximal purely wild extension (W,v) in accordance to Theorem 12.17. Our condition on the value group and the residue field yields that $(W|K,v)$ is immediate. But since (K,v) is assumed to be algebraically maximal, this extension must be trivial. This shows that $\tilde{K} = K^r$, i.e., (K,v) is a tame field. This proves 4)⇒1). □

If K has characteristic $p > 0$, then every extension by p-th roots is purely inseparable and thus purely wild. So the lemma yields:

Corollary 12.19 *A valued field (K,v) of characteristic $p > 0$ is tame if and only if it is algebraically maximal and perfect. Consequently, if (K,v) is an arbitrary valued field of characteristic $p > 0$, then every maximal immediate algebraic extension (W,v) of $(K^{1/p^\infty},v)$ is a tame field having the p-divisible hull of vK as its value group and the perfect hull of $\overline{K}$ as its residue field.*

For perfect valued fields of positive characteristic, "algebraically maximal" and "henselian defectless" are equivalent.

The next corollary shows how to construct tame fields with suitable prescribed value group and residue field:

Corollary 12.20 *Let p be a prime number, Γ a p-divisible ordered abelian group and k a perfect field of characteristic p. Then there exists a tame field K of characteristic p having Γ as its value group and k as its residue field such that $K|\mathbb{F}_p$ admits a valuation transcendence basis and the cardinality of K is equal to the maximum of the cardinalities of Γ and k.*

Proof: According to Theorem 7.57, there is a valued field (K_0,v) of characteristic p with value group Γ and residue field k, and admitting a valuation transcendence basis over its prime field. Now take (K,v) to be a maximal immediate algebraic extension of (K_0,v). Then (K,v) is algebraically maximal, and Lemma 12.18 shows that it is a tame field. Since it is an algebraic extension of (K_0,v), it still admits a valuation transcendence basis over its prime field. Hence, it follows from Example 7.6 that $|K| = \max\{|\Gamma|,|k|\}$. (If v is nontrivial, then K is infinite. If v is trivial, then $\Gamma = \{0\}$ and $K = k$.) □

Now we will prove an important lemma on tame fields that we will need in several instances.

Lemma 12.21 *Let (L, v) be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $\overline{L}|\overline{K}$ is an algebraic extension, then K is also a tame field and moreover, vK is pure in vL and $\overline{K} = \overline{L}$.*

Proof: Since (L, v) is tame, it is henselian and perfect. Since K is relatively algebraically closed in L, it is henselian and perfect too. Assume that $(K_1|K, v)$ is a finite purely wild extension; in view of Lemma 12.18, we have to show that it is trivial. The degree $[K_1 : K]$ is a power of p, say p^m. Since K is perfect, $L|K$ and $K_1|K$ are separable extensions. Since K is relatively algebraically closed in L, we know that L and K_1 are linearly disjoint over K. Thus, K_1 is relatively algebraically closed in $K_1.L$, and

$$[K_1.L : L] = [K_1 : K] = p^m \ .$$

Since L is assumed to be a tame field, the extension $(K_1.L|L, v)$ must be tame. This implies that

$$\overline{K_1.L} \,|\, \overline{L}$$

is a separable extension of degree p^m. On the other hand, $\overline{K_1.L} \,|\, \overline{K_1}$ is an algebraic extension since by hypothesis, $\overline{L} \,|\, \overline{K}$ and thus also $\overline{K_1.L} \,|\, \overline{K}$ are algebraic extensions. Furthermore, $(K_1.L, v)$ being a henselian field and K_1 being relatively algebraically closed in $K_1.L$, Hensel's Lemma shows that

$$\overline{K_1.L} \,|\, \overline{K_1}$$

must be purely inseparable. This yields that

$$\begin{aligned} p^m &= [\overline{K_1.L} : \overline{L}]_{\rm sep} \leq [\overline{K_1.L} : \overline{K}]_{\rm sep} = [\overline{K_1.L} : \overline{K_1}]_{\rm sep} \cdot [\overline{K_1} : \overline{K}]_{\rm sep} \\ &= [\overline{K_1} : \overline{K}]_{\rm sep} \leq [\overline{K_1} : \overline{K}] \leq [K_1 : K] = p^m \ , \end{aligned}$$

showing that

$$\overline{K_1} \,|\, \overline{K}$$

is separable of degree p^m. Since $K_1|K$ was assumed to be purely wild, we have $p^m = 1$ and the extension $K_1|K$ is trivial.

We have now shown that K is a tame field; hence by Lemma 12.18, vK is p-divisible and $\overline{K}$ is perfect. Since $\overline{L}|\overline{K}$ is assumed to be algebraic, we obtain from Lemma 9.24 that $\overline{K} = \overline{L}$ and that the torsion subgroup of vL/vK is a p-group. But vK is p-divisible since K is perfect. Thus, vL/vK has no p-torsion, showing that vL/vK has no torsion at all. □

The same lemma holds for separably tame fields, as stated in Lemma 12.29 below. The following corollaries will show some nice properties of the class of tame fields. They also possess generalizations to separably tame fields, see Corollary 12.30 below.

Corollary 12.22 *For every valued function field F with given transcendence basis $\mathcal{T}$ over a tame field K, there exists a tame subfield K_0 of K of finite rank with $\overline{K_0} = \overline{K}$ and vK_0 pure in vK, and furthermore a function field F_0 with transcendence basis $\mathcal{T}$ over K_0 such that*

$$F = K.F_0 \tag{12.5}$$

and

$$[F_0 : K_0(\mathcal{T})] = [F : K(\mathcal{T})] \ . \tag{12.6}$$

Proof: Let $F = K(\mathcal{T})(a_1, \ldots, a_n)$. There exists a finitely generated subfield K_1 of K such that $a_1, \ldots, a_n$ are algebraic over $K(\mathcal{T})$ and $[F : K(\mathcal{T})] = [K_1(\mathcal{T})(a_1, \ldots, a_n) : K_1(\mathcal{T})]$. This will also hold for every extension field of K_1 in K. As a finitely generated field, (K_1, v) has finite rank. Now let y_j, $j \in J$, be a system of elements in K such that the residues $\overline{y_j}$, $j \in J$, form a transcendence basis of $\overline{K}$ over $\overline{K_1}$. According to Lemma 7.31, the field $K_1(y_j | j \in J)$ has residue field $\overline{K_1}(\overline{y_j} | j \in J)$ and the same value group as K_1, hence it is again a field of finite rank. Let K_0 be the relative closure of this field within K. Since by construction, $\overline{K}|\overline{K_1}(\overline{y_j}|j \in J)$ and thus also $\overline{K}|\overline{K_0}$ are algebraic, we can infer from the preceding lemma that K_0 is a tame field with $\overline{K_0} = \overline{K}$ and vK_0 pure in vK. As an algebraic extension of a field of finite rank it is itself of finite rank. Finally, the function field $F_0 = K_0(\mathcal{T})(a_1, \ldots, a_n)$ has transcendence basis $\mathcal{T}$ over K_0 and satisfies assertions (12.5) and (12.6). □

Corollary 12.23 *For every extension $(L|K, v)$ with (L, v) a tame field, there exists a tame intermediate field L_0 such that the extension $(L_0|K, v)$ admits a valuation transcendence basis and the extension $(L|L_0, v)$ is immediate.*

Proof: Take $\mathcal{T}$ to be a maximal algebraically valuation independent set in $(L|K, v)$. With this choice, $vL/vK(\mathcal{T})$ is a torsion group and $\overline{L}|\overline{K(\mathcal{T})}$ is algebraic. Let L_0 be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by Lemma 12.21, we have that (L_0, v) is a tame field, that $\overline{L} = \overline{L_0}$ and that vL_0 is pure in vL and thus $vL_0 = vL$. This shows that the extension $(L|L_0, v)$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $(L_0|K, v)$ by construction. □

12.4 Separably tame fields

Recall that a valued field (K, v) is called **separably defectless** if every finite separable extension is defectless, and that it is called **separable-algebraically maximal** if it does not admit proper immediate separable algebraic extensions. Since the henselization of (K, v) is an immediate separable-algebraic extension of (K, v), a separable-algebraically maximal field (K, v) will coincide with its henselization and thus be henselian. Note that "henselian separably defectless" implies "separable-algebraically maximal".

Since every finite separable algebraic extension of a separably tame field is tame and thus defectless, a separably tame field is always henselian separably defectless. The converse is not true; it needs additional assumptions on the value group and the residue field. Under the assumptions that we are going to use frequently, the converse will even hold for "separable-algebraically maximal" in the place of "henselian separably defectless". Before proving this, we need a lemma which essentially makes use Theorem 12.17.

Lemma 12.24 *A henselian field (K, v) is defectless if and only if every finite purely wild extension of (K, v) is defectless. Similarly, (K, v) is separably defectless if and only if every finite separable purely wild extension of (K, v) is defectless.*

Proof: By Theorem 12.17, there exists a field complement W of K^r over K in $K^{\rm sep}$, and W^{1/p^∞} is a field complement of K^r in $\tilde{K}$. Consequently, given any finite extension (resp. finite separable extension) $(L|K, v)$, there is a finite subextension $N|K$ of $K^r|K$ and a finite (resp. finite separable) subextension $W_0|K$ of $W|K$ such that $L \subset N.W_0$. If $(N.W_0|K, v)$ is

defectless, then so is $(L|K,v)$; hence (K,v) is defectless (resp. separably defectless), if and only if every such extension $(N.W_0|K,v)$ is defectless. Since $(N|K,v)$ is a tame extension (by virtue of Theorem 12.8), Lemma 12.10 shows that

$$d(N.W_0|N,v) = d(W_0|K,v) \ .$$

Hence, $(L|K,v)$ is defectless if and only if $(W_0|K,v)$ is defectless. This yields our assertion. □

Lemma 12.25 *The following assertions are equivalent:*
1) (K,v) is separably tame,
2) Every separable algebraic purely wild extension $(L|K,v)$ is trivial,
3) (K,v) is separable-algebraically maximal and closed under purely wild Artin-Schreier extensions,
4) (K,v) is separable-algebraically maximal, vK is p-divisible and $\overline{K}$ is perfect.

Proof: Let (K,v) be a separably tame field, i.e., $K^r = K^{\rm sep}$. Then by Lemma 12.11, every separable algebraic purely wild extension of (K,v) must be trivial. This proves 1)⇒2).

Now suppose that every separable algebraic purely wild extension of (K,v) is trivial. Then in particular, (K,v) admits no purely wild Artin-Schreier extensions (because they are separable). Furthermore, (K,v) admits no proper separable algebraic immediate extension since they are also purely wild. Consequently, (K,v) is separable-algebraically maximal. This proves 2)⇒3).

If (K,v) is closed under purely wild Artin-Schreier extensions, then by Lemma 12.12, vK is p-divisible and $\overline{K}$ is perfect. This proves 3)⇒4).

Suppose that (K,v) is a separable-algebraically maximal field such that vK is p-divisible and $\overline{K}$ is perfect. By Lemma 8.29, (K,v) is henselian. Choose a maximal separable algebraic purely wild extension (W_s,v) in accordance to Theorem 12.17. Our condition on the value group and the residue field yields that $(W_s|K,v)$ is immediate. But since (K,v) is assumed to be separable-algebraically maximal, this extension must be trivial. This shows that $K^{\rm sep} = K^r$, i.e., (K,v) is a separably tame field. This proves 4)⇒1). □

Suppose that (K,v) separably tame. Choose (W_s,v) according to Theorem 12.17. Then by condition 2) of the lemma, the extension $(W_s|K,v)$ must be trivial. This yields that $(K^{1/p^\infty},v)$ is the unique maximal algebraic purely wild extension of $((K,v)$. Further, (K,v) also satisfies condition 3) of the lemma. From Lemma 12.12 it follows that (K,v) is dense in $(K^{1/p^\infty},v)$, i.e., K^{1/p^∞} lies in the completion of (K,v). This proves:

Corollary 12.26 *If (K,v) is separably tame, then the perfect hull K^{1/p^∞} of K is the unique maximal algebraic purely wild extension of $((K,v)$ and lies in the completion of (K,v). That is, every immediate algebraic extension of a separably tame field (K,v) is purely inseparable and included in the completion of (K,v), and every algebraic approximation type over (K,v) has distance ∞.*

Lemma 12.27 *(K,v) is a separably tame field if and only if $(K^{1/p^\infty},v)$ is a tame field. Consequently, if $(K^{1/p^\infty},v)$ is a tame field, then the extension (K,v) is dense in $(K^{1/p^\infty},v)$.*

Proof: Suppose that (K,v) is a separably tame field. Then $(K^{1/p^\infty},v)$ admits no purely wild algebraic extensions since otherwise, it would contain a proper separable algebraic purely wild subextension. Hence by Lemma 12.18, $(K^{1/p^\infty},v)$ is a tame field.

For the converse, suppose that $(K^{1/p^\infty},v)$ is a tame field. Observe that the extension $(K^{1/p^\infty}|K,v)$ is purely wild and contained in every maximal purely wild algebraic extension of (K,v). Consequently, if $(K^{1/p^\infty},v)$ admits no purely wild algebraic extension at all, then $(K^{1/p^\infty},v)$ is the unique maximal purely wild extension of (K,v). Then in view of Theorem 12.17, K^{1/p^∞} must be a field complement for K^r over K in $\tilde{K}$. This yields that $K^r = K^{\rm sep}$, i.e., $(K^{\rm sep}|K,v)$ is a tame extension by Theorem 12.8, showing that (K,v) is a separably tame field. By the foregoing corollary, it follows that (K,v) is dense in $(K^{1/p^\infty},v)$. □

The following lemma describes the behaviour of separably tame fields under a decomposition of their place.

Lemma 12.28 *Let (K,v) be a separably tame field and let P be the place associated with v. Assume $P = P_1P_2P_3$ where P_1 is a coarsening of P and P_2 is nontrivial. (P_3 may be trivial.) Then (KP_1,P_2) is a separably tame field. If also P_1 is nontrivial, then (KP_1,P_2) is a tame field.*

Proof: By Lemma 12.18, vK is p-divisible. The same is then true for $v_{P_2}KP_1$. We wish to show that the residue field KP_1P_2 is perfect. Indeed, assume that this were not the case. Then by Lemma 7.55 there is an Artin-Schreier extension of (K,P_1P_2) which adjoins a p-th root to the residue field KP_1P_2. Since already this residue field extension is purely inseparable, the induced extension of the residue field $\overline{K} = KP_1P_2P_3$ can not be separable of degree p. This shows that the constructed Artin-Schreier extension is a separable algebraic purely wild extension of (K,v), contrary to our assumption on (K,v).

By Lemma 12.25, (K,P) is separable-algebraically maximal. By Theorem 10.7, this yields that the same is true for (K,P_1P_2). If P_1 is trivial (hence w.l.o.g. equal to the identity map), then $(KP_1,P_2) = (K,P_1P_2)$ is separable-algebraically maximal, and it follows from Lemma 12.25 that (KP_1,P_2) is a separably tame field. If P_1 is nontrivial, then again by Theorem 10.7, we obtain that (KP_1,P_2) is an algebraically maximal field, and it follows from Lemma 12.18 that (KP_1,P_2) is a tame field. □

The following is an analogue of Lemma 12.21.

Lemma 12.29 *Let (K,v) be a separably tame field and $k \subset K$ a relatively algebraically closed subfield of K. If the residue field extension $\overline{K}|\overline{k}$ is algebraic, then (k,v) is also a separably tame field.*

Proof: Since k is relatively algebraically closed in K, it follows that also k^{1/p^∞} is relatively algebraically closed in K^{1/p^∞} (cf. Exercise 6.53). Since (K,v) is a separably tame field, $(K^{1/p^\infty},v)$ is a tame field by Lemma 12.27. From this lemma we also know that $\overline{K} == \overline{K^{1/p^\infty}}$ and $vK = vK^{1/p^\infty}$. Our assumption on $\overline{K}\,|\,\overline{k}$ yields that the extension $\overline{K^{1/p^\infty}}\,|\,\overline{k^{1/p^\infty}}$ is algebraic. From Lemma 12.21 we can now infer that $(k^{1/p^\infty},v)$ is a tame field with $\overline{k^{1/p^\infty}} = \overline{K^{1/p^\infty}} = \overline{K}$ and vk^{1/p^∞} pure in $vK^{1/p^\infty} = vK$. Again by Lemma 12.27, (k,v) is thus a separably tame field with $\overline{k} = \overline{k^{1/p^\infty}} = \overline{K}$ and $vk = vk^{1/p^\infty}$ pure in vK. □

Corollary 12.30 *Corollary 12.22 also holds for separably tame fields in the place of tame fields. More precisely, if $F|K$ is a separable extension, then F_0 and K_0 can be chosen such*

that $F_0|K_0$ (and thus also $F_0^h|K_0$) is a separable extension. Moreover, if vK is cofinal in vF then it can also be assumed that vK_0 is cofinal in vF_0.

Proof: Since the proof of Corollary 12.22 only involves Lemma 12.21, it can be adapted by use of Lemma 12.29. The first additional assertion can be shown using the fact that the finitely generated separable extension $F|K$ is separably generated. The second additional assertion is seen as follows. If vF admits a biggest proper convex subgroup, then let K_0 contain a nonzero element whose value does not lie in this subgroup. If vF and thus also vK does not admit a biggest proper convex subgroup, then first choose F_0 and K_0 as in the (generalized) proof of Lemma 12.22; since F_0 has finite rank, there exists some element in K whose value does not lie in the convex hull of vF_0 in vF, and adding this element to K_0 and F_0 will make vK_0 cofinal in vF_0. □

With the same proof as for Corollary 12.23, but using Lemma 12.29 in the place of Lemma 12.21, one shows:

Corollary 12.31 *Corollary 12.23 also holds for separably tame fields in the place of tame fields.*

12.5 Valuation independence of Galois groups

In this section, we will introduce a valuation theoretical property that characterizes the Galois groups of tame Galois extensions. Given a Galois extension $(L|K, v)$ of degree n of henselian fields, then its Galois group $\mathrm{Gal}\, L|K = \{\sigma_1, \ldots, \sigma_n\}$ will be called **valuation independent**, if for every choice of elements $d_1, \ldots, d_n \in \tilde{L}$ there exists an element $d_0 \in L$ such that (for the unique extension of the valuation v from L to $\tilde{L}$):

$$v\left(\sum_{i=1}^{n} \sigma_i(d_0) \cdot d_i\right) = \min_{1 \le i \le n} v(\sigma_i(d_0) \cdot d_i) \; . \tag{12.7}$$

We will show in this section that a Galois extension of a henselian field is tame iff it has a valuation independent Galois group. Note that we could give the above definition and the following results also for extensions which are not Galois, replacing automorphisms by embeddings; however, the normal hull of an algebraic extension $L|K$ of a henselian field K is a tame extension of K if and only if $L|K$ is a tame extension, so there is no loss of generality in restricting our scope to Galois extensions.

Lemma 12.32 *Every finite tame unramified Galois extension has a valuation independent Galois group.*

Proof: Let $(L|K, v)$ be a finite tame unramified Galois extension and $\{\sigma_1, \ldots, \sigma_n\}$ its Galois group; it induces the Galois group of the residue field extension $\overline{L}|\overline{K}$ through

$$\forall x \in \mathcal{O}_{\mathbf{L}} : \; \overline{\sigma_i}(\overline{a}) = \overline{\sigma_i(a)} \; .$$

Since the degree of $\overline{L}|\overline{K}$ is also equal to n, the automorphisms $\overline{\sigma_i} \in \mathrm{Gal}\, \overline{L}|\overline{K}$ are all distinct; thus they are distinct characters from $\overline{L}^\times$ into $\left(\overline{\tilde{L}}\right)^\times$. By a well known theorem of Artin we

know that these characters are linearly independent in $\overline{\tilde{L}}$. This means: for given elements $d_1, \ldots, d_n$ of $\mathcal{O}_{\tilde{\mathbf{L}}}$, not all in $\mathcal{M}_{\tilde{\mathbf{L}}}$, there is some $d_0 \in \mathcal{O}_{\mathbf{L}}^\times$ such that

$$\overline{\sum_{1\le i\le n} \sigma_i(d_0)\cdot d_i} = \sum_{1\le i\le n} \overline{\sigma_i}(\overline{d_0})\cdot \overline{d_i} \neq 0\ . \tag{12.8}$$

Now let be given arbitrary elements $d_1, \ldots, d_n \in \tilde{L}$. Choose some d_{i_0} of minimal value among them. Then (12.7) holds for $d_1, \ldots, d_n$ if and only if it holds for $d_{i_0}^{-1}d_1, \ldots, d_{i_0}^{-1}d_n$. Hence we may assume w.l.o.g. that $d_1, \ldots, d_n$ are elements of $\mathcal{O}_{\tilde{\mathbf{L}}}$, not all in $\mathcal{M}_{\tilde{L}}$. We choose some $d_0 \in L$ such that (12.8) holds, and we compute, using the fact that $v(\sigma_i(d_0)) = vd_0$ since K is henselian,

$$v\left(\sum_{1\le i\le n} \sigma_i(d_0)\cdot d_i\right) = 0 = \min_{1\le i\le n} v(\sigma_i(d_0)\cdot d_i)\ ;$$

this proves (12.7). □

Lemma 12.33 *Every tame Galois extension $(K(b)|K, v)$ with*

$$((vK + \mathbb{Z}vb) : vK) = [K(b) : K] =: n \quad \textit{and} \quad b^n \in K$$

has a valuation independent Galois group.

Proof: We may write $\operatorname{Gal} L|K = \{\sigma_1, \ldots, \sigma_n\}$ with $\sigma_i(b) = b\zeta^i$ where ζ is a primitive n-th root of unity. So $\sigma_i(b)/b = \zeta^i$, and since the characteristic of $\overline{L}$ does not divide n,

$$\overline{\sigma_i(b)/b} = \overline{\zeta^i} = \overline{\zeta}^i$$

where $\overline{\zeta}$ is also a primitive n-th root of unity in $\overline{L}$. The characters χ_i from the group of n-th roots of unity into $\left(\overline{\tilde{L}}\right)^\times$ where χ_i sends ζ to $\overline{\zeta}^i$ are all distinct ($1 \le i \le n$), hence they are linearly independent in $\overline{\tilde{L}}$ by the theorem of Artin cited in the foregoing proof. This means: given elements $d_1, \ldots, d_n$ of $\mathcal{O}_{\tilde{\mathbf{L}}}$, not all in $\mathcal{M}_{\tilde{L}}$, there is some j with $1 \le j \le n$ such that

$$\overline{\sum_{1\le i\le n} \zeta^{ij}\cdot d_i} = \sum_{1\le i\le n} \overline{\zeta}^{j\,i}\cdot \overline{d_i} \neq 0\ . \tag{12.9}$$

Now let be given arbitrary elements $d_1, \ldots, d_n \in L$ be given. As shown in the foregoing proof, we may assume w.l.o.g. that $d_1, \ldots, d_n$ are elements of $\mathcal{O}_{\tilde{\mathbf{L}}}$, not all in $\mathcal{M}_{\tilde{L}}$. We choose some $j \in \{1, \ldots, n\}$ such that (12.9) holds, and we compute, using the fact that $v(\sigma_i(b^j)) = vb$ since K is henselian,

$$\begin{aligned} v\left(\sum_{1\le i\le n} \sigma_i(b^j)\cdot d_i\right) &= v\left(\sum_{1\le i\le n} \frac{\sigma_i(b^j)}{b^j}\cdot d_i\right) + vb^j = v\left(\sum_{1\le i\le n} \zeta^{ij}\cdot d_i\right) + vb^j \\ &= vb^j = \min_{1\le i\le n} v(\sigma_i(b^j)\cdot d_i)\ ; \end{aligned}$$

this proves (12.7) for $d_0 = b^j$. □

Lemma 12.34 *Let $(N|K, v)$ a finite tame Galois extension of henselian fields, and let $L|K$ a Galois subextension of $N|K$. If $\mathrm{Gal}\, L|K$ and $\mathrm{Gal}\, N|L$ are valuation independent, then $\mathrm{Gal}\, N|K$ is valuation independent too.*

Proof: Let $m = [L : K]$ and $n = [N : L]$. We choose

$$\sigma_i, \tau_j \in \mathrm{Gal}\, N|K\ ,\ \ 1 \leq i \leq m\ ,\ 1 \leq j \leq n\ ,$$

such that

$$\begin{aligned} \mathrm{Gal}\, L|K &= \{\sigma_i|_L \mid 1 \leq i \leq m\} \\ \mathrm{Gal}\, N|L &= \{\tau_j \mid 1 \leq j \leq n\}\ . \end{aligned}$$

Then

$$\mathrm{Gal}\, N|K = \{\sigma_i\tau_j \mid 1 \leq i \leq m\,,\, 1 \leq j \leq n\}\ .$$

Note that since K and L are henselian, the automorphisms σ_i, τ_j do not change the value of elements when applied to them. Now we have to show that for any choice of elements $d_{ij} \in \tilde{N}$ there exists some $d_0 \in N$ such that

$$v\left(\sum_{i,j} \sigma_i\tau_j(d_0) \cdot d_{ij}\right) = \min_{i,j} v(\sigma_i\tau_j(d_0) \cdot d_{ij})\ ,$$

where the sum and the minimum run over $1 \leq i \leq m$, $1 \leq j \leq n$. Let i_0, j_0 be such that $d_{i_0j_0}$ is of minimal value among the d_{ij}. By the assumption of the lemma, we may choose some $d_0' \in N$ such that

$$v\left(\sum_j \tau_j(d_0') \cdot \sigma_{i_0}^{-1}(d_{i_0j})\right) = \min_j v(\tau_j(d_0') \cdot \sigma_{i_0}^{-1}(d_{i_0j})) = vd_0' + vd_{i_0j_0}\ .$$

For $i \neq i_0$ we have:

$$v\left(\sum_j \tau_j(d_0') \cdot \sigma_i^{-1}(d_{ij})\right) \geq \min_j v(\tau_j(d_0') \cdot \sigma_i^{-1}(d_{ij})) \geq vd_0' + vd_{i_0j_0}\ .$$

Again by the assumption, we may choose some $d_0'' \in L$ such that

$$v\left(\sum_i \sigma_i(d_0'') \cdot \sum_j \sigma_i\tau_j(d_0') \cdot d_{ij}\right) = \min_i\ v\left(\sigma_i(d_0'') \cdot \sum_j \sigma_i\tau_j(d_0') \cdot d_{ij}\right)\ . \tag{12.10}$$

Since $d_0'' \in L$ and thus $\tau_j(d_0'') = d_0''$ for every j, the left hand side of (12.10) is equal to

$$v\left(\sum_{i,j} \sigma_i(d_0'') \cdot \sigma_i\tau_j(d_0') \cdot d_{ij}\right) = v\left(\sum_{i,j} \sigma_i\tau_j(d_0''d_0') \cdot d_{ij}\right)\ .$$

The right hand side of (12.10) is equal to

$$\begin{aligned} \min_i\ v\left(\sigma_i(d_0'') \cdot \sigma_i(\sum_j \tau_j(d_0') \cdot \sigma_i^{-1}(d_{ij}))\right) &= \min_i\ v\left(d_0'' \cdot \sum_j \tau_j(d_0') \cdot \sigma_i^{-1}(d_{ij})\right) \\ &= vd_0'' + v\left(\sum_j \tau_j(d_0') \cdot \sigma_{i_0}^{-1}(d_{i_0j})\right) \\ &= vd_0'' + vd_0' + vd_{i_0j_0} \\ &= \min_{i,j} v(\sigma_i\tau_j(d_0''d_0') \cdot d_{ij})\ . \end{aligned}$$

Hence $d_0 = d_0'' d_0' \in N$ is the required element. □

Theorem 12.35 *Every finite tame Galois extension has a valuation independent Galois group.*

Proof: Let $(N|K, v)$ be a finite tame Galois extension. Let (L, v) be the inertia field of this extension. Then $(L|K, v)$ is a Galois extension of the same form as required in the assumption of Lemma 12.32, hence it has a valuation independent Galois group. By the foregoing lemma, it now suffices to prove that the (purely ramified) Galois extension $(N|K, v)$ has a valuation independent Galois group. If this extension is not trivial, then it admits a subextension of the form as required in the assumption of Lemma 12.33, which consequently has a valuation independent Galois group. In view of the foregoing lemma, our assertion now follows by induction on the degree of the extension. □

To show that tame Galois extensions are characterized by valuation independent Galois groups, it remains to prove the converse of the foregoing lemma. This is actually very easy.

Theorem 12.36 *A Galois extension of a henselian field is tame if and only if its Galois group is valuation independent.*

Proof: If the extension $(L|K, v)$ is not tame, then its ramification group is not trivial, i.e., its Galois group G contains an isomorphism σ such that

$$\forall x \in L : \ v(\sigma x - x) > vx = v\sigma x \ .$$

Assume w.l.o.g. that the elements of G are numbered such that $\sigma_1 = \sigma$ and $\sigma_2 = id$. Choosing $d_1 = 1$, $d_2 = -1$ and $d_i = 0$ for $i > 2$, we thus have that

$$\forall x \in L : \ v\left(\sum_{i=1}^{n} \sigma_i(x) \cdot d_i\right) = v(\sigma x - x) > \min_{1 \le i \le n} v(\sigma_i(x) \cdot d_i) \ ,$$

which shows that G is not valuation independent. □

12.6 Minimal purely wild extensions

Let (K, v) be a henselian field which is not tame and thus admits purely wild extensions. We will call a purely wild extension $(L|K, v)$ a **minimal purely wild extension** if it does not admit a proper nontrivial subextension. In this section, we will prove a result due to F. Pop (cf. [POP1]): every minimal purely wild extension is generated by the root ϑ of a polynomial $\mathcal{A}(X) - \mathcal{A}(\vartheta)$ where $\mathcal{A}(X)$ is an additive polynomial. This result will follow if we are able to show that the extension $L|K$ satisfies condition (11.12) which is the hypothesis of Lemma 11.26. As a natural candidate for an extension $K'|K$ which is Galois and linearly disjoint from $L|K$, we may take the maximal algebraic extension which is linearly disjoint from every purely wild extension, namely $K^r|K$ (cf. Lemma 12.11). By Lemma 12.2 we know that $L.K^r = L^r$. We set

$\mathcal{G}$ $:= \operatorname{Gal} K$
$\mathcal{N}$ $:= \operatorname{Gal} K^r$, which is a normal subgroup of $\mathcal{G}$ by Lemma 7.81 and a pro-p-group by Theorem 7.91
$\mathcal{H}$ $:= \operatorname{Gal} L$, which is a maximal proper subgroup of $\mathcal{G}$ since $L|K$ is a minimal nontrivial extension, and which satisfies $\mathcal{N}.\mathcal{H} = \mathcal{G}$ since $K^r|K$ is linearly disjoint from $L|K$
$\mathcal{D}$ $:= \mathcal{N} \cap \mathcal{H} = \operatorname{Gal} L.K^r = \operatorname{Gal} L^r$.

The next lemma examines this group theoretical situation.

Lemma 12.37 *Let $\mathcal{G}$ be a profinite group with maximal proper subgroup $\mathcal{H}$. Assume that the nontrivial pro-p-group $\mathcal{N}$ is a normal subgroup of $\mathcal{G}$ not contained in $\mathcal{H}$. Then $\mathcal{D} = \mathcal{N} \cap \mathcal{H}$ is a normal subgroup of $\mathcal{G}$ and the finite factor group $\mathcal{N}/\mathcal{D}$ is an elementary-abelian p-group. Further, $\mathcal{N}/\mathcal{D}$ is an irreducible right $\mathcal{G}/\mathcal{D}$-module.*

Proof: Since $\mathcal{N} \not\subset \mathcal{H}$, we have that $\mathcal{D}$ is a proper subgroup of $\mathcal{N}$. Since every maximal proper subgroup of a profinite group is of finite index, we have that $(\mathcal{N} : \mathcal{D}) = (\mathcal{G} : \mathcal{H})$ is finite. Observe that $\mathcal{D}$ is **$\mathcal{H}$-invariant** (which means that $\mathcal{D}^\sigma = \mathcal{D}$ for every $\sigma \in \mathcal{H}$). This is true since $\mathcal{N} \lhd \mathcal{G}$ and $\mathcal{H}$ are $\mathcal{H}$-invariant. Assume that $\mathcal{E}$ is an $\mathcal{H}$-invariant subgroup of $\mathcal{N}$ containing $\mathcal{D}$. Then $\mathcal{H}\mathcal{E}$ is a subgroup of $\mathcal{G}$ containing $\mathcal{H}$. From the maximality of $\mathcal{H}$ it follows that either $\mathcal{H}\mathcal{E} = \mathcal{H}$ or $\mathcal{H}\mathcal{E} = \mathcal{G}$, whence either $\mathcal{E} = \mathcal{D}$ or $\mathcal{E} = \mathcal{N}$ (this argument is as in the proof of Lemma 11.28). We have proved that $\mathcal{H}\mathcal{N} = \mathcal{G}$ and that $\mathcal{D}$ is a maximal $\mathcal{H}$-invariant subgroup of $\mathcal{N}$.

Now let $\Phi(\mathcal{N})$ denote the Frattini subgroup of $\mathcal{N}$. Since $\mathcal{D} \neq \mathcal{N}$, it follows that $\mathcal{D}.\Phi(\mathcal{N}) \neq \mathcal{N}$. Being a characteristic subgroup of $\mathcal{N}$, the Frattini subgroup $\Phi(\mathcal{N})$ is $\mathcal{H}$-invariant like $\mathcal{N}$. Consequently, also the group $\mathcal{D}.\Phi(\mathcal{N})$ is $\mathcal{H}$-invariant. From the maximality of $\mathcal{D}$ we deduce that $\mathcal{D}.\Phi(\mathcal{N}) = \mathcal{D}$, showing that

$$\Phi(\mathcal{N}) \subset \mathcal{D} .$$

On the other hand, for the p-group $\mathcal{N}$ we know by part a) of Corollary 6.56 that the factor group $\mathcal{N}/\Phi(\mathcal{N})$ is a (possibly infinite dimensional) $\mathbb{F}_p$-vector space. In view of $\Phi(\mathcal{N}) \subset \mathcal{D}$, this yields that also $\mathcal{D}$ is a normal subgroup of $\mathcal{N}$ and that also $\mathcal{N}/\mathcal{D}$ is an elementary-abelian p-group. Since $\mathcal{D}$ is $\mathcal{H}$-invariant, $\mathcal{D} \lhd \mathcal{N}$ implies that

$$\mathcal{D} \lhd \mathcal{H}\mathcal{N} = \mathcal{G} .$$

As a normal subgroup of $\mathcal{G}$, $\mathcal{N}$ is a $\mathcal{G}$-module, and in view of $\mathcal{D} \lhd \mathcal{G}$ it follows that $\mathcal{N}/\mathcal{D}$ is a $\mathcal{G}/\mathcal{D}$-module. If it were reducible then there would exist a proper subgroup $\mathcal{E}$ of $\mathcal{G}$ such that $\mathcal{E}/\mathcal{D}$ is a nontrivial $\mathcal{G}/\mathcal{D}$-module. But then, $\mathcal{E}$ must be a normal subgroup of $\mathcal{G}$ properly containing $\mathcal{D}$; in particular, $\mathcal{E}$ would be $\mathcal{H}$-invariant in contradiction to the maximality of $\mathcal{D}$. □

This lemma shows that $L^r|K$ is Galois and $L^r = L.K^r$ is a finite p-elementary extension of K^r. Hence $L|K$ satisfies (11.12) with $K' = K^r$. We may now apply Lemma 11.26 to obtain that there exist an additive polynomial $\mathcal{A}(X) \in K[X]$ and an element $\vartheta \in L$ such that $L = K(\vartheta)$ and $\mathcal{A}(\vartheta) \in K$. Since $L|K$ is a minimal purely wild extension by our assumption, it is in particular minimal with property (11.12) and thus satisfies the hypothesis of Lemma 11.28. Hence, $\mathcal{A}(X)$ may be chosen such that its coefficients lie in the ring $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H]$, where b is the generator of a normal basis of $K'|K$. Since

vK is cofinal in $v\tilde{K} = \widetilde{vK}$, we may choose some $c \in K$ such that $vcb \geq 0$. Since (K, v) is henselian by assumption, it follows that $v\sigma(cb) = vcb \geq 0$ for all $\sigma \in \operatorname{Gal} K$. On the other hand, cb is still the generator of a normal basis of $K'|K$. So we may replace b by cb, which yields that $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H] \subset \mathcal{O}_{\mathbf{K}}$ and consequently, that $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{K}}[X]$.

Now assume in addition that k is a subfield of K such that the extension $(K|k, v)$ is immediate. Then we may infer from Lemma 12.4 that $K^r = k^r.K$. So the Galois extension K' of K is the compositum of K with a suitable Galois extension k' of k. In this case, b may be chosen to be already the generator of a normal basis of k' over k; it will then also be the generator of a normal basis of K' over K. With this choice of b, we obtain that the ring $K \cap \mathbb{F}_p[\,\rho b \mid \rho \in H]$ is contained in $K \cap k' = k$, yielding that $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{k}}[X]$. Let us summarize what we have proved:

Theorem 12.38 *Let (K, v) be a henselian field and $(L|K, v)$ a minimal purely wild extension. Then $L^r|K$ is a Galois extension and $L^r|K^r$ is a p-elementary extension. Hence, $L|K$ satisfies condition (11.12), and there exist an additive polynomial $\mathcal{A}(X) \in \mathcal{O}_{\mathbf{K}}[X]$ and an element $\vartheta \in L$ such that $L = K(\vartheta)$ and $\mathcal{A}(\vartheta) \in K$. If k is a subfield of K such that the extension $(K|k, v)$ is immediate, then $\mathcal{A}(X)$ may already be chosen in $\mathcal{O}_{\mathbf{k}}[X]$.*

Let us conclude this section by discussing the following special case. Assume that the value group vK is divisible by all primes $q \neq p$. Then by Theorem 7.94, $K^r|K$ is an unramified extension. Consequently, $\operatorname{Gal} K'|K \cong \operatorname{Gal} \overline{K'}|\overline{K}$ and we may choose the element b such that $\overline{b}$ is the generator of a normal basis of $\overline{K'}|\overline{K}$. It follows that the residue map is injective on the ring $\mathbb{F}_p[\rho b \mid \rho \in H]$ and thus, also the map $\tau \mapsto \overline{\phi(\tau)}$ is injective. In this case, we obtain that

$$\overline{\mathcal{A}}(X) = \prod_{\tau \in N} (X - \overline{\phi(\tau)})$$

has no multiple roots and is thus separable.

12.7 Valued function fields

In this section, we wish to cite some results that we have proved in [KU1] and [KU2]. Let us mention that their proofs have initiated the systematic study of many notions that we have carried through in the earlier sections of this part of the thesis. The proofs can be reasonably simplified by an application of the results of this systematic treatment.

One of the main theorems on the behaviour of the defect is the following (cf. Theorem 3.1 of [KU1] and [KU2]):

Theorem 12.39 *Let $(F|K, v)$ be a function field without transcendence defect. If (K, v) is a defectless field, then (F, v) is a defectless field.*

Note that in case of residue characteristic, this theorem is a trivial consequence of the Lemma of Ostrowski (cf. Theorem 8.22).

Theorem 12.39 has many precursors. It is a generalization of a theorem of H. Grauert and R. Remmert [GR–RE] which works with completions instead of henselizations and is restricted to the case of algebraically closed complete ground fields of rank 1 (the value group is archimedean). A generalization of the Grauert–Remmert Theorem was given by L. Gruson [GRU2], an improved presentation of it can be found in the book [BO–G–RE] of

S. Bosch, U. Güntzer and Remmert. The proof uses methods of nonarchimedean analysis. Further generalizations are due to M. Matignon and J. Ohm; see also [GRE–MAT–POP1]. In [OHM3], Ohm arrived independently of [KU1] at a version of Theorem 12.39 which is restricted to the case $\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K})$. The idea of Ohm's proof is to reduce the general case of valuations of arbitrary rank to the case of rank 1 and to give an auxiliary argument for the transition from completions to henselizations. In this way, he is able to deduce his theorem from [BO–G–RE], Proposition 3, p. 215 (more precisely, from the generalized version of this proposition which is proved but not stated in [BO–G–RE]).

In contrast to this approach, we have given a new proof which replaces the analytic methods of [BO–G–RE] by valuation theoretical arguments. Such arguments seem to be more adequate for a theorem that appears to be of valuation theoretical nature.

To derive model theoretic results from Theorem 12.39, we will use an embedding lemma (Lemma 14.8) whose algebraic part is the following:

Lemma 12.40 (Embedding Lemma I)
Let (K,v) be a defectless field (the valuation is allowed to be trivial), $(F|K,v)$ a valued function field without transcendence defect and (K^*,v^*) a henselian extension of (K,v). Assume that vF/vK is torsion free and that $\overline{F}|\overline{K}$ is separable. If $\rho : vF \to v^*K^*$ is an embedding over vK and $\sigma : \overline{F} \to K^*v^*$ is an embedding over $\overline{K}$, then there exists an embedding $\iota : (F,v) \to (K^*,v^*)$ over K which respects ρ and σ.

Proof: From Corollary 7.32 we know that vF/vK and $\overline{F}|\overline{K}$ are finitely generated. Since vF/vK it is also torsion free by assumption, we have that

$$vF = vK \oplus \mathbb{Z}\alpha_1 + \ldots + \mathbb{Z}\alpha_r$$

for suitable elements $\alpha_i \in vF$. Since $\overline{F}|\overline{K}$ is separable by assumption, Lemma 6.44 shows that it is separably generated. Let $\zeta_1, \ldots, \zeta_s$ be a separating transcendence basis of $\overline{F}|\overline{K}$. Now choose $x_i\,,\ y_j \in F$ such that $vx_i = \alpha_i\,,\ 1 \le i \le r$, and $\overline{y}_j = \zeta_j\,,\ 1 \le j \le s$. Then $\mathcal{T} = \{x_i\,,\ y_j \mid 1 \le i \le r,\ 1 \le j \le s\}$ is a maximal algebraically valuation independent set in $(F|K,v)$. Since this extension is assumed to have no transcendence defect, it follows from Lemma 7.28 that the extension $(F|K(\mathcal{T}),v)$ is an extension without transcendence defect. Since in (F,v) there are no elements which are algebraically valuation independent over $K(\mathcal{T})$, we find that $F|K(\mathcal{T})$ must be algebraic. That is, $\mathcal{T}$ is a standard valuation transcendence basis of $(F|K,v)$. Note that $vK(\mathcal{T}) = vF$ by our choice of the values $vx_i = \alpha_i\,$. Further, $\overline{F}|\overline{K(\mathcal{T})}$ is separable algebraic. Choose a primitive element ζ such that $\overline{F} = \overline{K(\mathcal{T})}(\zeta)$. Let $\overline{f}$ be its minimal polynomial and choose a monic polynomial $f \in \mathcal{O}_{K(\mathcal{T})}[X]$ whose reduction is $\overline{f}$. Since $\overline{f}$ is separable, Hensel's Lemma shows that there exists an element a in the henselization $(F,v)^h$ such that $\overline{a} = \zeta$. We find that the extension $(F^h|K(\mathcal{T},a)^h,v)$ is immediate.

By Theorem 12.39, the field $(K(\mathcal{T},v)$ is a defectless field. By Theorem 8.9, also its henselization $(K(\mathcal{T},v)^h$ is a defectless field. Since $(K(\mathcal{T},a)^h,v)$ is a finite extension of $(K(\mathcal{T})^h,v)$, we find by part a) of Lemma 8.8 that also $(K(\mathcal{T},a),v)^h$ is a defectless field. Consequently, the immediate extension $(F^h|K(\mathcal{T},a)^h,v)$ must be trivial, that is,

$$(F,v)^h = (K(\mathcal{T},a),v)^h\ .$$

Now let (K^*,v^*) be as in the assumption of the lemma. From Lemma 7.33 we obtain an embedding of $(K(\mathcal{T}),v)$ in (K^*,v^*) over K which respects ρ and the corresponding

restriction of σ. By Theorem 7.107, this embedding can be extended to an embedding ι of $(K(\mathcal{T}), v)$ in (K^*, v^*), because the latter field is henselian by assumption.

Denote by ιf the polynomial over $\iota K(\mathcal{T})^h$ which is obtained from f by replacing every coefficient c_i by ιc_i. Then $\sigma\zeta$ is a simple root of $\overline{\iota f}$. By Hensel's Lemma, there is a root $a^* \in K^*$ of ιf with residue $\sigma\zeta$. The assignment $a \mapsto a^*$ extends ι to the field $K(\mathcal{T})^h(a) = K(\mathcal{T}, a)^h = (F, v)^h$, and this extension respects σ. We have thus embedded $(F, v)^h$ in (K^*, v^*) over K, and the required embedding of (F, v) is just a restriction of this embedding. □

To illustrate the use of the general notions developed in earlier chapters, let us show how the proof of Theorem 12.39 is stepwise reduced to the case of valued fields of finite rank.

- **Reduction to transcendence degree 1.**

Lemma 12.41 *To prove Theorem 12.39, it suffices to prove*

(R1) *Every function field of transcendence degree 1 without transcendence defect over a defectless ground field is a defectless field.*

Proof: The assertion is proved by induction on the transcendence degree of the function field. For transcendence degree 1, it is just **(R1)** which we assume to be true. Now let the assertion be proved for function fields of transcendence degree n. Assume that $(F|K, v)$ is a function field without transcendence defect of transcendence degree $n+1$ over the defectless field (K, v) and that **(R1)** is true. Let $\mathcal{T}$ be a valuation transcendence basis of $(F|K, v)$ and let $t \in \mathcal{T}$. Then $(K(t), v)$ is a defectless field by virtue of **(R1)**. Now $(F|K(t), v)$ is a function field of transcendence degree n, and it has no transcendence defect since it admits $\mathcal{T} \setminus \{t\}$ as a valuation transcendence basis. By our induction hypothesis, it follows that (F, v) is a defectless field. □

- **Reduction to algebraically closed ground fields.**

Lemma 12.42 *To prove* **(R1)**, *it suffices to prove*

(R2) *Every function field of transcendence degree 1 without transcendence defect over an algebraically closed field is a defectless field.*

Proof: Assume that $(F|K, v)$ is a function field without transcendence defect of transcendence degree 1 over the defectless field (K, v) and that assertion **(R2)** is true. By Lemma 7.29, $(F|K, v)$ admits a standard valuation transcendence basis $\{t\}$. Since $F|K(t)$ is finite, it suffices to prove that $(K(t), v)$ is a defectless field, cf. part a) of Lemma 8.8. Now $(\tilde{K}(t), v)$ is a defectless field by virtue of **(R2)**. Lemma 8.75 shows that $(K(\mathcal{T})|K, v)$ is a valuation regular extension. By Corollary 8.84 it follows that $(K(\mathcal{T}), v)$ is a defectless field. □

- **Reduction to finite rank.**

Lemma 12.43 *To prove* **(R2)**, *it suffices to prove*

(R3) *Every valued function field without transcendence defect of transcendence degree 1 over an algebraically closed ground field of finite rank is a defectless field.*

Proof: Assume that assertion **(R3)** is true. Let $(F|K,v)$ be a valued function field without transcendence defect of transcendence degree 1 over the algebraically closed field K. We wish to show that (F,v) is a defectless field. As in the preceding proof, we choose a standard valuation transcendence basis $\{t\}$, and we only have to show that $(K(t),v)$ is a defectless field. Let $(K(t,a_1,\ldots,a_n)|K(t),v)$ be an arbitrary finite extension; in view of Lemma 8.17, we have to show that it is h-defectless. Let k_0 be the prime field of K. We choose a finitely generated extension field k_1 of k_0 such that $a_1,\ldots,a_n$ are algebraic over $k_1(t)$. Now we take k to be the relative algebraic closure of k_1 in K. Since K is assumed to be algebraically closed, we find that also k is algebraically closed. Since k_1 is finitely generated over its prime field, the rank of k_1 and thus also of its algebraic closure k must be finite. By construction, $(k(t)|k,v)$ is a valued function field without transcendence defect of transcendence degree 1 over the algebraically closed ground field of (k,v) finite rank. So **(R3)** implies that $(k(t,a_1,\ldots,a_n)|k(t),v)$ is a defectless extension.

As k is algebraically closed, the extension $(K|k,v)$ is valuation regular. By Lemma 8.76, the same holds for $(K(t)|k(t),v)$. But this means that the algebraic extension $(k(t,a_1,\ldots,a_n)|k$ is valuation disjoint from $(K(t)|k(t),v)$. Now we apply Lemma 8.83 to obtain that $d_h(K(t,a_1,..$ $d_h(k(t,a_1,\ldots,a_n)|k(t),v)=1$. This completes the proof of our lemma. □

Now let us state some results about immediate valued function fields (function fields $(F|K,v)$ which are immediate extensions). The henselization $(F,v)^h$ of such a function field will be called **henselian rational** if admits a transcendence basis $\mathcal{T}$ over K such that $(F,v)^h=(K(\mathcal{T},v)^h$. In this case, we will say that **the henselian rational function field** $(F^h|K,v)$ **is generated by** $\mathcal{T}$.

Lemma 12.44 *Let (K,v) be a valued field of residue characteristic 0 or a finitely ramified field. Then the henselization of every immediate function field over (K,v) is henselian rational.*

Proof: Let $(F|K,v)$ be an immediate function field. Let $\mathcal{T}$ be an arbitrary transcendence basis of $F|K$. The henselization $(F,v)^h$ contains the henselization $(K(\mathcal{T}),v)^h$. The subextension $(K(\mathcal{T})|K,v)$ of $(F|K,v)$ is also immediate. Since the henselization is an immediate extension, we find that also $(F^h|K,v)$ and $(K(\mathcal{T})^h|K,v)$ are immediate extensions. Consequently, $(F^h|K(\mathcal{T})^h,v)$ is an immediate algebraic extension. If the residue characteristic of (K,v) is 0, then the same holds for $(K(\mathcal{T})^h,v)$ and thus, it is a defectless field by Theorem 8.22. It follows that the extension $(F^h|K(\mathcal{T})^h,v)$ must be trivial, which shows that $(F|K,v)$ is henselian rational.

If (K,v) is a finitely ramified field, then so is every immediate extension of (K,v) by virtue of Lemma 10.11. Now Theorem 10.13 shows that $(K(\mathcal{T})^h,v)$ is a defectless field, and it again follows that $F^h=K(\mathcal{T})^h$. □

The following theorem was proved in Chapter 7 of [KU2].

Theorem 12.45 *Let (K,v) be a tame field and $(F|K,v)$ an immediate function field. If its transcendence degree is 1, then $(F,v)^h$ is a henselian rational function field over K.*

Finally, let us prove:

Theorem 12.46 *Let (K,v) be a henselian field of arbitrary characteristic. If the valued function field $(F|K,v)$ is a separable subextension of the completion of (K,v), then $(F,v)^h$ is a henselian rational function field over (K,v), generated by every separating transcendence basis of $F|K$.*

Proof: Let $\mathcal{T}$ be a separating transcendence basis of $F|K$. Then (F, v) lies in the completion of $(K(\mathcal{T}), v)$ since it lies in the completion of (K, v). Consequently, also F^h lies in the completion of $K(\mathcal{T})^h$ (which is equal to K^c since the completion of a henselian field is again henselian, cf. Lemma 9.8). But $F^h|K(\mathcal{T})^h$ is a finite separable extension; by Theorem 7.114 it must be trivial. □

Part III

Model theory of valued fields

Chapter 13

Preliminaries from model theory

In this chapter, we provide the basic model theory which we will need when we deal with the model theory of valued fields. We develop it in view of the interested algebraist who has not yet been in a deeper contact with model theory. So we will give algebraic examples for many notions, and use a basically algebraic point of view in the constructions, as far as possible. We also try to reduce logical formalism as much as it seems justifiable. For further reading, we refer the reader to the following papers and books:

Bell and Slomson [BE–SL], G. Cherlin [CH1], C. C. Chang and H. J. Keisler [CHA–KE], W. Hodges [HO], C. U. Jensen and H. Lenzing [JE–LE], A. Macintyre [MAC4], A. Prestel [PR2], L. v. d. Dries [VDD10].

Our exposition uses material from [PR2], [VDD10] and [CHA–KE].

13.1 Structures, languages and axioms

The basic idea of model theoretic algebra is to analyze the assertions that an algebraist wants to prove, and to apply principles that are valid for certain types of assertions. Such principles prove once and for all facts that otherwise are proved over and over again in different settings. To state and apply such principles, it is necessary to make it precise what it is that we are talking about, and in which mathematical language we are talking. The reader may interpose that mathematicians are talking about mathematical structures, which are fixed by definitions, that is, by axioms. If for instance we are talking about a group, then we talk about a set of elements G and a binary function $G \times G \to G$ which associates with every two elements a third one. So besides the underlying set, we are using a **function symbol** for this function, which is $+$ if we write the group additively, and $\cdot$ if we write it multiplicatively. But a group also has a unit element, for which we use the **constant symbol** 0 in the additive and 1 in the multiplicative case. If we talk about ordered groups, then for expressing the ordering we need a further symbol, which might be $<$ or $\leq$. Although it is also binary like $+$ or $\cdot$, it is not a function symbol. Since the ordering is a relation between the elements of the group, it is called a **relation symbol**. The description of a mathematical object may need more than one constant, function or relation symbols. For a field, we need two binary functions, $+$ and $\cdot$, and two constants, 0 and 1. Further, we may need function symbols or relation symbols of any (fixed) number of entries. The number of entries is called the **arity** of the function or relation symbol. (That is, f resp. R is called n**-ary** if it has n entries.) For example, to talk about exponential

fields, we use the unary function symbol exp for the exponential. We have seen that an ordering can be given by a binary relation $<$ as well as by its positive cone, that is, by telling which elements are positive. The latter means that we are specifying a subset $\mathbf{P}$. This may also be viewed as a unary relation symbol: $\mathbf{P}(x)$ means that $x \in \mathbf{P}$. A unary relation symbol is called a **predicate symbol**. This already shows that we have some freedom in the choice of the language which we use for a mathematical object. Another example is given by the unary function symbols that are often used to denote inverses: $-$ for the additive case, and $^{-1}$ for the multiplicative case. But $-$ could also be used as a binary function symbol, so we see that it is important to specify the arity.

A **language** is defined to be a triple

$$\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$$

where

- $\mathcal{R}$ is a set of relation symbols,
- $\mathcal{F}$ is a set of function symbols,
- $\mathcal{C}$ is a set of constant symbols.

When we specify a language, then we will frequently use only one pair of set brackets. For example,

$$\mathcal{L}_{\mathrm{G}} := \{+, -, 0\}$$

is the **language of groups** (where $-$ is a unary function symbol), and

$$\mathcal{L}_{\mathrm{R}} := \{+, \cdot, -, 0, 1\}$$

is the **language of rings** (with 1). A language $\mathcal{L}' = (\mathcal{R}', \mathcal{F}', \mathcal{C}')$ is called an **extension** of $\mathcal{L}$ if $\mathcal{R} \subset \mathcal{R}'$, $\mathcal{F} \subset \mathcal{F}'$ and $\mathcal{C} \subset \mathcal{C}'$. For example, the language $\mathcal{L}_{\mathrm{G}}$ of groups is an extension of the language $\mathcal{L}_{\mathrm{M}} := \{+, 0\}$ of monoids, and the **language of fields**

$$\mathcal{L}_{\mathrm{F}} := \{+, \cdot, -, ^{-1}, 0, 1\}$$

is an extension of the language of rings $\mathcal{L}_{\mathrm{R}}$.

For a given language $\mathcal{L}$, an **$\mathcal{L}$-structure** is a quadruple

$$\mathfrak{A} = (A, \mathcal{R}_{\mathfrak{A}}, \mathcal{F}_{\mathfrak{A}}, \mathcal{C}_{\mathfrak{A}})$$

where

- A is a set, called the **universe of $\mathfrak{A}$**,
- $\mathcal{R}_{\mathfrak{A}} = \{R_{\mathfrak{A}} \mid R \in \mathcal{R}\}$ such that every $R_{\mathfrak{A}}$ is a relation on A of the same arity as the relation symbol R,
- $\mathcal{F}_{\mathfrak{A}} = \{f_{\mathfrak{A}} \mid f \in \mathcal{F}\}$ such that every $f_{\mathfrak{A}}$ is a function on A of the same arity as the function symbol f,
- $\mathcal{C}_{\mathfrak{A}} = \{c_{\mathfrak{A}} \mid f \in \mathcal{C}\}$ such that every $c_{\mathfrak{A}}$ is an element of A (called a **constant**).

We call $R_{\mathfrak{A}}$ the **interpretation of R on A**, and similarly for the functions $f_{\mathfrak{A}}$ and the constants $c_{\mathfrak{A}}$. The **cardinality** of the structure $\mathfrak{A}$ is simply the cardinality $|A|$ of its universe.

Any set A with a distinguished element 0 and a binary function $+ : A \times A \to A$ will be a structure for the language $\mathcal{L}_{\mathrm{M}} = \{+, 0\}$ of monoids. This does not say anything about

the behaviour of the function + and the element 0. For instance, 0 may not at all behave as a neutral element for +. Such properties of structures can not be fixed by the language. They have to be described by axioms, which we will discuss later.

If we have an $\mathcal{L}$-structure $\mathfrak{A}$ with nonempty universe A and we extend the language $\mathcal{L}$ to a language $\mathcal{L}'$, then it is always possible to interprete the new relation, function and constant symbols on A. For the new relation and function symbols, we just choose relations resp. functions on A of the same arity. Since we assume A to be nonempty, we can also choose arbitrary elements of A for the interpretation of new constant symbols. A structure $\mathfrak{A}'$ thus obtained is called an **$\mathcal{L}$-expansion of $\mathfrak{A}$**, and $\mathfrak{A}$ is called the $\mathcal{L}$-reduct of $\mathfrak{A}'$. For instance, if we extend the language $\mathcal{L}_{\mathrm{M}}$ of monoids to the language $\mathcal{L}_{\mathrm{G}} = \{+, -, 0\}$ of groups by adding the unary function symbol $-$, then we can expand every $\mathcal{L}_{\mathrm{M}}$-structure $\mathfrak{A}$ to an $\mathcal{L}_{\mathrm{G}}$-structure by interpreting $-$ by an arbitrary unary function on A. Observe that the reduct is always uniquely determined, while the expansion is not. For instance, if A contains more than one element, there is also more than one unary function on A. Or, if we add new constant symbols to the language, then every different choice for their interpretation yields a different expansion.

Given an $\mathcal{L}$-structure $\mathfrak{A}$ and a subset $S \subset A$, there is a very useful extension of the language $\mathcal{L}$ which enables us to give every element of S a name. This language is obtained by adjoining to $\mathcal{L}$ a constant symbol c_a for every $a \in S$. This extended language will be denoted by $\mathcal{L}(S)$. The interpretation of the symbol c_a in the structure $\mathfrak{A}$ is meant to be the element a. In this way, $\mathfrak{A}$ is expanded in a canonical way to an $\mathcal{L}(S)$-structure, which we will denote by $(\mathfrak{A}, S)$. An important special case is $S = A$ which means that we add to the language a name for every element of the universe of $\mathfrak{A}$. Later, when we do not have to care about interpretations formally, we will write a in the place of c_a , thereby identifying the constant symbol with its interpretation.

The universes of structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$ will always be denoted by A, B and S respectively. In this section, $\mathfrak{A}$ and $\mathfrak{B}$ will always be $\mathcal{L}$-structures.

$\mathfrak{A}$ is said to be a **substructure of** $\mathfrak{B}$ and $\mathfrak{B}$ is called an **extension** of $\mathfrak{A}$ if $A \subset B$ and

- the restriction of $R_{\mathfrak{B}}$ to A is equal to $R_{\mathfrak{A}}$, for every relation symbol $R \in \mathcal{R}$,
- the restriction of $f_{\mathfrak{B}}$ to A is equal to $f_{\mathfrak{A}}$, for every function symbol $f \in \mathcal{F}$,
- $c_{\mathfrak{A}} = c_{\mathfrak{B}}$ for every constant symbol $c \in \mathcal{C}$.

The notions of "substructure" and "extension" for $\mathcal{L}$-structures define in a unique way the notion of **extension** for all mathematical structures, provided that we use a fixed language for them (which is usually implicitly assumed). For instance, if we say that "the field L is an extension of the field K" then this gives us the information that L and K satisfy the field axioms and that K is a substructure of L with respect to the language $\mathcal{L}_{\mathrm{F}}$ of fields. When we know already that K is a field and we say that "L is an extension of K" then we implicitly add the information that also L is a field.

In the same way, the meaning of **homomorphism** and **embedding** are fixed through the language. A map $\sigma : A \to B$ is called a **homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$** if it satisfies

(HOMR) $R_{\mathfrak{A}}(a_1, \ldots, a_n) \Rightarrow R_{\mathfrak{B}}(\sigma a_1, \ldots, \sigma a_n)$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, \ldots, a_n) \in A^n$,

(HOMF) $\sigma f_{\mathfrak{A}}(a_1, \ldots, a_n) = f_{\mathfrak{B}}(\sigma a_1, \ldots, \sigma a_n)$ for all $n \in \mathbb{N}$, all n-ary function symbols $f \in \mathcal{F}$ and all $(a_1, \ldots, a_n) \in A^n$,

(HOMC) $\sigma c_{\mathfrak{A}} = c_{\mathfrak{B}}$ for all constant symbols $c \in \mathcal{C}$.

This definition does not exclude the case that for some relation R in $\mathcal{L}$, there are elements $a, b \in A$ such that $R(a, b)$ does not hold while $R(\sigma a, \sigma b)$ holds. We have algebraic examples for such homomorphisms. Let $(G, <)$ be an ordered abelian group, H a convex subgroup of G and $\sigma : G \to G/H$ the canonical epimorphism. Recall that the order on G induces an order on G/H. If H is nontrivial, then there will exist elements $a, b \in G$ such that $a > b$ and $a + H = b + H$, that is, $\sigma a = \sigma b$. With $R(x, y) :\Leftrightarrow x \leq y$, we find that $R(a, b)$ does not hold in $(G, <)$, but $R(\sigma a, \sigma b)$ holds in $(G/H, <)$.

At least for embeddings, we want to exclude such phenomenons. Thus, we will call a homomorphism σ a **strong homomorphism** if it satisfies

(HOMS) $R_{\mathfrak{A}}(a_1, \ldots, a_n) \Leftrightarrow R_{\mathfrak{B}}(\sigma a_1, \ldots, \sigma a_n)$ for all $n \in \mathbb{N}$, all n-ary relation symbols $R \in \mathcal{R}$ and all $(a_1, \ldots, a_n) \in A^n$.

Further, we define an **embedding** to be an injective strong homomorphism, and an **isomorphism** to be a surjective embedding. Note that a strong homomorphism is not automatically injective since equality is usually not contained as a relation in $\mathcal{R}$. It can in fact be interesting to distinguish between strong homomorphisms and embeddings. If for instance we replace the relation symbol for the order relation by a predicate $\mathbf{P}$ and interprete it on G/H by $\mathbf{P}(x) \Leftrightarrow x > 0$, and if the canonical epimorphism $G \to G/H$ is strong, then the interpretation of $\mathbf{P}$ will necessarily be $\mathbf{P}(x) \Leftrightarrow x > H$.

Having pointed out the difference between a symbol and its interpretation once and for all, we will from now on frequently drop the index "$\mathfrak{A}$".

The notion of an $\mathcal{L}$-structure does not yet tell much about the properties of the relations, functions and constants. For instance, a binary function on a set A can have various different properties. Although the symbols $+$ and $\cdot$ suggest that their interpretation have the properties of addition and multiplication, this is not expressed by the mere notion of a structure. The property of commutativity is not automatically connected with the symbol $\cdot$. Even worse, we know that for boolean algebras, $+$ and $\cdot$ are sometimes used in the place of $\cup$ and $\cap$. So if we work with mathematical objects, we have to state which properties the relations, functions and constants should have. This is done by axioms. For instance, groups, rings and fields are axiomatized by the usual axioms which describe the properties of addition, multiplication and the constants 0 and 1. For ordered and valued groups and fields, we have given the additional axioms (which describe the properties of the additional relations) in the first part of this book. For the additional function symbol exp, we have defined that its interpretation on an ordered field is an exponential if it is an isomorphism from the ordered additive group of the field onto the ordered multiplicative group of positive elements.

Now we have to consider the form in which these axioms are expressed. For example, let us take the axiom which states the existence of the additive inverse in an additively written group:

$$\forall x \, \exists y : \; x + y = 0 \, .$$

Besides the constant 0, there appear variables x and y and a term built up by x and y and the function $+$. This term will be an element of the group whenever we replace x and y by group elements. Then there are the logical symbols $\forall$ (the universal quantifier), $\exists$ (the existential quantifier), and $=$. Observe that we can eliminate the existential quantifier

if we introduce the unary function symbol $-$. Then the axiom can be expressed in the following way:

$$\forall x : \; x + (-x) = 0 \; .$$

The brackets are auxiliary symbols which indicate how the formula has to be read (cf. their use in the associative law). One can do without them if one follows exactly a given syntax, but this makes the formulas less readable. By introducing conventions, the formalism of mathematical logic becomes more acceptable for algebraists who wish to use it. For example, even if we use $-$ as a unary function, it is a common convention to write $x - y$ in the place of $x + (-y)$.

Often, the above axiom is given in the form

$$x - x = 0 \; ,$$

which means that we have to add the universal quantifier in front of it. That is, we pass to the **universal closure** of the formula. If more than one variables appear in such a formula, then for every one of it, a universal quantifier has to be added, unless the variable is already bounded by some quantifier. Those not bounded are called **free variables**.

We have to make precise what we mean by a **formula**. Let $\mathcal{L}$ be a language. Then an **$\mathcal{L}$-term** is a syntactically correct string built up by the constant and function symbols of $\mathcal{L}$ and symbols for variables (we will use x, y, z, occasionally with indeces or primes). In other words, the $\mathcal{L}$-terms form the smallest set of finite strings of function, constant and variable symbols which

1) contains all constant and variable symbols, and
2) is closed under forming strings of the form $f(t_1, \ldots, t_n)$ where f is a n-ary function symbol in $\mathcal{L}$ and $t_1, \ldots, t_n$ are already contained in the set.

This definition can also be viewed as a definition by induction over the complexity of terms (that is, the number of steps used to build them up, or equivalently, the number of function symbols appearing in them). It yields that whenever we replace the variables in a term by elements from the universe of an $\mathcal{L}$-structure, then we obtain an element in that universe. When we write $t(x_1, \ldots, x_n)$, then we mean that all variables appearing in the $\mathcal{L}$-term t are among the variables $x_1, \ldots, x_n$. It does not mean that every x_i necessarily appears in t. Furthermore, if we obtain a term from $t(x_1, \ldots, x_n)$ by just replacing a variable symbol by another which has not yet appeared in $t(x_1, \ldots, x_n)$, then we will identify both terms. Given n $\mathcal{L}$-terms $t_1, \ldots, t_n$, we can thus always write $t_i = t_i(x_1, \ldots, x_m)$ for $i = 1, \ldots, n$ and a suitable m. If an $\mathcal{L}$-term contains no variables, then we will call it a **constant $\mathcal{L}$-term**. The interpretation $t_{\mathfrak{A}}$ of a constant $\mathcal{L}$-term t in an $\mathcal{L}$-structure $\mathfrak{A}$ is defined by induction over the complexity of t. If t is a constant symbol $c \in \mathcal{C}$, then its interpretation was already defined to be $c_{\mathfrak{A}}$. If $t = f(t', t'', \ldots)$, then $t_{\mathfrak{A}} := f_{\mathfrak{A}}(t'_{\mathfrak{A}}, t''_{\mathfrak{A}}, \ldots)$. When we do not have to care about interpretations formally, we will drop the index "$\mathfrak{A}$", thereby identifying the term with its interpretation in $\mathfrak{A}$.

Strings of the form $t_1 = t_2$ and $R(t_1, \ldots, t_n)$ are called **atomic $\mathcal{L}$-formulas** if t_1, t_2 are $\mathcal{L}$-terms resp. $R \in \mathcal{L}$ is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms.

Now by an **(elementary) $\mathcal{L}$-formula** we mean a syntactically correct string built up by atomic $\mathcal{L}$-formulas and the logical symbols $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. We remark that the symbols $\exists$, $\vee$, $\rightarrow$, $\leftrightarrow$ are definable by $\forall$, $\neg$, $\wedge$. So we can define: The elementary $\mathcal{L}$-formulas form the smallest set which

1) contains all atomic $\mathcal{L}$-formulas, and
2) is closed under forming strings of the form $\neg\varphi$, $\varphi \wedge \psi$ and $\forall x \varphi$ where φ, ψ are already contained in the set and x stands for any variable symbol. An elementary formula is called an **(elementary) sentence** if every variable is bounded by some quantifier, that is, if it contains no free variables. Atomic $\mathcal{L}$-sentences are atomic $\mathcal{L}$-formulas which are sentences, that is, in which all terms are constant terms.

We will sometimes use induction over the **complexity** of formulas or sentences, which means the way that they are built up by a repeated application of 2) to atomic formulas. For our practical use, we can define the complexity of a formula to be just the number of the symbols $\neg$, $\wedge$, $\forall$ appearing in the formula.

How many essentially different formulas can be built from a given language $\mathcal{L}$? To answer this question, we define the **cardinality of** $\mathcal{L}$ to be

$$||\mathcal{L}|| := \max\{\aleph_0 , |\mathcal{R}| , |\mathcal{F}| , |\mathcal{C}|\} .$$

Formulas are *finite* strings of symbols from $\mathcal{L}$, the logical symbols and variables. If we identify formulas which differ only by the names for the variables (as we did already for terms), then we find that if at least one of the cardinalities $|\mathcal{R}|$, $|\mathcal{F}|$ or $|\mathcal{C}|$ is infinite, then there are just $\max\{|\mathcal{R}| , |\mathcal{F}| , |\mathcal{C}|\}$ many $\mathcal{L}$-formulas. On the other hand, if $|\mathcal{R}|$, $|\mathcal{F}|$ and $|\mathcal{C}|$ are finite or even empty, we can still build up countably many formulas by using variables and the logical symbols. Consequently, the number of essentially different $\mathcal{L}$-formulas (or $\mathcal{L}$-sentences) is just $||\mathcal{L}||$.

After what we have said so far, some of the axioms in the earlier parts of this book can be recognized as elementary formulas if one knows that the author has the habit of replacing $\rightarrow$ by $\Rightarrow$ and $\leftrightarrow$ by $\Leftrightarrow$ and of introducing " : " after the quantifiers in the beginning (which indicates that their range is the whole formula). The immediate examples are (OT), (OC), (OG) and (OM). In many other axioms, conventions are used. For instance, $x \not< x$ is an abbreviation of $\neg(x < x)$.

As we have already mentioned, free variables in axioms have to be understood as being bounded by universal quantifiers. So axioms are implicitly assumed to be sentences. But note that in arbitrary sentences or formulas, the quantifiers do not necessarily appear at the beginning. For instance, the (field) axiom which states that every nonzero element has an inverse, can be expressed by the elementary formula

$$\forall x : \ x \neq 0 \rightarrow \exists y : \ x \cdot y = 1 \tag{13.1}$$

(here, $x \neq 0$ is an abbreviation for $\neg(x = 0)$). But it can be shown that every formula is equivalent to a formula with all quantifiers in front (the so-called **prenex normal form**). See below for the definition of equivalence of formulas. In (13.1), the existential quantifier can just be moved to the front, behind the universal quantifier. A formula is called **quantifier free** if no quantifier appears in it. So a sentence in prenex normal form is a string of quantifiers of the form $\forall x$ and $\exists y$ (all appearing variables being distinct), followed by a quantifier free formula. If a sentence in prenex normal form has only universal quantifiers, then we call it a **universal sentence** (or a **universal axiom**, if it appears as an axiom), and similarly for "existential" in the place of "universal". If a sentence in prenex normal form is of the form $\forall x_1 \ldots \forall x_m \exists y_1 \ldots \exists y_n \varphi$ with φ a quantifier free formula, then it is called a **universal existential sentence**. The prenex normal form is

sufficiently unique for this notion to be well defined. The axioms (O$\not$R), (OT), (OC), (OG) and (OM) are universal, as well as the axioms expressing associativity, commutativity and distributivity for groups resp. fields. The reader may show that with the language $\{+,-,0\}$, all group axioms can be expressed by universal sentences, whereas this is not any more possible if one only uses the language $\{+,0\}$. In fact, the group axioms can already be written down by using only the symbol $+$, but it becomes much easier if one adds the constant 0. Similarly, the **field axioms**

(ASS+) $\forall x\,\forall y\,\forall z:\ x+(y+z)=(x+y)+z$

(COM+) $\forall x\,\forall y:\ x+y=y+x$

(NEU+) $\forall x:\ x+0=x$

(INV+) $\forall x:\ x-x=0$

(DIS $\cdot$ +) $\forall x\,\forall y\,\forall z:\ x\cdot(y+z)=x\cdot y\ +\ x\cdot z$

(ASS $\cdot$) $\forall x\,\forall y\,\forall z:\ x\cdot(y\cdot z)=(x\cdot y)\cdot z$

(COM $\cdot$) $\forall x\,\forall y:\ x\cdot y=y\cdot x$

(NEU $\cdot$) $\forall x:\ x\cdot 1=x$

(INV $\cdot$) $\forall x:\ x=0\ \vee\ x\cdot x^{-1}=1$

are all universal sentences in the language $\mathcal{L}_{\mathrm{F}}$ of fields. But in the language $\mathcal{L}_{\mathrm{R}}$ of rings, only the ring axioms (ASS+) – (NEU $\cdot$) can be written as universal sentences. Without use of the function symbol $^{-1}$, the existence of the inverse for every nonzero element can be expressed by the universal existential sentence

(INV $\cdot$)$'$ $\forall x\exists y:\ x=0\ \vee\ x\cdot y=1$

Axiom (NEU $\cdot$) can also be formulated in the language of rings without 1. In this case, it does not describe the property of a constant but just requires the existence of a neutral element for the multiplication:

$$\exists y\,\forall x:\ x\cdot y=x\ .$$

But this axiom is neither universal nor universal existential. To write down axiom a translation of axiom (INV $\cdot$) in this restricted language, one even has to combine it with this translation of (NEU $\cdot$).

While it is common use to employ the constant symbol 1 to axiomatize the theory of fields, there is a slight difficulty about the function symbol $^{-1}$. A quick look at the above axioms shows that they do not tell us what 0^{-1} might be. In algebra, one usually says that this is undefined. But our definition of structures is requiring that all functions are total. On the other hand, we do not want that 0^{-1} denotes different elements in different fields. We would not obtain the completeness result asserted in Corollary 13.50 below. So we are forced to fix the interpretation of 0^{-1}, for instance by

(INV0) $0^{-1}=0$

Since this additional axiom is not needed if we use the language of rings and replace (INV $\cdot$) by (INV $\cdot$)$'$, one may ask why we choose the language of fields. The advantage is that we obtain an axiom system that consists only of universal sentences. For a consequence of this, see Lemma 13.2 below.

Elementary axioms can appear in disguised form. To give an example, let us mention the axioms (PC+), (PC$\cap$), (PC$\cup$) and (PC$\,\cdot\,$) for a positive cone. We can view $\mathbf{P}$ as a predicate. For instance, axiom (PC+) can then be written as

$$\mathbf{P}(x) \wedge \mathbf{P}(y) \rightarrow \mathbf{P}(x+y)\,.$$

After having defined elementary $\mathcal{L}$-formulas and $\mathcal{L}$-sentences, we would like to know whether they hold for certain mathematical objects or not. This only makes sense if the mathematical object is given in the form of an $\mathcal{L}$-structure, so that we know how the relation, function and constant symbols are to be interpreted. With this interpretation at hand, it is intuitively clear to a mathematician what it means that a sentence **holds in** a given $\mathcal{L}$-structure $\mathfrak{A}$. If the elementary $\mathcal{L}$-sentence φ holds in $\mathfrak{A}$, then we will write $\mathfrak{A} \models \varphi$, and we will also say that $\mathfrak{A}$ **satisfies** φ. Given an elementary $\mathcal{L}$-formula, we pass to its universal closure (which we denote by $\forall\varphi$) and define that φ holds in $\mathfrak{A}$ if $\forall\varphi$ holds in $\mathfrak{A}$. That is, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \forall\varphi$.

A formal definition of $\mathfrak{A} \models \varphi$ can be given by induction over the complexity of the formula φ. It is convenient to define this notion from the start for the larger set of $\mathcal{L}(A)$-formulas. This enables us to do the following. Suppose we are given an $\mathcal{L}(A)$-term $t(x_1,\ldots,x_m)$. Then we can choose elements $a_1,\ldots,a_m \in A$ and replace every variable x_i by c_{a_i}. The result of this is the $\mathcal{L}(A)$-term $t(c_{a_1},\ldots,c_{a_m})$, whose interpretation in A will be denoted by $t_{\mathfrak{A}}(a_1,\ldots,a_m)$.

Given two $\mathcal{L}(A)$-terms $t = t(x_1,\ldots,x_m)$ and $t' = t'(x_1,\ldots,x_m)$, then we define $\mathfrak{A} \models t = t'$ to mean that $t_{\mathfrak{A}}(a_1,\ldots,a_m)$ and $t'_{\mathfrak{A}}(a_1,\ldots,a_m)$ are the same element of A, for all tuples $(a_1,\ldots,a_m) \in A^m$. Given an n-ary relation symbol $R \in \mathcal{L}$ and $\mathcal{L}$-terms $t(x_1,\ldots,x_m), t'(x_1,\ldots,x_m),\ldots$, then we define $\mathfrak{A} \models R(t,t',\ldots)$ to mean that

$$(t_{\mathfrak{A}}(a_1,\ldots,a_m), t'_{\mathfrak{A}}(a_1,\ldots,a_m)\ldots) \in R_{\mathfrak{A}} \quad \text{for all } (a_1,\ldots,a_m) \in A^m$$

(recall that an n-ary relation $R_{\mathfrak{A}}$ on a set A is just a subset $R_{\mathfrak{A}} \subset A^n$).

Further, if φ and ψ are $\mathcal{L}(A)$-formulas, then we define $\mathfrak{A} \models \neg\varphi$ to mean that it is not true that $\mathfrak{A} \models \varphi$, and $\mathfrak{A} \models \varphi \wedge \psi$ to mean that $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \psi$. Finally, we define $\mathfrak{A} \models \forall x\varphi(x)$ to mean that $\mathfrak{A} \models \varphi(c_a)$ for every $a \in A$. In this way, we have defined the meaning of $\mathfrak{A} \models \varphi$ by induction over the complexity of φ.

If φ and φ' are $\mathcal{L}$-formulas, then we will call them **equivalent** if $\forall(\varphi \leftrightarrow \varphi')$ holds in every $\mathcal{L}$-structure. If both φ and φ' are sentences, then this just means that for every $\mathcal{L}$-structure $\mathfrak{A}$, we have that $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \models \varphi'$. This does not hold in general if φ and φ' have free variables, since then, $\forall(\varphi \leftrightarrow \varphi')$ may not be equivalent to $\forall\varphi \leftrightarrow \forall\varphi')$. For example, if the language $\mathcal{L}$ contains two constant symbols c, c' then in every structure where they are interpreted by distinct elements, we will have that $\forall x\, x = c$ and $\forall x\, x = c'$ both do not hold. Then $(\forall x\, x = c) \leftrightarrow (\forall x\, x = c')$ holds and $\forall x\,(x = c \leftrightarrow x = c')$ does not hold.

If $\mathfrak{A} \models \varphi$ then we will say that $\mathfrak{A}$ **is a model of** φ. We leave it to the reader to deduce the following fact from the above formal definition of $\models$: If $\mathfrak{A}'$ is an expansion of the $\mathcal{L}$-structure $\mathfrak{A}$ and φ is an $\mathcal{L}$-sentence, then $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}' \models \varphi$.

The definition of $\mathfrak{A} \models \varphi$ assumes that in principle it can be decided whether a relation holds on A or whether the interpretation of two terms is really the same element of A. In reality, this can be a very hard task. The reader should not confuse "elementary" with

"easy". The author would not seriously demand from the reader as an exercise to prove (or disprove?) that the elementary sentence

$$\forall x \, \forall y \, \forall z : \; x^n + y^n = z^n \; \rightarrow \; x = 0 \vee y = 0 \vee z = 0$$

holds in the ring $(\mathbb{Z}, +, \cdot, 0)$ for all $n \geq 3$. And in fact, the meaning of model theory is to help proving that given sentences hold in certain structures, in cases where this can not easily be checked. As an example, we have already remarked in Section 11.5 that we do not know which elementary sentences hold in power series fields in positive characteristic. This remark will be made more precise in Section 14.11.

Observe that in the above sentence, we have again used a convention. In the language of rings, we have multiplication, but no n-th power function. However, x^n is just an abbreviation for

$$\underbrace{x \cdot \ldots \cdot x}_{n}$$

which can be written out as a usual term for every given n.

Note that we can not quantify over n in the above sentence. When we introduced terms, we saw already that variables denote the instances in a formula where elements of the universe can be substituted. Hence binding variables by quantifiers means that we quantify over the elements of universes. We do not quantify over the relations or functions which are given by the language (or in any other way). In particular, we do not quantify over subsets of the universes. This is the meaning of the notion **elementary** (also called **first order**).

Let us analyze some other axioms which are already familiar to us. Suppose that we are given an ordered abelian group $(G, <)$. The reader may prove that it is **archimedean** if it satisfies the axiom

(AR) $\qquad \forall x \, \forall y \, \exists n \in \mathbb{N} : \; x \leq 0 \; \vee \; y < nx \; .$

(Here, nx is an abbreviation for the additive case, similar to x^n for the multiplicative case.) Further, $(G, <)$ is **discrete** in the strong sense that it is trivial or isomorphic to $(\mathbb{Z}, <)$, if it satisfies the axiom

(DS) $\qquad \forall x \, \forall y \, \exists n \in \mathbb{N} \, \exists x_1 \, \ldots \, \exists x_n \, \forall z : \; x < z < y \; \rightarrow \; z = x_1 \vee \ldots \vee z = x_n \; .$

In both axioms, we have used quantification over natural numbers (which are not identified as elements of the group). In the second case, even the form of the axiom (in particular, the number of quantifiers) depends on this quantification. This does not exclude the possibility that these axioms can be reformulated as elementary $\mathcal{L}_{\mathrm{OG}}$-sentences or at least as a set of elementary $\mathcal{L}_{\mathrm{OG}}$-sentences. But we will see later that this is not the case.

We have mentioned the possibility of describing properties of mathematical objects by a set of elementary sentences. To give an example, let us fix the characteristic of a field by elementary $\mathcal{L}_{\mathrm{F}}$-sentences. This is easy for finite characteristic. We just say

(CHp) $\qquad \forall x : \; px = 0$

If p is a prime, it then will follow that it is the characteristic of every field in which this axiom holds. But if we want to express that the field has characteristic 0, then we have to say that $\forall x : \; px = 0$ does not hold for any prime p. This can be expressed by the countably infinite **(elementary) axiom scheme**

(CH0) $\exists x : nx \neq 0 \qquad (n \in \mathbb{N})$.

If all these sentences hold in a field (or ring), then it must be of characteristic 0. We have included an axiom for every $n \in \mathbb{N}$ instead of restricting to primes since in this form it also works for rings, and it gives a nice enumeration of the axioms. We will see later that the property described by this axiom scheme can not be expressed by a single (and hence also not a finite set) of elementary sentences in the field language. A further example for an axiom scheme is the property (VZ) of valued abelian groups. The use of the symbol v for the valuation will be discussed later.

If a property of $\mathcal{L}$-structures can be described by a set of elementary $\mathcal{L}$-sentences, then we call it an **elementary property**. Any set $\mathbf{T}$ of elementary $\mathcal{L}$-sentences will be called an **elementary $\mathcal{L}$-theory**. In the literature, there are also other definitions of this notion, but the additional properties required for theories are not of too much interest for us here.

Theories usually appear in two different forms. Firstly, an axiom system consisting of elementary $\mathcal{L}$-sentences is an $\mathcal{L}$-theory. If we use formulas with free variables in the axiom system (which we will frequently do to save us from writing a lot of quantifiers), then the axioms are meant to be their universal closures. Secondly, the set of $\mathcal{L}$-sentences which hold simultaneously in all members of a given class of $\mathcal{L}$-structures is a theory. An important example for this is the set $\mathbf{T}_{\mathrm{FF}}$ of all elementary $\mathcal{L}_{\mathrm{F}}$-sentences which hold in all finite fields.

Given a single $\mathcal{L}$-structure $\mathfrak{A}$, the set of all elementary $\mathcal{L}$-sentences which hold in $\mathfrak{A}$ is called the **(elementary) theory of $\mathfrak{A}$** and denoted by $\mathrm{Th}(\mathfrak{A})$. Note that for every $\mathcal{L}$-sentence φ, either φ or $\neg\varphi$ is in $\mathrm{Th}(\mathfrak{A})$.

We say that two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent** and write $\mathfrak{A} \equiv \mathfrak{B}$ if every $\mathcal{L}$-sentence φ holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$. Hence,

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{if and only if} \quad \mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B}) \; .$$

Lemma 13.1 *Isomorphic structures are elementarily equivalent.*

Proof: Let σ be any homomorphism from the $\mathcal{L}$-structure $\mathfrak{A}$ into the $\mathcal{L}$-structure $\mathfrak{B}$. Then $\mathfrak{B}$ can be made an $\mathcal{L}(A)$-structure by letting the interpretation of c_a in B just be σa. If $(\mathfrak{A}, A) \equiv (\mathfrak{B}, \sigma A)$, then it follows that $\mathfrak{A} \equiv \mathfrak{B}$. So we can assume from the start that $\mathcal{L} = \mathcal{L}(A)$, that is, that $\mathcal{L}$ contains a constant symbol for every $a \in A$.

Using (HOM2) and (HOM3) one shows by induction on the complexity of terms that for every constant $\mathcal{L}$-term t, the following holds:

$$t_{\mathfrak{B}} = \sigma t_{\mathfrak{A}} \; . \tag{13.2}$$

Now assume that σ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. We have to show that every $\mathcal{L}$-sentence φ holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$. The proof proceeds by induction on the complexity of φ. So we begin with atomic sentences. Since σ is an injective homomorphism, we have that

$$\mathfrak{A} \models t = t' \iff t_{\mathfrak{A}} = t'_{\mathfrak{A}} \iff \sigma t_{\mathfrak{A}} = \sigma t'_{\mathfrak{A}} \iff t_{\mathfrak{B}} = t'_{\mathfrak{B}} \iff \mathfrak{B} \models t = t'$$

for every two $\mathcal{L}$-terms t, t'.

If φ is of the form $\neg\psi$, then by induction hypothesis, $\mathfrak{A} \models \psi$ if and only if $\mathfrak{B} \models \psi$. Since every $\mathcal{L}$-structure is either a model of ψ or of $\neg\psi$, it follows that $\mathfrak{A} \models \neg\psi$ if and only

if $\mathfrak{B} \models \neg\psi$. If φ is of the form $\psi \wedge \psi'$, then by induction hypothesis, $\mathfrak{A} \models \psi$ if and only if $\mathfrak{B} \models \psi$, and $\mathfrak{A} \models \psi'$ if and only if $\mathfrak{B} \models \psi'$. It follows that $\mathfrak{A} \models \psi \wedge \psi'$ if and only if $\mathfrak{B} \models \psi \wedge \psi'$.

If φ is of the form $\forall x \psi(x)$, then by induction hypothesis, $\mathfrak{A} \models \psi(c_a)$ if and only if $\mathfrak{B} \models \psi(c_a)$, for every $a \in A$. Hence,

$$\mathfrak{A} \models \forall x \psi(x) \iff \mathfrak{A} \models \psi(c_a) \text{ for every } a \in A \iff \mathfrak{B} \models \psi(c_a) \text{ for every } a \in A\,.$$

But for every $b \in B$ there is some $a \in A$ such that $b = \sigma a$ since σ is surjective, and we have that $c_b = c_a$. Hence, $\mathfrak{B} \models \psi(c_a)$ for every $a \in A$ if and only if $\mathfrak{B} \models \forall x \psi(x)$. Observe that in general this is not true if σ is not surjective. □

In order to obtain the full information about the elementary properties of $\mathfrak{A}$, we will also consider $\mathrm{Th}(\mathfrak{A}, A)$ which is the set of all $\mathcal{L}(A)$-sentences which hold in $(\mathfrak{A}, A)$. A further theory connected with the $\mathcal{L}$-structure $\mathfrak{A}$ is its **(elementary) diagram** $\mathrm{Diag}(\mathfrak{A})$. It is defined to be the set of all atomic $\mathcal{L}(A)$-sentences and negated atomic $\mathcal{L}(A)$-sentences which hold in $\mathfrak{A}$. The diagram contains e.g. the information whether the interpretations of two $\mathcal{L}$-terms in $\mathfrak{A}$ are equal or not, and to which element of A they are equal. It also contains all information about the interpretation of the relation and function symbols on A. Thus, it determines the theory of $\mathfrak{A}$. This is because for every element of A there is a constant symbol in $\mathcal{L}(A)$. But if $\mathfrak{B}$ is an $\mathcal{L}(A)$-structure having elements which are not interpretations of $\mathcal{L}(A)$-terms, then even if it has the same diagram as $\mathfrak{A}$, its theory may be different from that of $\mathfrak{A}$. For an application of these theories, see Corollary 13.5 below.

If $\mathbf{T}$ is any $\mathcal{L}$-theory, then we say that the $\mathcal{L}$-structure $\mathfrak{A}$ **is a model of** $\mathbf{T}$ and write $\mathfrak{A} \models \mathbf{T}$ if $\mathfrak{A} \models \varphi$ for every $\varphi \in \mathbf{T}$. The theory $\mathbf{T}$ can be viewed as an axiom system for all of its models. For example, if $\mathbf{T}$ consists of all field axioms, then its models are precisely all fields. For an $\mathcal{L}$-sentence φ, we will write $\mathbf{T} \models \varphi$ if φ holds in every model of $\mathbf{T}$. If this is not the case, then we write $\mathbf{T} \not\models \varphi$. Note that $\mathbf{T} \not\models \varphi$ does in general not imply that $\mathbf{T} \models \neg\varphi$.

If for a class $\mathbf{C}$ of $\mathcal{L}$-structures there exists an $\mathcal{L}$-theory $\mathbf{T}$ such that $\mathbf{C}$ is precisely the class of all models of $\mathbf{T}$, then $\mathbf{C}$ is called an **elementary class**, and $\mathbf{T}$ is called an **(elementary) axiomatization** of $\mathbf{C}$. Hence, the class of all fields is an elementary class. An axiomatization of this class is given by the **theory of fields** which consists of the $\mathcal{L}_{\mathrm{F}}$-axioms (ASS+), (COM+), (NEU+), (INV+), (DIS$\cdot$+), (ASS$\cdot$), (COM$\cdot$), (NEU$\cdot$), (INV$\cdot$) and (INV0) (if we work in the language $\mathcal{L}_{\mathrm{R}}$ of rings, then we replace (INV$\cdot$) and (INV0) by (INV$\cdot$)′). In the language $\mathcal{L}_{\mathrm{F}}$, this theory consists only of universal sentences. Such a theory is called a **universal theory**. A second example of a universal theory is the **theory of abelian groups** which consists of the $\mathcal{L}_{\mathrm{G}}$-axioms (ASS+), (COM+), (NEU+) and (INV+). We leave it to the reader to define the universal theories of ordered abelian groups and of ordered fields.

We are interested in universal theories because of the following fact, which is intuitively clear, and whose proof is a formalism using the formal definition of $\models$.

Lemma 13.2 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\mathfrak{S}$ a substructure of $\mathfrak{A}$. If φ a universal $\mathcal{L}$-sentence which holds in $\mathfrak{A}$, then φ also holds in $\mathfrak{S}$. Consequently, if $\mathbf{T}$ is a universal $\mathcal{L}$-theory and $\mathfrak{A}$ is a model of $\mathbf{T}$, then also $\mathfrak{S}$ is a model of $\mathbf{T}$.*

Similarly, if ψ is an existential $\mathcal{L}$-sentence which holds in $\mathfrak{S}$, then it also holds in $\mathfrak{A}$.

In particular, every $\mathcal{L}_{\mathrm{G}}$-substructure of an abelian groups is again an abelian group, and every $\mathcal{L}_{\mathrm{F}}$-substructure of a field is again a field. In contrast to this, every $\mathcal{L}_{\mathrm{R}}$-substructure of a field need only be a commutative ring with 1. For instance, $\mathbb{Z}$ is an $\mathcal{L}_{\mathrm{R}}$-substructure of $\mathbb{Q}$. This does not really matter since if a ring is contained in a field, then also its quotient field is contained in the field. But we can avoid this argument if we work in the language of fields.

Since $\mathcal{L}_{\mathrm{F}}$ is an extension of $\mathcal{L}_{\mathrm{R}}$, every field with function $^{-1}$ can be viewed as an expansion of the same field without this function. Given an extension $\mathcal{L}'$ of a language $\mathcal{L}$, an $\mathcal{L}$-theory $\mathbf{T}$ and an $\mathcal{L}'$-theory $\mathbf{T}'$, then we will say that $\mathbf{T}'$ is an **expansion of T** if the $\mathcal{L}$-reduct of every model of $\mathcal{L}'$ is a model of $\mathbf{T}$. For example, the $\mathcal{L}_{\mathrm{F}}$-theory of fields is an expansion of the $\mathcal{L}_{\mathrm{R}}$-theory of fields. Both of them are expansions of the theory of abelian groups.

We will often use expansions to introduce auxiliary relations, functions or constants. If $\mathcal{L}$ is a language and R is a new n-ary relation symbol, then a **definition** of R is an $\mathcal{L}$-sentence

$$\forall x_1 \ldots \forall x_n : \; R(x_1, \ldots, x_n) \; \leftrightarrow \; \varphi(x_1, \ldots, x_n)$$

where $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula. Similarly, a definition of a new n-ary function symbol f is an $\mathcal{L}$-sentence

$$\forall x_1 \ldots \forall x_n \forall x : \; f(x_1, \ldots, x_n) = x \; \leftrightarrow \; \varphi(x_1, \ldots, x_n, x) \; ,$$

and a definition of a new constant symbol f is an $\mathcal{L}$-sentence

$$\forall x : \; x = c \; \leftrightarrow \; \varphi(x) \; .$$

For example, the function $^{-1}$ can be defined by the $\mathcal{L}_{\mathrm{R}}$-sentence

$$\forall x \forall y : \; x^{-1} = y \; \leftrightarrow \; x \cdot y = 1 \; \vee \; (x = 0 \; \wedge \; y = 0) \; .$$

Many expansions will just introduce abbreviations. For example, if we are working in the language $\mathcal{L}_{\mathrm{OG}}$ of ordered groups, we will often use the symbol $\leq$, which is defined by the $\mathcal{L}_{\mathrm{OG}}$-sentence

$$\forall x \forall y : \; x \leq y \; \leftrightarrow \; x < y \; \vee \; x = y \; ,$$

and we will also reverse the symbols, which is done by the trivial definition

$$\forall x \forall y : \; x > y \; \leftrightarrow \; y < x \; ,$$

and similarly for $\geq$. Such expansions by definable new symbols have no influence on the model theoretic properties of a theory, like model completeness and completeness (see below), since we can always replace the defined symbol by the defining sentence. The only exception is quantifier elimination, since it is possible to hide quantifiers in newly introduced predicates (cf. Theorem 13.71).

In contrast to expansions by definable symbols, we have examples like the theory of fields as an expansion of the theory of abelian groups. The multiplication in fields is not definable from addition *in general* (although this is possible in some fields). Such expansions $\mathbf{T}'$ of a theory $\mathbf{T}$ can have model theoretic properties which are very different from that of $\mathbf{T}$. (If this were not so, then we could just study the theory of abelian groups to find out about the theory of valued fields, and this book would probably never have been written).

13.2 Embeddings and elementary extensions

Let $\mathfrak{S}$ be a substructure of the $\mathcal{L}$-structure $\mathfrak{A}$, with universe S. Then $(\mathfrak{S}, S)$ is a substructure of the $\mathcal{L}(S)$-structure $(\mathfrak{A}, S)$. Now let $\mathfrak{S}$ be a common substructure of two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$. Then we will say that $\mathfrak{A}$ and $\mathfrak{B}$ are **elementarily equivalent over** $\mathfrak{S}$ and write $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ if $(\mathfrak{A}, S) \equiv (\mathfrak{B}, S)$. The constants inferred from a substructure are often called **parameters**. So if $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$, then one says that "$\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same elementary sentences with parameters from $\mathfrak{S}$". Note that $\equiv_{\mathfrak{S}}$ is an equivalence relation. Further, $\mathfrak{A}$ is said to be an **elementary substructure of** $\mathfrak{B}$ if $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \equiv_{\mathfrak{A}} \mathfrak{B}$. In this case, we write $\mathfrak{A} \prec \mathfrak{B}$ and call $\mathfrak{A} \subset \mathfrak{B}$ an **elementary extension**. An embedding of $\mathfrak{A}$ in $\mathfrak{B}$ is said to be an **elementary embedding** if the image of $\mathfrak{A}$ in $\mathfrak{B}$ under this embedding is an elementary substructure of $\mathfrak{B}$.

Let $\mathbf{S}$ be a set of $\mathcal{L}$-sentences and $\mathfrak{A}$, $\mathfrak{B}$ two $\mathcal{L}$-structures. Then we will write $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ if

$$\mathfrak{A} \models \varphi \implies \mathfrak{B} \models \varphi \quad \text{for all } \varphi \in \mathbf{S} .$$

If $\mathbf{S} = \{\varphi\}$, then we will just write $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$ instead of $\mathfrak{A} \Rightarrow_{\{\varphi\}} \mathfrak{B}$. If the $\mathcal{L}$-sentence ψ is equivalent to φ, then $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$ implies $\mathfrak{A} \Rightarrow_{\psi} \mathfrak{B}$.

Every $\mathcal{L}$-structure is either a model of φ or of $\neg\varphi$. From this it follows that $\mathfrak{B} \Rightarrow_{\varphi} \mathfrak{A}$ if and only if $\mathfrak{A} \Rightarrow_{\neg\varphi} \mathfrak{B}$. It also follows that $\neg\neg\varphi$ is equivalent to φ. With $\neg\mathbf{S} := \{\neg\varphi \mid \varphi \in \mathbf{S}\}$ we thus have that

$$\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B} \quad \text{if and only if} \quad \mathfrak{B} \Rightarrow_{\neg\mathbf{S}} \mathfrak{A} .$$

If we define $\mathbf{S}^{\neg}$ to be the closure of $\mathbf{S}$ under negations, then every sentence in $\mathbf{S}^{\neg}$ is equivalent to some sentence in $\mathbf{S} \cup \neg\mathbf{S}$. We thus obtain that

$$\mathfrak{A} \Rightarrow_{\mathbf{S}^{\neg}} \mathfrak{B} \quad \text{if and only if} \quad \mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B} \text{ and } \mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A} . \tag{13.3}$$

If $\mathbf{S}$ is closed under negations, then $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ if and only if $\mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A}$. With (13.3) it follows that $\mathfrak{A} \equiv \mathfrak{B}$ is equivalent to $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ and to $\mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{A}$, with $\mathbf{S}$ the set of all $\mathcal{L}$-sentences. Since $\mathrm{Th}(\mathfrak{A})$ is just the set of $\mathcal{L}$-sentences which hold in $\mathfrak{A}$, the latter conditions are equivalent to $\mathfrak{A} \Rightarrow_{\mathrm{Th}(\mathfrak{A})} \mathfrak{B}$ and to $\mathfrak{B} \Rightarrow_{\mathrm{Th}(\mathfrak{B})} \mathfrak{A}$. But this in turn is just expreesing that $\mathfrak{A}$ is a model of the theory of $\mathfrak{B}$, hence we have that

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{if and only if} \quad \mathfrak{A} \models \mathrm{Th}(\mathfrak{B}) .$$

It follows that

$$\mathfrak{A} \prec \mathfrak{B} \quad \text{if and only if} \quad \mathfrak{A} \subset \mathfrak{B} \text{ and } (\mathfrak{A}, A) \models \mathrm{Th}(\mathfrak{B}, A) .$$

As we have seen above, we can replace $(\mathfrak{A}, A) \models \mathrm{Th}(\mathfrak{B}, A)$ by $(\mathfrak{B}, A) \models \mathrm{Th}(\mathfrak{A}, A)$, by $(\mathfrak{B}, A) \Rightarrow_{\mathrm{Th}(\mathfrak{B},A)} (\mathfrak{A}, A)$, or by $(\mathfrak{A}, A) \Rightarrow_{\mathrm{Th}(\mathfrak{A},A)} (\mathfrak{B}, A)$.

Let us note the following. Assume that $\mathcal{L}'$ is an extension of the language $\mathcal{L}$. Then every $\mathcal{L}$-sentence is also an $\mathcal{L}'$-sentence. Consequently, if two $\mathcal{L}'$-structures are elementarily equivalent, then so are their $\mathcal{L}$-reducts. If $\mathfrak{A} \prec \mathfrak{B}$, then the same holds for their $\mathcal{L}$-reducts.

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures such that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ with $\mathbf{S}$ the set of all existential (resp. universal) $\mathcal{L}$-sentences, then we will write $\mathfrak{A} \Rightarrow_{\exists} \mathfrak{B}$ (resp. $\mathfrak{A} \Rightarrow_{\forall} \mathfrak{B}$). Note that

$$\mathfrak{A} \Rightarrow_{\exists} \mathfrak{B} \quad \text{if and only if} \quad \mathfrak{B} \Rightarrow_{\forall} \mathfrak{A} .$$

This is seen as follows. We have already shown that $\mathfrak{A} \Rightarrow_\varphi \mathfrak{B}$ if and only if $\mathfrak{B} \Rightarrow_{\neg\varphi} \mathfrak{A}$. Now if φ is of the form $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n)$ with ψ a quantifier free formula, then $\neg\varphi$ holds in a given structure if and only if $\forall x_1 \ldots \forall x_n \neg\psi(x_1, \ldots, x_n)$ holds. The same is true if we interchange the quantifiers $\exists$ and $\forall$. So we see that the negation of an existential sentence is equivalent to a universal sentence, and the negation of a universal sentence is equivalent to an existential sentence. Hence, $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$ implies $\mathfrak{A} \Rightarrow_{\neg\varphi} \mathfrak{B}$ for every existential sentence φ, and $\mathfrak{A} \Rightarrow_\exists \mathfrak{B}$ implies $\mathfrak{A} \Rightarrow_{\neg\varphi} \mathfrak{B}$ for every universal sentence φ. This yields our assertion.

Assume that $\mathfrak{A} \subset \mathfrak{B}$. Then every existential $\mathcal{L}$-sentence holds in $\mathfrak{B}$ if it holds in $\mathfrak{A}$ (cf. Lemma 13.2. On the other hand, every universal $\mathcal{L}$-sentence holds in $\mathfrak{A}$ if it holds in $\mathfrak{B}$. In other words:

$$\mathfrak{A} \subset \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \Rightarrow_\exists \mathfrak{B} \ \text{ and } \ \mathfrak{B} \Rightarrow_\forall \mathfrak{A} \,.$$

Note that quantifier free sentences are both existential and universal. Hence, they are preserved in both directions.

Let us also note that $\Rightarrow_{\mathbf{S}}$ is transitive:

$$\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B} \ \text{ and } \ \mathfrak{B} \Rightarrow_{\mathbf{S}} \mathfrak{C} \quad \text{imply} \quad \mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{C} \,.$$

It follows that also $\prec$, $\Rightarrow_\exists$ and $\Rightarrow_\forall$ are transitive.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and A_0 the subset of A which consists of all interpretations of constant $\mathcal{L}$-terms in $\mathfrak{A}$. Then A_0 is closed under all functions which are interpretations of function symbols $f \in \mathcal{F}$ and contains the interpretations of all constant symbols $c \in \mathcal{C}$. Consequently, if we interprete the functions and relations of $\mathcal{L}$ on A_0 by the restrictions of their interpretations on A, then we obtain a substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ with universe A_0. It is uniquely determined by $\mathfrak{A}$. Let us call it the **constant substructure of** $\mathfrak{A}$. Every substructure $\mathfrak{A}_1$ of $\mathfrak{A}$ will also contain $\mathfrak{A}_0$ as a substructure. Note that $\mathfrak{A}$ satisfies the same atomic $\mathcal{L}$-sentences as its constant substructure. (For atomic formulas, this in general not true.) The constant substructure of a field K with respect to the language of fields is its **prime field** (which is $\mathbb{Q}$ if $\operatorname{char} K = 0$, and $\mathbb{F}_p$ if $\operatorname{char} K = p > 0$). With respect to the language of rings, it is just $\mathbb{Z}$ if $\operatorname{char} K = 0$, and the prime field otherwise.

Let $\mathfrak{B}$ be a second $\mathcal{L}$-structure, with constant substructure $\mathfrak{B}_0$. We introduce a relation between A_0 and the universe B_0 of $\mathfrak{B}_0$ as follows. For all $a \in A_0$ and $b \in B_0$ we write $\operatorname{ctr}(a, b)$ if and only if there is a constant $\mathcal{L}$-term whose interpretation in A is a and whose interpretation in B is b. We call this relation the **constant term relation** between $\mathfrak{A}$ and $\mathfrak{B}$. Under certain conditions, this relation $\operatorname{ctr}(\mathfrak{A}, \mathfrak{B})$ is in fact a map from A_0 onto B_0. For example, if K, L are fields, then there exists a map from the prime field of K onto the prime field of L if and only if $\operatorname{char} K = \operatorname{char} L$ or $\operatorname{char} K = 0$.

Lemma 13.3 *a) Let* $\mathbf{S}$ *be the set of all atomic sentences* $t_1 = t_2$ *with* t_1, t_2 *constant* $\mathcal{L}$*-terms. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\operatorname{ctr}(\mathfrak{A}, \mathfrak{B})$ *is a map from* A_0 *onto* B_0.

b) Let $\mathbf{S}$ *be the set of all atomic* $\mathcal{L}$*-sentences. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\operatorname{ctr}(\mathfrak{A}, \mathfrak{B})$ *is a homomorphism from* $\mathfrak{A}_0$ *onto* $\mathfrak{B}_0$.

c) Let $\mathbf{S}$ *be the set of all atomic* $\mathcal{L}$*-sentences and their negations, or of all quantifier free* $\mathcal{L}$*-sentences. Then* $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\operatorname{ctr}(\mathfrak{A}, \mathfrak{B})$ *is an isomorphism from* $\mathfrak{A}_0$ *onto* $\mathfrak{B}_0$.

d) Let $\mathbf{S}$ *be the set of all* $\mathcal{L}$*-sentences. Then* $\mathfrak{A}_0 \Rightarrow_{\mathbf{S}} \mathfrak{B}$ *if and only if* $\operatorname{ctr}(\mathfrak{A}, \mathfrak{B})$ *is an elementary embedding of* $\mathfrak{A}_0$ *in* $\mathfrak{B}$.

Proof: a): Let $a_1, a_2 \in A_0$ and t_1, t_2 constant $\mathcal{L}$-terms whose interpretations are a_1, a_2 respectively. Let b_1, b_2 be the interpretations of t_1, t_2 in B. Then $a_1 = a_2$ implies $b_1 = b_2$ if and only if $\mathfrak{A} \models t_1 = t_2$ implies $\mathfrak{B} \models t_1 = t_2$. This shows that $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ is a map if and only if $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$. It is onto as soon as it is a map because every element in B_0 is the interpretation of a term t and thus the image of the interpretation of t in A_0.

b): If $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ is a homomorphism from $\mathfrak{A}_0$ onto $\mathfrak{B}_0$, then in view of part a), it remains to show that $\mathfrak{A} \models R(t_1, \ldots, t_n)$ implies $\mathfrak{B} \models R(t_1, \ldots, t_n)$ for every $R \in \mathcal{R}$. But this is just property (HOMR) of a homomorphism. Conversely, if $\mathfrak{A} \models \varphi$ implies $\mathfrak{B} \models \varphi$ for every atomic $\mathcal{L}$-sentence φ, then we obtain that $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$ satisfies property (HOMR), and by part a), that it is a map from A_0 onto B_0. By its definition, it saatisfies (HOMC), and it follows from the rules for the interpretation of terms that it satisfies property (HOMF).

c): If $\mathbf{S}$ is the set of all atomic $\mathcal{L}$-sentences and their negations, then the proof is similar to that of part b). To deal with injectivity and (HOMS), one just has to use (13.3). For the case of $\mathbf{S}$ being the set of all quantifier free $\mathcal{L}$-sentences, we only have to note the following: The set of quantifier free $\mathcal{L}$-sentences holding in an $\mathcal{L}$-structure $\mathfrak{A}$ is uniquely determined by the set of all atomic $\mathcal{L}$-sentences holding in $\mathfrak{A}$. (Note that this is not true for the set of all $\mathcal{L}$-sentences holding in $\mathfrak{A}$ unless $\mathcal{L}$ contains a constant symbol for every element in A.)

d): Use c), Lemma 13.1 and the fact that $\mathfrak{B}_0 \prec \mathfrak{B}$ if and only if $\mathfrak{B}_0 \Rightarrow_{\mathbf{S}} \mathfrak{B}$. □

Corollary 13.4 *For arbitrary $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, the following assertions are equivalent:*

1) $\mathfrak{A}$ and $\mathfrak{B}$ admit isomorphic substructures,

2) $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ with $\mathbf{S}$ the set of all quantifier free $\mathcal{L}$-sentences,

3) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic $\mathcal{L}$-sentences.

Proof: Observe that by (13.2), the restriction to $\mathfrak{A}_0$ of every homomorphism from some substructure $\mathfrak{A}_1$ into $\mathfrak{B}$ must coincide with $\mathrm{ctr}(\mathfrak{A}, \mathfrak{B})$. Hence, 1)⇔2) follows from part c) of the foregoing lemma. Equivalence 2)⇔3) follows from (13.3) and our remark in the proof of part c) of the foregoing lemma. □

If $\mathcal{L}$ contains a constant symbol for every element of A, then $\mathfrak{A}$ is its own constant substructure. To obtain this situation for an arbitrary $\mathcal{L}$-structure $\mathfrak{A}$, we can just pass to the extension $\mathcal{L}(A)$ of $\mathcal{L}$ and the expansion $(\mathfrak{A}, A)$ of $\mathfrak{A}$. From parts c) and d) of the above lemma, we then obtain:

Corollary 13.5 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. Then $\mathfrak{A}$ admits an embedding (resp. an elementary embedding) in $\mathfrak{B}$ if and only if for some $B' \subset B$, the expansion $(\mathfrak{B}, B')$ is a model of* $\mathrm{Diag}(\mathfrak{A})$ *(resp. of* $\mathrm{Th}(\mathfrak{A}, A)$*).*

We will further use a principle that we have already frequently applied in this book. Suppose that two structures $\mathfrak{A}$ and $\mathfrak{B}$ have substructures $\mathfrak{A}_0$ and $\mathfrak{B}_0$ which admit an isomorphism $\sigma : \mathfrak{A}_0 \to \mathfrak{B}_0$. We then would like to identify $\mathfrak{A}_0$ with $\mathfrak{B}_0$ and thus assume that $\mathfrak{A}$ and $\mathfrak{B}$ have a common substructure. To obtain this, one extends the isomorphism σ from $\mathfrak{A}_0$ to an isomorphism of $\mathfrak{A}$ onto some structure $\mathfrak{A}'$. The reader may show that this is always possible (the proof is a mere formalism). Now $\mathfrak{A}'$ and $\mathfrak{B}$ indeed have $\mathfrak{B}_0$ as a

common substructure. Since $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic, we do not loose anything: Lemma 13.1 has shown that they are elementarily equivalent. So whenever we find isomorphic substructures, we will identify them. Similarly, one can also find an isomorphic image $\mathfrak{B}'$ of $\mathfrak{B}$ such that $\mathfrak{A}$ and $\mathfrak{B}'$ have $\mathfrak{A}_0$ as a common substructure. We note:

Lemma 13.6 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$-structures. Assume that there is an isomorphism from a substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ onto a substructure $\mathfrak{B}_0$ of $\mathfrak{B}$. Then there exist $\mathcal{L}$-structures $\mathfrak{A}' \cong \mathfrak{A}$ and $\mathfrak{B}' \cong \mathfrak{B}$ such that $\mathfrak{A}'$ and $\mathfrak{B}$ have $\mathfrak{B}_0$ as a common substructure, and that $\mathfrak{A}$ and $\mathfrak{B}'$ have $\mathfrak{A}_0$ as a common substructure. Moreover, $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{B}' \equiv \mathfrak{B}$.*

Let be given a sequence $(\mathfrak{A}_\nu)_{\nu<\alpha}$ of $\mathcal{L}$-structures, indexed by all ordinals smaller than some fixed ordinal $\alpha > 0$, such that

(CHN1) $\mathfrak{A}_\mu \subset \mathfrak{A}_\nu$ whenever $\mu < \nu < \alpha$.

Then on the union $A_\alpha := \bigcup_{\nu<\alpha} A_\nu$ an $\mathcal{L}$-structure $\mathfrak{A}_\alpha$ can be defined such that $\mathfrak{A}_\nu \subset \mathfrak{A}_\alpha$ for every $\nu < \alpha$. Indeed, the interpretation of the constants is already determined by their interpretation in $\mathfrak{A}_0$, and the interpretation of the relation and function symbols on A_α is (necessarily) the set theoretical union of their interpretations in the structures $\mathfrak{A}_\nu$. Now $(\mathfrak{A}_\nu)_{\nu<\alpha}$ will be called a **chain** if it satisfies (CHN1) and

(CHN2) $\mathfrak{A}_\lambda = \bigcup_{\nu<\lambda} \mathfrak{A}_\nu$ for all limit ordinals $\lambda < \alpha$.

The union over a chain of $\mathcal{L}$-structures is again an $\mathcal{L}$-structure. Assume that an $\mathcal{L}$-sentence φ holds in all $\mathfrak{A}_\nu$ for $\nu < \alpha$. It will in general not follow that φ holds in the union. For instance, consider the chain

$$\mathbb{Z} \subset \frac{1}{2}\mathbb{Z} \subset \ldots \subset \frac{1}{2^n}\mathbb{Z} \subset \ldots$$

of $\mathcal{L}_{\mathrm{G}}$-structures, with $n \in \mathbb{N}$. Its union is the 2-divisible hull of $\mathbb{Z}$. All structures in the chain are models for the elementary $\mathcal{L}_{\mathrm{G}}$-sentence

(LPE) $\exists x \forall y : \ 0 \leq y \leq x \ \rightarrow \ y = 0 \vee y = x$

(note that "$0 \leq y \leq x$" is not itself an $\mathcal{L}_{\mathrm{OG}}$-formula; it is an abbreviation for the $\mathcal{L}_{\mathrm{OG}}$-formula $(0 = y \vee 0 < y) \wedge (y = x \vee y < x)$). But (LPE) does not hold in the union since it is a dense group. On the other hand, one sees at once that there are no problems with existential sentences since they carry over to every extension. We can prove a little bit more. (**In the following proof, like in most of the later proofs, we will not anymore distinguish between an element and its constant symbol.**)

Lemma 13.7 *Assume that $(\mathfrak{A}_\nu)_{\nu<\alpha}$ is a chain of $\mathcal{L}$-structures. If a universal existential $\mathcal{L}$-sentence φ holds in every $\mathfrak{A}_\nu$ for $\nu < \alpha$, then it holds in $\bigcup_{\nu<\alpha} \mathfrak{A}_\nu$.*

Proof: Let φ be a universal existential sentence

$$\forall x_1 \ldots \forall x_m \exists y_1, \ldots \exists y_n : \ \psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where ψ is a quantifier free $\mathcal{L}$-formula. Let $a_1, \ldots, a_m$ be arbitrary elements in $\bigcup_{\nu<\alpha} A_\nu$. Then there is already some $\nu < \alpha$ such that $a_1, \ldots, a_m \in A_\nu$. Since we have assumed that

φ holds in $\mathfrak{A}_\nu$, we know that there are elements $b_1, \ldots, b_n \in A_\nu$ such that the $\mathcal{L}$-sentence $\psi(a_1, \ldots, a_m, b_1, \ldots, b_n)$ holds in $\mathfrak{A}_\nu$. Since it is quantifier free, it also holds in the extension $\bigcup_{\nu<\alpha} \mathfrak{A}_\nu$. We conclude that φ holds in $\bigcup_{\nu<\alpha} \mathfrak{A}_\nu$. □

In view of the axiom systems of groups, rings and fields, which only consist of universal existential sentences, we conclude that unions of chains of groups, rings or fields are again groups, rings or fields respectively.

In order to obtain the preservation of all $\mathcal{L}$-sentences, we have to impose stronger conditions on the chain. A chain $(\mathfrak{A}_\nu)_{\nu<\alpha}$ of $\mathcal{L}$-structures is called an **elementary chain** if $\mathfrak{A}_\nu \prec \mathfrak{A}_{\nu+1}$ whenever $\nu + 1 < \alpha$.

Lemma 13.8 *The union over an elementary chain $(\mathfrak{A}_\nu)_{\nu<\alpha}$ is an elementary extension of every $\mathfrak{A}_\nu$, $\nu < \alpha$.*

Proof: The proof is done by induction on α. If $\alpha = 1$, then the assertion is trivial. Assume that we have proved the assertion for all chains of length $< \alpha$. Suppose that $\alpha = \beta + 1$. Then the union over $(\mathfrak{A}_\nu)_{\nu<\alpha}$ is just $\mathfrak{A}_\beta$. If β is a limit ordinal, then by (CHN2), $\mathfrak{A}_\beta = \bigcup_{\nu<\beta} \mathfrak{A}_\nu$, and by induction hypothesis, we obtain that $\mathfrak{A}_\nu \prec \mathfrak{A}_\beta$ for all $\nu < \beta$. If $\beta = \gamma + 1$, then by induction hypothesis, $\mathfrak{A}_\nu \prec \mathfrak{A}_\gamma$ for all $\nu < \gamma$. Since the chain is assumed to be elementary, we have that $\mathfrak{A}_\gamma \prec \mathfrak{A}_\beta$, and by the transitivity of $\prec$, it follows that $\mathfrak{A}_\nu \prec \mathfrak{A}_\beta$. This proves our assertion in the successor case $\alpha = \beta + 1$.

Now assume that α is a limit ordinal. In view of (CHN2), we write $\mathfrak{A}_\alpha$ for the union of $(\mathfrak{A}_\nu)_{\nu<\alpha}$. Given an arbitrary $\nu < \alpha$ and an arbitrary $\mathcal{L}(A_\nu)$-sentence φ, we have to show that φ holds in $(\mathfrak{A}_\nu, A_\nu)$ if and only if φ holds in $(\mathfrak{A}_\alpha, A_\alpha)$. The proof proceeds by induction on the complexity of φ. If φ has no quantifiers, then the assertion is true since $\mathfrak{A}_\alpha$ is an extension of $\mathfrak{A}_\nu$. For φ of the form $\neg\psi$ or $\psi_1 \wedge \psi_2$, the induction step is straightforward. Now assume that φ is of the form $\forall x \psi(x)$ and that the assertion is already proved for all $\nu < \alpha$ and all $\mathcal{L}(A_\nu)$-sentences which have less quantifiers than φ. Thus for every μ such that $\nu \leq \mu < \alpha$ and for every $a \in A_\mu$ we have that the $\mathcal{L}(A_\mu)$-sentence $\psi(a)$ holds in $(\mathfrak{A}_\mu, A_\mu)$ if and only if it holds in $(\mathfrak{A}_\alpha, A_\mu)$. (Here we use the fact that $A_\nu \subset A_\mu$ for $\nu \leq \mu$ and thus an $\mathcal{L}(A_\nu)$-formula $\psi(x)$ is also an $\mathcal{L}(A_\mu)$-formula.) If $\forall x \psi(x)$ holds in $(\mathfrak{A}_\alpha, A_\nu)$, then it also holds in the substructure $(\mathfrak{A}_\nu, A_\nu)$. For the converse, assume that $\forall x \psi(x)$ does not hold in $(\mathfrak{A}_\alpha, A_\nu)$. Then there is some b in $A_\alpha = \bigcup_{\nu<\alpha} A_\nu$ such that $\neg\psi(b)$ holds in $(\mathfrak{A}_\alpha, A_\alpha)$. We choose some μ such that $\nu \leq \mu < \alpha$ and $b \in A_\mu$. Now $\neg\psi(b)$ holds in $(\mathfrak{A}_\alpha, A_\mu)$, so it also holds in $(\mathfrak{A}_\mu, A_\mu)$ by our induction hypothesis. Hence, $\exists x \neg\psi(x)$ holds in $(\mathfrak{A}_\mu, A_\nu)$. But $\mathfrak{A}_\nu \prec \mathfrak{A}_\mu$ by our induction hypothesis (with respect to the length of the chain). Consequently, $\exists x \neg\psi(x)$ also holds in $(\mathfrak{A}_\nu, A_\nu)$, which yields that $\forall x \psi(x)$ does not hold in $(\mathfrak{A}_\nu, A_\nu)$, as required. This completes both inductions. □

From this lemma it follows that if $(\mathfrak{A}_\nu)_{\nu<\alpha}$ is an elementary chain, then $\mathfrak{A}_\mu \prec \mathfrak{A}_\nu$ whenever $\mu < \nu < \alpha$.

Exercise 13.1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ have a common constant substructure $\mathfrak{C}$. Prove that $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_{\mathfrak{C}} \mathfrak{B}$. Apply this to the case of fields.*

Exercise 13.2 *Let $\mathfrak{A}$ be a finite structure. Prove that $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

13.3 The Compactness Theorem

A basic question in model theory is whether a given theory has a model. If a theory is contradictory, we won't expect that it admits a model. For instance, if our theory contains two $\mathcal{L}_{\mathrm{F}}$-sentences, one saying that the characteristic is 2 and the other saying that it is 3, then there is certainly no field which is a model simultaneously for both sentences. But what exactly do we mean by contradictory? We do not want to go into details here and refer the reader to the general books on mathematical logic and model theory that we have already cited. But we want to give a brief sketch of the basic ideas.

Intuitively, we would call a theory contradictory if we can deduce a contradiction from it. Indeed, an $\mathcal{L}$-theory $\mathbf{T}$ is said to be **inconsistent** if there is an $\mathcal{L}$-sentence φ such that the $\mathcal{L}$-sentence $\varphi \wedge \neg\varphi$ can be deduced from $\mathbf{T}$. Otherwise, $\mathbf{T}$ is said to be **consistent**. We have not yet specified what is meant by **deduction**. For this notion, we refer the reader to the books about model theory or mathematical logic. There, the reader will find a logic calculus which defines how sentences can be deduced from others by use of the so-called logical axioms and logical rules. If an $\mathcal{L}$-sentence can be deduced in such a way (in finitely many steps!) from a given set $\mathbf{T}$ of $\mathcal{L}$-sentences, then one writes $\mathbf{T} \vdash \varphi$. Hence an $\mathcal{L}$-theory $\mathbf{T}$ is inconsistent if $\mathbf{T} \vdash \varphi \wedge \neg\varphi$ for some $\mathcal{L}$-sentence φ. Since for every deduction, only finitely many steps are allowed, only finitely many sentences from $\mathbf{T}$ can be used in the deduction. Consequently, if a contradiction can be deduced from $\mathbf{T}$, then it can already be deduced from a finite subset of $\mathbf{T}$. This proves:

Theorem 13.9 (Syntactical Compactness Theorem)
A set $\mathbf{S}$ of elementary $\mathcal{L}$-sentences is consistent if every of its finite subsets is consistent.

The converse is trivial.

Parallel to the syntactical relation of deduction, there is the semantical relation $\mathbf{T} \models \varphi$. Both $\mathbf{T} \models \varphi$ and $\mathbf{T} \vdash \varphi$ are just two different appearances of the same relation:

Theorem 13.10 (Gdel's Completeness Theorem)
Let $\mathbf{T}$ be an elementary $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence. Then $\mathbf{T} \models \varphi$ if and only if $\mathbf{T} \vdash \varphi$.

This theorem consists of two quite different parts. The implication "$\Leftarrow$" is sometimes called the **Soundness Theorem of First Order Logic** because it states that the formal notion of deduction reflects correctly the logic of our models. The proof of this part is done by induction on the length of formal deductions.

The implication "$\Rightarrow$" is the original completeness theorem. It says that every sentence can be syntactically deduced from $\mathbf{T}$ if it follows semantically from $\mathbf{T}$. In other words, our formal notion of deduction is not weaker than the relation between sentences that is seen in models. The idea of the proof of the completeness theorem is the following. If φ is an $\mathcal{L}$-sentence which can not be deduced from $\mathbf{T}$, then one shows that $\mathbf{T} \cup \{\neg\varphi\}$ is consistent. So if we can find a model for $\mathbf{T} \cup \{\neg\varphi\}$, then we have a model for $\mathbf{T}$ in which φ does not hold. That is, $\mathbf{T} \models \varphi$ can not be true. So the main ingredient of the proof is the following:

Theorem 13.11 (Model Existence Theorem)
Every consistent set of elementary $\mathcal{L}$-sentences has a model (of cardinality $\leq ||\mathcal{L}||$).

This theorem is commonly proved by constructing a model from $\mathcal{L}$-sentences. For the quite technical details, we refer the reader to the books on model theory.

The converse of this theorem is also true. Suppose that $\mathbf{T}$ has a model $\mathfrak{A}$. If φ is an $\mathcal{L}$-sentence, then φ and its negation $\neg\varphi$ can not hold in $\mathfrak{A}$ at the same time. That is, $\mathbf{T} \models \varphi \wedge \neg\varphi$ can not be true for any $\mathcal{L}$-sentence φ. By Gdel's Completeness Theorem, it follows that $\mathbf{T} \vdash \varphi \wedge \neg\varphi$ can not be true for any $\mathcal{L}$-sentence φ. But this just says that $\mathbf{T}$ is consistent. We have proved:

Lemma 13.12 *If a set of elementary $\mathcal{L}$-sentences has a model, then it is consistent.*

Hence if every finite subset of $\mathbf{T}$ has a model, then every finite subset of $\mathbf{T}$ is consistent. So by the Syntactical Compactness Theorem, $\mathbf{T}$ is consistent. But then by the Model Existence Theorem, $\mathbf{T}$ has a model. Conversely, every model of $\mathbf{T}$ is at the same time a model for every of its subsets. We have proved the semantical counterpart to the Syntactical Compactness Theorem:

Theorem 13.13 (Semantical Compactness Theorem)
A set of elementary $\mathcal{L}$-sentences has a model (of cardinality $\leq ||\mathcal{L}||$) if and only if every of its finite subsets has a model.

A "more algebraical" proof of this theorem (except for the limitation of the cardinality) can be given by means of ultrapowers, which we will introduce below (cf. Theorem 13.76). Together with the next theorem, the Semantical Compactness Theorem is in fact the only of the above results that we will need in the following.

Theorem 13.14 (Finiteness Theorem*)*
Assume $\mathbf{S}$ to be a nonempty set of $\mathcal{L}$-sentences. If φ is an $\mathcal{L}$-sentence such that $\mathbf{S} \models \varphi$, then there are $\mathcal{L}$-sentences $\psi_1, \ldots, \psi_n \in \mathbf{S}$ such that $\{\psi_1, \ldots, \psi_n\} \models \varphi$.

Proof: We have already mentioned that if $\mathbf{S} \vdash \varphi$, then there are $\mathcal{L}$-sentences $\psi_1, \ldots, \psi_n \in \mathbf{S}$ such that $\{\psi_1, \ldots, \psi_n\} \vdash \varphi$. By Gdel's Completeness Theorem, we can replace "$\vdash$" by "$\models$". The semantical version of this argument reads as follows: Suppose that $\mathbf{S}_0 \not\models \varphi$ for every finite subset $\mathbf{S}_0$ of $\mathbf{S}$, that is, there is a model for $\mathbf{S}_0 \cup \{\neg\varphi\}$. Then by the Semantical Compactness Theorem, there is a model for $\mathbf{S} \cup \{\neg\varphi\}$, which is impossible since we have assumed that $\mathbf{S} \models \varphi$. This shows that there is a finite subset $\{\psi_1, \ldots, \psi_n\} \subset \mathbf{S}$ such that $\{\psi_1, \ldots, \psi_n\} \models \varphi$. □

Let us illustrate the use of the Semantical Compactness Theorem by giving some applications.

Corollary 13.15 *There is a nonarchimedean ordered abelian group which is elementarily equivalent to the archimedean ordered abelian group $\mathbb{Z}$. Consequently, the properties "archimedean" and "isomorphic to $\mathbb{Z}$" are not elementary in the language of ordered groups.*

Proof: We adjoin two constant symbols c, d to the language $\mathcal{L}_{\mathrm{OG}}$ and consider the set

$$\mathbf{S} := \mathrm{Th}(\mathbb{Z}, <, +, -, 0) \cup \{0 < c\} \cup \{n \cdot c < d \mid n \in \mathbb{N}\}$$

of elementary $\mathcal{L}_{\mathrm{OG}}(\{c, d\})$-sentences. Every finite subset $\mathbf{S}_0$ of $\mathbf{S}$ contains only finitely many of the sentences $n \cdot c < d$. Let m be a natural number which is bigger than all n which occur in these finitely many sentences. If we then interprete c by 1 and d by m in $\mathbb{Z}$, we find that $\mathbb{Z}$ is a model of $\mathbf{S}_0$, because it is already a model of $\mathrm{Th}(\mathbb{Z})$ (with respect

to the language $\mathcal{L}_{\mathrm{OG}}$). Hence every finite subset of $\mathbf{S}$ admits a model. By Theorem 13.13, the set $\mathbf{S}$ itself admits a model. In this model, c must be interpreted by a positive element c, and d must be interpreted by an element which is bigger than all $n \cdot c$. So this model can not be archimedean. On the other hand, it is a model of $\mathrm{Th}(\mathbb{Z})$, showing that it has the same elementary properties as the ordered abelian group $\mathbb{Z}$ (with respect to the language $\mathcal{L}_{\mathrm{OG}}$). □

Let us give a further application from field theory.

Theorem 13.16 *Let $\mathbf{T}$ be an $\mathcal{L}_{\mathrm{F}}$-theory containing the theory of fields, and let φ be an $\mathcal{L}_{\mathrm{F}}$-sentence. If for every $n \in \mathbb{N}$ there is a prime p such that $p > n$ and φ holds in a field of characteristic p which is a model of $\mathbf{T}$, then there is a field of characteristic 0 in which φ holds and which is a model of $\mathbf{T}$. Equivalently, if ψ is an $\mathcal{L}_{\mathrm{F}}$-sentence which holds in all fields of characteristic 0 that are models of $\mathbf{T}$, then there is some prime $p = p(\psi)$ such that ψ holds in all fields of characteristic $\geq p$ that are models of $\mathbf{T}$.*

Proof: let us call a model of $\mathbf{T}$ a $\mathbf{T}$-field. Assume that for every $n \in \mathbb{N}$ there is a prime p such that $p > n$ and φ holds in a $\mathbf{T}$-field of characteristic p. We let $\mathbf{S} = \mathbf{T} \cup \{\varphi\} \cup (\mathrm{CH0})$. Recall that (CH0) is given by the infinite set $\{\exists x : \ n \cdot x \neq 0 \mid n \in \mathbb{N}\}$. Every finite subset $\mathbf{S}_0$ of $\mathbf{S}$ contains only finitely many of the sentences $\exists x : \ n \cdot x \neq 0$. Let m be the maximum all n which occur in these finitely many sentences. By assumption, there is a $\mathbf{T}$-field of characteristic $p > m$ such that φ holds in this field. This field is a model of $\{\exists x : \ n \cdot x \neq 0 \mid n < m\}$ and thus also of $\mathbf{S}_0$. Hence every finite subset of $\mathbf{S}$ admits a model, and by Theorem 13.13, the set $\mathbf{S}$ itself must have model. But this model is just a $\mathbf{T}$-field of characteristic 0 in which φ holds.

If ψ holds in all $\mathbf{T}$-fields of characteristic 0, then there is no $\mathbf{T}$-field of characteristic 0 in which $\neg\varphi$ holds. By what we just have proved, $\neg\varphi$ can thus not hold in $\mathbf{T}$-fields of arbitrarily high characteristic. This gives the second assertion. □

For $\mathbf{T}$ the theory of algebraically closed fields, we will apply this theorem in the proof of a result on polynomial maps (see Theorem 13.54 below).

If an elementary property is given by finitely many elementary sentences, then it can already be given by one elementary sentence (just take the conjunction of the finitely many sentences). Such an elementary property is called **finitely axiomatizable**. Now assume that the property "of characteristic 0" is finitely axiomatizable in the language of fields by the elementary sentence φ. Then φ holds in all fields of characteristic 0, but in no field of characteristic $p > 0$. This contradiction to the foregoing theorem shows:

Corollary 13.17 *The elementary property "of characteristic 0" is not finitely axiomatizable in the language of fields.*

Finally, let us apply the compactness theorem to obtain models of prescribed cardinality.

Lemma 13.18 *If the $\mathcal{L}$-theory $\mathbf{T}$ has an infinite model, then it has a model of every given cardinality $\kappa \geq ||\mathcal{L}||$.*

Proof: We adjoin κ many new constant symbols c_ν, $\nu < \kappa$, to the language $\mathcal{L}$. Further, we define

$$\mathbf{S} := \mathbf{T} \cup \{c_\mu \neq c_\nu \mid \mu < \nu < \kappa\} .$$

Every finite subset $\mathbf{S}_0$ of $\mathbf{S}$ contains only finitely many of the atomic sentences $c_\mu \neq c_\nu$. Let $c_{\nu_1}, \ldots, c_{\nu_n}$ be the constant symbols involved in these finitely many sentences. We take $\mathfrak{A}$ to be an infinite model of $\mathbf{T}$, which exists by hypothesis. Then we can choose distinct elements $a_1, \ldots, a_n \in A$ and interprete every c_{ν_i} by a_i. In this way, $(\mathfrak{A}, \{a_1, \ldots, a_n\})$ becomes a model of $\mathbf{S}_0$ (because it is already a model of $\mathbf{T}$). We have thus proved that every finite subset of $\mathbf{S}$ has a model. By the Compactness Theorem 13.13 it follows that $\mathbf{S}$ admits a model $\mathfrak{B}$ of cardinality $\leq ||\mathcal{L}(\{c_\nu \mid \nu < \kappa\})|| = \max\{||\mathcal{L}||, \kappa\} = \kappa$. But in this model $\mathfrak{B}$, all c_ν must be interpreted by distinct elements. Hence, $\mathfrak{B}$ is of cardinality at least κ. So it follows that its cardinality is precisely κ. Its $\mathcal{L}$-reduct is a model of $\mathbf{T}$ of cardinality κ, as required. □

If $\mathfrak{A}$ is an $\mathcal{L}$-structure, then we can take $\mathbf{T}$ to be $\mathrm{Th}(\mathfrak{A}, A)$. If $\mathfrak{A}$ is infinite, then $\mathrm{Th}(\mathfrak{A}, A)$ has an infinite model. Observe that $||\mathcal{L}(A)|| = \max\{||\mathcal{L}||, |\mathfrak{A}|\}$. Applying the foregoing lemma to this situation, we obtain a model $\mathfrak{B}$ of $\mathrm{Th}(\mathfrak{A}, A)$ of cardinality κ for every $\kappa \geq \max\{||\mathcal{L}||, |\mathfrak{A}|\}$. By Corollary 13.5, $(\mathfrak{A}, A)$ admits an elementary embedding in $\mathfrak{B}$. After replacing $\mathfrak{B}$ by an isomorphic image, we can assume that this embedding is the identity and that $\mathfrak{B}$ is an elementary extension of $(\mathfrak{A}, A)$ (cf. Lemma 13.6). It follows that the $\mathcal{L}$-reduct of $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ of cardinality κ. We have proved:

Theorem 13.19 (Upward Löwenheim-Skolem Theorem)
Let $\mathfrak{A}$ be an infinite $\mathcal{L}$-structure. Then for every cardinal $\kappa \geq \max\{||\mathcal{L}||, |\mathfrak{A}|\}$ there is an elementary extension of $\mathfrak{A}$ of cardinality κ.

13.4 Quantifier Elimination

Let $\mathbf{T}$ be an $\mathcal{L}$-theory. Then two $\mathcal{L}$-formulas φ, ψ are said to be **T-equivalent** if

$$\mathbf{T} \models \forall(\varphi \leftrightarrow \psi) .$$

The theory $\mathbf{T}$ is said to **admit quantifier elimination** if every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to a quantifier free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. There is a slight difficulty that we should discuss here. If φ is a sentence such that $\mathbf{T} \models \varphi$ or $\mathbf{T} \models \neg\varphi$, then it must be $\mathbf{T}$-equivalent to a quantifier free $\mathcal{L}$-sentence which holds in every resp. no model of $\mathbf{T}$. But if $\mathcal{L}$ contains no constant symbol, then there are no quantifier free $\mathcal{L}$-sentences at all. For this case, we add "or $\mathbf{T} \models \varphi$, or $\mathbf{T} \models \neg\varphi$" to our above condition for quantifier elimination.

$\mathbf{S}^\vee$ will denote the set $\{\psi_1 \vee \ldots \vee \psi_n \mid n \in \mathbb{N},\ \psi_i \in \mathbf{S}\}$ of all (finite) disjunctions of formulas of $\mathbf{S}$ (or in other words, the **closure** of $\mathbf{S}$ under taking finite disjunctions). Similarly, $\mathbf{S}^{\vee\wedge}$ denotes the set $(\mathbf{S}^\vee)^\wedge = \{\psi_1 \wedge \ldots \wedge \psi_n \mid n \in \mathbb{N},\ \psi_i \in \mathbf{S}^\vee\}$, which is the closure of $\mathbf{S}^\vee$ under taking finite conjunctions. Observe that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ implies that $\mathfrak{A} \Rightarrow_{\mathbf{S}^{\vee\wedge}} \mathfrak{B}$.

Lemma 13.20 *Let $\mathbf{T}$ be an $\mathcal{L}$-theory and $\mathbf{S}$ any set of $\mathcal{L}$-sentences. Then the following are equivalent for every $\mathcal{L}$-sentence φ:*

1) φ is $\mathbf{T}$-equivalent to a sentence in $\mathbf{S}^{\vee\wedge}$, or $\mathbf{T} \models \varphi$, or $\mathbf{T} \models \neg\varphi$,

2) for all $\mathbf{T}$-models $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$, the sentence φ satisfies $\mathfrak{A} \Rightarrow_\varphi \mathfrak{B}$.

Proof: Assume 1). If $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ and thus also $\mathfrak{A} \Rightarrow_{\mathbf{S}^{\vee\wedge}} \mathfrak{B}$ holds and if φ is $\mathbf{T}$-equivalent to a sentence in $\mathbf{S}^{\vee\wedge}$, then also $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$. If $\mathbf{T} \models \varphi$ or $\mathbf{T} \models \neg\varphi$, then $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}$ is automatically true.

Now assume 2) and suppose that $\mathbf{T} \not\models \neg\varphi$. Set $\mathbf{S}_\varphi := \{\psi \in \mathbf{S} \mid \mathbf{T} \models \varphi \to \psi\}^\vee$. Assume that $\mathbf{T} \cup \mathbf{S}_\varphi \not\models \varphi$. By definition, this means that there is a model $\mathfrak{B}$ of $\mathbf{T} \cup \mathbf{S}_\varphi$ such that $\mathfrak{B} \models \neg\varphi$. We set $\mathbf{T}' := \{\neg\psi \mid \psi \in \mathbf{S} \text{ and } \mathfrak{B} \models \neg\psi\}$. With this definition, for every model $\mathfrak{A}$ of $\mathbf{T} \cup \mathbf{T}'$ we obtain that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$, which by 2) implies that $\mathfrak{A} \models \neg\varphi$. We conclude that $\mathbf{T} \cup \mathbf{T}' \models \neg\varphi$. Hence in view of the Finiteness Theorem 13.14, there are $\neg\psi_1, \ldots, \neg\psi_n \in \mathbf{T}'$ such that $\mathbf{T} \cup \{\neg\psi_1, \ldots, \neg\psi_n\} \models \neg\varphi$. (Note that $\mathbf{T} \models \neg\varphi$ is excluded by assumption, hence $\mathbf{T}'$ cannot be empty.) It follows that in every model of $\mathbf{T}$ in which φ holds, at least one of the $\neg\psi_i$ does not hold. In other words, $\mathbf{T} \models \varphi \to (\psi_1 \vee \ldots \vee \psi_n)$. Since we have defined $\mathbf{S}_\varphi$ to be closed under disjunctions, it follows that $\psi_1 \vee \ldots \vee \psi_n \in \mathbf{S}_\varphi$. Since $\mathfrak{B} \models \mathbf{S}_\varphi$, we conclude that $\mathfrak{B} \models \psi_1 \vee \ldots \vee \psi_n$. But by our choice of the ψ_i, we have that $\mathfrak{B} \models \neg\psi_i$ for every i. This contradiction shows that our assumption was false, that is, $\mathbf{T} \cup \mathbf{S}_\varphi \models \varphi$ must hold.

If $\mathbf{S}_\varphi = \emptyset$, then we find that $\mathbf{T} \models \varphi$. If $\mathbf{S}_\varphi \neq \emptyset$, then again by Theorem 13.14, there are $\psi_1, \ldots, \psi_n \in \mathbf{S}_\varphi$ such that $\mathbf{T} \cup \{\psi_1, \ldots, \psi_n\} \models \varphi$. In other words, $\mathbf{T} \models \psi_1 \wedge \ldots \wedge \psi_n \to \varphi$. But by definition of $\mathbf{S}_\varphi$, we have that $\mathbf{T} \models \varphi \to \psi_i$ for all i. Hence, $\mathbf{T} \models \varphi \leftrightarrow \psi$ for $\psi := \psi_1 \wedge \ldots \wedge \psi_n \in \mathbf{S}_\varphi^\wedge \subset \mathbf{S}^{\vee\wedge}$. □

The reader should well note at which point of the proof we have made use of the assumption that φ be a *sentence*. From the assumption that $\mathbf{T} \cup \mathbf{S}_\varphi \not\models \varphi$, we have concluded that there is a model $\mathfrak{B}$ of $\mathbf{T} \cup \mathbf{S}_\varphi$ such that $\mathfrak{B} \models \neg\varphi$. This is in general not true if φ contains free variables. For example, let a be a constant symbol of $\mathcal{L}$ and $\varphi(x) :\Leftrightarrow x = a$. Then for every theory $\mathbf{T}$ which has models with more than one element, $\mathbf{T} \models \varphi(x)$ will not hold. But no model $\mathfrak{B}$ can satisfy $\mathfrak{B} \models \neg(x = a)$. The correct conclusion would be that there exists a model $\mathfrak{B}$ such that $\mathfrak{B} \models \neg\forall x\, x = a$. Here, we have worked with the universal closure $\forall x\, x = a$ of $x = a$, which is a sentence.

From working with polynomials we know the ambivalence of variables X and constants x (which in that case are transcendental over the ground field). Substituting a value a for the variable X can as well be understood as a homomorphism which sends the transcendental element x to a. This actually gives rise to places of function fields and is thus one origin of valuation theory. In model theory, we have a similar ambivalence. We are free to adjoin variable symbols as new constant symbols to the language. Let $\mathbf{T}$ be an $\mathcal{L}$-theory and $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula. When we extend $\mathcal{L}$ to $\mathcal{L}(\{x_1, \ldots, x_n\})$ (which we write shorter as $\mathcal{L}(x_1, \ldots, x_n)$), then $\varphi(x_1, \ldots, x_n)$ becomes an $\mathcal{L}(x_1, \ldots, x_n)$-sentence. Every $\mathcal{L}$-sentence φ is also a $\mathcal{L}(x_1, \ldots, x_n)$-sentence (we assume the bounded variables symbols appearing in φ to be different from $x_1, \ldots, x_n$!). Hence, $\mathbf{T}$ is also a $\mathcal{L}(x_1, \ldots, x_n)$-theory. The only point is that we have to show that the $\mathcal{L}$-theory $\mathbf{T}$ satisfies $\mathbf{T} \models \forall\varphi(x_1, \ldots, x_n)$ if and only if the $\mathcal{L}(x_1, \ldots, x_n)$-theory $\mathbf{T}$ satisfies $\mathbf{T} \models \varphi$. But every $\mathcal{L}(x_1, \ldots, x_n)$-structure arises from an $\mathcal{L}$-structure by expansion. Given an $\mathcal{L}$-structure $\mathfrak{A}$, it becomes an $\mathcal{L}(x_1, \ldots, x_n)$-structure by an arbitrary choice of the interpretation for the constant symbols $x_1, \ldots, x_n$. That is, for every n-tuple $a_1, \ldots, a_n$, there is an expansion of $\mathfrak{A}$ in which every x_i is interpreted by a_i. Consequently, $\mathfrak{A} \models \forall\varphi(x_1, \ldots, x_n)$ if and only if every expansion of $\mathfrak{A}$ to an $\mathcal{L}(x_1, \ldots, x_n)$-structure is a model of φ. This proves what we wanted. We will use this approach in the proof of the following lemma since we have to use Lemma 13.20, which only works with sentences.

Lemma 13.21 *Let* $\mathbf{T}$ *be an* $\mathcal{L}$*-theory. Then the following are equivalent for every* $\mathcal{L}$*-formula* $\varphi(x_1, \ldots, x_n)$*:*

1) $\varphi(x_1, \ldots, x_n)$ *is* $\mathbf{T}$*-equivalent to a quantifier free* $\mathcal{L}$*-formula* $\psi(x_1, \ldots, x_n)$*, or* $\mathbf{T} \models \varphi$*, or* $\mathbf{T} \models \neg\varphi$*,*

2) for all $\mathbf{T}$*-models* $\mathfrak{A}$*,* $\mathfrak{B}$*, every common substructure* $\mathfrak{S}$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}$*, and for all* n*-tuples* $(a_1, \ldots, a_n) \in S^n$*,*

$$(\mathfrak{A}, S) \models \varphi(a_1, \ldots, a_n) \implies (\mathfrak{B}, S) \models \varphi(a_1, \ldots, a_n) \, .$$

Proof: Assume 1) and the hypothesis of 2). If $\psi(x_1, \ldots, x_n)$ is a quantifier free $\mathcal{L}$-formula, then $\psi(a_1, \ldots, a_n)$ is a quantifier free $\mathcal{L}(S)$-sentence. Hence by Corollary 13.4, $(\mathfrak{A}, S) \models \psi(a_1, \ldots, a_n) \Rightarrow (\mathfrak{B}, S) \models \psi(a_1, \ldots, a_n)$. So if $\mathbf{T} \models \forall(\varphi \leftrightarrow \psi)$, then $(\mathfrak{A}, S) \models \varphi(a_1, \ldots, a_n) \leftrightarrow \psi(a_1, \ldots, a_n)$ and $(\mathfrak{B}, S) \models \varphi(a_1, \ldots, a_n) \leftrightarrow \psi(a_1, \ldots, a_n)$ and thus, $(\mathfrak{A}, S) \models \varphi(a_1, \ldots, a_n) \Rightarrow (\mathfrak{B}, S) \models \varphi(a_1, \ldots, a_n)$.

Now assume 2). By viewing $x_1, \ldots, x_n$ as new constant symbols, we can assume that φ is the $\mathcal{L}(x_1, \ldots, x_n)$-sentence $\varphi(x_1, \ldots, x_n)$. Let $\mathbf{S}$ be the set of quantifier free $\mathcal{L}(x_1, \ldots, x_n)$-sentences, and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}(x_1, \ldots, x_n)$-structures. Then by Corollary 13.4, $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic $\mathcal{L}(x_1, \ldots, x_n)$-substructures. Identifying them (cf. Lemma 13.6), we can assume that $\mathfrak{A}$ and $\mathfrak{B}$ have a common $\mathcal{L}(x_1, \ldots, x_n)$-substructure $\mathfrak{S}$. Hence by 2), $(\mathfrak{A}, S) \models \varphi(x_1, \ldots, x_n) \Rightarrow (\mathfrak{B}, S) \models \varphi(x_1, \ldots, x_n)$. Now we can apply the implication 2)⇒1) of Lemma 13.20 to deduce that $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to some $\psi(x_1, \ldots, x_n) \in \mathbf{S}^{\vee\wedge} = \mathbf{S}$, or $\mathbf{T} \models \varphi$ or $\mathbf{T} \models \neg\varphi$. □

An $\mathcal{L}$-theory $\mathbf{T}$ is called **substructure complete** if $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ for every two models $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathbf{T}$ and every common substructure $\mathfrak{S}$ of $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 13.22 *A theory* $\mathbf{T}$ *admits quantifier elimination if and only if it is substructure complete.*

Proof: Assume that $\mathbf{T}$ admits quantifier elimination, and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathbf{T}$ with common substructure $\mathfrak{S}$. We have to show that $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$ or equivalently, that $\mathfrak{A} \Rightarrow_{\mathrm{Th}(\mathfrak{A},S)} \mathfrak{B}$. So let φ be an $\mathcal{L}(S)$-sentence. We can write it as $\varphi(a_1, \ldots, a_n)$ with $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula. Since $\mathbf{T}$ admits quantifier elimination, the implication 1)⇒2) of Lemma 13.21 shows that $(\mathfrak{A}, S) \models \varphi(a_1, \ldots, a_n) \Rightarrow (\mathfrak{B}, S) \models \varphi(a_1, \ldots, a_n)$, as required.

For the converse, assume that $\mathbf{T}$ is substructure complete. Then condition 2) of Lemma 13.21 holds for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ because $\varphi(a_1, \ldots, a_n)$ is an $\mathcal{L}(S)$-sentence. So condition 1) of Lemma 13.21 holds for every $\mathcal{L}$-formula, i.e., $\mathbf{T}$ admits quantifier elimination. □

The proof of the following lemma works in the same way as that of Lemma 13.21, now taking $\mathbf{S}$ to be the set of all atomic $\mathcal{L}$-formulas. The closure $\mathbf{S}^{\vee\wedge}$ of $\mathbf{S}$ under disjunctions and conjunctions is the set of **positive quantifier free $\mathcal{L}$-formulas**.

Lemma 13.23 *Let* $\mathbf{T}$ *be an* $\mathcal{L}$*-theory. Then the following are equivalent for every* $\mathcal{L}$*-formula* $\varphi(x_1, \ldots, x_n)$*:*

1) $\varphi(x_1, \ldots, x_n)$ *is* $\mathbf{T}$*-equivalent to a positive quantifier free* $\mathcal{L}$*-formula* $\psi(x_1, \ldots, x_n)$*, or* $\mathbf{T} \models \varphi$*, or* $\mathbf{T} \models \neg\varphi$*,*
2) for all $\mathbf{T}$*-models* $\mathfrak{A}$*,* $\mathfrak{B}$*, every substructure* $\mathfrak{S}$ *of* $\mathfrak{A}$*, every homomorphism of* $\mathfrak{S}$ *into* $\mathfrak{B}$ *and all* n*-tuples* $(a_1, \ldots, a_n) \in S^n$*,*

$$\mathfrak{A} \models \varphi(a_1, \ldots, a_n) \implies \mathfrak{B} \models \varphi(\sigma a_1, \ldots, \sigma a_n) \, .$$

This lemma has nice applications to algebraic and semialgebraic geometry, as L. v. d. Dries has worked out in the paper [VDD5].

For the conclusion of this section, we apply Theorem 13.22 and the Upward Löwenheim–Skolem Theorem (Theorem 13.19) in order to give two examples of theories which admit quantifier elimination. We have already met abelian groups and the $\mathcal{L}_{\mathrm{G}}$-sentences by which they are axiomatized. An additively written group G is **nontrivial** if $G \neq \{0\}$, that is, if it satisfies

(NTG) $\quad \exists x : \, x \neq 0 \, .$

The property of an abelian group to be **torsion free** can be expressed by the axiom scheme

(TF) $\quad \forall x : \, nx = 0 \, \rightarrow \, x = 0 \qquad (n \in \mathbb{N}) \, .$

Further, the property of being divisible is expressed by

(DIV) $\quad \forall x \, \exists y : \, x = ny \qquad (n \in \mathbb{N}) \, .$

The **theory of nontrivial torsion free divisible abelian groups** is the $\mathcal{L}_{\mathrm{G}}$-theory which consists of the axioms for abelian groups together with (NTG) and the axiom schemes (TF) and (DIV).

For fields, the analogue to the divisibility of abelian groups is the property of being algebraically closed. This is expressed by the following scheme of universal existential $\mathcal{L}_{\mathrm{R}}$-sentences:

(AC) $\quad \forall x_1 \ldots \forall x_n \, \exists y : \, y^n + x_1 y^{n-1} + \ldots x_{n-1} y + x_n = 0 \qquad (n \in \mathbb{N}) \, .$

Here, the variables x_i stand for the coefficients of all monic polynomials of degree n. The **theory of algebraically closed fields** is the $\mathcal{L}_{\mathrm{F}}$-theory which consists of the axioms for fields together with the axiom scheme (AC).

Theorem 13.24 *The theory of nontrivial torsion free divisible abelian groups and the theory of algebraically closed fields admit quantifier elimination.*

Proof: Torsion free abelian groups G and fields K have the following properties in common:

1) the divisible hull of G (resp. the algebraic closure of K) is unique up to isomorphism over G (resp. over K),

2) If $G'|G$ (resp. $K'|K$) is an extension such that $|G'| > |G| \geq \aleph_0$ (resp. $|K'| > |K| \geq \aleph_0$), then every transcendence basis of the extension has cardinality $|G'|$ (resp. $|K'|$),

3) bijections between transcendence bases yield isomorphisms of the divisible hulls (resp. of the algebraic closures).

We give the proof of our theorem for the case of groups. The analogous proof for the case of fields, following the above comparison, is left to the reader.

Let G_1 and G_2 be nontrivial torsion free divisible abelian groups, and let H be a common subgroup of both. Then the divisible hull $\tilde{H}$ of H is contained in both G_1 and G_2 (up to isomorphism, cf. Lemma 13.6). So we can assume that $\tilde{H}$ is a substructure of both G_1 and G_2. Now let $\kappa > \max\{|G_1|, |G_2|, ||\mathcal{L}_{\mathrm{G}}||\} = \max\{|G_1|, |G_2|\}$. Note that G_1 and G_2 are infinite since they are nontrivial and torsion free. Hence by Theorem 13.19 we can choose elementary extensions G_1^* and G_2^* of G_1 and G_2 respectively, both of cardinality κ. We choose maximal sets $\mathcal{T}_i \subset G_i^*$ of elements which are rationally independent over $\tilde{H}$, $i = 1, 2$ (this is what we call "transcendence bases" of $G_i^*|\tilde{H}$). Since $|G_i^*| = \kappa > |G_i| \geq |\tilde{H}|$, we have that $|\mathcal{T}_1| = |G_1^*| = |G_2^*| = |\mathcal{T}_2|$. That is, there is a bijection between $\mathcal{T}_1$ and $\mathcal{T}_2$. This bijection induces an isomorphism of G_1^* and G_2^* over $\tilde{H}$, because G_1^* and G_2^* are torsion free divisible abelian groups (being elementary extensions of G_1 and G_2 respectively). It follows by Theorem 13.1 that $G_1^* \equiv_{\tilde{H}} G_2^*$. Since $G_i \prec G_i^*$ and $\tilde{H}$ is a substructure of G_i, $i = 1, 2$, it follows that $G_1 \equiv_{\tilde{H}} G_1^* \equiv_{\tilde{H}} G_2^* \equiv_{\tilde{H}} G_2$. Since $H \subset \tilde{H}$, it follows that $G_1 \equiv_H G_2$. This proves that the theory of nontrivial torsion free divisible abelian groups is substructure complete. Our assertion now follows from Theorem 13.22. □

13.5 Types and saturation

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and S a subset of A. Further, let x be a constant name that does not appear in $\mathcal{L}$. We will write $\mathcal{L}(S, x)$ in the place of $\mathcal{L}(S \cup \{x\})$. Let $\mathbf{t}$ be a set of $\mathcal{L}(S, x)$-sentences. Then $\mathbf{t}$ is called a **type over $\mathfrak{A}$ in** S if every finite subset of

$$\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, A)$$

has a model. In the literature, one often finds the additional condition that $\mathbf{t}$ be maximal with this property. In this case, every $\mathcal{L}(S, x)$-sentence or its negation will belong to $\mathbf{t}$. Such types may be called **complete types**. But in model theoretic algebra, it is sometimes more convenient to use the **partial types** that we have defined. By use of Zorn's Lemma, every partial type can be extended to a complete type. An important example for a complete type is the **type of an element** in a given structure: If $a \in A$, then its type (in $\mathfrak{A}$) is defined to be

$$\mathbf{t}_{\mathfrak{A}}(a) := \{\varphi(x) \mid \varphi(x) \text{ is an } \mathcal{L}(x)\text{-sentence such that } (\mathfrak{A}, a) \models \varphi(a)\} .$$

Interpreting x by a, we can turn $\mathfrak{A}$ into an $\mathcal{L}(x)$-structure. Then $\mathbf{t}_{\mathfrak{A}}(a) = \mathrm{Th}(\mathfrak{A}, x)$. In this sense, the type of a in $\mathfrak{A}$ is the theory $\mathrm{Th}(\mathfrak{A}, a)$.

Let $\mathfrak{A}^*$ be an elementary extension of $\mathfrak{A}$. Then $\mathrm{Th}(\mathfrak{A}, A) = \mathrm{Th}(\mathfrak{A}^*, A)$ and thus also $\mathrm{Th}(\mathfrak{A}, S) = \mathrm{Th}(\mathfrak{A}^*, S)$ for $S \subset A$. Consequently, a set $\mathbf{t}$ of $\mathcal{L}(S, x)$-sentences is a type over $\mathfrak{A}$ if and only if it is a type over $\mathfrak{A}^*$.

A structure $\mathfrak{B}$ is said to **realize the type $\mathbf{t}$** if $\mathfrak{B}$ is an $\mathcal{L}(S)$-structure and can be expanded to an $\mathcal{L}(S, x)$-structure $(\mathfrak{B}, x)$ which is a model of $\mathbf{t}$. This means that there is some $b \in B$ (namely the interpretation of x) such that $(\mathfrak{B}, b) \models \varphi(b)$ for *all* $\varphi(x) \in \mathbf{t}$. In this case, we say that b **realizes $\mathbf{t}$ in $\mathfrak{B}$**. Note that if $\mathbf{t}$ is realized in $\mathfrak{B}$, then it is also realized in every elementary extension $\mathfrak{B}^*$ of $\mathfrak{B}$. Indeed, if $b \in B$ such that $(\mathfrak{B}, b) \models \varphi(b)$, then also $(\mathfrak{B}^*, b) \models \varphi(b)$ because $(\mathfrak{B}^*, B)$ is a model of $\mathrm{Th}(\mathfrak{B}, B)$. We will not be too rigid

in handling our definition of realization. If $\mathfrak{B}$ is an $\mathcal{L}(S')$-structure with $S \subset S'$, then we will view $\mathfrak{B}$ also as an $\mathcal{L}(S)$-structure, so that we can still say that "$\mathfrak{B}$ realizes $\mathbf{t}$". For example, if $\mathfrak{B}$ is an extension of $\mathfrak{A}$ and thus B contains S, then we will also say "$(\mathfrak{B}, A)$ realizes $\mathbf{t}$" or "$(\mathfrak{B}, B)$ realizes $\mathbf{t}$" if $(\mathfrak{B}, S)$ realizes $\mathbf{t}$.

If the type $\mathbf{t}_{\mathfrak{A}}(a)$ of a in $\mathfrak{A}$ is realized in $\mathfrak{B}$ by an element $b \in B$, then it follows that $(\mathfrak{A}, a) \equiv (\mathfrak{B}, b)$ (which means that $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ are equivalent in the language $\mathcal{L}(\{c\})$ if a and b are the interpretations of c in $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ respectively). Indeed, if φ is an $\mathcal{L}(a)$-sentence, then we can write it as $\varphi(a)$ with $\varphi(x)$ an $\mathcal{L}$-formula. If $(\mathfrak{A}, a) \models \varphi(a)$, then $\varphi(x) \in \mathbf{t}_{\mathfrak{A}}(a)$ and if $\mathbf{t}_{\mathfrak{A}}(a)$ is realized by b in $\mathfrak{B}$, then $(\mathfrak{B}, b) \models \varphi(b)$.

Every subset of $\mathbf{t}$ is again a type, and we will call it a **subtype of $\mathbf{t}$**. We will say that $\mathbf{t}$ is **finitely realizable in $\mathfrak{B}$** if $\mathfrak{B}$ realizes every finite subtype of $\mathbf{t}$.

Lemma 13.25 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $S \subset A$. For a set $\mathbf{t}$ of $\mathcal{L}(S, x)$-sentences, the following assertions are equivalent:*

1) $\mathbf{t}$ is a type over $\mathfrak{A}$,

2) $\mathbf{t}$ is finitely realizable in $(\mathfrak{A}, S)$,

3) there is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $(\mathfrak{B}, S)$ realizes $\mathbf{t}$.

Proof: 1)⇒3): If $\mathbf{t}$ is a type, then by definition, every finite subset of $\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, S)$ has a model. Hence by the Compactness Theorem 13.13, there exists a model $\mathfrak{B}'$ of $\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, A)$. We take $\mathfrak{B}$ to be the $\mathcal{L}$-reduct of $\mathfrak{B}'$. Since $\mathfrak{B}'$ is a model of $\mathrm{Th}(\mathfrak{A}, A)$, Corollary 13.5 shows that $\mathfrak{A}$ admits an elementary embedding in $\mathfrak{B}$. We replace $\mathfrak{B}'$ by an isomorphic image such that $\mathfrak{A}$ becomes an elementary substructure of $\mathfrak{B}$ (cf. Lemma 13.6). Finally, $(\mathfrak{B}, S)$ realizes $\mathbf{t}$ since its expansion $\mathfrak{B}'$ is a model of $\mathbf{t}$.

3)⇒2): Let $\mathbf{t}$ be a type and $\{\varphi_1(x), \ldots, \varphi_n(x)\} \subset \mathbf{t}$ a finite subtype. Let $\mathfrak{B}$ be the elementary extension of $\mathfrak{A}$ such that $(\mathfrak{B}, S)$ realizes $\mathbf{t}$. Then in particular, $(\mathfrak{B}, S) \models \exists x \varphi_1(x) \wedge \ldots \wedge \varphi_n(x)$. By $\mathfrak{A} \prec \mathfrak{B}$ it follows that $(\mathfrak{A}, S) \models \exists x \varphi_1(x) \wedge \ldots \wedge \varphi_n(x)$. But this just means that $(\mathfrak{A}, S)$ realizes $\{\varphi_1(x), \ldots, \varphi_n(x)\}$.

2)⇒1): If $\mathbf{t}$ is finitely realizable in $(\mathfrak{A}, S)$, then every finite subset of $\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, A)$ has a model (which is an expansion of $(\mathfrak{A}, A)$). Hence by definition, $\mathbf{t}$ is a type. □

Let κ be any cardinal. An $\mathcal{L}$-structure $\mathfrak{A}$ is called **κ-saturated** if for every subset $S \subset A$ of cardinality $|S| < \kappa$, $(\mathfrak{A}, S)$ realizes every type over $\mathfrak{A}$ in S. If $\mathfrak{A}$ is κ-saturated then trivially, it is also λ-saturated for every cardinal $\lambda \leq \kappa$. Further, we note:

Lemma 13.26 *If $\mathfrak{A}$ is κ-saturated and if $S_0 \subset A$ is of cardinality $|S_0| < \kappa$, then also $(\mathfrak{A}, S_0)$ is κ-saturated.*

Indeed, if $S \subset A$ is of cardinality $|S| < \kappa$, then $|S_0 \cup S| < \kappa$, and a type over $(\mathfrak{A}, S_0)$ in S can also be viewed as a type over $\mathfrak{A}$ in $S_0 \cup S$.

Our goal now is to construct κ^+-saturated structures of cardinality $\leq 2^\kappa$, using elementary chains as a tool.

Lemma 13.27 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\kappa \geq ||\mathcal{L}||$ a cardinal such that $|A| \leq 2^\kappa$. Then there exists an elementary extension $\mathfrak{A}^*$ of cardinality $|A^*| \leq 2^\kappa$ such that for all $S \subset A$ with $|S| \leq \kappa$, $(\mathfrak{A}, S)$ realizes every type over $\mathfrak{A}$ in S.*

Proof: Let $S \subset A$ of cardinality $|S| \leq \kappa$. Every type over $\mathfrak{A}$ in S is a subset of the set of all $\mathcal{L}(S,x)$-sentences. Because of our assumption $\kappa \geq ||\mathcal{L}||$, the cardinality of the set of all $\mathcal{L}(S,x)$-sentences is $||\mathcal{L}(S,x)|| = \max\{||\mathcal{L}||, |S \cup \{x\}|\} \leq \max\{||\mathcal{L}||, \kappa\} = \kappa$. Consequently, there are at most 2^κ many distinct types over $\mathfrak{A}$ in S. Further, there are at most 2^κ many distinct subsets of A of cardinality at most κ. Hence, there are at most $2^\kappa \cdot 2^\kappa = 2^\kappa$ many distinct types over $\mathfrak{A}$ in some $S \subset A$ of cardinality $|S| \leq \kappa$. So we can put these types in a sequence $(\mathbf{t}_\nu)_{\nu<\lambda}$ with λ a cardinal $\leq 2^\kappa$.

We construct $\mathfrak{A}^*$ as the union over an elementary chain. We set $\mathfrak{A}_0 = \mathfrak{A}$. Assume that $\mu < \lambda$ and that we have already constructed $\mathfrak{A}_\nu$ of cardinality $\leq 2^\kappa$ for all $\nu < \mu$ such that $\mathfrak{A}_\nu$ realizes $\mathbf{t}_\alpha$ for every $\alpha < \nu$. If μ is a limit ordinal, then we set $\mathfrak{A}_\mu := \bigcup_{\nu<\mu} \mathfrak{A}_\nu$. Then by Lemma 13.8, $\mathfrak{A}_\mu$ is an elementary extension of every $\mathfrak{A}_\nu$ with $\nu < \mu$. Consequently, every $\mathbf{t}_\nu$ is a type over $\mathfrak{A}_\mu$, and $(\mathfrak{A}_\mu, A)$ realizes $\mathbf{t}_\alpha$ for every $\alpha < \mu$ (since there is already an earlier member of the chain which does it). For the cardinality of $\mathfrak{A}_\mu$ we find that

$$|A_\mu| = |\bigcup_{\nu<\mu} A_\nu| \leq \max\{\mu, 2^\kappa\} = 2^\kappa \,. \tag{13.4}$$

Now assume that $\mu = \nu + 1$ is a successor ordinal. Then we choose $\mathfrak{A}_{\nu+1}$ to be an elementary extension of $\mathfrak{A}_\nu$ of cardinality $|A_{\nu+1}| \leq 2^\kappa$ such that $(\mathfrak{A}_\nu, A)$ realizes $\mathbf{t}_\nu$. Its existence follows from Lemma 13.25. As before, one shows that every $\mathbf{t}_\nu$ is a type over $\mathfrak{A}_\mu$, and that $(\mathfrak{A}_\mu, A)$ realizes $\mathbf{t}_\alpha$ for every $\alpha < \nu$.

The union $\mathfrak{A}^*$ over the elementary chain $(\mathfrak{A}_\nu)_{\nu<2^\kappa}$ is an elementary extension of $\mathfrak{A}$ by virtue of Lemma 13.8. For its cardinality, we employ (13.4) with $\mu = \lambda$ to deduce that it is $\leq 2^\kappa$. As before, it follows that $(\mathfrak{A}^*, A)$ realizes every $\mathbf{t}_\nu$ for $\nu < \lambda$. □

Theorem 13.28 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\kappa \geq ||\mathcal{L}||$ a cardinal such that $|A| \leq 2^\kappa$. Then there exists a κ^+-saturated elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ of cardinality 2^κ.*

Proof: We construct an elementary chain $(\mathfrak{A}_\nu)_{\nu<\kappa^+}$ of $\mathcal{L}$-structures such that for all $\nu < \kappa^+$,

1) $|A_\nu| \leq 2^\kappa$,
2) for all $S \subset A_\nu$, every type over $\mathfrak{A}_\nu$ in S is realized in $(\mathfrak{A}_{\nu+1}, S)$.

We choose $\mathfrak{A}_0 = \mathfrak{A}$. Assume that $\mu < \kappa^+$ and that we have already constructed $\mathfrak{A}_\nu$ for all $\nu < \mu$. If μ is a limit ordinal, then we set $\mathfrak{A}_\mu := \bigcup_{\nu<\mu} \mathfrak{A}_\nu$. Then by Lemma 13.8, $\mathfrak{A}_\mu$ is an elementary extension of every $\mathfrak{A}_\nu$ with $\nu < \mu$. Now assume that $\mu = \nu + 1$ is a successor ordinal. Then we choose $\mathfrak{A}_{\nu+1}$ to be the elementary extension of $\mathfrak{A}_\nu$ whose existence is assured by the foregoing lemma.

The union $\mathfrak{A}^*$ over the elementary chain $(\mathfrak{A}_\nu)_{\nu<\kappa^+}$ is an elementary extension of $\mathfrak{A}$ by virtue of Lemma 13.8. We use (13.4) with $\mu = \kappa^+$ to conclude that the cardinality of $\mathfrak{A}^*$ is $\leq 2^\kappa$.

We show that $\mathfrak{A}^*$ is κ^+-saturated. First we remark that every successor cardinal κ^+ is regular, that is, it is not the supremum of any set $\mathcal{S}$ of ordinals $< \kappa^+$ such that $|\mathcal{S}| < \kappa^+$. (The reader will find the details in every good book on set theory.) Now assume that $S \subset A^*$ is of cardinality $|S| < \kappa$. Let $\mathcal{S}$ be the set of those $\nu < \kappa^+$ such that A_ν contains an element of S which is not contained in any A_μ for $\mu < \nu$. It follows that $|\mathcal{S}| \leq |S| < \kappa^+$. Hence, κ^+ is not the supremum of $\mathcal{S}$, that is, there is some $\nu < \kappa^+$ such that $S \subset A_\nu$. But then in view of $\mathfrak{A}_\nu \prec \mathfrak{A}^*$, every type $\mathbf{t}$ over $\mathfrak{A}^*$ in S is already a type over $\mathfrak{A}_\nu$ in S. So by

2), it is realized in $(\mathfrak{A}_{\nu+1}, S)$. Because of $\mathfrak{A}_{\nu+1} \prec \mathfrak{A}^*$, we find that $(\mathfrak{A}^*, S)$ realizes $\mathbf{t}$. This proves that $\mathfrak{A}^*$ is κ^+-saturated. □

We leave it to the reader to show that every κ^+-saturated structure cardinality $\geq \kappa^+$. Hence if we assume the **Generalized Continuum Hypothesis**, which says that $\kappa^+ = 2^\kappa$ for every infinite cardinal κ, then every κ^+-saturated structure of cardinality $\leq 2^\kappa$ will automatically be of cardinality $\kappa^+ = 2^\kappa$.

We will now discuss some properties of κ-saturated structures which will lead us to a general embedding lemma. If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures and $a_\nu \in A$, $b_\nu \in B$, then by writing $(\mathfrak{A}, (a_\nu)_{\nu<\kappa}) \equiv (\mathfrak{B}, (b_\nu)_{\nu<\kappa})$ we indicate equivalence in the language $\mathcal{L}(c_\nu \mid \nu < \kappa)$ such that c_ν is interpreted by a_ν in $(\mathfrak{A}, (a_\nu)_{\nu<\kappa})$ and by b_ν in $(\mathfrak{B}, (b_\nu)_{\nu<\kappa})$ for every ν.

Lemma 13.29 *Let $\mathfrak{A}$ be κ-saturated, $\mathfrak{A} \equiv \mathfrak{B}$ and $b_\nu \in B$, $\nu < \kappa$. Then there are elements $a_\nu \in A$, $\nu < \kappa$, such that*

$$(\mathfrak{A}, (a_\nu)_{\nu<\kappa}) \equiv (\mathfrak{B}, (b_\nu)_{\nu<\kappa}) \ .$$

Proof: The elements $a_\nu \in A$ are chosen by induction on $\nu < \kappa$. Suppose that $\mu < \kappa$ and that we have already chosen a_ν for $\nu < \mu$ such that $(\mathfrak{A}, (a_\nu)_{\nu<\mu}) \equiv (\mathfrak{B}, (b_\nu)_{\nu<\mu})$. Then let $\mathbf{t}$ be the type of b_μ in $(\mathfrak{B}, (b_\nu)_{\nu<\mu})$. In view of $\mathrm{Th}(\mathfrak{A}, (a_\nu)_{\nu<\mu}) = \mathrm{Th}(\mathfrak{B}, (b_\nu)_{\nu<\mu})$, we find that $\mathbf{t}$ is a type over $\mathfrak{A}$ in $\{a_\nu \mid \nu < \mu\}$. Since $\mathfrak{A}$ is κ-saturated and $\mu < \kappa$, also $(\mathfrak{A}, (a_\nu)_{\nu<\mu})$ is κ-saturated. Thus, we find that $\mathbf{t}$ is realized in $(\mathfrak{A}, (a_\nu)_{\nu<\mu})$ by some element $a_\mu \in A$. It follows that $(\mathfrak{A}, (a_\nu)_{\nu\leq\mu}) = (\mathfrak{A}, (a_\nu)_{\nu<\mu}, a_\mu) \equiv (\mathfrak{B}, (b_\nu)_{\nu<\mu}, a_\mu) = (\mathfrak{B}, (b_\nu)_{\nu\leq\mu})$. □

The concept of types can be generalized to several variables. Let α be an arbitrary cardinal and x_ν, $\nu < \alpha$, constant names that do not appear in $\mathcal{L}$. Let $\mathbf{t}$ be a set of $\mathcal{L}(S \cup \{x_\nu \mid \nu < \alpha\})$-sentences. Then $\mathbf{t}$ is called an **α-type over $\mathfrak{A}$ in** S if every finite subset of $\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, A)$ has a model. A 1-type is just a type as defined above. The notion of realization is generalized to α-types in a straightforward way.

If $\mathbf{t}$ is an α-type over $\mathfrak{A}$ in S, then by the Compactness Theorem 13.13 there is an $\mathcal{L}(S \cup \{x_\nu \mid \nu < \alpha\})$-structure $\mathfrak{B}'$ which is a model of $\mathbf{t} \cup \mathrm{Th}(\mathfrak{A}, A)$. Its $\mathcal{L}(S)$-reduct $\mathfrak{B}$ is elementarily equivalent to $(\mathfrak{A}, S)$ since it is a model of $\mathrm{Th}(\mathfrak{A}, S)$. Let b_ν the interpretation of x_ν in $\mathfrak{B}'$, for every $\nu < \alpha$. Now assume that $\mathfrak{A}$ is κ-saturated and that $|S| < \kappa$. Then also $(\mathfrak{A}, S)$ is κ-saturated. Hence by the foregoing lemma, there are elements $a_\nu \in A$, $\nu < \alpha$, such that $(\mathfrak{A}, S, (a_\nu)_{\nu<\kappa}) \equiv (\mathfrak{B}, (a_\nu)_{\nu<\kappa})$. But this means that $(\mathfrak{A}, S)$ realizes the type $\mathbf{t}$. We have proved:

Corollary 13.30 *Let $\mathfrak{A}$ be κ-saturated, $S \subset A$ of cardinality $|S| < \kappa$ and α a cardinal $\leq \kappa$. Then $(\mathfrak{A}, S)$ realizes every α-type over $\mathfrak{A}$ in S.*

Now we are able to give the following important criteria for the existence of embeddings in sufficiently saturated structures:

Theorem 13.31 (General Embedding Theorem)
Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. Further, let $\kappa \geq |B|$ an infinite cardinal and assume $\mathfrak{A}$ to be κ-saturated.

a) If $\mathfrak{B} \Rightarrow_\exists \mathfrak{A}$, then there is an embedding of $\mathfrak{B}$ in $\mathfrak{A}$.

b) If $\mathfrak{B} \equiv \mathfrak{A}$, then there is an elementary embedding of $\mathfrak{B}$ in $\mathfrak{A}$.

Proof: We set $\alpha := |B|$ and write $B = \{b_\nu \mid \nu < \alpha\}$.
a): We set

$$\begin{aligned}\mathbf{t} \;=\; &\{\varphi(x_{\nu_1}, \ldots, x_{\nu_n}) \mid \varphi(x_{\nu_1}, \ldots, x_{\nu_n}) \text{ is a quantifier free} \\ &\mathcal{L}(x_\nu \mid \nu < |B|)\text{-sentence such that } (\mathfrak{B}, B) \models \varphi(b_{\nu_1}, \ldots, b_{\nu_n})\} \,.\end{aligned}$$

Then $\mathbf{t}$ is an α-type over $\mathfrak{B}$ (in $\emptyset$) since it is realized in $\mathfrak{B}$ if we interprete every x_ν by b_ν. We wish to show that it is also an α-type over $\mathfrak{A}$. So let $\mathbf{t}_0$ be a finite subtype of $\mathbf{t}$, and take $\varphi(x_{\nu_1}, \ldots, x_{\nu_n})$ to be the conjunction of all sentences in $\mathbf{t}_0$ (note that it is quantifier free). Using again the ambivalence between constant and variable names, we form the existential $\mathcal{L}$-sentence $\exists x_{\nu_1} \ldots \exists x_{\nu_n} \varphi(x_{\nu_1}, \ldots, x_{\nu_n})$. It holds in $\mathfrak{B}$ by our definition of $\mathbf{t}$. In view of the hypothesis $\mathfrak{B} \Rightarrow_\exists \mathfrak{A}$, we find that it also holds in $\mathfrak{A}$. That is, $\mathbf{t}_0$ is realized in $\mathfrak{A}$, as desired. Thus, the foregoing corollary yields that $\mathbf{t}$ is realized in $\mathfrak{A}$ by elements $a_\nu \in A$, $\nu < \alpha$. But this just means that $(\mathfrak{A}, \{a_\nu \mid \nu < \alpha\})$ is a model of $\mathrm{Diag}(\mathfrak{B})$. (Observe that $\mathbf{t}$ is indeed the diagram of $\mathfrak{B}$, with the constant symbol for b_ν replaced by the variable symbol x_ν.) Now it follows from Corollary 13.5 that $\mathfrak{B}$ admits an embedding in $\mathfrak{A}$.

b): We set

$$\begin{aligned}\mathbf{t} \;=\; &\{\varphi(x_{\nu_1}, \ldots, x_{\nu_n}) \mid \varphi(x_{\nu_1}, \ldots, x_{\nu_n}) \text{ is an} \\ &\mathcal{L}(x_\nu \mid \nu < |B|)\text{-sentence such that } (\mathfrak{B}, B) \models \varphi(b_{\nu_1}, \ldots, b_{\nu_n})\} \,.\end{aligned}$$

As in the proof of a), $\mathbf{t}$ is an α-type over $\mathfrak{B}$ (in $\emptyset$). To show that $\mathbf{t}$ is also an α-type over $\mathfrak{A}$, we proceed as in the proof of part a). Here, the formula $\varphi(x_{\nu_1}, \ldots, x_{\nu_n})$ is not necessarily quantifier free, so that $\exists x_{\nu_1} \ldots \exists x_{\nu_n} \varphi(x_{\nu_1}, \ldots, x_{\nu_n})$ is not necessarily a universal sentence. But now, we have by assumption that $\mathfrak{B} \equiv \mathfrak{A}$. Hence again, we find that $\exists x_{\nu_1} \ldots \exists x_{\nu_n} \varphi(x_{\nu_1}, \ldots, x_{\nu_n})$ holds in $\mathfrak{A}$. So indeed, $\mathbf{t}$ is also an α-type over $\mathfrak{A}$. By the foregoing corollary, it is realized in $\mathfrak{A}$ by elements $a_\nu \in A$, $\nu < \alpha$. This means that $(\mathfrak{A}, \{a_\nu \mid \nu < \alpha\})$ is a model of $\mathrm{Th}(\mathfrak{B}, B)$. (Observe that $\mathbf{t}$ is indeed the theory of $(\mathfrak{B}, B)$, up to the replacement of the constant symbols by variable symbols.) Now Corollary 13.5 shows that $\mathfrak{B}$ admits an elementary embedding in $\mathfrak{A}$. □

The converses of assertions a) and b) are also true, without the assumption on the saturation of $\mathfrak{A}$. If $\mathfrak{B}$ can be embedded in $\mathfrak{A}$ by an embedding ι, then $\iota\mathfrak{B}$ is a substructure of $\mathfrak{A}$, which yields that $\mathfrak{B} \Rightarrow_\exists \iota\mathfrak{A}$. By Theorem 13.1, we obtain that $\iota\mathfrak{A} \Rightarrow_\exists \mathfrak{A}$. It follows that $\mathfrak{B} \Rightarrow_\exists \mathfrak{A}$. Similarly, one proceeds for the converse of b), replacing $\Rightarrow_\exists$ by $\Rightarrow_{\mathrm{Th}(\mathfrak{B},A)}$.

There is also a compactness principle for embeddings. Before stating it, we need a definition. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and $S \subset A$. Then we denote by $\langle S \rangle_\mathfrak{A}$ the smallest $\mathcal{L}$-substructure of $\mathfrak{A}$ whose universe contains S. For example, $\langle \emptyset \rangle_\mathfrak{A}$ is just the constant substructure of $\mathfrak{A}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is called **finitely generated** if it is equal to $\langle S \rangle_\mathfrak{A}$ for a suitable finite set $S \subset A$.

Now let us observe that if an existential $\mathcal{L}$-sentence holds in $\mathfrak{B}$, then it already holds in some finitely generated substructure of $\mathfrak{B}$. Indeed, if $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n)$ (where ψ is quantifier free) holds in $\mathfrak{B}$, then there are $a_1, \ldots, a_n \in B$ such that the quantifier free $\mathcal{L}$-sentence $\psi(a_1, \ldots, a_n)$ holds in $(\mathfrak{B}, \{a_1, \ldots, a_n\})$. Consequently, it also holds in the substructure $(\langle \{a_1, \ldots, a_n\} \rangle_\mathfrak{B}, \{a_1, \ldots, a_n\})$. So $\exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n)$ holds in the finitely generated substructure $\langle \{a_1, \ldots, a_n\} \rangle_\mathfrak{B}$ of $\mathfrak{B}$.

Since $\mathfrak{S} \subset \mathfrak{B}$ implies $\mathfrak{S} \Rightarrow_\exists \mathfrak{B}$, we also have the converse: If an existential $\mathcal{L}$-sentence holds in some finitely generated substructure of $\mathfrak{B}$, then it also holds in $\mathfrak{B}$. We note:

Lemma 13.32 *An existential $\mathcal{L}$-sentence holds in the $\mathcal{L}$-structure $\mathfrak{B}$ if and only if it holds in some finitely generated substructure of $\mathfrak{B}$.*

From this lemma it follows that $\mathfrak{B} \Rightarrow_\exists \mathfrak{A}$ if and only if $\mathfrak{S} \Rightarrow_\exists \mathfrak{A}$ for every finitely generated substructure $\mathfrak{S}$ of $\mathfrak{B}$. On the other hand, we have already seen that $\mathfrak{S} \Rightarrow_\exists \mathfrak{A}$ holds if $\mathfrak{S}$ is embeddable in $\mathfrak{A}$. We conclude:

Lemma 13.33 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. Further, let $\kappa \geq |B|$ an infinite cardinal and assume $\mathfrak{A}$ to be κ-saturated. If every finitely generated substructure of $\mathfrak{B}$ can be embedded in $\mathfrak{A}$, then $\mathfrak{B}$ itself can be embedded in $\mathfrak{A}$.*

Exercise 13.3 *Use the idea of the proof of Corollary 13.15 to show: if $(G, <)$ is a nontrivial ordered abelian group and $(G, <)^*$ is a $|G|^+$-saturated elementary extension of $(G, <)$, then there is some $\alpha \in G^*$ such that $\alpha > G$.*

13.6 Strong limit models

Let κ be an arbitrary cardinal $\geq ||\mathcal{L}||$. We set $\kappa_0 := \kappa$ and by induction, we define $\kappa_{\nu+1} = 2^\kappa$ for all $\nu < \omega$. An $\mathcal{L}$-structure $\mathfrak{A}^*$ will be called a **strong limit model of basis** κ if it is the union over an elementary chain $(\mathfrak{A}_\nu)_{\nu<\omega}$ with the following properties:

1) $\mathfrak{A}_0$ is an arbitrary $\mathcal{L}$-structure of cardinality $\leq \kappa$,

2) $\mathfrak{A}_{\nu+1}$ is a κ_ν^+-saturated elementary extension of $\mathfrak{A}_\nu$ of cardinality $\leq 2^{\kappa_\nu} = \kappa_{\nu+1}$, for every $\nu < \omega$.

We have already remarked that the cardinality of a κ^+-saturated structure must be $\geq \kappa^+$. Hence, $\kappa_\nu < \kappa_\nu^+ \leq |\mathfrak{A}_{\nu+1}| \leq \kappa_{\nu+1}$ for all $\nu < \omega$. This shows that the cardinality of the union $\mathfrak{A}^*$ is equal to the supremum of the κ_ν , $\nu < \omega$. Such a supremum λ is a **strong limit cardinal**, that is, $2^\alpha < \lambda$ for every $\alpha < \lambda$. Indeed, if $\alpha < \lambda$, then there is some $\beta \in \mathcal{S}_\kappa$ such that $\alpha \leq \beta$, which yields that $2^\alpha \leq 2^\beta < 2^{2^\beta} \leq \lambda$. In fact, λ is the least strong limit cardinal above κ. In particular, it follows that there exist arbitrarily large strong limit cardinals. But note that they need not be regular (the ones that we have constructed are not!) and so it does not follow that $\mathfrak{A}^*$ be λ-saturated.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure of cardinality $\leq \kappa$. An elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ will be called a **strong limit extension of $\mathfrak{A}$ of basis** κ if it is of the above form with $\mathfrak{A}_0 = \mathfrak{A}$. To produce a strong limit extension of basis κ of $\mathfrak{A}$, we can start with $\mathfrak{A}_0 = \mathfrak{A}$ and construct the elementary extensions $\mathfrak{A}_\nu$ by induction on $\nu < \omega$, using Theorem 13.28. This shows:

Lemma 13.34 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure. Then for every cardinal $\kappa \geq \max\{|A|, ||\mathcal{L}||\}$ there exists a strong limit extension of $\mathfrak{A}$ of basis κ.*

The concept of strong limit models is derived from that of **special models**. For the theory of special models, we refer the reader to [CHA–KE]. Strong limit models are special models, but the converse is not necessarily true. The notion of strong limit models is the most suitable for those situations where we will work with elementary chain constructions (cf. Lemma 14.9, Lemma 14.36 and Section 14.10). With special models (and ultrapowers), they have the following property in common:

Theorem 13.35 (Isomorphism Theorem)
Equivalent strong limit models with equal basis are isomorphic.

Proof: Let $\mathfrak{A}^* = \bigcup_{\nu<\omega} \mathfrak{A}_\nu$ and $\mathfrak{B}^* = \bigcup_{\nu<\omega} \mathfrak{B}_\nu$ be equivalent strong limit models with equal basis κ. We will show by a **back and forth method** that they are isomorphic. Since the chains are elementary, we have that every $\mathfrak{A}_\nu$ is elementarily equivalent to every $\mathfrak{B}_\mu$. Since $\mathfrak{A}_1$ is κ^+-saturated and since $|B_0| \leq \kappa$, it follows from part b) of Theorem 13.31 that $\mathfrak{B}_0$ admits an elementary embedding ι_0 in $\mathfrak{A}_1$. In other words, $(\mathfrak{B}_0, B_0) \equiv (\mathfrak{A}_1, \iota_0 B_0)$. Since $|B_0| \leq \kappa < \kappa_1$, we know that not only $\mathfrak{B}_2$, but also $(\mathfrak{B}_2, B_0)$ is κ_1^+-saturated. Since $\mathfrak{B}_2$ is an elementary extension of $\mathfrak{B}_0$, we have that $(\mathfrak{B}_2, B_0)$ and $(\mathfrak{A}_1, \iota_0 B_0)$ are elementarily equivalent. Hence, it again follows from part b) of Theorem 13.31 that $(\mathfrak{A}_1, \iota_0 B_0)$ admits an elementary embedding ι_1' in $(\mathfrak{B}_2, B_0)$. We obtain that $(\mathfrak{B}_2, \iota_1' A_1) \equiv (\mathfrak{A}_1, A_1) \equiv (\mathfrak{A}_3, A_1)$. As before, we obtain an elementary embedding ι_2 of $(\mathfrak{B}_2, \iota_1' A_1)$ in $(\mathfrak{A}_3, A_1)$. We show that ι_2 extends ι_0. Let $b \in B_0$. Since ι_1' is an embedding of $(\mathfrak{A}_1, \iota_0 B_0)$ in $(\mathfrak{B}_2, B_0)$, it sends $\iota_0 b$ to b. Since ι_2 is an embedding of $(\mathfrak{B}_2, \iota_1' A_1)$ in $(\mathfrak{A}_3, A_1)$, it sends $b = \iota_1' \iota_0 b$ to $\iota_0 b$.

By induction, we construct elementary embeddings

$$\begin{aligned} \iota_\nu : &\quad (\mathfrak{B}_\nu, \iota'_{\nu-1} A_{\nu-1}) \longrightarrow (\mathfrak{A}_{\nu+1}, A_{\nu-1}) \\ \iota'_{\nu+1} : &\quad (\mathfrak{A}_{\nu+1}, \iota_\nu B_\nu) \longrightarrow (\mathfrak{B}_{\nu+2}, B_\nu) \end{aligned}$$

for every even ν, such that $\iota_{\nu+2}$ extends ι_ν and $\iota'_{\nu+3}$ extends $\iota'_{\nu+1}$. Then we take ι to be the set theoretical union over the embeddings ι_ν, $\nu < \omega$. Now ι is an embedding of $\mathfrak{B}^*$ in $\mathfrak{A}^*$. It is onto since every A_ν lies in the image of $\iota_{\nu+1}$, for every odd ν. □

Since strong limit extensions are elementary extensions and since isomorphic structures are elementarily equivalent (Theorem 13.1), the following is an immediate consequence of the foregoing theorem:

Corollary 13.36 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two infinite $\mathcal{L}$-structures and $\kappa \geq \max\{||\mathcal{L}||, |A|, |B|\}$ a cardinal. Then the following assertions are equivalent:*

1) $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent,

2) $\mathfrak{A}$ and $\mathfrak{B}$ admit isomorphic strong limit extensions with basis κ,

3) every two strong limit extensions of $\mathfrak{A}$ and $\mathfrak{B}$ with basis κ are isomorphic.

13.7 Existentially closed substructures

Let $\mathfrak{A}$ be a substructure of the $\mathcal{L}$-structure $\mathfrak{B}$. If $(\mathfrak{B}, A) \Rightarrow_\exists (\mathfrak{A}, A)$ (or equivalently, if $(\mathfrak{A}, A) \Rightarrow_\forall (\mathfrak{B}, A)$), then we will say that $\mathfrak{A}$ is **existentially closed** in $\mathfrak{B}$, and we will write $\mathfrak{A} \prec_\exists \mathfrak{B}$. In other words, $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ if and only if the following holds: If an existential $\mathcal{L}$-sentence with parameters from $\mathfrak{A}$ holds in $\mathfrak{B}$, then it holds in $\mathfrak{A}$. Note that $\prec_\exists$ is transitive. As an illustration of this notion, we prove the following lemma:

Lemma 13.37 *a) Let $G|H$ be an extension of torsion free abelian groups, and consider it as an extension of $\mathcal{L}_{\mathrm{G}}$-structures. If H is existentially closed in G, then G/H is torsion free.*

b) Let $L|K$ be a field extension, and consider it as an extension of $\mathcal{L}_{\mathrm{R}}$- or $\mathcal{L}_{\mathrm{F}}$-structures. If K is existentially closed in L, then $L|K$ is regular.

Proof: a): Suppose that G/H is not torsion free. Since G is supposed to be torsion free, it follows that there is some $g \in G \setminus H$ and some $n \in \mathbb{N}$ such that $0 \neq ng \in H$. Setting

$h := ng$, we obtain that the existential $\mathcal{L}_{\mathrm{G}}(H)$-sentence $\exists x\, nx = h$ holds in G. But it does not hold in H since $g \notin H$ is the unique solution of $nx = h$. We conclude that H is not existentially closed in G.

b): Suppose that K is not relatively algebraically closed in L. Then there exists an irreducible polynomial $f(X) = X^n + c_{n-1}X^{n-1} + \ldots + c_0 \in K[X]$ which has a root in L. The existential $\mathcal{L}_{\mathrm{R}}(K)$-sentence

$$\textbf{(EXR}_f\textbf{)} \qquad \exists x :\ x^n + c_{n-1}x^{n-1} + \ldots + c_0 = 0$$

expresses the existence of a root of f. By our choice of f, it holds in L but not in K. Consequently, K is not existentially closed in L.

Now assume that $L|K$ is not separable. Then we infer from Lemma 6.41 that $K|K^p$ is not linearly disjoint from $L^p|K^p$. Choose K^p-linearly independent elements $a_1, \ldots, a_n \in K$ which are not L^p-linearly independent. Then the existential $\mathcal{L}_{\mathrm{R}}(K)$-sentence

$$\exists x_1\, \exists x_2\, \ldots\, \exists x_n :\ x_1^p a_1 + \ldots + x_n^p a_n = 0$$

holds in L but not in K, showing again that K is not existentially closed in L. In view of Lemma 6.48 we have now proved that $L|K$ is regular if $K \prec_\exists L$. □

The converses do not hold in general. For instance, let $H = \mathbb{Z}$ and G be the subgroup $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ of $\mathbb{R}$. Then G/H is torsion free. But H satisfies the sentence (LPE), which expresses the existence of a least positive element. In the language $\mathcal{L}(\mathbb{Z})$, this sentence can be reformulated as

$$\forall y :\ 0 \leq y \leq 1 \ \rightarrow\ y = 0 \vee y = 1\ ,$$

which is just the negation of the existential $\mathcal{L}(\mathbb{Z})$-sentence

$$\exists y :\ 0 < y < 1\ .$$

This holds in G because G is dense. Since this sentence does not hold in H, it follows that H is not existentially closed in G.

To give an example for fields, we take the regular extension $\mathbb{F}_p(t)|\mathbb{F}_p$ where t is transcendental over $\mathbb{F}_p$. The universal sentence

$$\textbf{(EX}_n\textbf{)} \qquad \exists x_1 \ldots \exists x_n :\ \textstyle\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

expresses the fact that there exist n distinct elements. Here, $\bigwedge$ indicates the conjunction over a number of formulas, in the present case the sentences $x_i \neq x_j$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Observe that a structure is infinite if and only if it is a model of the axiom scheme (EX_n), $n \in \mathbb{N}$. In particular, all sentences (EX_n) hold in $\mathbb{F}_p(t)$. But for $n > p$, (EX_n) does not hold in $\mathbb{F}_p$. This shows that $\mathbb{F}_p$ is not existentially closed in $\mathbb{F}_p(t)$. Observe that the sentences (EX_n) do not use any language symbols, hence the argument can always be applied, showing that a finite substructure $\mathfrak{A}$ of a structure $\mathfrak{B}$ can only be existentially closed in $\mathfrak{B}$ if it is equal to $\mathfrak{B}$.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures and that $\mathfrak{S}$ is a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$. Then $(\mathfrak{S}, S)$ is the constant substructure of both $(\mathfrak{A}, S)$ and $(\mathfrak{B}, S)$. Consequently, since every homomorphism from $(\mathfrak{A}, S)$ in $(\mathfrak{B}, S)$ induces the constant term relation on the constant substructures of $(\mathfrak{A}, S)$ and $(\mathfrak{B}, S)$, it induces the identity on S. A homomorphism from $(\mathfrak{A}, S)$ in $(\mathfrak{B}, S)$ will thus also be called a **homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ over $\mathfrak{S}$**, following the more algebraic terminology. We prove a very useful lemma, which we will employ at several instances.

Lemma 13.38 *Let $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \subset \mathfrak{A}^*$ be extensions of $\mathcal{L}$-structures. If $\mathfrak{B}$ can be embedded over $\mathfrak{A}$ in $\mathfrak{A}^*$ and if $\mathfrak{A} \prec \mathfrak{A}^*$, then $\mathfrak{A} \prec_\exists \mathfrak{B}$. Conversely, if $\mathfrak{A} \prec_\exists \mathfrak{B}$ holds and if $\mathfrak{A}^*$ is $|B|^+$-saturated, then $\mathfrak{B}$ can be embedded over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

Proof: Suppose that σ is an embedding of $\mathfrak{B}$ over $\mathfrak{A}$ in $\mathfrak{A}^*$. Then every $\mathcal{L}(A)$-sentence will hold in $(\mathfrak{B}, A)$ if and only if it holds in $(\sigma\mathfrak{B}, A)$ (because isomorphic structures are equivalent, cf. Theorem 13.1). Every existential $\mathcal{L}(A)$-sentence φ which holds in $(\mathfrak{B}, A)$ will then also hold in $(\mathfrak{A}^*, A)$ since $\mathfrak{A}^*$ is an extension of $\sigma\mathfrak{B}$. If in addition $\mathfrak{A} \prec \mathfrak{A}^*$, then φ will also hold in $(\mathfrak{A}, A)$. This proves our first assertion.

Now suppose that $\mathfrak{A} \prec_\exists \mathfrak{B}$. Then $(\mathfrak{B}, A) \Rightarrow_\exists (\mathfrak{A}, A)$. Since $\mathfrak{A}^*$ is an extension of $\mathfrak{A}$, we have that $(\mathfrak{A}^*, A)$ is an $\mathcal{L}(A)$-structure and that $(\mathfrak{A}, A) \Rightarrow_\exists (\mathfrak{A}^*, A)$. So we find that $(\mathfrak{B}, A) \Rightarrow_\exists (\mathfrak{A}^*, A)$. Now assume in addition that $\mathfrak{A}^*$ is $|B|^+$-saturated. Since $|A| \leq |B| < |B|^+$, we find that also $(\mathfrak{A}^*, A)$ is $|B|^+$-saturated. Hence by part a) of Theorem 13.31, $(\mathfrak{B}, A)$ can be embedded in $(\mathfrak{A}^*, A)$ or in other words, $\mathfrak{B}$ can be embedded in $\mathfrak{A}^*$ over $\mathfrak{A}$. □

If we have an extension $\mathfrak{A} \subset \mathfrak{B}$ of $\mathcal{L}$-structures and want to show that $\mathfrak{A} \prec_\exists \mathfrak{B}$, then by our lemma it suffices to show that $\mathfrak{B}$ can be embedded over $\mathfrak{A}$ in some elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$. This is the motivation for **embedding lemmas**, which will play an important role in the next chapter. When we look for such embeddings, we can use the very helpful principle which is formulated by Lemma 13.33. This lemma tells us that if $\mathfrak{A}^*$ is $|B|^+$-saturated, then to show the existence of an embedding of $(\mathfrak{B}, A)$ in $(\mathfrak{A}^*, A)$, it suffices to show that every finitely generated substructure of $(\mathfrak{B}, A)$ can be embedded in $(\mathfrak{A}^*, A)$. This motivates the following definition. If $\mathfrak{A} \subset \mathfrak{B}$, then a substructure of $\mathfrak{B}$ is called **finitely generated over** $\mathfrak{A}$ if it is equal to $\langle S \cup A \rangle_\mathfrak{B}$ for a suitable finite set $S \subset B$. Every such substructure of $\mathfrak{B}$ contains $\mathfrak{A}$ as a substructure. We have proved:

Lemma 13.39 *Let $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \subset \mathfrak{A}^*$ be extensions of $\mathcal{L}$-structures. Assume that $\mathfrak{A}^*$ is $|B|^+$-saturated. If every substructure of $\mathfrak{B}$ which is finitely generated over $\mathfrak{A}$ can be embedded over $\mathfrak{A}$ in $\mathfrak{A}^*$, then also $\mathfrak{B}$ can be embedded over $\mathfrak{A}$ in $\mathfrak{A}^*$.*

We know already that every existential sentence which holds in $\mathfrak{B}, A)$ will also hold in a finitely generated substructure of $\mathfrak{B}, A)$. But the $\mathcal{L}$-reduct of such a substructure is just a substructure of $\mathfrak{B}$ which is finitely generated extension over $\mathfrak{A}$. Such a substructure will also be called a **finitely generated subextension of** $\mathfrak{A} \subset \mathfrak{B}$. We obtain:

Lemma 13.40 *Let $\mathfrak{A} \subset \mathfrak{B}$ be an extension of $\mathcal{L}$-structures. Then $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ if and only if $\mathfrak{A}$ is existentially closed in every finitely generated subextension of $\mathfrak{A} \subset \mathfrak{B}$.*

By use of Lemma 13.38, one can also prove this lemma from Lemma 13.39, and vice versa.

Given elements $a_1, \ldots, a_n \in A$, then

$$\begin{aligned} \mathbf{t}^{\mathrm{qf}}_\mathfrak{A}(a_1, \ldots, a_n) \;:=\; & \{\varphi(x_1, \ldots, x_n) \mid \varphi(x_1, \ldots, x_n) \text{ is a quantifier free} \\ & \mathcal{L}(x_1, \ldots, x_n)\text{-sentence such that } (\mathfrak{A}, a_1, \ldots, a_n) \models \varphi(a_1, \ldots, a_n)\} \,. \end{aligned}$$

is called the **quantifier free type of** $a_1, \ldots, a_n$ **in** $\mathfrak{A}$. We leave it to the reader to prove the following:

Lemma 13.41 *Let $\mathfrak{A} \subset \mathfrak{B}$ be an extension of $\mathcal{L}$-structures. Then $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$ if and only if for every $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in B$, the quantifier free type $\mathbf{t}^{\mathrm{qf}}_{(\mathfrak{B},A)}(b_1, \ldots, b_n)$ is finitely realizable in $(\mathfrak{A}, A)$.*

If the condition holds for one n, then this means that all existential $\mathcal{L}$-sentences with at most n quantifiers hold in $(\mathfrak{A}, A)$ if they hold in $(\mathfrak{B}, A)$. Then one says that $\mathfrak{A}$ is **n-existentially closed in $\mathfrak{B}$**. The type $\mathbf{t}^{\mathrm{qf}}_{(\mathfrak{B},A)}(b_1, \ldots, b_n)$ will also be called the **quantifier free type of $b_1, \ldots, b_n$ over $\mathfrak{A}$**.

13.8 Model complete theories

An $\mathcal{L}$-theory $\mathbf{T}$ is called **model complete** if for every three models $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{S}$ of $\mathbf{T}$ such that $\mathfrak{S}$ is a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$, we have that $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{B}$. From this definition and Theorem 13.22, we immediately obtain:

Corollary 13.42 *If the theory $\mathbf{T}$ admits quantifier elimination, then it is model complete.*

The converse is not true; a counterexample is provided by the theory of $\mathbb{Z}$-groups, which we will treat in Section 13.11. By use of this corollary and Theorem 13.24, we can now infer our first examples for model complete theories:

Corollary 13.43 *The theory of nontrivial torsion free divisible abelian groups and the theory of algebraically closed fields are model complete.*

If $\mathfrak{A}$ and $\mathfrak{S}$ are models of $\mathbf{T}$ and $\mathfrak{S} \subset \mathfrak{A}$, then we call $\mathfrak{S}$ a **submodel** of $\mathfrak{A}$ (if it is clear which theory $\mathbf{T}$ we are talking about).

An alternative definition for an $\mathcal{L}$-theory $\mathbf{T}$ to be model complete is the following: if $\mathfrak{S} \subset \mathfrak{A}$ are models of $\mathbf{T}$, then $\mathfrak{S} \prec \mathfrak{A}$. The latter means that $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{S}$, which we obtain from the original definition by setting $\mathfrak{B}$ equal to $\mathfrak{S}$. For the converse, assume that the condition of the present definition holds. If $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{S}$ are models of $\mathbf{T}$ such that $\mathfrak{S}$ is a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$, then it follows that $\mathfrak{S} \prec \mathfrak{A}$ and $\mathfrak{S} \prec \mathfrak{B}$, that is, $\mathfrak{A} \equiv_{\mathfrak{S}} \mathfrak{S} \equiv_{\mathfrak{S}} \mathfrak{B}$. This shows that both definitions are equivalent.

Let us give an example for the application of model completeness. Using the model completeness of the theory of algebraically closed fields, one can prove **Hilbert's Nullstellensatz**. As a first step, we prove:

Lemma 13.44 *Let K be an algebraically closed field and $f_1, \ldots, f_r \in K[X_1, \ldots, X_s]$. If these polynomials have no common zero in $\tilde{K}$, then the $K[X_1, \ldots, X_s]$-ideal generated by them is the whole ring $K[X_1, \ldots, X_s]$.*

Proof: Let us assume that the ideal I generated by $f_1, \ldots, f_r$ is not the whole ring $K[X_1, \ldots, X_s]$. Let J be a maximal ideal containing I. The field $L := K[X_1, \ldots, X_s]/J$ contains (an isomorphic copy of) the field K. The residues $f_i + J$ of the polynomials f_i under the canonical epimorphism $K[X_1, ..., X_s] \to L$ are all equal to 0. But this epimorphism is an epimorphism over K, so the image $(X_1 + J, \ldots, X_s + J)$ of $X_1, \ldots, X_s$ is a common zero of the polynomials f_i. This remains true in the algebraic closure $\tilde{L}$. The existential $\mathcal{L}_R(K)$-sentence stating the existence of a common zero for the f_i thus holds in $\tilde{L}$. By the model completeness of the theory of algebraically closed fields, it must also hold in $\tilde{K}$. $\square$

Theorem 13.45 (Hilbertscher Nullstellensatz)
Let K be a field and $g, f_1, \ldots, f_k \in K[X_1, \ldots, X_n]$. If all common zeros of $f_1, \ldots, f_n$ in $\tilde{K}$ are also zeros of g, then g is of the form

$$g^m = \sum_{i=1}^{k} h_i f_i$$

with $m \in \mathbb{N}$, $m \geq 1$, and $h_1, \ldots, h_k \in K[X_1, \ldots, X_n]$, i.e., g lies in the radical of the $K[X_1, \ldots, X_n]$-ideal generated by the f_i.

Proof: We introduce a new variable Z. By hypothesis, the polynomials

$$f_1, \ldots, f_k, 1 - g \cdot Z \in K[X_1, \ldots, X_n, Z]$$

have no common zero in $\tilde{K}$. Hence the foregoing lemma shows that they generate the ring $K[X_1, \ldots, X_n, Z]$. That is, there exist polynomials $g_1, \ldots, g_{k+1} \in K[X_1, \ldots, X_n, Z]$ such that

$$1 = \sum_{i=1}^{k} g_i \cdot f_i + g_{k+1} \cdot (1 - g \cdot Z) \; .$$

Let m be the maximal natural number such that Z^m appears in one of the polynomials $g_1, \ldots, g_k$. If we set $Z := g^{-1}$, the last summand vanishes, and we multiply by g^m to obtain that

$$g^m = \sum_{i=1}^{k} h_i \cdot f_i \; ,$$

with polynomials $h_1, \ldots, h_k \in K[X_1, \ldots, X_n]$. □

Analyzing the proof of Lemma 13.44, we see that we only have used that $\tilde{K}$ is existentially closed in $\tilde{L}$. The assertion that this is true for *every* extension of algebraically closed fields is in fact *equivalent* to the model completeness of the theory of algebraically closed fields. This follows from Robinson's Test, which we will introduce now. We need the following lemma, which presents a further application of Lemma 13.20.

Lemma 13.46 *Let $\mathbf{T}$ be an $\mathcal{L}$-theory. Then the following are equivalent for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$:*

1) $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$, or $\mathbf{T} \models \varphi$,

2) for all $\mathbf{T}$-models $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \subset \mathfrak{B}$, and for all n-tuples $(a_1, \ldots, a_n) \in A^n$,

$$(\mathfrak{A}, A) \models \varphi(a_1, \ldots, a_n) \implies (\mathfrak{B}, A) \models \varphi(a_1, \ldots, a_n) \; .$$

Proof: Assume 1). Then for every $(a_1, \ldots, a_n) \in A^n$, $\varphi(a_1, \ldots, a_n)$ is $\mathbf{T}$-equivalent to the existential sentence $\psi(a_1, \ldots, a_n)$. If the latter holds in $(\mathfrak{A}, A)$, then it also holds in the extension $(\mathfrak{B}, A)$ of $(\mathfrak{A}, A)$. Hence if $\varphi(a_1, \ldots, a_n)$ holds in $(\mathfrak{A}, A)$, then it holds in $(\mathfrak{B}, A)$.

Now assume 2). As in the proof of Lemma 13.21, we view $x_1, \ldots, x_n$ as new constant symbols and thus assume that φ is the $\mathcal{L}(x_1, \ldots, x_n)$-sentence $\varphi(x_1, \ldots, x_n)$. Let $\mathbf{S}$ be the set of all existential $\mathcal{L}(x_1, \ldots, x_n)$-sentences. Further, let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}(x_1, \ldots, x_n)$-structures such that $\mathfrak{A} \Rightarrow_{\mathbf{S}} \mathfrak{B}$. Then by part a) of Theorem 13.31, we can embed $\mathfrak{A}$ in an $|A|$-saturated elementary extension $\mathfrak{B}'$ of $\mathfrak{B}$, which exists by Theorem 13.28. Applying 2) with $\mathfrak{B}'$ in the place of $\mathfrak{B}$, we obtain that $\mathfrak{A} \Rightarrow_{\varphi} \mathfrak{B}'$. But $\mathfrak{B}' \Rightarrow_{\varphi} \mathfrak{B}$ since $\mathfrak{B}'$ is an

elementary extension of $\mathfrak{B}$. Hence $\mathfrak{A} \Rightarrow_\varphi \mathfrak{B}$. Now we apply the implication 2)⇒1) of Lemma 13.20 to deduce that $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to some $\psi(x_1, \ldots, x_n) \in \mathbf{S}^{\vee\wedge}$, or $\mathbf{T} \models \varphi$ or $\mathbf{T} \models \neg\varphi$. We leave it to the reader to show that every sentence in $\mathbf{S}^{\vee\wedge}$ is equivalent to a sentence in $\mathbf{S}$. Finally, if $\mathbf{T} \models \neg\varphi$, then φ is equivalent to the existential $\exists x \neg(x = x)$, which also satisfies $\mathbf{T} \models \neg\exists x \neg(x = x)$. □

The implication b)⇒a) of the following theorem is known as **Robinson's Test**:

Theorem 13.47 *For an $\mathcal{L}$-theory $\mathbf{T}$, the following assertions are equivalent:*

1) $\mathbf{T}$ is model complete,

2) $\mathfrak{A} \prec_\exists \mathfrak{B}$ for every two models $\mathfrak{A}$, $\mathfrak{B}$ of $\mathbf{T}$ such that $\mathfrak{A} \subset \mathfrak{B}$,

3) every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$,

4) every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to a universal $\mathcal{L}$-formula $\psi'(x_1, \ldots, x_n)$.

Proof: 1)⇒2): If $\mathfrak{A} \subset \mathfrak{B}$ are models of the model complete theory $\mathbf{T}$, then $\mathfrak{A} \prec \mathfrak{B}$. This implies that $\mathfrak{A} \prec_\exists \mathfrak{B}$.

2)⇒3): First, let $\varphi(x_1, \ldots, x_n)$ be any universal $\mathcal{L}$-formula. If $\mathfrak{A} \subset \mathfrak{B}$ are models of $\mathbf{T}$, then it follows from 2) that $\mathfrak{A} \prec_\exists \mathfrak{B}$. Recall that $\mathfrak{A} \prec_\exists \mathfrak{B}$ if and only if $(\mathfrak{A}, A) \Rightarrow_\forall (\mathfrak{B}, A)$. Hence for all $(a_1, \ldots, a_n) \in A^n$, the universal $\mathcal{L}$-sentence $\varphi(a_1, \ldots, a_n)$ satisfies $(\mathfrak{A}, A) \models \varphi(a_1, \ldots, a_n) \Rightarrow (\mathfrak{B}, A) \models \varphi(a_1, \ldots, a_n)$. Thus, Lemma 13.46 yields that $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$.

Now let $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}$-formula. We wish to show that it is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula. We proceed by induction on the complexity of φ. But this time, we use the symbols $\neg$, $\wedge$ and $\exists$ to build the formulas. (This is possible since $\forall x \psi(x)$ is equivalent to $\neg\exists x \neg\psi(x)$.) For quantifier free $\mathcal{L}$-formulas, there is nothing to show. If φ is of the form $\psi_1 \wedge \psi_2$, then the induction step is straightforward. Suppose that φ is of the form $\exists x \psi(x, x_1, \ldots, x_n)$. By induction hypothesis, there is an existential $\mathcal{L}$-formula $\psi(x, x_1, \ldots, x_n)$ such that $\mathbf{T} \models \forall(\psi(x, x_1, \ldots, x_n) \leftrightarrow \psi'(x, x_1, \ldots, x_n)$. It follows that $\mathbf{T} \models \forall(\exists x\, \psi(x, x_1, \ldots, x_n) \leftrightarrow \exists x\, \psi'(x, x_1, \ldots, x_n))$, and we are done.

Now assume that φ is of the form $\neg\psi(x_1, \ldots, x_n)$. By induction hypothesis, there is an existential $\mathcal{L}$-formula $\psi'(x_1, \ldots, x_n)$ such that $\mathbf{T} \models \forall(\psi(x_1, \ldots, x_n) \leftrightarrow \psi'(x_1, \ldots, x_n))$. Consequently, $\mathbf{T} \models \forall(\neg\psi(x_1, \ldots, x_n) \leftrightarrow \neg\psi'(x_1, \ldots, x_n))$. But $\neg\psi'(x_1, \ldots, x_n)$ is equivalent to a universal $\mathcal{L}$-formula. By what we have shown in the beginning, this in turn is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula $\psi''(x_1, \ldots, x_n)$. Hence, also $\neg\psi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to $\psi''(x_1, \ldots, x_n)$. This completes our proof by induction.

3)⇒4): Let $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}$-formula. By 3), $\neg\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to an existential $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. Hence, $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to $\neg\psi(x_1, \ldots, x_n)$, and this in turn is equivalent to a universal $\mathcal{L}$-formula.

4)⇒1): Let $\mathfrak{A} \subset \mathfrak{B}$ be models of $\mathbf{T}$ and φ an $\mathcal{L}(A)$-sentence. We can write φ in the form $\varphi(a_1, \ldots, a_n)$ with $\varphi(x_1, \ldots, x_n)$ an $\mathcal{L}$-formula. By 4), $\varphi(x_1, \ldots, x_n)$ is $\mathbf{T}$-equivalent to a universal $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. It follows that $\varphi(a_1, \ldots, a_n)$ is $\mathbf{T}$-equivalent to the universal $\mathcal{L}(A)$-sentence $\psi(a_1, \ldots, a_n)$. If $\psi(a_1, \ldots, a_n)$ holds in $(\mathfrak{B}, A)$, then it also holds in the substructure $(\mathfrak{A}, A)$. Hence if φ holds in $(\mathfrak{B}, A)$, then it holds in $(\mathfrak{A}, A)$. This proves that $(\mathfrak{B}, A) \Rightarrow_{\mathrm{Th}(\mathfrak{B}, A)} (\mathfrak{A}, A)$, that is, $\mathfrak{A} \prec \mathfrak{B}$. So we have $\mathfrak{A} \prec \mathfrak{B}$ for every two models $\mathfrak{A} \subset \mathfrak{B}$ of $\mathbf{T}$, that is, $\mathbf{T}$ is model complete. □

The implication 2)⇒1) can also be proved by use of strong limit models and a back and forth method. To show this, assume that 2) holds and that $\mathfrak{A} \subset \mathfrak{B}$ are models of $\mathbf{T}$. Then $\mathfrak{A} \prec_\exists \mathfrak{B}$. We take $\mathfrak{A}^*$ and $\mathfrak{B}^*$ to be strong limit extensions of equal basis $\kappa \geq \max\{||\mathcal{L}||, |A|, |B|\}$. Now we carry through a back and forth construction similar to that in the proof of Theorem 13.35. We have that $\mathfrak{B}_0 = \mathfrak{B}$. All $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are elementary extensions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, so it follows that they are models of $\mathbf{T}$. Since $\mathfrak{A}_1$ is κ^+-saturated and since $|A_0| \leq \kappa$, it follows from part Lemma 13.38 that $\mathfrak{B}_0$ admits an embedding ι_0 in $\mathfrak{A}_1$. Like $\mathfrak{B}_0$, also $\iota_0\mathfrak{B}_0$ is a model of $\mathbf{T}$ (cf. Theorem 13.1). Since $\iota_0\mathfrak{B}_0 \subset \mathfrak{A}_1$, we have that $\iota_0\mathfrak{B}_0 \subset \mathfrak{A}_1$ by hypothesis. Since $|B_0| \leq \kappa < \kappa_1$, we know that not only $\mathfrak{B}_2$, but also $(\mathfrak{B}_2, B_0)$ is κ_1^+-saturated. Hence, it again follows from Lemma 13.38 that $(\mathfrak{A}_1, \iota_0 B_0)$ admits an elementary embedding ι_1' in $(\mathfrak{B}_2, B_0)$. Now we proceed by induction, like in the proof of Theorem 13.35. We obtain an isomorphism of $\mathfrak{A}^*$ and $\mathfrak{B}^*$ over $\mathfrak{A}$, which shows that $\mathfrak{A}^* \equiv_\mathfrak{A} \mathfrak{B}^*$. This implies that $\mathfrak{A} \equiv_\mathfrak{A} \mathfrak{B}$ or in other words, that $\mathfrak{A} \prec \mathfrak{B}$.

Condition 2) of the theorem may be viewed as a **Nullstellensatz** for the theory $\mathbf{T}$, in the widest sense of the word. So the notion of "Nullstellensatz" is closely related to model completeness. And in fact, there are further model completeness theories which satisfy a more or less interesting Nullstellensatz. For instance, the model complete **theory of real closed fields** satisfies a Nullstellensatz which is an analogue to Hilbert's Nullstellensatz, but even more interesting is that **Hilbert's 17. Problem**, which was solved by E. Artin, can also be solved by an application of model completeness. This was shown by A. Tarski [TA]. Similarly, for the proof of a **p-adic Nullstellensatz**, the model completeness of the **theory of p-adically closed fields** (in an appropiate language, cf. Section 14.8) has been used by M. Jarden and P. Roquette [JA–ROQ]. See also V. Weispfenning [WEI2,4].

13.9 Complete theories

A theory $\mathbf{T}$ is called **complete** if every two models of $\mathbf{T}$ are elementarily equivalent. Similarly, an elementary class $\mathbf{C}$ is called **complete** if every two structures in $\mathbf{C}$ are elementarily equivalent. If $\mathbf{T}$ is a complete $\mathcal{L}$-theory and if φ is an $\mathcal{L}$-sentence which holds in some model of $\mathbf{T}$, then it holds in all models of $\mathbf{T}$. So if we have an axiom system for some class of algebraic structures and this axiom system turns out to be a complete theory, then we know that it determines completely the elementary properties of the structures in this class. Then we can transfer our (elementary) knowledge from one structure to another. This gives a transfer principle similar to that in the case of model complete theories. Syntactically, the completeness of $\mathbf{T}$ means that every $\mathcal{L}$-sentence or its negation can be deduced from $\mathbf{T}$.

In the following, we will give two criteria for the completeness of a theory. A model $\mathfrak{A}_p$ of $\mathbf{T}$ is called a **prime model for $\mathbf{T}$** if it can be embedded in every other model of $\mathbf{T}$.

Lemma 13.48 (Prime Model Test) *Let $\mathbf{T}$ be a model complete theory. If it admits a prime model, then it is complete.*

Proof: Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathbf{T}$, and let $\mathfrak{A}_p$ be a model of $\mathbf{T}$ which can be embedded in both $\mathfrak{A}$ and $\mathfrak{B}$. Since $\mathbf{T}$ is assumed to be model complete, both embeddings are elementary. Hence, $(\mathfrak{A}, A_p) \equiv (\mathfrak{A}_p, A_p) \equiv (\mathfrak{B}, A_p)$, which yields that $\mathfrak{A} \equiv \mathfrak{B}$. □

Let $\mathbf{T}$ be the theory of nontrivial torsion free divisible abelian groups. Every such group G contains a copy of $\mathbb{Q}$. Indeed, pick a nonzero element in G. The subgroup generated by it is isomorphic to $\mathbb{Z}$ since the groups are torsion free. Since G is divisible, it contains the divisible hull of this subgroup, which is isomorphic to $\mathbb{Q}$. Hence, the theory of nontrivial torsion free divisible abelian groups admits $\mathbb{Q}$ as its prime model (up to isomorphism, it is the only one). From Corollary 13.43 and Lemma 13.48, we can thus conclude:

Corollary 13.49 *The theory of nontrivial torsion free divisible abelian groups is complete.*

For the theory $\mathbf{T}$ of algebraically closed fields, the case is slightly different. This theory can not be complete, because it has models of arbitrary characteristic. So none of the $\mathcal{L}_{\mathrm{R}}$-sentences (CH0) or (CHp) with p a prime holds in every model of $\mathbf{T}$. This is reflected by the fact that $\mathbf{T}$ has no prime model. Indeed, given any field, then it can not be embedded in a field of different characteristic. We see that at least we have to fix the characteristic to obtain a complete theory. Now as soon as we fix the characteristic by adding one of the axioms (CH0) or (CHp) to $\mathbf{T}$, the corresponding prime field will be embeddable in every model of the theory thus obtained. That is, the prime field then is the prime model that we are looking for. Hence, Corollary 13.43 and Lemma 13.48 show :

Corollary 13.50 *Let p be a prime or $p = 0$. Then the theory of algebraically closed fields of characteristic p is complete.*

Recall that the information about the characteristic is contained in the diagram of every field. That is, if we choose a field K of characteristic p, then the $\mathcal{L}_{\mathrm{F}}(K)$-theory $\mathbf{T} \cup \mathrm{Diag}(K)$ will contain the complete theory of algebraically closed fields of characteristic p (together with the information about the field K). It is itself complete by virtue of the following lemma:

Lemma 13.51 *An $\mathcal{L}$-theory $\mathbf{T}$ is model complete if and only if for every model $\mathfrak{A}$ of $\mathbf{T}$, the $\mathcal{L}(A)$-theory $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$ is complete.*

Proof: Assume that $\mathbf{T}$ is model complete and that $\mathfrak{A}$ is a model of $\mathbf{T}$. Further, let $\mathfrak{B}'$ and $\mathfrak{C}'$ be models of $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$, and $\mathfrak{B}$ and $\mathfrak{C}$ their $\mathcal{L}$-reducts. Then by Corollary 13.5, $\mathfrak{A}$ admits an embedding in both $\mathfrak{B}$ and $\mathfrak{C}$. By passing to isomorphic images (cf. Lemma 13.6), we can assume that $\mathfrak{A}$ is a common substructure of $\mathfrak{B}$ and $\mathfrak{C}$. Then $\mathfrak{B}' = (\mathfrak{B}, A)$ and $\mathfrak{C}' = (\mathfrak{C}, A)$. Since $\mathbf{T}$ is model complete, we have that $(\mathfrak{B}, A) \equiv (\mathfrak{C}, A)$. This proves that $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$ is complete.

For the converse, assume that $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$ is complete. Let $\mathfrak{B}$ and $\mathfrak{C}$ be models of $\mathbf{T}$ with $\mathfrak{A}$ as a common submodel. Then by Corollary 13.5, $(\mathfrak{B}, A)$ and $(\mathfrak{C}, A)$ are models of $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$. It follows that $(\mathfrak{B}, A) \equiv (\mathfrak{C}, A)$, showing that $\mathbf{T}$ is model complete. □

Since the theory of algebraically closed fields is model complete, our lemma shows that the theory $\mathbf{T} \cup \mathrm{Diag}(K)$ is complete if K is an algebraically closed field. We will see below that this already holds for every field K.

We turn to the second criterion for completeness. Let $\mathbf{T}$ be an $\mathcal{L}$-theory. An $\mathcal{L}$-structure $\mathfrak{S}$ is called a **prime structure for $\mathbf{T}$** if it can be embedded in every model of $\mathbf{T}$.

Lemma 13.52 (Prime Structure Test) *Let $\mathbf{T}$ be a theory which admits quantifier elimination. If it admits a prime structure, then it is complete.*

Proof: Similar to the proof of Lemma 13.48, using the substructure completeness in the place of the model completeness (cf. Theorem 13.22). □

For the theory of nontrivial torsion free divisible abelian groups, the prime structure is $\mathbb{Z}$, as we have seen above. To show that this theory is complete, we thus could also have used the foregoing lemma together with Theorem 13.24 and Theorem 13.22.

Instead of using prime models or prime structures which embed in *every* model of a given theory, one can also use models or structures which embed in every sufficiently saturated model. We will call them **weak prime models** resp. **weak prime structures**. We leave it to the reader to show that the Prime Model Test and the Prime Structure Test remain true for "weak prime" in the place of "prime".

The following lemma is the analogue of Lemma 13.51, for substructure completeness in the place of model completeness. Its proof is similar to that of Lemma 13.51.

Lemma 13.53 *An $\mathcal{L}$-theory $\mathbf{T}$ is substructure complete if and only if for every substructure $\mathfrak{S}$ of a model of $\mathbf{T}$, the $\mathcal{L}(S)$-theory $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{S})$ is complete.*

Since the theory of algebraically closed fields is substructure complete by virtue of Theorem 13.24 and Theorem 13.22, this lemma shows that the theory $\mathbf{T} \cup \mathrm{Diag}(K)$ is complete already for every field K. From this example, it may seem as if completeness implies model completeness and even substructure completeness. But this is not true in general. Certainly, if an $\mathcal{L}$-theory is complete and contained in an $\mathcal{L}$-theory $\mathbf{S}$, then also $\mathbf{S}$ is complete. But in our case, with $\mathfrak{A}$ an $\mathcal{L}$-structure, the $\mathcal{L}(A)$-theory $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$ is in general not an $\mathcal{L}$-theory (it may only be viewed as such if $\mathcal{L}$ already contains a constant symbol for every element of A). In this situation, it does not necessarily follow from the completeness of $\mathbf{T}$ that also the $\mathcal{L}(A)$-theory $\mathbf{T} \cup \mathrm{Diag}(\mathfrak{A})$ is complete. A counterexample will be provided by the theory of $\mathbb{Z}$-groups, cf. Section 13.11.

The completeness of a theory yields a **Transfer Principle** for that theory. For example, the completeness of the theory of algebraically closed fields of fixed characteristic tells us that every elementary $\mathcal{L}_{\mathrm{F}}$-sentence which holds in one algebraically closed field will also hold in all other algebraically closed fields of the same characteristic. This reminds of the **Lefschetz Principle** which was stated and partially proved by Weil and Lefschetz and states (roughly speaking) that algebraic geometry over all algebraically closed fields of a fixed characteristic is the same ("there is but one algebraic geometry in characteristic p"). As for the elementary $\mathcal{L}_{\mathrm{F}}$-sentences of algebraic geometry, this indeed follows from the completeness. However, Lefschetz and Weil did not have in mind only the elementary sentences. That is why Weil worked with so-called "universal domains" which are algebraically closed and of infinite transcendence degree over their prime field. So the assertion was that there is but one algebraic geometry over universal domains of characteristic p. A satisfactory formalization and model theoretic proof by use of an **infinitary language** is due to P. Eklof [EK]. Infinitary languages admit the conjunction of infinitely many elementary sentences. With such infinitary sentences, one can also express the fact that a field has infinite transcendence degree over their prime field. This can not be done by elementary sentences. Indeed, algebraically closed fields are elementarily equivalent to the algebraic closure of their prime field, even if they have infinite transcendence degree.

One of the most famous and useful Transfer Principles is the one used by J. Ax and S. Kochen to prove a deep number theoretic result. It works with the elementary theory of henselian field of residue characteristic 0. We will discuss it in Section 14.9.

Here, we want to give a quick example for a Transfer Principle which works with algebraically closed fields. A **polynomial map** is a function of the form

$$f(\underline{x}) = (f_1(\underline{x}), \ldots, f_n(\underline{x})) \ ,$$

where $\underline{x} = (x_1, \ldots, x_n)$ and the f_i are polynomials in n variables.

Theorem 13.54 *Let*

$$f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

be an injective polynomial map. Then f is onto. (Here, $\mathbb{C}$ can be replaced by every algebraically closed field.)

Proof: Let the maximum of the degrees of the f_i's be called the degree of f. We leave it to the reader to formulate the injectivity of a polynomial map as a universal elementary sentence with the coefficients of the polynomial as parameters. Similarly, the surjectivity can be stated as a universal existential sentence. Let the maximum of the degrees of the f_i's be called the degree of f. Then the assertion that every polynomial map of degree at most n is surjective if it is injective, can be stated as a universal existential sentence φ (we replace the coefficients by variables, just as in axiom (AC)). This sentence holds in every finite field since there, injectivity of a map always implies surjectivity. The algebraic closure of a finite field of characteristic p is the union over all finite fields of characteristic p. Being universal existential, φ thus also holds in all algebraic closures of finite fields, by virtue of Lemma 13.7. By the completeness of the theory of algebraically closed fields of fixed characteristic $p > 0$, we obtain that φ holds in every algebraically closed field of positive characteristic. Further, we can infer from Theorem 13.16 (where we take $\mathbf{T}$ to be the theory of algebraically closed fields) that φ holds in some algebraically closed field of characteristic 0. By the completeness of the theory of algebraically closed fields of characteristic 0, it thus holds in $\mathbb{C}$ and in every other algebraically closed field of characteristic 0. □

Connected with the completeness of elementary theories, there is a further notion of model theory, which we will occasionally use but which we will not explain in detail here. An $\mathcal{L}$-theory $\mathbf{T}$ is called **decidable** if there is a ("recursive") algorithm which tells us for every $\mathcal{L}$-sentence φ whether $\mathbf{T} \models \varphi$ or not. We refer the reader to the books on model theory and recursion theory. We will only use the following fact: If a theory $\mathbf{T}$ consists of a set of axioms which can be produced by such an algorithm and if $\mathbf{T}$ is complete, then $\mathbf{T}$ is decidable. This follows, via Gdel's Completeness Theorem, from the fact that all there is an algorithm that tells us whether a sentence φ or its negation $\neg\varphi$ can be deduced from $\mathbf{T}$ by the logic calculus that we have mentioned in Section 13.3. If $\mathbf{T}$ is complete, then for every sentence φ, either φ or $\neg\varphi$ can be deduced from $\mathbf{T}$, so we can decide about every sentence φ as soon as we can produce the axioms of $\mathbf{T}$ by an algorithm. This is trivial if $\mathbf{T}$ is a finite axiom system, like that of abelian groups and fields (but these are not complete). But also the axiom schemes that we have introduced, like (DIV) and (AC), are recursive, that is, they can be produced by an algorithm. It follows that the theories of nontrivial torsion free divisible abelian groups and of algebraically closed fields of fixed characteristic are decidable.

13.10 Model companion, model completion and hulls

We have seen that the theory of fields is not model complete, but the theory of algebraically closed fields is. So we just had to add the axiom scheme (AC) to the theory of fields to obtain a model complete theory (for the same language). In the same way, we obtain a model complete theory by adding the divisibility to the theory of nontrivial torsion free abelian groups. We introduce notions that reflect this connection between theories. Let $\mathbf{T}$ and $\mathbf{T}'$ be two $\mathcal{L}$-theories. Then $\mathbf{T}'$ is called a **model companion of T** if

(MC1) every model of $\mathbf{T}'$ is a model of $\mathbf{T}$,

(MC2) every model of $\mathbf{T}$ is embeddable in a model of $\mathbf{T}'$,

(MC3) $\mathbf{T}'$ is model complete.

If in addition,

(MC4) $\mathbf{T}' \cup \mathrm{Diag}(\mathfrak{A})$ is complete for every model $\mathfrak{A}$ of $\mathbf{T}$,

then $\mathbf{T}'$ is called a **model completion** of $\mathbf{T}$. In view of Lemma 13.53, we immediately obtain:

Lemma 13.55 *Assume that every model of* $\mathbf{T}$ *is embeddable in a model of* $\mathbf{T}'$*, and that every substructure of* $\mathbf{T}'$ *is a model of* $\mathbf{T}$*. Then* $\mathbf{T}'$ *admits quantifier elimination if and only if it is a model completion of* $\mathbf{T}$*.*

Now we see that the theory of algebraically closed fields is the model completion of the theory of fields, and that the theory of nontrivial divisible torsion free abelian groups is the model completion of the theory of nontrivial torsion free abelian groups. The reason for this strong property in either case is the existence of algebraic closures (= divisible hulls in the abelian group case) which are unique up to isomorphism and which are models of $\mathbf{T}^*$. So let us say that $\mathbf{T}$ **admits** $\mathbf{T}^*$**-hulls** if for every model $\mathfrak{S}$ of $\mathbf{T}$ there exists a model $\mathfrak{A}_\mathfrak{S}$ of $\mathbf{T}^*$ with the property

(HULL) for every model $\mathfrak{B}$ of $\mathbf{T}^*$ with substructure $\mathfrak{S}$, there is an embedding of $\mathfrak{A}_\mathfrak{S}$ in $\mathfrak{B}$ over $\mathfrak{S}$.

If the embedding is unique, then the $\mathbf{T}^*$-hull $\mathfrak{A}_\mathfrak{S}$ of $\mathfrak{S}$ is unique up to isomorphism over $\mathfrak{S}$. Indeed, if $\mathfrak{A}_\mathfrak{S}$ and $\mathfrak{A}'_\mathfrak{S}$ are $\mathbf{T}^*$-hulls of $\mathfrak{S}$ and $\iota : \mathfrak{A}_\mathfrak{S} \to \mathfrak{A}'_\mathfrak{S}$ and $\iota' : \mathfrak{A}'_\mathfrak{S} \to \mathfrak{A}_\mathfrak{S}$ are the embeddings according to (HULL), then both the identity and $\iota' \circ \iota$ are embeddings of $\mathfrak{A}_\mathfrak{S}$ into itself. Hence it follows that $\iota' \circ \iota$ is the identity on $\mathfrak{A}_\mathfrak{S}$, showing that ι is injective and ι' is surjective. By symmetry, it follows that ι is an isomorphism.

In view of the fact that the $\mathbf{T}^*$-hull of $\mathfrak{S}$ can be embedded over $\mathfrak{S}$ into every model $\mathfrak{B}$ of $\mathbf{T}^*$ which contains $\mathfrak{S}$, we speak of "a $\mathbf{T}^*$-hull of $\mathfrak{S}$ in $\mathfrak{B}$".

Let us call a theory $\mathbf{T}^*$ a **hull extension** of $\mathbf{T}$ if the following holds:

(HUE1) every substructure of a model of $\mathbf{T}'$ is a model of $\mathbf{T}$,

(HUE2) $\mathbf{T}$ admits $\mathbf{T}'$-hulls.

We note important examples for hull extensions. Some of the theories have not yet been axiomatized, but this is of no importance here.

Lemma 13.56 *The following hulls have property (HULL):*

a) the divisible hulls of torsion free abelian groups (i.e., the theory of divisible torsion free abelian groups is a hull extension of the theory of torsion free abelian groups),

b) the algebraic closures of fields (i.e., the theory of algebraically closed fields is a hull extension of the theory of fields),

c) the divisible hulls of ordered abelian groups (i.e., the theory of divisible ordered abelian groups is a hull extension of the theory of ordered abelian groups),

d) the real closures of ordered fields (i.e., the theory of real closed ordered fields is a hull extension of the theory of ordered fields),

e) the divisible hulls of valued abelian groups which satisfy (NVZ) (i.e., the theory of divisible valued abelian groups satisfying (NVZ) is a hull extension of the theory of valued abelian groups satisfying (NVZ)),

f) the algebraic closures of valued fields (i.e., the theory of algebraically closed valued fields is a hull extension of the theory of valued fields),

g) the henselizations of valued fields (i.e., the theory of henselian fields is a hull extension of the theory of valued fields),

h) the perfect hulls of fields (i.e., the theory of perfect fields is a hull extension of the theory of fields).

In the group cases and cases g) and h), we have unique embeddings.

Proof: a), b): This has already been mentioned.

c): This follows from a) and Lemma 3.31.

d): This was proved by E. Artin and O. Schreier [AR–SCHR]. Cf. A. Prestel [PRE1], Theorem 3.10 and Corollary 3.11 .

e): This follows from a) and the fact that axiom (NVZ) yields uniqueness of the extension of the valuation to the divisible hull.

f): Let (K, v) be a valued field and (L, v) an algebraically closed valued extension field. Let w be any extension of v to $\tilde{K}$. Further, let ι be any embedding of $\tilde{K}$ in L. Then w and $v \circ \iota$ are valuations on $\tilde{K}$. By Theorem 7.68, there exists an automorphism σ of $\tilde{K}|K$ such that $w \circ \sigma = v \circ \iota$. It follows that $v = w \circ \sigma\iota^{-1}$ on $\iota\sigma^{-1}\tilde{K} = \iota\tilde{K} \subset L$, that is, $\iota\sigma^{-1}$ is an embedding of $(\tilde{K}, w)$ in (L, v).

g): Cf. Lemma 7.107.

h): Cf. Lemma 6.28. □

Hulls can serve to deduce quantifier elimination from model completeness:

Lemma 13.57 *Assume that* $\mathbf{T}'$ *is a hull extension of* $\mathbf{T}$*. If* $\mathbf{T}'$ *is model complete, then* $\mathbf{T}'$ *admits quantifier elimination.*

Proof: Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $\mathbf{T}'$ with a common substructure $\mathfrak{S}$. Then $\mathfrak{S}$ is a model of $\mathbf{T}$ by hypothesis. Again by hypothesis, there is a $\mathbf{T}'$-hull $\mathfrak{A}_{\mathfrak{S}}$ which is a model of $\mathbf{T}'$ and embeds in both $\mathfrak{A}$ and $\mathfrak{B}$ over $\mathfrak{S}$. By passing to isomorphic images (cf. Lemma 13.6), we can identify $\mathfrak{A}_{\mathfrak{S}}$ with its images in $\mathfrak{A}$ and $\mathfrak{B}$. Now $\mathfrak{A}_{\mathfrak{S}}$ is a common submodel of $\mathfrak{A}$ and $\mathfrak{B}$ and contains $\mathfrak{S}$. If $\mathbf{T}'$ is model complete, then $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent over $\mathfrak{A}_{\mathfrak{S}}$ and thus

also over $\mathfrak{S}$. This proves that $\mathbf{T}'$ is substructure complete if it is model complete. Now the assertion follows from Theorem 13.22. □

Hulls can also serve to facilitate the proof of model completeness. The key lemma is the following:

Lemma 13.58 *Assume that* $\mathbf{T}'$ *is a hull extension of* $\mathbf{T}$. *Let* $\mathfrak{A} \subset \mathfrak{B}$ *are models of* $\mathbf{T}'$ *and* $\mathfrak{S}$ *a substructure of* $\mathfrak{B}$ *containing* $\mathfrak{A}$ *such that* $\mathfrak{B}$ *is the* $\mathbf{T}'$*-hull of* $\mathfrak{S}$. *If* $\mathfrak{A}$ *is existentially closed in* $\mathfrak{S}$, *then it is also existentially closed in* $\mathfrak{B}$.

Proof: If $\mathfrak{A}$ is existentially closed in $\mathfrak{S}$, then by Lemma 13.38, $\mathfrak{S}$ is embeddable over $\mathfrak{A}$ in every $|S|^+$-saturated elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$. By passing to isomorphic images (cf. Lemma 13.6), we identify $\mathfrak{S}$ with its image in $\mathfrak{A}^*$. But then by (HULL), the $\mathbf{T}'$-hull $\mathfrak{B}$ of $\mathfrak{B}$ embeds in $\mathfrak{A}^*$ over $\mathfrak{S}$. By Lemma 13.38, it follows that $\mathfrak{A}$ is existentially closed in $\mathfrak{B}$. □

Lemma 13.59 *Assume that* $\mathbf{T}'$ *is a hull extension of* $\mathbf{T}$. *Suppose that the following holds: If* $\mathfrak{A} \subset \mathfrak{B}$ *are models of* $\mathbf{T}'$, *then* $\mathfrak{A}$ *is existentially closed in every substructure* $\mathfrak{S}$ *of* $\mathfrak{B}$ *which is generated over* $\mathfrak{A}$ *by one element of* $\mathfrak{B}$. *Then* $\mathbf{T}'$ *admits quantifier elimination.*

Proof: In view of Lemma 13.57 and Robinson's Test (Theorem 13.47), we have to show that $\mathfrak{A} \prec_\exists \mathfrak{B}$ for every two models $\mathfrak{A} \subset \mathfrak{B}$ of $\mathbf{T}'$. For this, it suffices to show that $\mathfrak{A}$ is existentially closed in every substructure $\mathfrak{S}$ of $\mathfrak{B}$ which is finitely generated over $\mathfrak{A}$ (cf. Lemma 13.40). We proceed by induction on the number of generators. So assume that we have already shown that for every two models $\mathfrak{A} \subset \mathfrak{B}$ of $\mathbf{T}'$ and every substructure $\mathfrak{S}'$ of $\mathfrak{B}$ which is generated over $\mathfrak{A}$ by n elements, we have that $\mathfrak{A}$ is existentially closed in $\mathfrak{S}'$. (For $n = 1$, this is the hypothesis.) Now let $\mathfrak{S}$ be a substructure of $\mathfrak{B}$ which is generated over $\mathfrak{A}$ by $n+1$ elements $b_1, \ldots, b_n, b \in B$. We let $\mathfrak{S}'$ be the substructure of $\mathfrak{S}$ generated by $b_1, \ldots, b_n$ over $\mathfrak{A}$, and $\mathfrak{B}'$ its $\mathbf{T}'$-hull in $\mathfrak{B}$. By induction hypothesis, $\mathfrak{A}$ is existentially closed in $\mathfrak{S}'$, and thus also in $\mathfrak{B}'$ by the foregoing lemma. Now let $\mathfrak{S}''$ be the substructure of $\mathfrak{B}$ generated by b over $\mathfrak{B}'$. Then by hypothesis, $\mathfrak{B}'$ is existentially closed in $\mathfrak{S}''$. By the transitivity of $\prec_\exists$, also $\mathfrak{A}$ is existentially closed in $\mathfrak{S}''$. Since $\mathfrak{S}''$ contains $\mathfrak{S}$, it follows that $\mathfrak{A}$ is also existentially closed in $\mathfrak{S}$. This establishes the induction step and thereby proves our lemma. □

The idea of using induction over the number of generators of a finitely generated extension can also be used restricted to one given extension $\mathfrak{A} \subset \mathfrak{B}$ of models of $\mathbf{T}'$. Assume that $\mathbf{T}'$ is a hull extension of $\mathbf{T}$. Given elements $b_1, \ldots, b_n \in B$, we construct a chain of models of $\mathbf{T}'$ as follows. We take $\mathfrak{S}_1$ to be the substructure of $\mathfrak{B}$ which is generated by b_1 over $\mathfrak{A}$, and $\mathfrak{B}_1$ a $\mathbf{T}'$-hull of $\mathfrak{S}_1$ in $\mathfrak{B}$. Then we take $\mathfrak{S}_2$ to be the substructure of $\mathfrak{B}$ which is generated by b_2 over $\mathfrak{B}_1$, and $\mathfrak{B}_2$ a $\mathbf{T}'$-hull of $\mathfrak{S}_2$ in $\mathfrak{B}$. By induction, we proceed until we have found $\mathfrak{S}_n$ and $\mathfrak{B}_n$. We set $\mathfrak{B}_0 := \mathfrak{A}$. To show that $\mathfrak{A}$ is existentially closed in $\mathfrak{S}_n$, it suffices to show that every $\mathfrak{B}_i$ is existentially closed in $\mathfrak{B}_{i+1}$, for $0 \leq i < n$. For this, by Lemma 13.58 it suffices to show that every $\mathfrak{B}_i$ is existentially closed in $\mathfrak{S}_{i+1}$. We have thus proved:

Lemma 13.60 *Assume that* $\mathbf{T}'$ *is a hull extension of* $\mathbf{T}$*. Let* $\mathfrak{A} \subset \mathfrak{B}$ *be models of* $\mathbf{T}'$*, and suppose that the following holds: If* $\mathfrak{B}'$ *is a model of* $\mathbf{T}'$ *such that* $\mathfrak{A} \subset \mathfrak{B}' \subset \mathfrak{B}$*, then* $\mathfrak{B}'$ *is existentially closed in every substructure of* $\mathfrak{B}$ *which is generated over* $\mathfrak{B}'$ *by one element of* $\mathfrak{B}$*. Then* $\mathfrak{A} \prec_\exists \mathfrak{B}$*.*

The meaning of this lemma is that we are **slicing** the extension $\mathfrak{A} \subset \mathfrak{B}$ to obtain smaller extensions which we can handle more easily. For instance, if $\mathbf{T}'$ is some theory of fields, then extensions generated by one element are of transcendence degree at most one. If in this situation the $\mathbf{T}'$-hull is always an algebraic extension, then also the extensions $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ of models of $\mathbf{T}'$ are all of transcendence degree at most one. In the case of groups, the analogue are extensions $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ such that the rational rank of $\mathfrak{B}_{i+1}/\mathfrak{B}_i$ is at most one. Even more importantly, to find an embedding of $\mathfrak{B}_{i+1}$ over $\mathfrak{B}_i$ in some sufficiently saturated elementary extension $\mathfrak{B}_i^*$ of $\mathfrak{B}_i$, we only have to realize the quantifier free type $\mathbf{t}^{\mathrm{qf}}_{(\mathfrak{B}_{i+1},B_i)}(b_{i+1})$ in $\mathfrak{B}_i^*$, or equivalently, we have to show that $\mathbf{t}^{\mathrm{qf}}_{\mathfrak{B}_i}(b_{i+1})$ is finitely realizable in $(\mathfrak{B}_i, B_i)$. That is, we just have to show that $\mathfrak{B}_i$ is 1-existentially closed in $\mathfrak{B}_{i+1}$. The reduction of "existentially closed" to "1-existentially closed" is a principle that we will use several times, but it can not be expected that all instances of model completeness of elementary theories can be proved by such a reduction.

To illustrate the fact that "1-existentially closed" is (in general) weaker than "existentially closed", consider the following example.

Example 13.61 Let K be a field which is not perfect. In Example 6.52 we have shown that there is a nontrivial extension $L|K$ such that K is relatively algebraically closed in L, but $L|K$ is not separable. By the former property, every element $b \in L$ is transcendental over K. So the subextension $K(b)$ generated by b over K is embeddable over K in every extension of K which is not algebraic. If we take K^* to be a $|K|^+$-saturated elementary extension of K, then it must have higher cardinality than K, so it can not be algebraic over K (this can also be inferred from Lemma 13.37). So $K(b)$ can be embedded in K^*, which shows that $K \prec_\exists K(b)$. This proves that K is 1-existentially closed in L. But since $L|K$ is not separable, Lemma 13.37 shows that K is not existentially closed in L. ♢

13.11 Ordered abelian groups

In this section, we shall apply the model theoretic machinery as developed so far, to treat two distinguished theories of ordered abelian groups. Recall that every ordered abelian group is torsion free, as follows from (NVZ). We have already treated the theory of nontrivial torsion free abelian groups. Let us now add the order. We use the language $\mathcal{L}_{\mathrm{OG}} = \{<, +, -, 0\}$ of ordered groups. The **theory of nontrivial ordered abelian groups** is given by the axioms for abelian groups together with (NTG), (DIV), (O$\not$R), (OT), (OC) and (OG).

We need two key lemmas. When we write $(H, <) \to (H', <)$, then we mean an **isomorphism of ordered groups**. If it fixes a common subgroup G of H and H' elementwise, then it is an **isomorphism over** G.

Lemma 13.62 *Let* $(G, <) \subset (H, <)$ *and* $(G, <) \subset (G^*, <)$ *be two extensions of ordered abelian groups. Further, let* $a \in H \setminus G$ *and* $b \in G^* \setminus G$*. Assume that* a *and* b *induce the*

same cut in G and that G is divisible. Then the assignment $a \mapsto b$ induces an isomorphism

$$(G + \mathbb{Z}a, <) \longrightarrow (G + \mathbb{Z}b, <)$$

over G.

Proof: Since $a \notin G$, $b \notin G$ and G is divisible, we know that a and b are torsion free over G. Thus, $a \mapsto b$ induces an isomorphism of the abelian groups $G + \mathbb{Z}a$ and $G + \mathbb{Z}b$. An element $g + na$, $g \in G$, $n \in \mathbb{Z}$, is sent to $g + nb$. We have to show that this isomorphism is order preserving. So for $g' \in G$ and $n' \in \mathbb{Z}$, we have to show that $g + na < g' + n'a$ if and only if $g + nb < g' + n'b$. If $n = n'$, then $g + na < g' + n'a \Leftrightarrow g < g' \Leftrightarrow g + nb < g' + n'b$. If $n \neq n'$, then

$$g + na < g' + n'a \iff (n - n')a < g' - g \iff a < \frac{g' - g}{n - n'} ,$$

and similarly with b in the place of a. Since G is divisible, $\frac{g'-g}{n-n'}$ is an element of G. Consequently, $g + na < g' + n'a \Leftrightarrow g + nb < g' + n'b$ holds for all g, g' and n, n' if and only if

$$\{g \in G \mid a < g\} = \{g \in G \mid b < g\} .$$

But this is the same as saying that a and b induce the same cut in G. □

The condition that G be divisible implies that $\frac{g'-g}{n-n'} \in G$, and this is a crucial point in the proof. For example, the cut $(-\mathbb{N} \cup \{0\}, \mathbb{N})$ in the ordered abelian group $\mathbb{Z}$ is induced by all irrational numbers in the interval $(0, 1)$. Although irrationals are torsion free over $\mathbb{Z}$ by definition, we will have that $(\mathbb{Z} + a\mathbb{Z}, <)$ and $(\mathbb{Z} + b\mathbb{Z}, <)$ are only isomorphic if $a = b$. So the cut that they induce in $\mathbb{Z}$ contains not enough information, in contrast to the cut that they induce in $\mathbb{Q}$.

The assertion of the foregoing lemma can also be read as follows:

Corollary 13.63 *Let $(H, <)$ be an extension of the divisible ordered abelian group $(G, <)$. Then for every $a \in H \setminus G$, its quantifier free type over $(G, <)$ is determined by the cut that a induces in G and the assertion that $a \notin G$.*

Indeed, if $b \in H' \setminus G$ induces the same cut in $(G, <)$, then by Lemma 13.62 there is an isomorphism $(G + \mathbb{Z}a, <) \to (G + \mathbb{Z}b, <)$ over G, which shows that a and b have the same quantifier free type over $(G, <)$.

A further consequence of the foregoing lemma is the following **embedding lemma**:

Lemma 13.64 *Let $(H, <)$ and $(G^*, <)$ be extensions of the divisible ordered abelian group $(G, <)$. Assume $(G^*, <)$ to be $|G|^+$-saturated. Then for every $a \in H \setminus G$, there is an embedding of $(G + \mathbb{Z}a, <)$ in $(G^*, <)$ over G.*

Proof: In view of Lemma 13.62, we just have to find an element $b \in G^* \setminus G$ which induces the same cut as a in G. Since $a \notin G$, we have that $\{g \in G \mid g \leq a\} = \{g \in G \mid g < a\}$. We consider the following subtype of the quantifier free type of a over $(G, <)$:

$$\mathbf{t}^{\mathrm{cut}}_{(G,<)}(a) := \{g < x \mid g \in G \wedge g < a\} \cup \{x < g \mid g \in G \wedge a < g\} .$$

Let $\mathbf{t}_0$ be a finite subtype of $\mathbf{t}^{\mathrm{cut}}_{(G,<)}(a)$. Then we take $g_1 \in G$ to be the maximum of the g's that appear in sentences of the form $g < x$ in $\mathbf{t}_0$, and $g_2 \in G$ to be the minimum of the g's

that appear in sentences of the form $x < g$ in $\mathbf{t}_0$. We have that $g_1 < a < g_2$. Since G is divisible, it is dense and thus, there is $g_0 \in G$ such that $g_1 < g_0 < g_2$. For every sentence $g < x$ in $\mathbf{t}_0$, we have that $g \leq g_1 < g_0$. For every sentence $x < g$ in $\mathbf{t}_0$, we have that $g_0 < g_2 \leq g$. This proves that g_0 realizes $\mathbf{t}_0$ in $(G, <)$.

By the saturation of G^*, we can choose an element $b \in G^*$ which realizes $\mathbf{t}^{\mathrm{cut}}_{(G,<)}(a)$ in G^*. But this just means that b induces the same cut as a in G. For every $g \in G$, we have that $g < b$ or $b < g$, whence $b \notin G$. □

If we take $(G^*, <)$ to be a $|G|^+$-saturated elementary extension of $(G, <)$, then in view of Lemma 13.38 we find that $(G, <)$ is existentially closed in $(G + \mathbb{Z}a, <)$, for every $a \in H$. This shows that $(G, <)$ is existentially closed in every extension generated by one element of an arbitrary divisible ordered abelian group which contains $(G, <)$. Since by part c) of Lemma 13.56, the theory of divisible ordered abelian groups is a hull extension of the theory of ordered abelian groups, we can now apply our slicing approach. By the Prime Structure Test (Lemma 13.59) and by Corollary 13.42, we obtain:

Theorem 13.65 *The theory of nontrivial divisible ordered abelian groups admits quantifier elimination and is model complete.*

If $(G, <)$ is a nontrivial ordered abelian group, then we can pick a nonzero positive element a in G. The assignment $1 \mapsto a$ then induces an embedding of the ordered abelian group $(\mathbb{Z}, <)$ in $(G, <)$. This shows that $(\mathbb{Z}, <)$ is a prime structure for the theory of nontrivial divisible ordered abelian groups. By Lemma 13.52 and in view of the recursive axioms, we thus obtain:

Theorem 13.66 *The theory of nontrivial divisible ordered abelian groups is complete and decidable.*

Now we turn to a second important theory of ordered abelian groups. We wish to investigate the elementary theory of $(\mathbb{Z}, <)$ and to determine the ordered abelian groups which are elementarily equivalent to $(\mathbb{Z}, <)$. Whenever we have $(\mathbb{Z}, <) \prec_\exists (G, <)$, then the element 1 of $\mathbb{Z}$ must also be the least positive element of $(G, <)$. It does not suffice to require by an axiom that there exists a least positive element. For instance, $(\mathbb{Z}, <)$ is not existentially closed in $(\frac{1}{2}\mathbb{Z}, <)$, although both are isomorphic and thus also elementarily equivalent. To obtain model completeness, it is necessary to add a constant symbol for the least positive element to the language of ordered groups so that we are able to say by an axiom that this element (which remains the same in every extension) is the least positive element. So we define the **language of $\mathbb{Z}$-groups** to be $\mathcal{L}_{\mathrm{ZG}} := \mathcal{L}_{\mathrm{OG}} \cup \{1\} = \{<, +, -, 0, 1\}$. The **theory of $\mathbb{Z}$-groups** is given by the axioms for abelian groups together with (O$\not{\mathbb{R}}$), (OT), (OC), (OG) and

(ZG1) $\quad \forall x : \ 0 < 1 \ \wedge \ \neg(0 < x < 1) \ .$

(ZG2) $\quad \forall x \, \exists y : \ ny = x \ \vee \ ny = x + 1 \ \vee \ldots \vee \ ny = x + n - 1 \qquad (n \in \mathbb{N}) \, .$

Axiom (ZG1) says that 1 is the least positive element of the group. By the condition that $0 < 1$, the group is automatically nontrivial and therefore, we have dropped axiom (NTG). If $(G, <)$ is an ordered abelian group satisfying (ZG1), then $(\mathbb{Z}1, <)$ is a convex subgroup of $(G, <)$ (we will denote it just by $\mathbb{Z}$). Axiom scheme (ZG2) now says that the ordered

abelian group $G/\mathbb{Z}$ is divisible. The simplest example for a $\mathbb{Z}$-group not isomorphic to $\mathbb{Z}$ is $\mathbb{Q} \amalg \mathbb{Z}$. But note that in general, $G/\mathbb{Z}$ is not necessarily a summand of the $\mathbb{Z}$-group G (see our discussion preceding to Lemma 3.33. For this reason, the theory of $\mathbb{Z}$-groups is not a hull extension of the theory of its substructures (which is the theory of all ordered abelian groups which satisfy (ZG1)). To give an example, we start with the ordered abelian group $G_0 := \mathbb{Z} \amalg \mathbb{Z}$. Now we take the element $a := (\frac{1}{2}, \frac{1}{2})$ and define $G_1 := G_0 + \mathbb{Z}a$. This group still satisfies (ZG1). By Zorn's Lemma, one can extend G_1 to a subgroup of the divisible hull of G_0 which is maximal with respect to the property (ZG1). Such a group $(G, <)$ can be shown to be a $\mathbb{Z}$-group. On the other hand, there is no $\mathbb{Z}$-group extending $\mathbb{Z} \amalg \mathbb{Z}$ and embedding over $\mathbb{Z} \amalg \mathbb{Z}$ in both $\mathbb{Q} \amalg \mathbb{Z}$ and G. Indeed, in $\mathbb{Q} \amalg \mathbb{Z}$ there is an element b such that $2b = (1, 0)$ whereas in G there can not be such an element since otherwise, $(0, \frac{1}{2}) = a - b \in G$. Since $\exists x\, 2x = (1, 0)$ is a sentence with parameter $(1, 0)$ from $\mathbb{Z} \amalg \mathbb{Z}$, we also see that $(\mathbb{Q} \amalg \mathbb{Z}, <)$ and $(G, <)$ are not equivalent over $(\mathbb{Z} \amalg \mathbb{Z}, <)$. We conclude that the theory of $\mathbb{Z}$-groups does not admit quantifier elimination. Nevertheless, we can show that it is model complete and complete.

To show the model completeness, we take up again the idea of slicing given extensions of $\mathbb{Z}$-groups. We can use Lemma 3.33 which tells us that the relative divisible closure of a substructure in a $\mathbb{Z}$-group is again a $\mathbb{Z}$-group. Note that since we are using the language of $\mathbb{Z}$-groups, a substructure of a $\mathbb{Z}$-group also contains 1. Now if $(G, <) \subset (H, <)$ is an extension of $\mathbb{Z}$-groups and $b_1, \ldots, b_n \in H$, then by Lemma 3.33 we find a chain of $\mathbb{Z}$-groups $(G_i, <) \subset (H, <)$ such that $G_0 = G$ and every G_{i+1} lies in the divisible hull of $G_i + \mathbb{Z}b_{i+1}$. It thus suffices to show that $(G_i, <) \prec_\exists (G_{i+1}, <)$ for $i = 0, \ldots, n-1$. But since we do not have uniqueness for the relative divisible closure in a $\mathbb{Z}$-group, the assertion of Lemma 13.60 fails. Hence it does not suffice to show that $(G_i, <) \prec_\exists (G_i + \mathbb{Z}b_{i+1})$. Nevertheless, we can reduce again to extensions which are generated by just one element. Indeed, Lemma 13.40 shows that it suffices to prove that $(G_i, <) \prec_\exists (G_i', <)$ for every subgroup G_i' of G_{i+1} which is finitely generated over G_i. But G_{i+1} lies in the divisible hull of $G_i + \mathbb{Z}b_{i+1}$, which yields that the rational rank of G_{i+1}/G_i and hence also of G_i'/G_i is at most one. Since the latter is finitely generated, and since G_{i+1}/G_i is torsion free by virtue of Lemma 3.33, it follows that G_i'/G_i is isomorphic to $\mathbb{Z}$ (or that $G_i' = G_i$). Choose $a \in G_i'$ such that $a + G_i$ generates G_i'/G_i. Then a generates G_i' over G_i. Hence it indeed suffices to show that $(G_i, <)$ is existentially closed in $(G_i + \mathbb{Z}a, <)$ for every $a \in G_{i+1}$. This is settled by use of the following analogue of Lemma 13.62.

Lemma 13.67 *Let $(G, <) \subset (H, <)$ and $(G, <) \subset (H', <)$ be two extensions of $\mathbb{Z}$-groups. Further, let $a \in H \setminus G$ and $b \in H' \setminus G$. Assume that a and b induce the same cut in G. Then the assignment $a \mapsto b$ induces an isomorphism $(G + \mathbb{Z}a, <) \to (G + \mathbb{Z}b, <)$ over G.*

Proof: Since $(G, <)$, $(H, <)$ and $(H', <)$ all have the same least positive element 1, we know that $\mathbb{Z}$ is a common convex subgroup of all of them, and that $(G/\mathbb{Z}, <)$, $(H/\mathbb{Z}, <)$ and $(H'/\mathbb{Z}, <)$ are divisible ordered abelian groups. Since $a, b \notin G$, it follows that $\overline{a} := a + \mathbb{Z} \notin G/\mathbb{Z}$ and $\overline{b} := b + \mathbb{Z} \notin G/\mathbb{Z}$. Since a, b induce the same cut in $(G, <)$, it follows that $\overline{a}, \overline{b}$ induce the same cut in $(G/\mathbb{Z}, <)$. We can thus infer from Lemma 13.62 that the assignment $\overline{a} \mapsto \overline{b}$ induces an isomorphism $(G/\mathbb{Z} + \mathbb{Z}\overline{a}, <) \to (G/\mathbb{Z} + \mathbb{Z}\overline{b}, <)$ over $G/\mathbb{Z}$. If $g \in G$ such that $g + \mathbb{Z} =: \overline{g} < n\overline{a}$, then $g + \mathbb{Z} < na$, for every $n \in \mathbb{Z}$. The same holds with $>$ in the place of $<$, and with b in the place of a. This shows that the order on $(G + \mathbb{Z}a, <)$ is uniquely determined by the order on $(G/\mathbb{Z} + \mathbb{Z}\overline{a}, <)$, and similarly for b in the place of a. Hence, also $a \mapsto b$ induces an isomorphism $(G + \mathbb{Z}a, <) \to (G + \mathbb{Z}b, <)$ over G. □

The proofs of Lemma 13.64 and the argument subsequent to that lemma can be taken over to show:

Corollary 13.68 *Let $(G,<) \subset (H,<)$ be an extension of $\mathbb{Z}$-groups (with respect to the language $\mathcal{L}_{\mathrm{ZG}}$ of $\mathbb{Z}$-groups), and $a \in H$. Then $(G + \mathbb{Z}a,<)$ can be embedded over G in every $|G|^+$-saturated extension of $(G,<)$. Consequently, $(G,<) \prec_\exists (G + \mathbb{Z}a,<)$.*

We have now proved that $(G,<) \prec_\exists (H,<)$ for every extension $(G,<) \subset (H,<)$ of $\mathbb{Z}$-groups (with respect to the language $\mathcal{L}_{\mathrm{ZG}}$ of $\mathbb{Z}$-groups). Robinson's Test (Theorem 13.47) allows us to deduce:

Theorem 13.69 *The theory of $\mathbb{Z}$-groups (in the language $\mathcal{L}_{\mathrm{ZG}}$) is model complete.*

Since $(\mathbb{Z},<)$ is the prime model of the theory of $\mathbb{Z}$-groups, we can use Lemma 13.48 to conclude:

Theorem 13.70 *The theory of $\mathbb{Z}$-groups is complete (in the language $\mathcal{L}_{\mathrm{OG}}$).*

(Here, we do not need the constant symbol for the least positive element, because $\mathbb{Z}$ can always be embedded by sending 1 to the least positive element.)

We have already seen above that we have a problem concerning quantifier elimination for $\mathbb{Z}$-groups. To obtain quantifier elimination, we have to get rid of those common substructures of two $\mathbb{Z}$-groups which do not have the same relative divisible closure in both groups. This can be done by introducing for every $n \in \mathbb{N}$, $n > 1$, a new predicate symbol P_n to the language $\mathcal{L}_{\mathrm{ZG}}$ and fixing its interpretations by the additional axiom scheme

(PDIV) $\qquad P_n(x) \Leftrightarrow \exists y\, x = ny$

Since "$P_n(x)$" can always be replaced by "$\exists y\, x = ny$", we obtain model completeness and completeness also for the theory of $\mathbb{Z}$-groups together with (PDIV) in the language $\mathcal{L}_{\mathrm{ZG}} \cup \{P_n \mid 1 < n \in \mathbb{N}\}$. The important difference is that now we have only well-behaved common substructures. To see this, let $(H,<,P_n \mid 1 < n \in \mathbb{N})$ and $(H',<,P_n \mid 1 < n \in \mathbb{N})$ be $\mathbb{Z}$-groups satisfying (PDIV), and let $(G,<,P_n \mid 1 < n \in \mathbb{N})$ a common substructure of $(H,<,P_n \mid 1 < n \in \mathbb{N})$ and $(H',<,P_n \mid 1 < n \in \mathbb{N})$. Then by the definition of a substructure, the restrictions of P_n from $(H,<,P_n \mid 1 < n \in \mathbb{N})$ and $(H',<,P_n \mid 1 < n \in \mathbb{N})$ to G both must coincide with the interpretation of P_n on G. That is, for every element $g \in G$, the sentence $P_n(g)$ holds in $(H,<,P_n \mid 1 < n \in \mathbb{N})$ if and only if it holds in $(H',<,P_n \mid 1 < n \in \mathbb{N})$. Since $(H,<,P_n \mid 1 < n \in \mathbb{N})$ and $(H',<,P_n \mid 1 < n \in \mathbb{N})$ are models of (PDIV), this just tells us that g/n lies in H if and only if it lies in H' (up to isomorphism). We conclude that the relative divisible closure H_0 of G in H is (up to isomorphism) the same as the relative divisible closure H_0' of G in H'. By Lemma 3.33, $(H_0,<)$ is a $\mathbb{Z}$-group. Since we know already that the theory of $\mathbb{Z}$-groups is model complete, we conclude that $(H,<)$ and $(H',<)$ are equivalent over $(H_0,<)$. It follows that $(H,<,P_n \mid 1 < n \in \mathbb{N})$ and $(H',<,P_n \mid 1 < n \in \mathbb{N})$ are equivalent over $(H_0,<,P_n \mid 1 < n \in \mathbb{N})$ and thus also over $(G,<,P_n \mid 1 < n \in \mathbb{N})$. We have proved:

Theorem 13.71 *The theory of $\mathbb{Z}$-groups together with the axiom scheme (PDIV) admits quantifier elimination in the language $\mathcal{L}_{\mathrm{ZG}} \cup \{P_n \mid 1 < n \in \mathbb{N}\}$.*

The model theoretic results of this section are due to A. Robinson and E. Zakon [ROB–Z]. In their terminology, a $\mathbb{Z}$-group is called **regularly discrete group**.

For further information on the model theory of ordered abelian groups, see V. Weispfenning [WEI5,9,11].

13.12 Ultraproducts

Suppose that we are given a set of $\mathcal{L}$-structures. Is it possible to combine their elementary properties in a single $\mathcal{L}$-structure? One might be tempted to form the product over these $\mathcal{L}$-structures. But it is not clear from the start how it can be made an $\mathcal{L}$-structure. For example, if we take the product over a set of fields, then this product will be a ring (and thus an $\mathcal{L}_{\mathrm{R}}$-structure) if we define addition and multiplication componentwise. But we have difficulties with the existence of multiplicative inverses. It is necessary that the neutral element for the multiplication is the tuple with 1 at every entry. But then, a tuple with a 0 at an arbitrary entry is not invertible. So if we take a product over more than one field, that is, if we are working with tuples having at least two entries, then all of a sudden, we have more than one element which is not invertible. But a field only allows one single element to be not invertible, namely the neutral element for the addition. Everything would be fine again if we were able to replace the ring by a quotient in such a way that all non-invertible elements are identified (loosely speaking, we would like to "mod out the difficulties"). This is, at the level of a very simple example, the main idea of ultraproducts.

The suitable quotients of the products are found by use of ultrafilters, which we are going to define now. Let I be a nonempty set and $\mathcal{D}$ a subset of the power set $\mathcal{P}(I) = \{J \mid J \subset I\}$. Then $\mathcal{D}$ is called a **filter on** I if it is nonempty and satisfies

(FI1) $\quad \emptyset \notin \mathcal{D}\,,$

(FI2) $\quad J, J' \in \mathcal{D} \implies J \cap J' \in \mathcal{D}\,,$

(FI3) $\quad J \in \mathcal{D} \wedge J \subset J' \subset I \implies J' \in \mathcal{D}\,.$

If the filter $\mathcal{D}$ satisfies also the condition

(UFI) $\quad J \subset I \wedge J \notin \mathcal{D} \implies I \setminus J \in \mathcal{D}\,,$

then it is called an **ultrafilter on** I. Note that for an ultrafilter $\mathcal{D}$ on I and every $J \subset I$, *either* J *or* $I \setminus J$ is in $\mathcal{D}$. Indeed, it follows from (FI2) and (FI1) that not both sets can simultaneously be in $\mathcal{D}$. A filter on I is called **principal** if it is of the form $\{J \subset I \mid i \in J\}$ for some $i \in I$; otherwise, it is called **nonprincipal**. Note that every principal filter is an ultrafilter. The reader may show that every filter on a finite set I is principal. If $\mathcal{D}$ is a nonprincipal ultrafilter, then for every $i \in I$, the set $I \setminus \{i\}$ is contained in $\mathcal{D}$. The reader may show that a nonprincipal ultrafilter can not contain finite sets.

To give an example for a nonprincipal filter, we let I be infinite. A subset $J \subset I$ is called **cofinite** if $I \setminus J$ is finite. The set of all cofinite subsets of I is a filter on I. But since I admits infinite subsets whose complements in I are also infinite, this filter is not an ultrafilter. But it can be extended to an ultrafilter:

Lemma 13.72 *Let $\mathcal{D}_0 \subset \mathcal{P}(I)$ satisfy* (FI1) *and* (FI2)*. Then there exists an ultrafilter $\mathcal{D} \subset \mathcal{P}(I)$ which contains $\mathcal{D}_0$. If $\mathcal{D}_0$ contains all cofinite subsets of I, then $\mathcal{D}$ is nonprincipal.*

Proof: Using Zorn's Lemma, one shows the existence of a maximal subset $\mathcal{D} \subset \mathcal{P}(I)$ containing $\mathcal{D}_0$ and satisfying (FI1) and (FI2). We have to show that $\mathcal{D}$ satisfies (FI3) and (UFI). It is straightforward to show that the set

$$\mathcal{D}' := \{J' \subset I \mid \exists J \in \mathcal{D} :\ J \subset J'\}$$

satisfies (FI1) and (FI2). By the maximality of $\mathcal{D}$ it follows that $\mathcal{D} = \mathcal{D}'$, showing that $\mathcal{D}$ satisfies (FI3). In particular, $I \in \mathcal{D}$.

To show that $\mathcal{D}$ satisfies (UFI), let $\emptyset \neq J \subset I$. We have to show that $J \in \mathcal{D}$ or $I \setminus J \in \mathcal{D}$. It is again straightforward to prove that the set

$$\mathcal{D} \cup \{J \cap J' \mid J' \in \mathcal{D}\}$$

satisfies (FI2). If it does not satisfy (FI1), then $J \cap J' = \emptyset$ for some $J' \in \mathcal{D}$. But this means that $J' \subset I \setminus J$. Since we have already shown that $\mathcal{D}$ satisfies (FI3), we obtain that $I \setminus J \in \mathcal{D}$, and we are done. If on the other hand the set satisfies (FI1), then it must be equal to $\mathcal{D}$ in view of the maximality of the latter. Because of $I \in \mathcal{D}$, we then obtain that $J = I \cap J \in \mathcal{D}$.

Now assume that $\mathcal{D}_0$ contains all cofinite subsets of I. Then for every $i \in I$, it contains $I \setminus \{i\}$. Thus, $\mathcal{D}$ can not contain $\{i\}$. This proves that $\mathcal{D}$ must be non-principal. □

Corollary 13.73 *Let J be an infinite subset of I. Then there is a nonprincipal ultrafilter $\mathcal{D}$ on I such that $J \in \mathcal{D}$.*

Proof: Let $\mathcal{D}_0$ be the set of all cofinite subsets of I. It satisfies (FI1) and (FI2). Also the set $\mathcal{D}' := \mathcal{D}_0 \cup \{J \cap J' \mid J' \in \mathcal{D}_0\}$ satisfies (FI2). Since J is assumed to be infinite, also $J \cap J'$ is infinite, for every $J' \in \mathcal{D}_0$. This shows that $\mathcal{D}'$ satisfies (FI1). By the foregoing lemma, there is an ultrafilter $\mathcal{D}$ containing $\mathcal{D}'$, and it is nonprincipal by our choice of $\mathcal{D}_0 \subset \mathcal{D}'$. Moreover, it contains J because $I \in \mathcal{D}_0$ and $J = J \cap I \in \mathcal{D}'$. □

Let I be a nonempty set, $(\mathfrak{A}_i)_{i \in I}$ a family of nonempty $\mathcal{L}$-structures and $\mathcal{D}$ an ultrafilter on I. On the product

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid \forall i \in I : \ a_i \in A_i\}$$

we define an equivalence relation by use of $\mathcal{D}$ as follows:

$$(a_i)_{i \in I} \sim_{\mathcal{D}} (a'_i)_{i \in I} \quad \text{if and only if} \quad \{i \in I \mid a_i = a'_i\} \in \mathcal{D} \ .$$

The reflexivity follows from the fact that $I \in \mathcal{D}$. The symmetry is trivial. To prove the transitivity, observe that

$$\{i \in I \mid a_i = a'_i\} \cap \{i \in I \mid a'_i = a''_i\} \subset \{i \in I \mid a_i = a''_i\} \ .$$

If the first two sets are in $\mathcal{D}$, then also their intersection, because of (FI2). But then also the third set lies in $\mathcal{D}$, because of (FI3). On the set

$$A^* = \prod_{i \in I} A_i \,/\, \mathcal{D} \ := \ \prod_{i \in I} A_i \,/\, \sim_{\mathcal{D}} \ ,$$

we interprete the relation, function and constant symbols of $\mathcal{L}$ as follows. For every $(a_i) := (a_i)_{i \in I} \in \prod_{i \in I} A_i$, we denote its $\sim_{\mathcal{D}}$-equivalence class by $\overline{(a_i)}$. For every relation symbol $R \in \mathcal{R}$, we set

$$R\,(\,\overline{(a_i)}\,,\,\overline{(b_i)}\,,\,\ldots\,) \quad \text{if and only if} \quad \{i \in I \mid R(a_i, b_i, \ldots)\} \in \mathcal{D} \ .$$

We have to show that this definition does not depend upon the representatives $(a_i), (b_i), \ldots$ of the equivalence classes $\overline{(a_i)}, \overline{(b_i)}, \ldots$. So let $(a_i) \sim_{\mathcal{D}} (a'_i)$, $(b_i) \sim_{\mathcal{D}} (b'_i)$, $\ldots$, and let J be

the intersection of the finitely many sets $\{i \in I \mid a_i = a_i'\} \in \mathcal{D}$, $\{i \in I \mid b_i = b_i'\} \in \mathcal{D}, \ldots$. Then $J \in \mathcal{D}$ and

$$\{i \in I \mid R(a_i, b_i, \ldots)\} \cap J \subset \{i \in I \mid R(a_i', b_i', \ldots)\} .$$

So if the former set lies in $\mathcal{D}$, then also the latter set lies in $\mathcal{D}$. Since the situation is symmetrical, this shows that our definition is indeed independent of the choice of representatives.

Analogously, for every function symbol $f \in \mathcal{F}$ we set

$$f(\overline{(a_i)}, \overline{(b_i)}, \ldots) := \overline{(f(a_i), f(b_i), \ldots)} .$$

Finally, if the constant symbol $c \in \mathcal{C}$ is interpreted by c_i in $\mathfrak{A}_i$ for every $i \in I$, then we let its interpretion in A^* be

$$\overline{(c_i)} .$$

The independence of the choice of representatives is shown as before. By these interpretations, the set A^* becomes an $\mathcal{L}$-structure. We call it the **ultraproduct of the structures** $\mathfrak{A}_i$, $i \in I$, **with respect to** $\mathcal{D}$, and we denote it by $\prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$.

Since we have defined the functions and constants componentwise, and since the canonical surjection $^- : \prod_{i\in I} A_i \to \prod_{i\in I} A_i/\mathcal{D}$ is a homomorphism, it follows that for every $\mathcal{L}$-term $t(x_1, \ldots, x_n)$,

$$t(\overline{(a_i)}, \overline{(b_i)}, \ldots) = \overline{(t(a_i, b_i, \ldots))} . \tag{13.5}$$

Note that the definition of the $\mathcal{L}$-structure $\prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$ has only used that $\mathcal{D}$ is a filter. For such, one calls this structure the **reduced product of the structures** $\mathfrak{A}_i$, $i \in I$, **with respect to** $\mathcal{D}$. But property (UFI) will be necessary for the assertion of the following very important theorem, which is due to J. Łos.

Theorem 13.74 (Fundamental Theorem of Ultraproducts)
Let $\mathfrak{A}^*$ be the ultraproduct of the $\mathcal{L}$-structures $\mathfrak{A}_i$, $i \in I$, with respect to the ultrafilter $\mathcal{D}$. Then for every $\mathcal{L}$-sentence φ,

$$\mathfrak{A}^* \models \varphi \quad \textit{if and only if} \quad \{i \in I \mid \mathfrak{A}_i \models \varphi\} \in \mathcal{D} .$$

Proof: Before we begin with the proof, we want to expand every $\mathfrak{A}_i$ to an $\mathcal{L}(A^*)$-structure. In order to do so, we choose for every $a \in A^*$ suitable elements $a_i \in A_i$ such that $a = (a_i)$. Then we add the constant symbol c_a to $\mathcal{L}$ and interprete it by a in $\mathfrak{A}^*$ and by a_i in $\mathfrak{A}_i$. So we can assume from the start that $\mathcal{L}$ contains a constant symbol for every $a \in A^*$.

The proof proceeds by induction on the complexity of φ. The proof for atomic $\mathcal{L}$-sentences works with (13.5) and the same idea that we have used to show that our definition of the relations on A^* is independent of the choice of representatives. So we leave it to the reader.

Now assume that we have proved the assertion for all $\mathcal{L}$-sentences of lower complexity than φ. First, let φ be of the form $\neg\psi$. At this point, we have to use the property (UFI) of an ultrafilter. Using also the induction hypothesis, we obtain

$$\begin{aligned} \mathfrak{A}^* \models \varphi &\iff \mathfrak{A}^* \not\models \psi \iff \{i \in I \mid \mathfrak{A}_i \models \psi\} \notin \mathcal{D} \iff I \setminus \{i \in I \mid \mathfrak{A}_i \models \psi\} \in \mathcal{D} \\ &\iff \{i \in I \mid \mathfrak{A}_i \not\models \psi\} \in \mathcal{D} \iff \{i \in I \mid \mathfrak{A}_i \models \varphi\} \in \mathcal{D} . \end{aligned}$$

Now let φ be of the form $\psi_1 \wedge \psi_2$. Using the induction hypothesis as well as (FI2) and (FI3), we obtain

$$\begin{aligned}
\mathfrak{A}^* \models \varphi &\iff \mathfrak{A}^* \models \psi_1 \text{ and } \mathfrak{A}^* \models \psi_2 \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi_1\} \in \mathcal{D} \text{ and } \{i \in I \mid \mathfrak{A}_i \models \psi_2\} \in \mathcal{D} \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi_1\} \cap \{i \in I \mid \mathfrak{A}_i \models \psi_2\} \in \mathcal{D} \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi_2 \text{ and } \mathfrak{A}_i \models \psi_2\} \in \mathcal{D} \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi_1 \wedge \psi_2\} \in \mathcal{D} \iff \{i \in I \mid \mathfrak{A}_i \models \varphi\} \in \mathcal{D} .
\end{aligned}$$

Finally, assume that φ is of the form $\forall x : \psi(x)$. Then

$$\begin{aligned}
\mathfrak{A}^* \models \varphi &\iff \mathfrak{A}^* \models \psi(a) \text{ for all } a = (a_i) \in A \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi(a_i)\} \in \mathcal{D} \text{ for all } a = (a_i) \in A \qquad (13.6)
\end{aligned}$$

and

$$\begin{aligned}
\{i \in I \mid \mathfrak{A}_i \models \varphi\} \in \mathcal{D} &\iff \{i \in I \mid \mathfrak{A}_i \models \forall x : \psi(x)\} \in \mathcal{D} \\
&\iff \{i \in I \mid \mathfrak{A}_i \models \psi(a_i) \text{ for all } a_i \in A_i\} \in \mathcal{D} . \qquad (13.7)
\end{aligned}$$

So we have to show that (13.6) and (13.7) are equivalent. Since for every fixed $b = (b_i) \in A$,

$$\{i \in I \mid \mathfrak{A}_i \models \psi(a_i) \text{ for all } a_i \in A_i\} \subset \{i \in I \mid \mathfrak{A}_i \models \psi(b_i)\} ,$$

it follows from the property (FI3) of $\mathcal{D}$ that (13.7) implies (13.6).

For the converse, assume that (13.7) does not hold. Then by property (UFI) of $\mathcal{D}$ we obtain that

$$J := \{i \in I \mid \mathfrak{A}_i \not\models \psi(a_i) \text{ for some } a_i \in A_i\} \in \mathcal{D} .$$

For every $i \in J$, we choose $a_i \in A_i$ such that $\mathfrak{A}_i \not\models \psi(a_i)$, and for every $i \notin J$, we choose an arbitrary $a_i \in A_i$. Then

$$\{i \in I \mid \mathfrak{A}_i \models \psi(a_i)\} \subset I \setminus J .$$

But $I \setminus J \notin \mathcal{D}$, hence also $\{i \in I \mid \mathfrak{A}_i \models \psi(a_i)\} \notin \mathcal{D}$ by (FI3), showing that (13.6) does not hold. This completes our proof. □

If we apply the assertion of the theorem to φ if $\mathfrak{A}^* \models \varphi$, and to $\neg\varphi$ if $\mathfrak{A}^* \models \neg\varphi$, we obtain:

$$\{i \in I \mid \mathfrak{A}^* \models \varphi \text{ if and only if } \mathfrak{A}_i \models \varphi\} \in \mathcal{D} . \qquad (13.8)$$

An important special case of the theorem is obtained if all $\mathfrak{A}_i$ are equal to a given $\mathcal{L}$-structure $\mathfrak{A}$. In this case,

$$\mathfrak{A}^I/\mathcal{D} := \prod_{i \in I} \mathfrak{A} \,/\, \mathcal{D}$$

is called the **ultrapower of $\mathfrak{A}$ with respect to $\mathcal{D}$**. It can be viewed as an elementary extension of $\mathfrak{A}$:

Corollary 13.75 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\mathcal{D}$ an ultrafilter on the set I. Then the map ι defined by $\iota(a) := \overline{(a)_{i \in I}}$ for every $a \in A$ is an elementary embedding of $\mathfrak{A}$ in the ultrapower $\mathfrak{A}^* = \mathfrak{A}^I/\mathcal{D}$.*

Proof: There is a canonical way to endow $\mathfrak{A}^I/\mathcal{D}$ with an $\mathcal{L}(A)$-structure. According to our definition of the interpretation of a constant symbol in $\prod A_i/\mathcal{D}$, the interpretation of the constant symbol for $a \in A$ in $\prod A/\mathcal{D}$ must be $\overline{(a)_{i\in I}}$. Hence by hypothesis, ι is just the constant term relation between $(\mathfrak{A}, A)$ and $(\mathfrak{A}^*, A)$. In view of part d) of Lemma 13.3, we have to show that $(\mathfrak{A}, A) \models \varphi \Rightarrow (\mathfrak{A}^*, A) \models \varphi$ for every $\mathcal{L}(A)$-sentence φ. But $(\mathfrak{A}^*, A)$ is just the ultrapower $(\mathfrak{A}, A)^I/\mathcal{D}$, and thus we obtain from the foregoing theorem that

$$(\mathfrak{A}, A) \models \varphi \implies \{i \in I \mid (\mathfrak{A}, A) \models \varphi\} = I \in \mathcal{D} \implies (\mathfrak{A}^*, A) \models \varphi \, .$$

□

As a further corollary to the above theorem, we give the ultraproduct version of the **Compactness Theorem** 13.13. It proves the Compactness Theorem without using the Completeness Theorem.

Theorem 13.76 *Let* $\mathbf{S}$ *be a set of* $\mathcal{L}$*-sentences and* $\mathcal{P}_\omega(\mathbf{S})$ *the set of all finite subsets of* $\mathbf{S}$*. Further, for every* $\mathbf{s} \in \mathcal{P}_\omega(\mathbf{S})$*, let* $\mathfrak{A}_\mathbf{s}$ *be a model of* $\mathbf{s}$*. Then there exists an ultrafilter* $\mathcal{D}$ *on* $\mathcal{P}_\omega(\mathbf{S})$ *such that*

$$\mathfrak{A}^* = \prod_{\mathbf{s}\in\mathcal{P}_\omega(\mathbf{S})} \mathfrak{A}_\mathbf{s} \,/\, \mathcal{D}$$

is a model of $\mathbf{S}$*.*

Proof: For every $\mathbf{s} \in \mathcal{P}_\omega(\mathbf{S})$, we define a nonempty subset

$$J_\mathbf{s} := \{\mathbf{s}' \in \mathcal{P}_\omega(\mathbf{S}) \mid \mathbf{s} \subset \mathbf{s}'\} \subset \mathcal{P}_\omega(\mathbf{S}) \, .$$

The set $\mathcal{D}_0 = \{J_\mathbf{s} \mid \mathbf{s} \in \mathcal{P}_\omega(\mathbf{S})\}$ satisfies (FI1), and it also satisfies (FI2) since for all $\mathbf{s}, \mathbf{s}' \in \mathcal{P}_\omega(\mathbf{S})$,

$$J_\mathbf{s} \cap J_{\mathbf{s}'} = J_{\mathbf{s}\cup\mathbf{s}'} \, .$$

Hence by Lemma 13.72 there exists an ultrafilter $\mathcal{D}$ containing $\mathcal{D}_0$.

Now let $\varphi \in \mathbf{S}$. Then $\varphi \in \mathbf{s}$ implies $\mathfrak{A}_\mathbf{s} \models \varphi$. Hence,

$$J_{\{\varphi\}} = \{\mathbf{s} \in \mathcal{P}_\omega(\mathbf{S}) \mid \varphi \in \mathbf{s}\} \subset \{\mathbf{s} \in \mathcal{P}_\omega \mid \mathfrak{A}_\mathbf{s} \models \varphi\} \, .$$

Since $J_{\{\varphi\}} \in \mathcal{D}_0 \subset \mathcal{D}$, by (FI3) it follows that

$$\{\mathbf{s} \in \mathcal{P}_\omega \mid \mathfrak{A}_\mathbf{s} \models \varphi\} \in \mathcal{D} \, .$$

By Theorem 13.74, this means that $\mathfrak{A}^* \models \varphi$. □

For the conclusion of this section, let us cite a theorem whose proof is beyond the scope of our introduction to model theory. We won't need this theorem since Theorem 13.35 does the same job in our applications. On the other hand, this theorem and the subsequent corollary illustrate nicely the power of ultraproducts and give a somewhat purely algebraic criterion for elementary equivalence and elementary classes.

Theorem 13.77 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* $\mathcal{L}$*-structures. Then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent if and only if they have isomorphic ultrapowers.*

For the proof, see [CHA–KE], Theorem 6.1.15 . Let be given a class $\mathbf{C}$ of $\mathcal{L}$-structures of $\mathcal{L}$-structures. We say that $\mathbf{C}$ is **closed under ultraproducts** if whenever $\mathfrak{A}_i \in \mathbf{C}$, $i \in I$, then every ultraproduct over the $\mathfrak{A}_i$ belongs to $\mathbf{C}$. Further, we say that $\mathbf{C}$ is **closed under descendence from ultrapowers** if whenever an ultrapower of the $\mathcal{L}$-structure $\mathfrak{A}$ belongs to $\mathbf{C}$, then also $\mathfrak{A}$ belongs to $\mathbf{C}$.

Corollary 13.78 *A class of $\mathcal{L}$-structures is elementary if and only if it is closed under isomorphisms, ultraproducts and descendence from ultrapowers.*

Proof: Assume that $\mathbf{C}$ is an elementary class, axiomatized by the $\mathcal{L}$-theory $\mathbf{T}$. Isomorphic structures are always elementarily equivalent (Theorem 13.1). If $\mathfrak{A}_i \models \mathbf{T}$, $i \in I$, then by Theorem 13.74 also every ultraproduct over the $\mathfrak{A}_i$ is a model of $\mathbf{T}$. If an ultrapower $\mathfrak{A}^I/\mathcal{D}$ is a model of $\mathbf{T}$, then also $\mathfrak{A}$ is a model of $\mathbf{T}$, by virtue of Corollary 13.75. Hence, $\mathbf{C}$ is closed under isomorphisms, ultraproducts and descendence from ultrapowers.

For the converse, assume that the class $\mathbf{C}$ of $\mathcal{L}$-structures is closed under ultraproducts and descendence from ultrapowers. We let $\mathbf{T}$ be the set of all $\mathcal{L}$-sentences which hold in every member of $\mathbf{C}$. We have to show that every model $\mathfrak{B}$ of $\mathbf{T}$ belongs to $\mathbf{C}$. Let $\mathbf{S} = \mathrm{Th}(\mathfrak{B})$. Every finite subset $\mathbf{s} = \{\varphi_1, \ldots, \varphi_n\} \subset \mathbf{S}$ has a model $\mathfrak{A}_\mathbf{s} \in \mathbf{C}$ since otherwise, the sentence $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$ would belong to $\mathbf{T}$ but would not hold in $\mathfrak{B}$. By Theorem 13.76, there is an ultraproduct $\mathfrak{A}$ over the $\mathfrak{A}_\mathbf{s}$ which is a model of $\mathbf{S}$. Since $\mathbf{C}$ is closed under ultraproducts, it follows that $\mathfrak{A} \in \mathbf{C}$. By our choice of $\mathbf{S}$, we have that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$. Hence there are isomorphic ultrapowers $\mathfrak{A}^*$ and $\mathfrak{B}^*$ of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Again since $\mathbf{C}$ is closed under ultraproducts, $\mathfrak{A} \in \mathbf{C}$. Since $\mathbf{C}$ is closed under isomorphisms, we obtain that $\mathfrak{B}^* \in \mathbf{C}$. Finally, since $\mathbf{C}$ is closed under descendence from ultrapowers, we conclude that $\mathfrak{B} \in \mathbf{C}$. □

Remark 13.79 Theorem 13.76 can be proved by showing that there are ultrapowers which are sufficiently saturated (this is the hard part), and then employing a back and forth method which is (more or less) similar to that used in the proof of the Isomorphism Theorem 13.35. It is not too hard to prove that if $\mathcal{L}$ is a countable language, $\mathfrak{A}$ an $\mathcal{L}$-structure and $\mathcal{D}$ a nonprincipal ultrafilter on $\mathbb{N}$ (which exists by Lemma 13.72), then $\mathfrak{A}^\mathbb{N}/\mathcal{D}$ is $\aleph_1$-saturated.

13.13 Pairs of models

If we have an $\mathcal{L}$-theory $\mathbf{T}$, we would also like to talk about the relations between two models $\mathfrak{A} \subset \mathfrak{B}$ of $\mathbf{T}$. This can be done by adding a predicate $\mathcal{P}$ for the substructure to our language. Then the range of $\mathcal{P}$ on $\mathfrak{B}$ can be interpreted to be the set A. Whenever we will talk of an **extension of** $(\mathfrak{A} \subset \mathfrak{B})$ then we will mean an extension with respect to the extended language $\mathcal{L} \cup \{\mathcal{P}\}$, where $\mathcal{P}$ is interpreted on $\mathfrak{B}$ as described above. If $(\mathfrak{A} \subset \mathfrak{B})^*$ is an extension of $(\mathfrak{A} \subset \mathfrak{B})$, then we will write $\mathfrak{B}^*$ for its $\mathcal{L}$-reduct. If $(\mathfrak{A} \subset \mathfrak{B}) \prec (\mathfrak{A} \subset \mathfrak{B})^*$, then also $\mathfrak{B} \prec \mathfrak{B}^*$. The range of $\mathcal{P}$ on $\mathfrak{B}$ may be anything; we only know that its intersection with B must be A (this follows from the definition of "extension", applied to the predicate $\mathcal{P}$). But using the predicate $\mathcal{P}$, one can state that its range in $\mathfrak{B}$ is a substructure of $\mathfrak{B}$. This is done by the elementary sentences $\mathcal{P}(c)$ for all constant symbols c, and $\forall(\mathcal{P}(x_1) \wedge \ldots \wedge \mathcal{P}(x_n) \to \mathcal{P}(f(x_1, \ldots, x_n))$ for every $n \in \mathbb{N}$ and every n-ary function symbol f. So we obtain: If $(\mathfrak{A} \subset \mathfrak{B})^*$ is an elementary extension of $(\mathfrak{A} \subset \mathfrak{B})$, then the range of $\mathcal{P}$ on $\mathfrak{B}^*$ will be a substructure of $\mathfrak{B}^*$, and we will denote it by $\mathfrak{A}^*$.

We can talk about the elementary properties of the substructure $\mathfrak{A}$ in the language $\mathcal{L} \cup \{\mathcal{P}\}$ by restricting all quantifiers to the range of $\mathcal{P}$. By induction on the complexity of sentences, we just have to replace $\forall x\, \varphi(x)$ by $\forall x\, (\mathcal{P}(x) \to \varphi(x))$. From this, we conclude:

Lemma 13.80 *Let $(\mathfrak{A} \subset \mathfrak{B})^*$ be an elementary extension of $(\mathfrak{A} \subset \mathfrak{B})$. Then $\mathfrak{B}^*$ is an elementary extension of $\mathfrak{B}$ and $\mathfrak{A}^*$ is an elementary extension of $\mathfrak{A}$.*

We have shown how to restrict a sentence φ to the substructure $\mathfrak{A}$. Let us denote the resulting sentence by $\varphi_{\mathcal{P}}$. Now we see that the relations $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{A} \prec_\exists \mathfrak{B}$ are elementary properties in the language $\mathcal{L} \cup \{\mathcal{P}\}$. Indeed, the former is axiomatized by an axiom scheme of elementary $\mathcal{L} \cup \{\mathcal{P}\}$-sentences

$$\varphi \to \varphi_{\mathcal{P}}$$

running over all $\mathcal{L}(A)$-sentences φ, whereas in the second case we only let it run over all existential $\mathcal{L}(A)$-sentences φ. This proves:

Lemma 13.81 *Let $(\mathfrak{A} \subset \mathfrak{B})^*$ be an elementary extension of $(\mathfrak{A} \subset \mathfrak{B})$. If $\mathfrak{A} \prec \mathfrak{B}$, then $\mathfrak{A}^* \prec \mathfrak{B}^*$. If $\mathfrak{A} \prec_\exists \mathfrak{B}$, then $\mathfrak{A}^* \prec_\exists \mathfrak{B}^*$.*

Elementary extensions of pairs show certain disjointness properties. We leave it to the reader to prove:

Lemma 13.82 *a) Let $L|K$ be an extension of fields and $(L|K)^* = L^*|K^*$ an elementary extension of $L|K$. Then $L|K$ is linearly disjoint from $K^*|K$ in L^*.*

b) Let $G|H$ be an extension of groups and $(G|H)^* = G^*|H^*$ an elementary extension of $G|H$. Then $G|H$ is disjoint from $H^*|H$ in G^*.

(Note: the second assertion is already proved.)

The degree of finite extensions is fixed. Again, we leave it to the reader to prove:

Lemma 13.83 *a) Let $L|K$ be a finite extension of fields and $(L|K)^* = L^*|K^*$ an elementary extension of $L|K$. Then every basis of $L|K$ is also a basis of $L^*|K^*$ and thus, $[L : K] = [L^* : K^*]$.*

b) Let $G|H$ be a finite extension of abelian groups and $(G|H)^* = G^*|H^*$ an elementary extension of $G|H$. Then every set of representatives of G/H is also a set of representatives of G^*/H^* and thus, $(G : H) = (G^* : H^*)$.

Pairs of real fields were studied by A. Macintyre [MAC1] and W. Baur [BAUR1], [BAUR2]. Pairs of valued field were by F. Delon [DEL5].

13.14 Ultrametric spaces

In this section, we wish to discuss the model theoretic aspects of approximation types. For this, we need to fix the language and theory of ultrametric spaces. The language $\mathcal{L}_{\mathrm{U}}$ consists of just one relation symbol R of arity 4, which is to be interpreted in ultrametric spaces as

$$R(x, y, x', y') \iff u(x, y) \geq u(x', y') \; .$$

Instead of R, we will use the term on the right hand side. For convenience, we define the following relations:

$$\begin{aligned} u(x,y) > u(x',y') &:\Longleftrightarrow u(x,y) \geq u(x',y') \wedge \neg(u(x',y') \geq u(x,y)) \\ u(x,y) = u(x',y') &:\Longleftrightarrow u(x,y) \geq u(x',y') \wedge u(x',y') \geq u(x,y)\,. \end{aligned}$$

Now we restate the three axioms for ultrametric spaces in the following way:

(UM 0) $(\forall x' \forall y'\, u(x,y) \geq u(x',y')) \Leftrightarrow x = y$

(UMT) $u(x,z) \geq u(x,y) \vee u(x,z) \geq u(y,z)$

(UMS) $u(x,y) = u(y,x)$

It still remains to fix by axioms in the language of ultrametric spaces that the value set is a (totally) ordered set. This is done as follows:

(UMV$\not$R) $\neg(u(x,y) < u(x,y))$

(UMVT) $u(x,y) < u(x',y') \wedge u(x',y') < u(x'',y'') \rightarrow u(x,y) < u(x'',y'')$

(UMVC) $u(x,y) < u(x',y') \vee u(x,y) = u(x',y') \vee u(x,y) > u(x',y')$

We define the **theory of ultrametric spaces** $\mathbf{T}_{\mathrm{U}}$ to consist of the axioms (UM 0), (UMT), (UMS), (UMV$\not$R), (UMVT) and (UMVC).

As the name **approximation type** already indicates, approximation types are closely related to model theoretic types. In the following, we will discuss this relation. We have to adjust our use of symbols to the model theoretic setting. We will consider approximation types over the ultrametric space (X,u), but we will denote constants from X by $a,b,c,d,\ldots$, as we did in the last chapter. Similarly, we will use the symbol x ambivalently as a variable symbol or as a constant symbol for an element in an extension of (X,u). As for types over (X,u), we work in the language $\mathcal{L}_{\mathrm{U}}(X,x)$ to find sentences which describe a given approximation type over (X,u). We will ask for sets $\mathbf{t}$ of $\mathcal{L}_{\mathrm{U}}(X,x)$-sentences which describe a given approximation type $\mathbf{A}$ completely, in a sense that we have to make precise now. We say that $\mathbf{t}$ **determines** $\mathbf{A}$ if the following holds: If $(X,u) \subset (X \cup \{x\},u)$ is an extension of ultrametric spaces such that every sentence of $\mathbf{t}$ holds in $(X \cup \{x\},u)$, then $\mathbf{A} = \mathrm{appr}(x,X)$. We can rephrase the condition as: "If $(X,u) \subset (Y,u)$ is an extension of ultrametric spaces such that $x \in Y$ realizes $\mathbf{t}$". But note that we will treat cases where $\mathbf{t}$ is not a type over (X,u). However, we will show that immediate approximation types are always determined by suitable types.

In the following, we will put $\mathcal{L}_{\mathrm{U}}(X,x)$-sentences in quotation marks, in order to distinguish them from the formulas in our computations.

We start with immediate approximation types. We know from Corollary 2.33 that for an immediate approximation type $\mathbf{A}$, we can actually forget about the second part of the map $\mathbf{A}$ which sends $\alpha \in uX\infty$ to $\mathbf{A}^\circ_\alpha$. All information is carried already by the first part of the map. Hence, an immediate approximation type $\mathbf{A}$ is determined by the set

$$\mathbf{t}_{\geq}(\mathbf{A}) = \{\text{“}u(x,c) \geq u(a,b)\text{”} \mid a,b,c \in X \wedge c \in \mathbf{A}_{u(a,b)}\}$$

of $\mathcal{L}_{\mathrm{U}}(X,x)$-sentences.

Lemma 13.84 *For every approximation type* $\mathbf{A}$ *over* (X,u)*, the set* $\mathbf{t}_{\geq}(\mathbf{A})$ *is finitely realizable in* (X,u)*. Thus, it is a type over* (X,u) *in* X*.*

Proof: Let $\mathbf{t}_0 = \{\text{“}u(x,c_i) \geq u(a_i,b_i)\text{”} \mid 1 \leq i \leq n\} \subset \mathbf{t}_{\geq}(\mathbf{A})$ be a finite subtype. Without loss of generality, we can assume that $u(a_1,b_1)$ is the maximum of the values $\alpha_i := u(a_i,b_i)$. Consequently, $c_1 \in \mathbf{A}_{\alpha_i}$ for all i. That is, $u(c_1,c_i) \geq u(a_i,b_i)$, showing that c_1 realizes $\mathbf{t}_0$ in (X,u). □

Let $\mathfrak{A}$ be any expansion of the ultrametric space (X,u), and let $\mathbf{A}$ be an immediate approximation type over (X,u). By the preceding lemma, for every immediate approximation type $\mathbf{A}$ the set $\mathbf{t}_{\geq}(\mathbf{A})$ is finitely realizable in $\mathfrak{A}$. If we take $\mathfrak{A}^*$ to be an $|A|^+$-saturated elementary extension of $\mathfrak{A}$, which exists by Theorem 13.28, then $\mathbf{t}_{\geq}(\mathbf{A})$ is realized by some $x \in A^*$. If $\mathfrak{A}$ is a valued abelian group or a valued field, then Lemma 13.37 shows that $\mathfrak{A}^*/\mathfrak{A}$ is torsion free resp. transcendental. In particular, x is torsion free resp. transcendental over $\mathfrak{A}$. If $\mathfrak{S}$ is the substructure of $\mathfrak{A}^*$ which is generated by x over $\mathfrak{A}$, then it follows from Lemma 13.38 that $\mathfrak{A} \prec_\exists \mathfrak{S}$. We summarize what we have proved:

Lemma 13.85 *Let* $\mathfrak{A}$ *be any expansion of the ultrametric space* (X,u) *and* $\mathbf{A}$ *an immediate approximation type over* $\mathfrak{A}$*. Then* $\mathbf{A}$ *is realized in every* $|A|^+$*-saturated extension of* $\mathfrak{A}$*. In particular, there is an elementary extension* $\mathfrak{A}^*$ *of* $\mathfrak{A}$ *in which* $\mathbf{A}$ *is realized. That is, there is some* $x \in A^*$ *such that* $\mathbf{A} = \operatorname{appr}(x,X)$*. If* $\mathfrak{A}$ *is an abelian group, then* $\mathfrak{A}^*/\mathfrak{A}$ *is torsion free. If* $\mathfrak{A}$ *is a field, then* $\mathfrak{A}^*|\mathfrak{A}$ *is regular. If* $\mathfrak{S}$ *is the substructure of* $\mathfrak{A}^*$ *which is generated by* x *over* $\mathfrak{A}$*, then* $\mathfrak{A} \prec_\exists \mathfrak{S}$*.*

Under additional assumptions, it is possible to show the same for approximation types which are not immediate. Let us begin with the trivial observation that every approximation type $\mathbf{A}$ over (X,u) is determined by the set

$$\mathbf{t}(\mathbf{A}) := \mathbf{t}_{>}(\mathbf{A}) \cup \mathbf{t}_{=}(\mathbf{A}) \cup \mathbf{t}_{<}(\mathbf{A})$$

of $\mathcal{L}_{\mathrm{U}}(X,x)$-sentences, where

$$\begin{aligned}
\mathbf{t}_{>}(\mathbf{A}) &:= \{\text{“}u(x,c) > u(a,b)\text{”} \mid a,b,c \in X \wedge c \in \mathbf{A}^\circ_{u(a,b)}\} \\
\mathbf{t}_{=}(\mathbf{A}) &:= \{\text{“}u(x,c) = u(a,b)\text{”} \mid a,b,c \in X \wedge c \in \mathbf{A}_{u(a,b)} \setminus \mathbf{A}^\circ_{u(a,b)}\} \\
\mathbf{t}_{<}(\mathbf{A}) &:= \{\text{“}u(x,c) < u(a,b)\text{”} \mid a,b,c \in X \wedge c \notin \mathbf{A}_{u(a,b)}\}\ .
\end{aligned}$$

This holds even in the following strong sense: if $\mathbf{A}$ and $\mathbf{A}'$ are approximation types, then

$$\mathbf{t}(\mathbf{A}) \subset \mathbf{t}(\mathbf{A}') \Longrightarrow \mathbf{A} = \mathbf{A}'\ .$$

Let us now consider an approximation type $\mathbf{A}$ which is not residue-immediate. Then by Corollary 2.26, $\gamma := \operatorname{dist}\mathbf{A} \in uX$, and $\mathbf{A}$ is uniquely determined by the ball $\mathbf{A}_\gamma$ and the information that $\mathbf{A}^\circ_\gamma = \emptyset$. Hence, $\mathbf{A}$ is already determined by the set

$$\mathbf{t}_{=\gamma}(\mathbf{A}) := \{\text{“}u(x,c) = u(a,b)\text{”} \mid a,b,c \in X \wedge \gamma = u(a,b) \wedge c \in \mathbf{A}_\gamma \setminus \mathbf{A}^\circ_\gamma\}$$

of $\mathcal{L}_{\mathrm{U}}(X,x)$-sentences. Indeed, if x realizes this set of sentences, then $\operatorname{appr}(x,X)_\gamma = \mathbf{A}_\gamma$, but also $\operatorname{appr}(x,X)^\circ_\gamma = \emptyset$ because every $c \in \mathbf{A}_\gamma$ satisfies $u(x,c) = \gamma$.

Inspired by the paper "Espaces ultramétriques" by F. Delon [DEL3], we call an ultrametric space (X,u) **rich** if all its α-polygons are infinite, for all $\alpha \in uX$. Note that if (X,u) is the underlying ultrametric space of a valued abelian group (G,v), then (X,u) is rich if and only if all components of (G,v) are infinite. Similarly, if (X,u) is the underlying ultrametric space of a valued field (K,v), then (X,u) is rich if and only if the residue field $\overline{K}$ is infinite. For ultrametric spaces, the property of being rich is axiomatized by the axiom scheme

(RICH) $\forall x_1 \ldots \forall x_n \forall x' \forall x'' \exists y :$
$\bigwedge_{1 \le i < j \le n} u(x_i,x_j) = u(x',x'') \;\Rightarrow\; \bigwedge_{1 \le i \le n} u(x_i,x) = u(x',x'')$ $(n \in \mathbb{N})$.

Lemma 13.86 *Let* $\mathbf{A}$ *be an approximation type over the rich ultrametric space* (X,u). *Then for every* $\gamma \in \Lambda(\mathbf{A})$, *the set* $\mathbf{t}_{=\gamma}(\mathbf{A})$ *is finitely realizable in* (X,u).

Proof: Let $\mathbf{t}_0$ be a finite subset of $\mathbf{t}_{=\gamma}(\mathbf{A})$. Since

$$\{c \in X \mid \text{“}u(x,c) = u(a,b)\text{”} \in \mathbf{t}_0\} \; .$$

is finite and by hypothesis, every α-polygon of (X,u) is infinite, there exists an element $c_0 \in X$ such that $u(c_0,c) = \alpha = u(a,b)$ for all $c \in \mathcal{C}$. That is, c_0 realizes $\mathbf{t}_0$ in (X,u). □

Since richness is an elementary property, it is inherited by elementary extensions. So from the foregoing lemma, we obtain:

Corollary 13.87 *Let* $\mathfrak{A}$ *be any expansion of the rich ultrametric space* (X,u) *and* $\mathbf{A}$ *a value-immediate approximation type over* $\mathfrak{A}$. *Then* $\mathbf{A}$ *is realized in every* $|A|^+$*-saturated extension of* $\mathfrak{A}$.

In view of the fact that every approximation type is residue immediate if it is not value immediate (cf. Corollary 2.25), it just remains to deal with the case of a residue immediate approximation type. Let us consider an approximation type $\mathbf{A}$ over (X,u) which is not value-immediate. From Corollary 2.32 we know that there is some $c \in X$ such that

$$(\, \{\alpha \mid c \notin \mathbf{A}_\alpha\} \,,\, \{\alpha \mid c \in \mathbf{A}^\circ_\alpha\} \,) \tag{13.9}$$

is a cut in uX. We claim that $\mathbf{A}$ is determined by the set

$$\begin{aligned} \mathbf{t}_c(\mathbf{A}) \;:=\; & \{\text{“}u(x,c) < u(a,b)\text{”} \mid a,b,c \in X \wedge c \notin \mathbf{A}_{u(a,b)}\} \\ & \cup \{\text{“}u(x,c) > u(a,b)\text{”} \mid a,b,c \in X \wedge c \in \mathbf{A}^\circ_{u(a,b)}\} \; . \end{aligned}$$

Indeed, if x realizes this set in some extension of (X,u), then

$$(\, \{\alpha \mid c \notin \operatorname{appr}(x,X)_\alpha\} \,,\, \{\alpha \mid c \in \operatorname{appr}(x,X)^\circ_\alpha\} \,)$$

is equal to the cut (13.9), and Corollary 2.32 shows that $\operatorname{appr}(x,X) = \mathbf{A}$.

Lemma 13.88 *Let* $\mathbf{A}$ *be a residue immediate approximation type over the ultrametric space* (X,u) *whose value set* uX *is densely ordered without endpoints. Assume further that* $x_0 \in X$ *and that*

$$u(\,.\,,x_0) : \; X \;\to\; uX\infty \tag{13.10}$$

is surjective. Then the set $\mathbf{t}_c(\mathbf{A})$ *is finitely realizable in* (X,u).

Proof: We have to show that every finite set $\mathbf{t}_0 \subset \mathbf{t}_c(\mathbf{A})$ is realized in (X, u). Since (13.9) is a cut, we have that

$$\{u(a,b) \mid \text{“}u(x,c) > u(a,b)\text{”} \in \mathbf{t}_0\} \; < \; \{u(a,b) \mid \text{“}u(x,c) < u(a,b)\text{”} \in \mathbf{t}_0\} \; .$$

Since both sets are finite and uX is densely ordered without endpoints by assumption, there exists an element $\alpha_0 \in uX$ which lies between these sets. We can write $\alpha_0 = u(c_0, c)$ because we have assumed that the map (13.10) is surjective. Now c_0 realizes $\mathbf{t}_0$. □

For “densely ordered without endpoints”, we will write shorter **dense w.e.p.** For ultrametric spaces, this property of their value sets can be axiomatized by

(DWEP) $\forall x_1 \ldots \forall x_4 \exists y_1 \ldots \exists y_6 :$
$u(y_1, y_2) > u(x_1, x_2) > u(y_3, y_4) > u(x_3, x_4) > u(y_5, y_6)$

Consequently, if the value set of (X, u) is dense w.e.p., then this also holds for the value set of every elementary extension of (X, u)

If (X, u) is a rich ultrametric space then in particular, no α-polygon is a singleton. This means that for every $c \in X$, the map (13.10) is surjective. So we obtain the following assertion from Lemma 13.85, Corollary 13.87 and the foregoing lemma:

Corollary 13.89 *Let $\mathfrak{A}$ be any expansion of the rich ultrametric space (X, u) with dense value set uX w.e.p. Further, let $\mathbf{A}$ be an arbitrary approximation type over $\mathfrak{A}$. Then $\mathbf{A}$ is realized in every $|A|^+$-saturated extension of $\mathfrak{A}$. Consequently, there exists an extension $\mathfrak{S}$ generated by one element x over $\mathfrak{A}$ such that $\mathbf{A} = \mathrm{appr}(x, X)$ and $\mathfrak{A} \prec_\exists \mathfrak{S}$.*

We use this corollary to prove the following result, which is due to Delon.

Lemma 13.90 *Let (X, u) be a rich ultrametric space. Then (X, u) is existentially closed in every extension (Y, u) which satisfies $uY = uX$. If in addition uX is dense w.e.p., then (X, u) is existentially closed in every extension (Y, u).*

Proof: Let $(X, u) \subset (Y, u)$ be an extension of ultrametric spaces. According to Lemma 13.40, to show that $(X, u) \prec_\exists (Y, u)$ it suffices to assume that Y is finitely generated over X, which just means that $Y \setminus X$ is finite. Let $Y \setminus X = \{y_1, \ldots, y_n\}$. Further, let (X^*, u) be an $|X|^+$-saturated elementary extension of (X, u). Note that as an elementary extension of (X, u), the ultrametric space (X^*, u) is again rich, and if uX is dense w.e.p., then so is uY. We wish to embed (Y, u) in (X^*, u) over X. We start with the element y_1. If uY is dense w.e.p., then $\mathrm{appr}\,(y_1, X)$ can always be realized by some element y_1^* in (X^*, u), and by Lemma 2.38 we know that $y_1' \to y_1^*$ induces an isomorphism $(X \cup \{y_1\}, u) \to (X \cup \{y_1^*\}, u)$ over X. If uY is not dense w.e.p., then by assumption we have that $uY = uX$, which yields that $\mathrm{appr}\,(y_1, X)$ must be value-immediate. In this case, we can use Corollary 13.87 to find $y_1^* \in X^*$ which realizes $\mathrm{appr}\,(y_1, X)$.

Now we proceed by induction. Suppose that we have embedded $(X \cup \{y_1, \ldots, y_m\}, u)$ with $m < n$. By passing to isomorphic images (cf. Lemma 13.6), we identify $X \cup \{y_1, \ldots, y_m\}$ with its image in X^*. The approximation type $\mathrm{appr}\,(y_{m+1}, X \cup \{y_1, \ldots, y_m\})$ is realized in (X^*, u) for the same reasons as before, giving rise to an embedding of $(X \cup \{y_1, \ldots, y_{m+1}\}, u)$ over $X \cup \{y_1, \ldots, y_m\}$. We just have to note two facts. Firstly, since (X^*, u) is $|X|^+$-saturated, it is also $|X \cup \{y_1, \ldots, y_m\}|^+$-saturated (since (X, u) is assumed to be rich, it is infinite, which yields that $|X| = |X \cup \{y_1, \ldots, y_m\}|$). Secondly, if $uY = uX$,

then also $uY = u(X \cup \{y_1, \ldots, y_m\})$, which again yields that appr $(y_{m+1}, X \cup \{y_1, \ldots, y_m\})$ is value-immediate. □

Employing Robinson's Test (Theorem 13.47), we obtain:

Theorem 13.91 *The theory of rich ultrametric spaces with dense value sets w.e.p. is model complete.*

The model theory of ultrametric spaces was treated in detail by F. Delon in her thesis [DEL1] and in the article [DEL3]. Among other results, Delon shows the following:

Theorem 13.92 *Let Γ be an ordered set and T_Γ the elementary theory of all rich ultrametric spaces (X, u) such that $uX \equiv \Gamma$. Then T_Γ is complete resp. is model complete resp. admits quantifier elimination if the same holds for* Th(Γ).

Let us mention two interesting points in the proof of this theorem. For the proof concerning completeness, Delon shows that two rich ultrametric spaces with equal value sets are equivalent even in an infinitary language, which roughly speaking means that also infinite conjunctions of sentences are admitted. This equivalence is consequently stronger than elementary equivalence. A similar result can be proved for valued vector spaces with value preserving scalar multiplication, cf. S. Kuhlmann [KUS1]. This originates from the fact that by Lemma 2.38, we obtain isomorphisms already when the approximation types are equal (so we did not have to work much to prove Lemma 13.90). We have already seen in Section 7.10 that for valued fields the situation is different.

For the proof concerning model completeness and quantifier elimination, Delon shows that for every sentence φ in the language of ultrametric space there is a corresponding formula φ' in the language of ordered sets (which is just $\{<\}$) such that for every rich ultrametric space (X, u), the sentence φ holds in (X, u) if and only if the sentence φ' holds in uX. This is a consequence of the simplicity of the language of ultrametric spaces and a close connection between rich ultrametric spaces and their value sets.

To show that a given approximation type $\mathbf{A}$ over (X, u) can always be realized in some extension of (X, u), one has to show that every ultrametric space admits a rich extension (Y, u) with dense value set w.e.p. We leave this as an exercise to the reader (Exercise 13.6). We do not know whether the same is true for every expansion of the theory of ultrametric spaces, that is, whether for a given model $\mathfrak{A}$ there is always an extension $\mathfrak{S}$ such that the underlying ultrametric space (Y, u) of $\mathfrak{S}$ is rich with dense value set w.e.p. For the case of valued abelian groups, this is true (see Exercise 13.4). Let us show here that every approximation type over a valued field is realized in some simple extension.

Corollary 13.93 *For every approximation type $\mathbf{A}$ over a valued field (K, v), there exists a simple valued field extension $(K(x)|K, v)$ such that $\mathbf{A} =$* appr(x, K).

Proof: The field $(\tilde{K}, v)$ has the infinite residue field $\widetilde{\widetilde{K}}$ and the divisible value group $\widetilde{vK}$, which is dense w.e.p. Let (L, v) be a $|K|^+$-saturated elementary extension of $(\tilde{K}, v)$. The underlying ultrametric space of (L, v) is rich and its value set is dense w.e.p., just like the underlying ultrametric space of (K, v). Hence, Lemma 13.89 guarantees that $\mathbf{A} =$ appr(x, K) for some $x \in L$. □

At this point, let us introduce an observation which is a corollary to Lemma 2.30 and will be useful for the next lemma:

Corollary 13.94 *For every approximation type* $\mathbf{A}$ *over an ultrametric space* (X,u) *there exists an immediate approximation type* $\mathbf{A}'$ *over* (X,u) *such that* $\mathbf{t}_{\geq}(\mathbf{A}) \subset \mathbf{t}_{\geq}(\mathbf{A}')$ *and* $\mathbf{t}_{>}(\mathbf{A}) \subset \mathbf{t}_{>}(\mathbf{A}')$.

Proof: If $\mathbf{A}$ is already immediate, there is nothing to show. If this is not the case, then we know by Lemma 2.30 that there exists a universal element $c_0 \in X$ for $\mathbf{A}$. We put $\mathbf{A}' = \mathrm{appr}\,(c_0, X)$. Assume that "$u(x,c) \geq u(a,b)$" $\in \mathbf{t}_{\geq}(\mathbf{A})$, i.e., $c \in \mathbf{A}_{u(a,b)}$. Hence also c_0 lies in the nonempty ball $\mathbf{A}_{u(a,b)}$ and by (at 2), we obtain that $u(c,c_0) \geq u(a,b)$. This yields that $c \in \mathbf{A}'_{u(a,b)}$, showing that "$u(x,c) \geq u(a,b)$" $\in \mathbf{t}_{\geq}(\mathbf{A}')$. We have thus proved that $\mathbf{t}_{\geq}(\mathbf{A}) \subset \mathbf{t}_{\geq}(\mathbf{A}')$. Similarly, replacing "$\geq$" by "$>$" and $\mathbf{A}_{u(a,b)}$ by $\mathbf{A}^{\circ}_{u(a,b)}$, we obtain a proof for the inclusion $\mathbf{t}_{>}(\mathbf{A}) \subset \mathbf{t}_{>}(\mathbf{A}')$. □

The following lemma is helpful for the construction of immediate approximation types:

Lemma 13.95 *Let* (X,u) *be an ultrametric space and* $\mathbf{t}$ *a set of* $\mathcal{L}_{\mathrm{U}}(X,x)$*-sentences of the form "*$u(x,c) \geq u(a,b)$*" (*$a,b,c \in X$*). Assume that* $\mathbf{t}$ *is finitely realizable in* (X,u)*. Then there exists an immediate approximation type* $\mathbf{A}$ *over* (X,u) *such that* $\mathbf{t} \subset \mathbf{t}_{\geq}(\mathbf{A})$.

Proof: Since $\mathbf{t}$ is finitely realizable in $\mathfrak{A}$, it is realized in every $|X|^+$-saturated extension of (X,u). Let x be an element realizing $\mathbf{t}$. Then $\mathbf{t} \subset \mathbf{t}_{\geq}(\mathrm{appr}(x,X))$. If $\mathrm{appr}(x,X)$ is not already immediate, then according to the preceding corollary we can replace it by an immediate approximation type $\mathbf{A}'$ such that

$$\mathbf{t}_{\geq}(\mathrm{appr}(x,X)) \subset \mathbf{t}_{\geq}(\mathbf{A}')$$

and hence also $\mathbf{t} \subset \mathbf{t}_{\geq}(\mathbf{A}')$. □

Exercise 13.4 *Prove an analogue of Corollary 13.93 for an approximation type over the valued abelian group* (G,v)*. Show that the divisible hull of* G *is not a good substitute for the algebraic closure of the field* K *in this situation; construct an extension group that will do the job.*

Exercise 13.5 *Let* $\mathbf{A}$ *be an approximation type over* (X,u) *and let* (Y,u) *be an extension of* (X,u)*. Prove that there exists an approximation type* $\mathbf{A}_Y$ *over* (Y,u) *whose restriction to* (X,u) *coincides with* $\mathbf{A}$*. Moreover, show that* $\mathbf{A}_Y$ *can be chosen to be immediate if* $\mathbf{A}$ *is immediate.*

Exercise 13.6 *a) Let* (X,u) *be an ultrametric space. Construct a rich extension* (Y,u) *of* (X,u) *such that* uY *is dense w.e.p.*

b) Prove: If (X,u) *is existentially closed in every extension* (Y,u) *such that* $uY = uX$*, then* (X,u) *is rich. If* (X,u) *is existentially closed in every extension* (Y,u)*, then* (X,u) *is rich and* uX *is dense w.e.p.*

c) Show that the theory of rich ultrametric spaces with dense value sets w.e.p. admits a prime model, and conclude that it is complete. (Hint: The dense linear ordering $\mathbb{Q}$ *w.e.p. is the prime model of the theory of dense linear orderings w.e.p. Now take one point* c *and choose, for every* $\alpha \in \mathbb{Q}$*, countably many points* $c_{\alpha,i}$ *such that* $u(c,c_{\alpha,i}) = \alpha$*.)*

13.15 Valued abelian groups and vector spaces

If we want to talk about the model theory of valued groups, we need to formalize the valuation by a symbol which we add to the language of groups. It is convenient to formalize it by a binary predicate R_v which in every valued group is to be interpreted as

$$R_v(x,y) \iff vx \geq vy \ ,$$

as we will do for valued fields. We will use "$vx \geq vy$" in the place of "$R_v(x,y)$". We leave it as an excercise to the reader to axiomatize the **theory of valued abelian groups** and to give axioms for the additional properties that we have treated. We will give the axioms for valued fields in the next section. Here, we want to point out that the theory of valued groups and thus also its expansions (valued modules, vector spaces and fields) can be viewed as expansions of the theory of ultrametric spaces. Indeed, the relation that we have used to formalize the ultrametric can be defined from the relation R_v as follows:

$$u(x,y) \geq u(x',y') \ \leftrightarrow \ v(x-y) \geq v(x'-y') \ .$$

Consequently, the results of the foregoing section become available for valued groups and its expansions.

To our knowledge, there is not much literature on the model theory of valued abelian groups. The model theory of valuated groups was studied by P. Schmitt. In his article [SCHTT1], Schmitt treats p-valuations which in addition satisfy axiom (V 0). He shows that their elementary theory is undecidable. The theory of a more special class of groups with a p-valuation, called the **tamely p-valuated groups**, was shown to be decidable in [SCHTT2].

Valued vector spaces are formalized by adding predicates predicates for the scalar multiplication $c\cdot$ to the language of valued groups, for every element c of the ground field. (For valued vector spaces as well as for valued fields, one can also use many sorted languages, but we do not want to introduce them here). For results on the model theory of valued (and ordered) vector spaces, see L. van den Dries [VDD4] and S. Kuhlmann [KUS1], [KUS2].

Chapter 14

Model theory of valued fields

14.1 Axioms for valued fields

We have seen in Section 5.1 that a valuation v on a field K can be given in several ways. To fit in our concept of languages and structures, two possible concepts are available. We can take the **valuation divisibility relation** and formalize it as a binary predicate R_v which in every valued field is to be interpreted as

$$R_v(x,y) \iff vx \geq vy \ .$$

But we can also take the valuation ring and formalize it as a predicate $\mathcal{O}$ which in every valued field is to be interpreted as

$$\mathcal{O}(x) \iff x \in \mathcal{O} \ .$$

This predicate can be defined from the valuation divisibility relation by

$$\mathcal{O}(x) \leftrightarrow R_v(x,1) \ .$$

If we are working in the language of fields (what we usually do), then the valuation divisibility relation can be defined from the predicate $\mathcal{O}$ by

$$R_v(x,y) \leftrightarrow (y \neq 0 \wedge \mathcal{O}(xy^{-1})) \vee x = 0 \ ,$$

whereas in general, it can not be defined using $\mathcal{O}$ and the language of rings without the use of quantifiers like in

$$R_v(x,y) \leftrightarrow (\exists z \, yz = 1 \wedge \mathcal{O}(xz)) \vee x = 0 \ .$$

This fact is only of importance for questions of quantifier elimination, and only if one has decided to work in the language of rings. Note that two fields are equivalent in the language of rings if and only if they are equivalent in the language of fields. A similar assertion holds for valued fields in the respective languages, and it also holds for the notions "elementary extension" and "existentially closed in" in the place of "equivalent" (cf. Exercise 14.2).

We prefer to write "$vx \geq vy$" in the place of "$R_v(x,y)$". For convenience, we define the following relations:

$$\begin{aligned} vx > vy &\leftrightarrow vx \geq vy \wedge \neg(vy \geq vx) \\ vx = vy &\leftrightarrow vx \geq vy \wedge vy \geq vx \ . \end{aligned}$$

The definitions for the reversed relations $vx \leq vy$ and $vx < vy$ are left to the reader.

We will work in the **language of valued fields** $\mathcal{L}_{\mathrm{VF}}$ which is the language $\mathcal{L}_{\mathrm{F}}$ of fields together with the binary relation symbol $vx \geq vy$. The **theory of valued fields** is the theory of fields (in the language of fields) together with the axioms

(V0) $\quad (\forall y\, vx \geq vy) \;\Leftrightarrow\; x = 0$

(VT) $\quad v(x-y) \geq vx \;\vee\; v(x-y) \geq vy$

and the axioms which state that the value group is an ordered abelian group:

(VV$\not$R) $\quad \neg(vx < vx)$

(VVT) $\quad vx < vy \wedge vy < vz \;\Rightarrow\; vx < vz$

(VVC) $\quad vx < vy \vee vx = vy \vee vx > vy$

(VVG) $\quad vx < vy \;\Rightarrow\; vxz < vyz$

(the group axioms for the value group follow from the group axioms for the multiplicative group of the field). Valued fields can be viewed as expansions of their underlying ultrametric spaces and of their underlying valued abelian groups (cf. our remarks in Section 13.15).

Although we do not see the value group and the residue field explicitly in the formal structure of a valued field, they are implicitly present in that structure. The following lemma shows what we mean by this:

Lemma 14.1 *Let (K, v) be a valued field.*

a) For every sentence φ in the language of ordered groups there is a sentence φ' in the language of valued fields such that for every valued field (K, v), φ holds in vK if and only if φ' holds in (K, v).

b) For every sentence φ in the language of rings there is a sentence φ' in the language of valued fields such that for every valued field (K, v), φ holds in $\overline{K}$ if and only if φ' holds in (K, v).

Proof: a): First, we translate terms from the value group to the valued field. The constant 0 is translated to $v1$, every variable x is translated to vx and if t_1 and t_2 are $\mathcal{L}_{\mathrm{OG}}$-terms with translations t_1' and t_2', then the $\mathcal{L}_{\mathrm{OG}}$-terms $t_1 + t_2$ and $-t_1$ are translated to $vt_1't_2'$ and vt_1^{-1} respectively. From this, we know how to translate atomic formulas: $t_1 = t_2$ is translated to $t_1' = t_2'$, and $t_1 < t_2$ is translated to $vt_1' < vt_2'$. If φ_1, φ_2 are translated to φ_1', φ_2', then $\neg\varphi_1$ is translated to $\neg\varphi_1$, and $\varphi_1 \wedge \varphi_2$ is translated to $\varphi_1' \wedge \varphi_2'$. Finally, if $\varphi(x)$ is translated to $\varphi'(x)$, then $\forall x\varphi(x)$ is translated to $\forall x(x \neq 0 \to \varphi'(x))$. It is straightforward to check by induction on the complexity that φ holds in vK if and only if its translation φ' holds in (K, v).

b): We translate terms from the language of fields to the language of valued fields literally (this is possible since the only constants to be translated are 0 and 1). The atomic sentence $t_1 = t_2$ is translated to $v(t_1 - t_2) > 0$. Negation and conjunction are handled as before. Finally, if $\varphi(x)$ is translated to $\varphi'(x)$, then $\forall x\varphi(x)$ is translated to $\forall x(vx \geq 0 \to \varphi'(x))$. Again, it is checked by induction on the complexity that φ holds in $\overline{K}$ if and only if its translation φ' holds in (K, v). For the atomic formula $t_1 = t_2$ one uses the fact that the residue map is a homomorphism from the valuation ring onto the residue field and that its

kernel is the valuation ideal. This is why we have restricted quantification in translated formulas to elements of value ≥ 0, and this is why we only use the language of rings for the residue field. □

As immediate consequences of this lemma, we obtain:

Corollary 14.2 *If (K,v) and (L,v) are valued fields such that $(K,v) \equiv (L,v)$ in the language of valued fields, then $vK \equiv vL$ in the language of ordered groups, and $\overline{K} \equiv \overline{L}$ in the language of rings (and thus also in the language of fields). The same holds with $\prec$ or $\prec_\exists$ in the place of $\equiv$.*

Corollary 14.3 *If (K,v) is κ-saturated, then so are vK (in the language of ordered groups) and $\overline{K}$ (in the language of fields).*

Exercise 14.1 *Let $(L|K,v)$ be an extension of valued fields and $(L|K,v)^*$ an extension of $(L|K,v)$, in the sense of Section 13.13. Prove that $(L|K,v)$ is valuation disjoint from $(K^*|K,v)$ in (L^*,v).*

Exercise 14.2 *Let $\mathbf{T}$ be the $\mathcal{L}_{\mathrm{F}}$-theory of fields and $\mathbf{T}'$ any expansion of the theory of fields in a language $\mathcal{L}'$ which is an extension of $\mathcal{L}_{\mathrm{F}}$. Let $\mathcal{L}' = \mathcal{L}'' \cup \{^{-1}\}$, where $\mathcal{L}''$ is an extension of $\mathcal{L}_{\mathrm{R}}$. Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be models of $\mathbf{T}'$. Prove that $\mathfrak{A} \prec_\exists \mathfrak{B}$ resp. $\mathfrak{A} \equiv \mathfrak{B}$ resp. $\mathfrak{A} \equiv_{\mathfrak{C}} \mathfrak{B}$ if and only if the same holds for their $\mathcal{L}''$-reducts. What is the general principle behind this fact?*

14.2 The Ax–Kochen–Ershov Principle

From Corollary 14.2 we see that for (K,v) and (L,v) to be elementarily equivalent, it is necessary that vK and vL are elementarily equivalent, and that $\overline{K}$ and $\overline{L}$ are elementarily equivalent. Our main concern in the next sections will be to find criteria for that these necessary conditions are also sufficient.

As it follows from our exposition of the preliminaries from model theory in the last chapter, our basic approach to model theoretic algebra is to ask for criteria for a structure to be existentially closed in a given extension. Replacing $\equiv$ by $\prec_\exists$, we can express our "main concern" in the following notion. We will say that a valued field extension $(L|K,v)$ satisfies the **Ax–Kochen–Ershov Principle** if

$$vK \prec_\exists vL \ \wedge\ \overline{K} \prec_\exists \overline{L} \quad \Longrightarrow \quad (K,v) \prec_\exists (L,v)\ . \tag{14.1}$$

The conditions

$$vK \prec_\exists vL \ \text{ and } \ \overline{K} \prec_\exists \overline{L} \tag{14.2}$$

will be called **side conditions**. Recall that by Lemma 13.37 they imply that vL/vK is torsion free and that $\overline{L}|\overline{K}$ is regular.

Let us use the abbreviation **AKE** for "Ax–Kochen–Ershov". A valued field (K,v) will be called an **AKE field** if every valued field extension $(L|K,v)$ satisfies the AKE Principle.

In many applications (such as the proof of a Nullstellensatz), only existential sentences play a role. In these cases, it suffices to have an AKE principle at hand. We will also meet situations where we can not expect more than this principle (see the next section). On the other hand, there are important applications where we need more (cf. Section 14.9). So we will also be interested in the implication (14.1) with $\prec_\exists$ replaced by $\prec$ or $\equiv$. Let $\mathbf{C}$ be an elementary class of valued fields. If for every two fields $(K,v),(L,v) \in \mathbf{C}$, the side conditions $vK \prec vL$ and $\overline{K} \prec \overline{L}$ imply that $(K,v) \prec (L,v)$, then $\mathbf{C}$ will be called

relatively model complete. If the same holds with $\equiv$ in the place of $\prec$, then we will call **C relatively complete**. Here, "relatively" means "relative to the value groups and residue fields". We have not used our original definition of "model complete" for the translation into the relative situation since there is no adequate translation (unless the value groups and residue fields of the members of **C** form model complete elementary classes). For a similar notion, see our definition of "relatively subcomplete" in Section 14.4 below.

Let us now discuss the axiomatization of important properties of valued fields. In particular, let us ask for those properties that a valued field must have if it is an AKE field. An **immediate** consequence of (14.1) is the following observation:

Lemma 14.4 *Every AKE field is algebraically maximal.*

Proof: Let (K,v) be a valued field which admits an immediate algebraic extension (L,v). Then by Lemma 13.37, K is not existentially closed in L. Hence, (K,v) is not existentially closed in (L,v). But $vK = vL$ and $\overline{K} = \overline{L}$, so that the conditions $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$ hold. This shows that (K,v) is not an AKE field. □

In particular, this lemma shows that every AKE field must be henselian. The property of being henselian is axiomatized by the following axiom scheme:

(HENS) $$\begin{aligned} vy \geq 0 \ \wedge \ \textstyle\bigwedge_{1\leq i\leq n} vx_i \geq 0 \ \wedge \ & v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) > 0 \\ & \wedge \ v(ny^{n-1} + (n-1)x_1y^{n-2} + \ldots + x_{n-1}) = 0 \\ \Rightarrow \ \exists z \ v(y-z) > 0 \ \wedge \ & z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n = 0 \qquad (n \in \mathbb{N})\,. \end{aligned}$$

Here we use Hensel's Lemma, i.e., assertion 4) of Theorem 9.1 to characterize henselian fields.

But in view of Theorem 7.134, also the property of being algebraically maximal is easily axiomatized by axiom scheme (HENS) together with the following axiom scheme:

(MAXP) $$\exists y \forall z : \ v(y^n + x_1y^{n-1} + \ldots + x_{n-1}y + x_n) \geq v(z^n + x_1z^{n-1} + \ldots + x_{n-1}z + x_n) \qquad (n \in \mathbb{N})\,.$$

A special case of the AKE Principle is given if a an extension $(L|K,v)$ is immediate. Then, the side conditions are trivially true. We conclude that an AKE field must in particular be existentially closed in every immediate extension (L,v). (We have used this idea already in the proof of the foregoing lemma.) We can exploit this fact by taking (L,v) to be a maximal immediate extension of (K,v) and asking which properties of (L,v) are inherited by (K,v) if $(K,v) \prec_\exists (L,v)$. We know that (L,v) has strong properties since by Theorem 7.124, it is spherically complete. Nevertheless, we will see later that if (K,v) is henselian of residue characteristic 0, then we even have that $(K,v) \prec (L,v)$, which means that the elementary properties of (L,v) are not stronger than that of (K,v). For other classes of valued fields, the situation can be much different. Let us prove that every AKE field is henselian defectless:

Lemma 14.5 *Let (K,v) be a valued field and assume that there is some maximal immediate extension (L,v) of (K,v) which satisfies $(K,v) \prec_\exists (L,v)$. Then (K,v) is henselian defectless. In particular, every AKE field is henselian defectless.*

Proof: Let $(E|K,v)$ be an arbitrary finite extension. We take $(E|K,v)^*$ to be a κ-saturated elementary extension of $(E|K,v)$ (cf. Section 13.13). Then (E^*,v^*) and (K^*,v^*)

are κ-saturated elementary extensions of (E,v) and (K,v) respectively. We choose κ such that $\kappa \geq |L|^+$. Since by assumption (K,v) is existentially closed in (L,v), Lemma 13.38 shows that we can embed (L,v) over (K,v) in (K^*,v^*). We identify it with its image in (K^*,v^*). Since $(E^*|K^*,v^*)$ is an elementary extension of $(E|K,v)$ and $n=[E:K]<\infty$ we have that $[E^*:K^*]=n$ and thus also $[E.L:L]=n$.

The extension $(L|K,v)$ is immediate, and we will prove the same for $(E.L,v^*)|(E,v)$. Since $E.L|L$ is algebraic and $vL=vK$, we know that $v^*(E.L)/vK$ and thus also $v^*(E.L)/vE$ is a torsion group. For the same reason $\overline{L}=\overline{K}$ yields that $\overline{E.L}|\overline{K}$ and hence also $\overline{E.L}|\overline{E}$ is algebraic. On the other hand, since (E^*,v^*) is an elementary extension of (E,v) we know by Lemma 13.37 that v^*E^*/vE is torsion free and that $\overline{E}$ is relatively algebraically closed in $\overline{E^*}$. Combining these facts, we get that

$$v^*(E.L)=vE \quad \text{and} \quad \overline{E.L}=\overline{E}$$

showing that $(E.L,v^*)|(E,v)$ is immediate, as contended.

Since (L,v) is maximal, it is a defectless field by Corollary 8.28. Consequently,

$$[E:K]=n=[E.L:L]=(v^*(E.L):vL)\cdot[\overline{E.L}:\overline{L}]=(vE:vK)\cdot[\overline{E}:\overline{K}]$$

which shows that $(E|K,v)$ is defectless and that the extension of the valuation v from K to E is unique. Since (E,v) was an arbitrary finite extension of (K,v), this shows that (K,v) is a henselian defectless field. □

In contrast to the property of being henselian, we do not know of a sufficiently handy axiom scheme which axiomatizes the property of being defectless. Nevertheless, with a little work (which we will mainly leave to the reader), one can prove:

Lemma 14.6 *The property of being henselian defectless is elementary.*

Proof: We have to talk about finite extensions of a given field. We first observe that this can be encoded in the field itself. Let $L|K$ be a field extension of degree n, and pick a basis $b_1,\ldots,b_n$ of $L|K$. Then every element of L is of the form $c_1b_1+\ldots+c_nb_n$ with $c_i\in K$, and the assignment $c_1b_1+\ldots+c_nb_n\mapsto(c_1,\ldots,c_n)\in K^n$ defines an isomorphism of K-vector spaces. Hence as a K-vector space, we can describe L by talking about n-tuples of elements of K (we need n variables $x_1,\ldots,x_n$ for that). The multiplication induced by L on K^n can be described by an n-tuple $(c_{1ij},\ldots,c_{nij})\in K^n$ for every product $b_ib_j=c_{1ij}b_1+\ldots+c_{nij}b_n$. If we want to quantify over all possible multiplications on K^n, then we have to introduce variables y_{ijk} for the elements in these tuples. Since the set of field axioms is finite (it contains no infinite schemes), we can take the conjunction over all axioms to obtain a single sentence. In this sentence, we translate the addition to the addition of n-tuples in K^n, and the multiplication to the multiplication in K^n which is determined by the choice of the y_{ijk}. By the so obtained sentence φ_{F}, we can say that a given choice of elements in K for the variables y_{ijk} induces a field structure on K^n. Now we can talk about the properties of extensions of K of degree n by saying $\forall y_{ijk}:\varphi_{\mathrm{F}}(y_{ijk})\rightarrow\psi$ or $\exists y_{ijk}:\varphi_{\mathrm{F}}(y_{ijk})\rightarrow\psi$. We can also define the norm $\mathrm{N}_{L|K}$ (and the trace $\mathrm{Tr}_{L|K}$). To see this, for every divisor m of n we take a formula $\psi(x_1,\ldots,x_n,z_1,\ldots,z_m)$ such that for $c_i,d_i\in K$, $\psi(c_1,\ldots,c_n,d_1,\ldots,d_m)$ holds if and only if $X^m+d_1X^{m-1}+\ldots+d_m$ is the minimal polynomial over K of the element represented by $(c_1,\ldots,c_n)$. We leave it to the

reader to define ψ, using the already defined multiplication on K^n. The norm N is then defined by

$$\bigwedge_{m|n} \Big((\psi(x_1, \ldots, x_n, z_1, \ldots, z_m) \ \rightarrow \ N(x_1, \ldots, x_n) = (-1)^n z_m^{n/m} \Big) \ .$$

Now we are prepared to talk about valued field extensions (L, w) of (K, v) of degree n, provided that (K, v) is henselian (which we state by a translated version of axiom (HENS)). Then the value of an element $a \in L$ is given by $\frac{1}{n} v \mathrm{N}_{L|K} a$ (cf. Exercise 7.13). Consequently, the valuation divisibility relation on K^n with respect to a given multiplication can be defined by

$$w(x_1, \ldots, x_n) \geq w(x'_1, \ldots, x'_n) \ \leftrightarrow \ vN(x_1, \ldots, x_n) \geq vN(x'_1, \ldots, x'_n) \ .$$

After these preparations, it remains to find a sentence which states that with the given structure of a valued field extension of (K, v), there is a valuation basis in K^n over (K, v). For this, we note that $b_1, \ldots, b_n \in L$ is a valuation basis over K if and only if for all $d_1, \ldots, d_n \in K$ we have that

$$\bigwedge_{1 \leq i \leq n} \left((\bigwedge_{1 \leq j \leq n} v d_i b_i \leq v d_j b_j)) \ \rightarrow v d_i b_i = v(d_1 b_1 + \ldots + d_n b_n) \right) \ .$$

We leave it to the reader to formalize this in K^n. □

For the conclusion of this section, let us observe that Lemma 14.1 shows that also the elementary properties of value groups and residue fields of valued fields can be axiomatized in the language of valued fields. For instance, the **theory of henselian fields of residue characteristic** 0 is the theory of valued fields together with axiom scheme (HENS) and the axiom scheme

(RC0) $\qquad v(n \cdot 1) = 0 \qquad\qquad (n \in \mathbb{N}) \, .$

Similarly, residue characteristic p (a prime) can be expressed by

(RC0) $\qquad v(p \cdot 1) > 0 \ .$

So by adding the axiom(s) which describe the characteristic of the field itself, we can state whether a valued field is of equal or of mixed characteristic.

14.3 Extensions without transcendence defect

Let (K, v) be a henselian defectless field and $(L|K, v)$ an extension without transcendence defect. We choose (K^*, v^*) to be an $|L|^+$-saturated elementary extension of (K, v). Since "henselian" is an elementary property, (K^*, v^*) is henselian like (K, v). Further, it follows from Corollary 14.3 that $K^* v^*$ is an $|\overline{L}|^+$-saturated elementary extension of $\overline{K}$ and that $v^* K^*$ is a $|vL|^+$-saturated elementary extension of vK. Assume that the side conditions $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$ hold. Then by Lemma 13.38, there exist embeddings

$$\rho : \ vL \longrightarrow v^* K^*$$

over vK and

$$\sigma : \overline{L} \longrightarrow K^*v^*$$

over $\overline{K}$. By definition of "without transcendence defect", every finitely generated subextension $(F|K, v)$ of $(L|K, v)$ is a function field without transcendence defect. At this point, let us recall the assertion of Embedding Lemma I (Lemma 12.40):

Let (K, v) be a defectless field (the valuation is allowed to be trivial), $(F|K, v)$ a valued function field without transcendence defect and (K^*, v^*) a henselian extension of (K, v). Assume that vF/vK is torsion free and that $\overline{F}|\overline{K}$ is separable. If $\rho : vF \to v^*K^*$ is an embedding over vK and $\sigma : \overline{F} \to K^*v^*$ is an embedding over $\overline{K}$, then there exists an embedding $\iota : (F, v) \to (K^*, v^*)$ over K which respects ρ and σ.

By assumption, $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$, which implies that $vK \prec_\exists vF$ and $\overline{K} \prec_\exists \overline{F}$ because $vF|vK$ is a subextension of $vL|vK$, and $\overline{F}|\overline{K}$ is a subextension of $\overline{L}|\overline{K}$. So we can infer from Lemma 13.37 that the conditions "vF/vK is torsion free" and "$\overline{F}|\overline{K}$ is separable" are satisfied. Hence there is an embedding

$$\iota : (F, v) \longrightarrow (K^*, v^*)$$

over K that respects ρ and σ. By Lemma 13.38, this shows that (K, v) is existentially closed in (F, v). Since this holds for every finitely generated subextension $(F|K, v)$ of $(L|K, v)$, we can now use Lemma 13.40 to conclude:

Theorem 14.7 *Every extension without transcendence defect of a henselian defectless field satisfies the Ax–Kochen–Ershov Principle.*

Note that it is not in general true that for henselian defectless fields (K, v) and (L, v), the strengthened side conditions

$$vK \prec vL\ , \quad \overline{K} \prec \overline{L}$$

would imply that $(K, v) \prec (L, v)$. See Corollary 14.48 below.

For further use, we have to make our result more precise:

Lemma 14.8 (Embedding Lemma II)
Let (K, v) be a defectless field (the valuation is allowed to be trivial), $(L|K, v)$ an extension without transcendence defect and (K^*, v^*) an $|L|^+$-saturated henselian extension of (K, v). Assume that vL/vK is torsion free and that $\overline{L}|\overline{K}$ is separable. If

$$\rho : vL \longrightarrow v^*K^*$$

is an embedding over vK and

$$\sigma : \overline{L} \longrightarrow K^*v^*$$

is an embedding over $\overline{K}$, then there exists an embedding

$$\iota : (L, v) \longrightarrow (K^*, v^*)$$

over K which respects ρ and σ.

Proof: We have already shown that every finitely generated subextension of $(L|K, v)$ can be embedded over (K, v) in (K^*, v^*) respecting both embeddings ρ and σ. Using the saturation property of (K^*, v^*) we have to deduce our assertion from this. To do so, we will work in an extended language $\mathcal{L}'$ consisting of the language $\mathcal{L}_{\rm VF}$ of valued fields together with the predicates

$$\begin{aligned} \mathcal{P}_\alpha\,, &\qquad \alpha \in \rho v L \\ \mathcal{Q}_\zeta\,, &\qquad \zeta \in \sigma\overline{L} \end{aligned}$$

which are interpreted in (K^*, v^*) such that

$$\begin{aligned} \mathcal{P}_\alpha(a) &\iff v^*a = \alpha \\ \mathcal{Q}_\zeta(a) &\iff av^* = \zeta \end{aligned}$$

for all $a \in K^*$ and in (L, v) such that

$$\begin{aligned} \mathcal{P}_\alpha(b) &\iff \rho v b = \alpha \\ \mathcal{Q}_\zeta(b) &\iff \sigma\overline{b} = \zeta \end{aligned}$$

for all $b \in L$. Note that these interpretations coincide on K.

We show that (K^*, v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}'$. To this end, we choose a subset $S_v \subset K^*$ of representatives for all values α in $\rho v L$, and a subset $S_r \subset K^*$ of representatives for all residues ζ in $\sigma\overline{L}$. We compute

$$\begin{aligned} |S_v| &= |\rho v L| = |vL| \leq |L| < |L|^+\,, \\ |S_r| &= |\sigma\overline{L}| = |\overline{L}| \leq |L| < |L|^+\,, \end{aligned}$$

hence $|S_v \cup S_r| < |L|^+$. Consequently, it follows from Lemma 13.26 that (K^*, v^*) remains $|L|^+$-saturated in the extended language $\mathcal{L}_{\rm VF}(S_v \cup S_r)$ (the new constants are interpreted in K^* by the corresponding elements from $S_v \cup S_r$). Now the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ become definable in the language $\mathcal{L}^*$. Indeed, if $\alpha \in \rho v L$, then we choose $b_\alpha \in S_v$ such that $vb_\alpha = \alpha$ and define $\mathcal{P}_\alpha(x) :\Leftrightarrow vx = vb_\alpha$. If $\zeta \in \sigma\overline{L}$, then we choose $b_\zeta \in S_r$ such that $b_\zeta v^* = \zeta$ and define $\mathcal{Q}_\zeta(x) :\Leftrightarrow v^*(x - b_\zeta) > 0$. Since (K^*, v^*) is $|L|^+$-saturated in the language $\mathcal{L}_{\rm VF}(S_v \cup S_r)$, it follows that it is also $|L|^+$-saturated in the language $\mathcal{L}'(S_v \cup S_r)$ and thus also in the language $\mathcal{L}'$, as asserted.

An embedding ι of an arbitrary subextension (L_0, v) of $(L|K, v)\,)$ in (K^*, v^*) over K respects the predicates $\mathcal{P}_\alpha$ and $\mathcal{Q}_\zeta$ if and only if it satisfies, for all $b \in L_0\,$,

$$\begin{aligned} &\rho v b = \alpha \iff (L_0, v) \models \mathcal{P}_\alpha(b) \iff (K^*, v^*) \models \mathcal{P}_\alpha(\iota b) \iff v^*\iota b = \alpha\,, \\ &\sigma\overline{b} = \zeta \iff (L_0, v) \models \mathcal{Q}_\zeta(b) \iff (K^*, v^*) \models \mathcal{Q}_\zeta(\iota b) \iff \iota b/v^* = \zeta\,, \end{aligned}$$

which expresses the property of ι to respect the embeddings ρ and σ. We know that for every finitely generated subextension of $(L|K, v)$ there exists such an embedding ι. The saturation property of (K^*, v^*) now yields an embedding of (L, v) in (K^*, v^*) over K which respects the predicates and thus the embeddings ρ and σ. This completes the proof of our lemma. □

14.4 The Relative Embedding Property

Inspired by the assertion of Lemma 14.8, we define a property that will play a key role in our treatment of the model theory of valued fields. Let $\mathbf{C}$ be a class of valued fields. We will say that $\mathbf{C}$ has the **Relative Embedding Property**, if the following holds: If $(L,v),(K^*,v^*) \in \mathbf{C}$ and (K,v) is a defectless valued subfield of (L,v) and (K^*,v^*) such that (K^*,v^*) is $|L|^+$-saturated, vL/vK is torsion free and $\overline{L}|\overline{K}$ is separable, and if there are embeddings $\rho: vL \longrightarrow v^*K^*$ over vK and $\sigma: \overline{L} \longrightarrow K^*v^*$ over $\overline{K}$, then there exists an embedding $\iota: (L,v) \longrightarrow (K^*,v^*)$ over K which respects ρ and σ.

We will show that the Relative Embedding Property implies the following important property. If for every two fields $(L,v),(F,v) \in \mathbf{C}$ and every common defectless subfield (K,v) of (L,v) and (F,v) such that vL/vK is torsion free and $\overline{L}|\overline{K}$ is separable, the side conditions $vL \equiv_{vK} vF$ and $\overline{L} \equiv_{\overline{K}} \overline{F}$ imply that $(L,v) \equiv_{(K,v)} (F,v)$, then we will call $\mathbf{C}$ **relatively subcomplete**. Note that if $\mathbf{C}$ is a relatively subcomplete class of defectless fields, then $\mathbf{C}$ is relatively model complete because by Lemma 13.37, the side conditions $vK \prec vL$ and $\overline{K} \prec \overline{L}$ imply that $(K,v) \prec (L,v)$, and they imply that $vK \equiv_{vK} vL$ and $\overline{K} \equiv_{\overline{K}} \overline{L}$. But relative model completeness is weaker than relative subcompleteness, because $vL \equiv_{vK} vF$ does not imply that $vK \prec vL$, and $\overline{L} \equiv_{\overline{K}} \overline{F}$ does not imply that $\overline{K} \prec \overline{L}$.

The following lemma shows that the Relative Embedding Property is a very powerful property:

Lemma 14.9 *Let $\mathbf{C}$ be an elementary class of defectless valued fields which has the Relative Embedding Property. Then $\mathbf{C}$ is relatively subcomplete and relatively model complete, and the AKE Principle is satisfied by all extensions $(L|K,v)$ such that both $(K,v),(L,v) \in \mathbf{C}$. If moreover all fields in $\mathbf{C}$ are of fixed equal characteristic, then $\mathbf{C}$ is relatively complete.*

Proof: We can assume that all fields in the class $\mathbf{C}$ are infinite since otherwise, $\mathbf{C}$ only consists of one trivially valued field (up to isomorphism, cf. Exercise 13.2), and all assertions are trivial.

Let us first show that $(L|K,v)$ satisfies the AKE Principle whenever $(K,v),(L,v) \in \mathbf{C}$. So assume that $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$. We take (K^*,v^*) to be an $|L|^+$-saturated elementary extension of (K,v). Since $\mathbf{C}$ is assumed to be an elementary class, $(K,v) \in \mathbf{C}$ implies that $(K^*,v^*) \in \mathbf{C}$. Because of $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$, there are embeddings $vL \to v^*K^*$ over vK and $\overline{L} \to K^*v^*$ over $\overline{K}$, and moreover, vL/vK is torsion free and $\overline{L}|\overline{K}$ is separable by Lemma 13.37. So by the Relative Embedding Property there is an embedding of (L,v) in (K^*,v^*) over K, which by Lemma 13.38 shows that $(K,v) \prec_\exists (L,v)$.

Now assume that $(L,v),(F,v) \in \mathbf{C}$ with common valued subfield (K,v) such that vL/vK is torsion free, $\overline{L}|\overline{K}$ is separable, $vL \equiv_{vK} vF$ and $\overline{L} \equiv_{\overline{K}} \overline{F}$. We have to show that $(L,v) \equiv_{(K,v)} (F,v)$. We take (L^*,v) and (F^*,v) to be strong limit extensions of (L,v) and (F,v) respectively, with basis $\kappa \geq \max\{|L|,|F|\}$ (note that $||\mathcal{L}_{\mathrm{VF}}|| = \aleph_0$). The fields in the chains will be denoted by L_ν and F_ν respectively, and we have that $L_0 = L$ and $F_0 = F$. All (L_i,v) and (F_i,v) are elementary extensions of (L,v) and (F,v) respectively, so it follows that they lie in $\mathbf{C}$. Now we carry out a back and forth construction similar to that in the proof of Theorem 13.35. Since $(F,v) \prec (F_1,v)$, it follows from Corollary 14.2 that $vF \prec vF_1$ and $\overline{F} \prec \overline{F}_1$. Hence, the side conditions $vL \equiv_{vK} vF$ and $\overline{L} \equiv_{\overline{K}} \overline{F}$ imply that $vL \equiv_{vK} vF_1$ and $\overline{L} \equiv_{\overline{K}} \overline{F}_1$. Since (F_1,v) is κ^+-saturated, it follows from Corollary 14.3 that the same

is true for vF_1 and $\overline{F}_1$. On the other hand, $|vL| \leq |L| \leq \kappa$ and $|\overline{L}| \leq |L| \leq \kappa$. Hence by Theorem 13.31, we obtain elementary embeddings $vL \to vF_1$ over vK and $\overline{L} \to \overline{F}_1$ over $\overline{K}$. By the Relative Embedding Property, we find an embedding ι_0 of (L_0, v) in (F_1, v) over K which respects the embeddings of the value group and the residue field. That is, we have that $v\iota_0 L_0 \prec vF_1$ and $\overline{\iota_0 L_0} \prec \overline{F}_1$.

The isomorphism $\iota_0^{-1} : \iota_0 L_0 \to L_0$ can be extended to an isomorphism $\iota_0^{-1} : \iota_0 F_1 \to \iota_0^{-1} F_1$. It induces isomorphism on the value groups and the residue fields, so that we obtain that $vL_0 \prec v\iota_0^{-1} F_1$ and $\overline{L_0} \prec \overline{\iota_0^{-1} F_1}$. Since $(L_0, v) \prec (L_2, v)$, it follows from Corollary 14.2 that $vL_0 \prec vL_2$ and $\overline{L}_0 \prec \overline{L}_2$. Hence, we obtain that $vL_2 \equiv_{vL_0} v\iota_0^{-1} F_1$ and $\overline{L}_2 \equiv_{\overline{L}_0} \overline{\iota_0^{-1} F_1}$. Since (L_2, v) is κ_1^+-saturated, the same is true for vL_2 and $\overline{L}_2$. On the other hand, $|v\iota_0^{-1} F_1| \leq |\iota_0^{-1} F_1| = |F_1| \leq \kappa_1$ and $|\overline{\iota_0^{-1} F_1}| \leq |F_1| \leq \kappa_1$. Hence, we obtain elementary embeddings $v \to vL_2$ over vL_0 and $\overline{\iota_0^{-1} F_1} \to \overline{L}_2$ over $\overline{L}_0$. By the Relative Embedding Property, we find an embedding $\tilde{\iota}_1$ of $(\iota_0^{-1} F_1, v)$ in (L_2, v) over L_0 which respects the embeddings of the value group and the residue field. That is, we obtain an embedding $\iota_1' := \tilde{\iota}_1 \iota^{-1}$ such that $v\iota_1' F_1 \prec vL_2$ and $\overline{\iota_1' F_1} \prec \overline{L}_2$.

Now we proceed by induction, like in the proof of Theorem 13.35. We obtain an isomorphism from (L^*, v) onto (F^*, v) over K, which shows that $(L^*, v) \equiv_{(K,v)} (F^*, v)$. Since $(L, v) \prec (L^*, v)$ and $(F, v) \prec (F^*, v)$, this implies that $(L, v) \equiv_{(K,v)} (F, v)$, as required. We have proved that $\mathbf{C}$ is relatively subcomplete, and we know already that this implies that $\mathbf{C}$ is relatively model complete.

Finally, assume in addition that all fields in $\mathbf{C}$ are of fixed equal characteristic. We wish to show that $\mathbf{C}$ is relatively complete. So let $(L, v), (F, v) \in \mathbf{C}$ such that $vL \equiv vF$ and $\overline{L} \equiv \overline{F}$. Fixed characteristic means that L and F have a common prime field K. The assumption that both (L, v) and (F, v) are of equal characteristic means that the restrictions of their valuations to K is trivial. Hence, $vK = 0$ and consequently, vL/vK is torsion free and $vL \equiv vF$ implies that $vL \equiv_{vK} vF$. Further, $K = \overline{K}$ is also the prime field of $\overline{L}$ and $\overline{F}$, so $\overline{L} \equiv \overline{F}$ implies that $\overline{L} \equiv_{\overline{K}} \overline{F}$ (cf. Exercise 13.1). Since a prime field is always perfect, we also have that $\overline{L}|\overline{K}$ is separable. From what we have already proved, we obtain that $(L, v) \equiv_{(K,v)} (F, v)$, which implies that $(L, v) \equiv (F, v)$. □

Now we look for a criterion for an elementary class of valued fields to have the Relative Embedding Property. Somehow, we have to improve Embedding Lemma II (Lemma 14.8) to cover the case of extensions $(L|K, v)$ with transcendence defect. Loosely speaking, these contain an immediate part. The idea is to require that this part can be treated separately, that is, that we find an intermediate field $(L', v) \in \mathbf{C}$ such that $(L|L', v)$ is immediate and $(L'|K, v)$ has no transcendence defect. The immediate part has then to be settled by the following auxiliary embedding lemma. Note that by Theorem 7.131, the condition "appr(x, K) is transcendental" does automatically hold if (K, v) is algebraically maximal.

Lemma 14.10 (Embedding Lemma III)
Let $(K(x)|K, v)$ be a nontrivial immediate extension of valued fields. If appr(x, K) *is transcendental, then $(K(x), v)^h$ can be embedded over K in every $|K|^+$-saturated henselian extension $(K, v)^*$ of (K, v).*

Proof: Since $(K, v)^*$ is $|K|^+$-saturated, Lemma 13.85 shows that appr(x, K) is realized by an element x' in $(K, v)^*$. By Theorem 7.120, the homomorphism induced by $x \mapsto x'$ is an embedding of $(K(x), v)$ over K in $(K, v)^*$ because appr(x, K) is transcendental. By the

universal property of the henselization (Theorem 7.107), this embedding can be extended to an embedding of $(K(x),v)^h$ over K in $(K,v)^*$, since the latter is henselian by hypothesis. □

Note that the lemma fails if the condition on the approximation type of x over K is omitted, even if we require in addition that (K,v) is henselian. By Theorem 8.41 and Theorem 8.45 there exist nontrivial finite immediate extensions $(K(x)|K,v)$ of henselian fields. On the other hand, K^* may be a regular extension of K (e.g., this is always the case if $(K,v)^*$ is an elementary extension of (K,v)), and then, $K(x)$ does certainly not admit an embedding over K in K^*.

The model theoretic application of Embedding Lemma III is:

Corollary 14.11 *Let (K,v) be a henselian field and $(K(x)|K,v)$ an immediate extension such that* appr (x,K) *is transcendental. Then $(K,v) \prec_\exists (K(x),v)^h$. In particular, an algebraically maximal field (K,v) is existentially closed in every immediate henselian rational function field $(K(x),v)^h$.*

Proof: Choose $(K,v)^*$ to be a $|K|^+$-saturated elementary extension of (K,v). Since "henselian" is an elementary property, $(K,v)^*$ will also be henselian. Apply Embedding Lemma III and Lemma 13.38. □

Now we are able to give the announced criterion:

Lemma 14.12 *Let* **C** *be an elementary class of valued fields which satisfies*

(CHDL) *every field in* **C** *is henselian defectless,*

(CRAC) *given $(L,v) \in \mathbf{C}$ and K' relatively algebraically closed in L such that $\overline{L}|\overline{K'}$ is algebraic and vL/vK' is a torsion group, then $(K',v) \in \mathbf{C}$ with $\overline{L} = \overline{K'}$ and $vL = vK'$,*

(CIMM) *if $(K,v) \in \mathbf{C}$, then every immediate henselian function field of transcendence degree 1 over (K,v) is henselian rational.*

Then **C** *has the Relative Embedding Property and is relatively subcomplete and relatively model complete, and the AKE Principle is satisfied by all extensions $(L|K,v)$ such that both (K,v) and (L,v) are members of* **C**.

Proof: Let **C** satisfy (CHDL), (CRAC) and (CIMM). Further, let $(L,v),(K^*,v^*) \in \mathbf{C}$ with (K^*,v^*) being $|L|^+$-saturated, (K,v) a valued subfield of (L,v) and (K^*,v^*) such that vL/vK is torsion free and $\overline{L}|\overline{K}$ is separable, $\rho : vL \to v^*K^*$ an embedding over vK and $\sigma : \overline{L} \to K^*v^*$ an embedding over $\overline{K}$. We have to show that there exists an embedding $\iota : (L,v) \to (K^*,v^*)$ over K which respects ρ and σ.

Take $\mathcal{T} = \{x_i\,,\, y_j \mid i \in I\,,\, j \in J\}$ to be a maximal standard algebraically valuation independent set in $(L|K,v)$. Then in view of Lemma 7.26, the values $vx_i\,,\ i \in I$, form a transcendence basis of $vL|vK$, and the residues $\overline{y}_j\,,\ j \in J$, form a transcendence basis of $\overline{L}|\overline{K}$. Consequently, $vL/vK(\mathcal{T})$ is a torsion group and $\overline{L}|\overline{K(\mathcal{T})}$ is algebraic. Let K' be the relative algebraic closure of $K(\mathcal{T})$ within L. It follows that also vL/vK' is a torsion group and $\overline{L}|\overline{K'}$ is algebraic. Hence by condition (CRAC), we have that $(K',v) \in \mathbf{C}$ with $\overline{L} = \overline{K'}$ and $vL = vK'$, which shows that the extension $L|K'$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $(K'|K,v)$ by construction, hence

according to Corollary 7.30, this extension has no transcendence defect. Since (K, v) is defectless by assumption and (K^*, v^*) is henselian by condition (CHDL), Lemma 14.8 gives an embedding of (K', v) in (K^*, v^*) over K which respects ρ and σ. Now we have to look for an extension of this embedding to (L, v). Since $(L|K', v)$ is immediate, such an extension will automatically respect ρ and σ.

We identify K' with its image in K^*. In view of Lemma 13.39, it remains to show that every finitely generated subextension (L', v) of $(L|K', v)$ can be embedded in (K^*, v^*) over K'. We apply our slicing approach. Since L' is finitely generated over K', it has a finite transcendence basis $\{t_1, \ldots, t_n\}$ over K'. Let us put $K_0 = K'$ and K_i to be the relative algebraic closure of $K(t_1, \ldots, t_i)$ in L for $1 \leq i \leq n$. Then K_n contains L', and by condition (CRAC), every (K_i, v) is a member of $\mathbf{C}$. Moreover, $\mathrm{trdeg}(K_{i+1}|K_i) = 1$ for $0 \leq i < n$. We proceed by induction on i. If we have shown that (K_i, v) can be embedded in (K^*, v^*) over K', then we identify it with its image. Hence it now remains to show that the immediate extension (K_{i+1}, v) of transcendence degree 1 can be embedded in (K^*, v^*) over K_i. Since (K^*, v^*) is $|L|^+$-saturated, it is also $|K_{i+1}|^+$-saturated. Hence again, Lemma 13.39 shows that it suffices to prove the existence of an embedding for every finitely generated subextension (K'_{i+1}, v) of $(K_{i+1}|K_i, v)$. The henselization of (K'_{i+1}, v) is an immediate henselian function field of transcendence degree 1 over (K_i, v), so by condition (CIMM), it is henselian rational. Since (K_i, v) is defectless by condition (CHDL), Embedding Lemma III now yields that there is an embedding of (K'_{i+1}, v) in (K^*, v^*) over K_i. This completes our proof by induction. □

14.5 Model theory of tame fields

In this section we wish to treat the model theory of tame fields. We have already shown in Lemma 12.18 that in positive characteristic, the class of tame fields coincides with the class of algebraically maximal perfect fields. Let us show that the property of being a tame field of fixed residue characteristic is elementary. If the residue characteristic is fixed to 0 then by Lemma 12.14, "tame" is equivalent to "henselian" which is axiomatized by the axiom scheme (HENS). Now assume that the residue characteristic is fixed to be a positive prime. By Lemma 12.18, a valued field of positive residue characteristic is tame if and only if it is an algebraically maximal field having p-divisible value group and perfect residue field. A valued field (K, v) has p-divisible value group if and only if it satisfies the following elementary axiom:

($\mathbf{VGD}_p$) $\quad \forall x \, \exists y : \; vxy^p = 0 \; .$

Furthermore, (K, v) has perfect residue field if and only if it satisfies:

($\mathbf{RFD}_p$) $\quad \forall x \, \exists y : \; vx = 0 \; \rightarrow \; v(xy^p - 1) > 0 \; .$

Finally, the property of being algebraically maximal is axiomatized by the axiom schemes (HENS) and (MAXP). We summarize: The **theory of tame fields of residue characteristic** 0 is just the theory of henselian fields of residue characteristic 0. If p is a prime, then the **theory of tame fields of residue characteristic** p is the theory of valued fields together with axioms (VGD_p), (RFD_p), (HENS) and (MAXP). Now we also see how

we have to proceed to axiomatize the theory of all tame fields. Indeed, for residue characteristic 0 there are no conditions on the value group and the residue field. For residue characteristic $p > 0$, we have to require (VGD_p) and (RFD_p). We can do this by the axiom scheme

(TAD) $\quad v(p \cdot 1) > 0 \ \rightarrow \ (\mathrm{VGD}_p) \wedge (\mathrm{RFD}_p) \qquad\qquad (p \text{ prime})\,.$

So the **theory of tame fields** is the theory of valued fields together with axioms (TAD), (HENS) and (MAXP).

Recall that by Corollary 12.19, a valued field of positive characteristic is tame if and only if it is algebraically maximal and perfect. We have already seen that every AKE field must be henselian defectless and in particular algebraically maximal. So this condition is necessary to obtain good model theoretical results. In view of this fact, it is also the model theory of perfect field in positive characteristic that we will develop now.

Let $\mathbf{C}$ be the elementary class of all tame fields. By definition, all tame fields are henselian defectless, so $\mathbf{C}$ satisfies condition (CHDL) of Lemma 14.12. By Lemma 12.21, it also satisfies condition (CRAC). Finally, it satisfies (CIMM) by virtue of Theorem 12.45. Hence, we can infer from Lemma 14.12 and Lemma 14.9:

Theorem 14.13 *The elementary class of tame fields has the Relative Embedding Property and is relatively subcomplete and relatively model complete. Every elementary class of tame fields of fixed equal characteristic is relatively complete.*

Lemma 14.12 does not give the full information about the AKE principle. Therefore, we need the following lemma:

Lemma 14.14 *If Γ is a p-divisible ordered abelian group and $\Gamma \prec_\exists \Delta$, then Γ is also existentially closed in the p-divisible hull of Δ. If k is a perfect field and $k \prec_\exists \ell$, then k is also existentially closed in the perfect hull of ℓ.*

If (K, v) is a tame field and $(L|K, v)$ an extension with $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$, then every maximal purely wild algebraic extension (W, v) of (L, v) is a tame field satisfying $vK \prec_\exists vW$ and $\overline{K} \prec_\exists \overline{W}$.

Proof: By Lemma 13.38, $\Gamma \prec_\exists \Delta$ implies that Δ is embeddable over Γ in every $|\Delta|^+$-saturated elementary extension of Γ. Such an elementary extension is p-divisible like Γ. Hence, the embedding can be extended to an embedding of $\frac{1}{p^\infty}\Delta$, which by Lemma 13.38 shows that $\Gamma \prec_\exists \frac{1}{p^\infty}\Delta$.

Again by Lemma 13.38, $k \prec_\exists \ell$ implies that ℓ is embeddable over k in every $|\ell|^+$-saturated elementary extension of k. Such an elementary extension is perfect like k. Hence, the embedding can be extended to an embedding of ℓ^{1/p^∞}, which by Lemma 13.38 shows that $k \prec_\exists \ell^{1/p^\infty}$.

By Theorem 12.17, vW is the p-divisible hull $\frac{1}{p^\infty}vL$ of vL, and $\overline{W}$ is the perfect hull $\overline{L}^{1/p^\infty}$ of $\overline{L}$. So our assertion since we have just proved that vK (which is itself p-divisible by Lemma 12.18) is existentially closed in $\frac{1}{p^\infty}vL$ and that $\overline{K}$ (which is itself perfect by Lemma 12.18) is existentially closed in the perfect hull $\overline{L}^{1/p^\infty}$ of $\overline{L}$. □

Assume that (K, v) is a tame field and $(L|K, v)$ an extension such that $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$. We choose the tame field (W, v) according to the foregoing lemma. If $(K, v) \prec_\exists (W, v)$, then also $(K, v) \prec_\exists (L, v)$. In view of Lemma 14.12, this proves:

Theorem 14.15 *Every tame field is an AKE field.*

This theorem was applied in Chapter 10 of [KU1] (which is Chapter 9 of [KU2] to generalize the results of [KU–PR] to the case of perfect ground fields.

Now let $\mathbf{C}$ be an elementary class of valued fields. We define

$$v\mathbf{C} := \{vK \mid (K,v) \in \mathbf{C}\} \quad \text{and} \quad \overline{\mathbf{C}} := \{\overline{K} \mid (K,v) \in \mathbf{C}\} .$$

If both $v\mathbf{C}$ and $\overline{\mathbf{C}}$ are elementary model complete classes, then the side conditions $vK \prec vL$ and $\overline{K} \prec \overline{L}$ will hold for every two members $(K,v) \subset (L,v)$ of $\mathbf{C}$. Similarly, if $v\mathbf{C}$ and $\overline{\mathbf{C}}$ are elementary complete classes, then the side conditions $vK \equiv vL$ and $\overline{K} \equiv \overline{L}$ will hold for all $(K,v), (L,v) \in \mathbf{C}$. So we obtain from the foregoing theorems:

Theorem 14.16 *If $\mathbf{C}$ is an elementary class consisting of tame fields and if $v\mathbf{C}$ and $\overline{\mathbf{C}}$ are elementary model complete classes, then $\mathbf{C}$ is model complete. If $\mathbf{C}$ is an elementary class consisting of tame fields of fixed equal characteristic, and if $v\mathbf{C}$ and $\overline{\mathbf{C}}$ are elementary complete classes, then $\mathbf{C}$ is complete.*

The converses are true by virtue of Corollary 14.2, provided that $v\mathbf{C}$ and $\overline{\mathbf{C}}$ are elementary classes.

As an immediate consequence of the foregoing Theorem, we get the following criterion for decidability:

Corollary 14.17 *Let (K,v) be a tame field of equal characteristic. Assume that the theories $\mathrm{Th}(vK)$ of its value group (as an ordered group) and $\mathrm{Th}(\overline{K})$ of its residue field (as a field) both admit recursive elementary axiomatizations. Then also the theory of (K,v) as a valued field admits a recursive elementary axiomatization, and it is decidable.*

Indeed, the axiomatization of $\mathrm{Th}(K,v)$ can be taken to consist of the axioms of tame fields of equal characteristic $\mathrm{char}\, K$, together with the translations of the axioms of $\mathrm{Th}(vK)$ and $\mathrm{Th}(\overline{K})$ to the language of valued fields (cf. Lemma 14.1).

Corollary 14.18 *Let $\mathbf{T}$ be an elementary theory consisting of perfect valued fields of equal characteristic. Assume that the value groups and the residue fields of the models of $\mathbf{T}$ form model complete elementary classes. Then the theory $\mathbf{T}^*$ of algebraically maximal valued fields satisfying $\mathbf{T}$ is the model companion of $\mathbf{T}$.*

Proof: It follows from Theorem 14.16 that $\mathbf{T}^*$ is model complete. For every model K of $\mathbf{T}$, any maximal immediate algebraic extension is a model of $\mathbf{T}^*$ (by Lemma 12.18); note that it is an extension of K having the same value group and residue field. □

Note that in the case of positive characteristic, $\mathbf{T}^*$ is in general not a model completion since there exist perfect valued fields of positive characteristic which admit two nonisomorphic maximal immediate algebraic extension, both being models of the model companion. In the case of equal characteristic 0, the algebraically maximal fields are just the henselian fields, and in view of part g) of Lemma 13.56, we find that $\mathbf{T}^*$ is a model completion of $\mathbf{T}$.

Elementary classes of tame fields of equal characteristic admit weak prime models if the elementary classes of their value groups and their residue fields do:

Theorem 14.19 *Let* $\mathbf{C}$ *be an elementary class consisting of tame fields of equal characteristic. Suppose that there exists an infinite cardinal* κ*, an ordered group* $\Gamma \in v\mathbf{C}$ *and a field* $k \in \overline{\mathbf{C}}$*, both of cardinality* $\leq \kappa$*, such that* Γ *admits an elementary embedding in every* κ^+*-saturated member of* $v\mathbf{C}$ *and* k *admits an elementary embedding in every* κ^+*-saturated member of* $\overline{\mathbf{C}}$*. Then there exists* $(F, v) \in \mathbf{C}$ *of cardinality* $\leq \kappa$*, having value group* Γ *and residue field* k*, such that* (F, v) *admits an elementary embedding in every* κ^+*-saturated member of* $\mathbf{C}$*. Moreover, we can assume that* (F, v) *admits a valuation transcendence basis over its prime field.*

Proof: By Lemma 12.18 we know that k is perfect. Since k is embeddable in every κ^+-saturated member of $\overline{\mathbf{C}}$, all of them must have the same characteristic exponent p. Since every κ^+-saturated member of $\mathbf{C}$ has a κ^+-saturated residue field (cf: Corollary 14.3), it follows that every such member is a tame field of characteristic exponent p. Now let $(E, v) \in \mathbf{C}$ such that $vE = \Gamma$, and let $(E, v)^*$ be a κ^+-saturated elementary extension of (E, v). Since $\mathbf{C}$ is an elementary class, we find that $(E, v)^* \in \mathbf{C}$. Consequently, $(E, v)^*$ is a tame field of characteristic exponent p. By Lemma 12.18, its value group is p-divisible. By Corollary 14.2, v^*E^* is an elementary extension of $vE = \Gamma$, which shows that also Γ is p-divisible.

Now by Lemma 12.20, there exists a tame field (F, v) of cardinality at most κ having value group Γ and residue field k and admitting a valuation transcendence basis over its prime field. If (K^*, v^*) is a κ^+-saturated model of $\mathbf{C}$, then v^*K^* and K^*v^* are κ^+-saturated models of $v\mathbf{C}$ and $\overline{\mathbf{C}}$ respectively. Hence by hypothesis, there exists an elementary embedding of Γ in v^*K^* over the trivial group $\{0\}$, and an elementary embedding of k in K^*v^* over the prime field k_0 of k. Now k_0 is at the same time the prime field of F and K^*, and the valuation v is trivial on k_0. We have that that vF/vk_0 is torsion free and $\overline{F}|\overline{k}_0$ is separable. Now Embedding Lemma II shows the existence of an embedding of (F, v) in (K^*, v^*) over k_0. By virtue of Theorem 14.13, this embedding is elementary (because the embeddings of value group and residue field are). This shows that (F, v) is elementarily embeddable in every κ^+-saturated model of $\mathbf{C}$. This in turn shows that (F, v) is a model of $\mathbf{C}$ and thus a weak prime model of $\mathbf{C}$. □

The weak prime models that we have constructed in the foregoing proof have the special property that they admit a valuation transcendence basis over their prime field. The following corollary confirms the representative role of models with this property.

Corollary 14.20 *For every tame field* (L, v) *of arbitrary characteristic, there exists a subfield* $(K, v) \prec (L, v)$ *such that* (K, v) *admits a valuation transcendence basis over its prime field and* $(L|K, v)$ *is immediate.*

Proof: According to Lemma 12.23, for every tame field (L, v) there exists a subfield (K, v) of (L, v) admitting a valuation transcendence basis over its prime field, such that $(L|K, v)$ is immediate. In view of Theorem 14.13, the latter fact shows that $(K, v) \prec (L, v)$. □

As a final example, we want to treat here the **theory of tame fields of fixed positive characteristic with divisible value groups and fixed finite residue field** (which is the theory of tame fields together with an axiom fixing the characteristic of the field, a

translation of the axiom scheme (DIV) to the language of valued fields and an axiom of the form (RESF$_f$) which we will introduce in Section 14.8 below).

Corollary 14.21 *Every elementary class* $\mathbf{C}$ *of tame fields of fixed positive characteristic with divisible value group and fixed residue field* $I\!F_q$ *(*$q = p^r$*) is model complete, complete and decidable. Moreover, it possesses a model having transcendence degree 1 over* $I\!F_q$ *and admits an elementary embedding in every* $\aleph_1$*-saturated member of* $\mathbf{C}$*.*

Proof: The theory of divisible ordered abelian groups is model complete, complete and decidable (cf. Theorem 13.65 and Theorem 13.66). The same holds trivially for the theory of the finite field $I\!F_q$ which has only $I\!F_q$ as a model. Hence, model completeness, completeness and decidability follow readily from Theorem 14.13 and Corollary 14.17. The prime model is constructed as follows: The valued field $(I\!F_q(t), v_t)$ has value group $\mathbb{Z}$ and residue field $I\!F_q$. By Theorem 7.57, it admits an algebraic extension (F', v_t) with value group $\mathbb{Q}$ and residue field $I\!F_q$. Now we let (F, v_t) be a maximal immediate algebraic extension of (F', v_t). By Lemma 12.18, it is a tame field. Moreover, it admits $\{t\}$ as a valuation transcendence basis over its prime field. Note that $|F| = \aleph_0$. Since $\mathbb{Q}$ is a prime model of the theory of nontrivial divisible ordered abelian groups, Embedding Lemma II (Lemma 14.8) shows that (F, v_t) admits an embedding in every $\aleph_1$-saturated member of $\mathbf{C}$. By the model completeness that we have already proved, this embedding is elementary. □

Exercise 14.3 *Let* (K, v) *be a henselian field and* $(K, v) \prec_\exists (L, v)$*. Show that* $(K, v) \prec_\exists (L, v)^h$*. State other examples for the same principle.*

14.6 Separably defectless and separably tame fields

We will first deal with the completion of a valued field.

Theorem 14.22 *Let* (K, v) *be a henselian field. Assume that* $(L|K, v)$ *is a separable subextension of* $(K^c|K, v)$*. Then* (K, v) *is existentially closed in* (L, v)*. In particular, every henselian inseparably defectless field is existentially closed in its completion.*

Proof: By Lemma 13.40, it suffices to show that (K, v) is existentially closed in every subfield (F, v) of (L, v) which is finitely generated over K. Equivalently, it suffices to show that (K, v) is existentially closed in $(F, v)^h$ (note that $(F, v)^h \subset (K, v)^c$ since the completion of a henselian field is again henselian, cf. Lemma 9.8). By Theorem 12.46,

$$(F, v)^h = (K(x_1, \ldots, x_n), v)^h \ ,$$

where $x_1, \ldots, x_n$ is any separating transcendence basis of $F|K$. Let $F_0 = K$ and $(F_i, v) = (K(x_1, \ldots, x_i), v)^h$, $1 \leq i \leq n$, where the henselization is taken within F^h. Now it suffices to show that $(F_{i-1}, v) \prec_\exists (F_i, v)$ for $1 \leq i \leq n$. As x_i is an element of the completion K^c of (F_{i-1}, v) and is transcendental over F_{i-1}, its approximation type over F_{i-1} must be transcendental too, according to Lemma 7.127. Hence by Corollary 14.11, $(F_{i-1}, v) \prec_\exists (F_{i-1}(x_i), v)^h$ for $1 \leq i \leq n$, which in view of $(F_{i-1}(x_i), v)^h = (F_i, v)^h$ proves our assertion.

The second assertion of our theorem follows from the first and the fact that if (K, v) is inseparably defectless, then the immediate extension $K^c|K$ is separable by virtue of Corollary 8.81. □

We wish to give an application to power series fields. Recall that $(K((t)), v_t)$ is henselian (by Theorem 7.112). So we can choose the henselization $(K(t), v_t)^h$ in $(K((t)), v_t)$ (by Theorem 7.107). Since already $(K((t))|K(t), v_t)$ is immediate, the same is true for $(K((t))|K(t)^h, v_t)$. Further, (K, v_t) is trivially valued and thus defectless. By Theorem 12.39, it follows that $(K(t), v_t)^h$ is henselian defectless. Now Corollary 8.81 shows:

Lemma 14.23 *The extension $K((t))|K(t)^h$ is regular.*

From this lemma together with Theorem 14.22, we conclude:

Theorem 14.24 *Let K be an arbitrary field. Then $(K(t), v_t)^h \prec_\exists (K((t)), v_t)$.*

To give a further application, we need another lemma.

Lemma 14.25 *Let t be transcendental over K and v_t denote the t-adic valuation on $K(t)$. Suppose that K admits a nontrivial henselian valuation v. Then $(K, v) \prec_\exists (K(t), v_t \circ v)^h$.*

Proof: Let (K^*, v^*) be a $|K(t)^h|^+$-saturated elementary extension of (K, v). Then by Corollary 14.3, v^*K^* is a $|vK|^+$-saturated elementary extension of vK. Hence, there exists an element $\alpha \in v^*K^*$ such that $\alpha > vK$ (cf. Exercise 13.3). We also have that $(v_t \circ v)t > vK$. Now if $\Gamma \subset \Delta$ is an extension of ordered abelian groups and $\alpha \in \Delta$ such that $\alpha > \Gamma$, then the order on $\mathbb{Z}\alpha + \Gamma$ is uniquely determined. Indeed, $\mathbb{Z}\alpha + \Gamma$ is the lexicographic product $\mathbb{Z}\alpha \amalg \Gamma$. So we see that the assignment $(v_t \circ v)t \mapsto \alpha$ induces an embedding of $(v_t \circ v)K(t) = \mathbb{Z}(v_t \circ v)t \amalg vK$ in v^*K^* over vK as ordered groups. Now choose $t^* \in K^*$ such that $v^*t^* = \alpha$. Lemma 7.33 shows that the assignment $t \mapsto t^*$ induces an embedding of $(K(t), v_t \circ v)$ in (K^*, v^*) over K. Since (K, v) is henselian, so is the elementary extension (K^*, v^*). By the universal property of the henselization, the embedding can thus be extended to an embedding of $(K(t), v_t \circ v)^h$ in (K^*, v^*). By Lemma 13.38, this gives our assertion. □

Now we are able to prove:

Theorem 14.26 *If the field K admits a nontrivial henselian valuation, then $K \prec_\exists K((t))$ (as fields).*

Proof: Let v be the nontrivial valuation on K such that (K, v) is henselian. By Lemma 14.25, we have that $(K, v) \prec_\exists (K(t), v_t \circ v)^h$. By the foregoing lemma, $K((t))|K(t)^h$ is separable. Since $(K((t)), v_t)$ is the completion of $(K(t), v_t)$, it follows from Lemma 10.8 that $(K((t)), v_t \circ v)$ is the completion of $(K(t), v_t \circ v)$. Hence, Theorem 14.22 shows that $(K(t), v_t \circ v)^h \prec_\exists (K((t)), v_t \circ v)$. It follows that $(K, v) \prec_\exists (K((t)), v_t \circ v)$. In particular, $K \prec_\exists K((t))$, as asserted. □

We turn to extensions without transcendence defect and prove an analogue of Theorem 14.7. We do not know whether the cofinality condition can be dropped.

Theorem 14.27 *Let (K, v) be a henselian separably defectless field and $(L|K, v)$ a separable extension without transcendence defect. Let vK be cofinal in vL. Then $(L|K, v)$ satisfies the AKE Principle.*

Proof: Assume that $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$. Since vK is supposed to be cofinal in vL, we know that $(K,v)^c$ is contained in $(L,v)^c$. The compositum $(L.K^c, v)$, taken in the completion $(L,v)^c$, is an immediate extension of (L,v). Thus, $vK^c = vK \prec_\exists vL = vL.K^c$ and $\overline{K^c} = \overline{K} \prec_\exists \overline{L} = \overline{L.K^c}$. By Corollary 8.60, $(K,v)^c$ is a henselian defectless field. Since $(L|K,v)$ is an extension without transcendence defect, the same holds for $(L.K^c|K^c,v)$. By Theorem 14.7, it follows from our side conditions that

$$(K^c,v) \prec_\exists (L.K^c,v) \ .$$

Let us now take an $|L.K^c|^+$-saturated elementary extension $(K^c|K,v)^*$ of the valued field extension $(K^c|K,v)$. We note that $(K^c,v)^*$ is a subfield of the completion of $(K,v)^*$ since the property of K to be dense in K^c is elementary in the language of valued fields with the predicate $\mathcal{P}$ for the subfield:

$$\forall x \forall y \exists z : \ \mathcal{P}(z) \ \wedge \ (\, y \neq 0 \ \rightarrow \ v(x-z) > vy \,)$$

expresses this property.

Since $(K^c,v) \prec_\exists (L.K^c,v)$, Lemma 13.38 shows that $(L.K^c,v)$ can be embedded over K^c in $(K^c,v)^*$ and can thus be considered to be a subfield of the completion of $(K,v)^*$, and so the same holds for the smaller field $(L.K^*,v)$. Since $L|K$ is assumed to be separable, it follows that also $L.K^*|K^*$ is separable. Now Theorem 14.22 shows that

$$(K,v)^* \prec_\exists (L.K^*,v) \ .$$

Since $(K,v) \prec (K,v)^*$, we obtain that $(K,v) \prec_\exists (L.K^*,v)$, which yields that $(K,v) \prec_\exists (L,v)$, as asserted. □

Theorem 14.28 *Let (K,v) be a separably tame field. Then every separable extension satisfies the AKE Principle.*

Proof: Assume that $(L|K,v)$ is a separable extension with $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$. The perfect hull K^{1/p^∞} of K admits a unique extension v of the valuation of K, and with this valuation it is a subextension of the completion of K, according to Lemma 12.26. By Lemma 12.27, $(K^{1/p^\infty},v)$ is a tame field. Both K^{1/p^∞} and $L.K^{1/p^\infty}$ are valued subfields of the perfect hull $(L^{1/p^\infty},v)$ of (L,v), whose value group is the p-divisible hull of vL and whose residue field is the perfect hull of $\overline{L}$ (Lemma 7.62). Now $vK^{1/p^\infty} = vK$ is p-divisible and $\overline{K}^{1/p^\infty} = \overline{K}$ is perfect, hence by our side conditions, we obtain that $vK^{1/p^\infty} \prec_\exists v(L.K^{1/p^\infty})$ and $\overline{K}^{1/p^\infty} \prec_\exists \overline{L.K^{1/p^\infty}}$ (cf. Lemma 14.14). According to the AKE Principle for tame fields (Theorem 14.15), this yields that

$$(K^{1/p^\infty},v) \prec_\exists (L.K^{1/p^\infty},v) \ .$$

Now take a $|L.K^{1/p^\infty}|^+$-saturated elementary extension $(K^{1/p^\infty}|K,v)^*$ of $(K^{1/p^\infty}|K,v)$. From this point on, the proof is just an analogue of the proof of the foregoing theorem. □

Related to this result are results of F. Delon [DEL1]. She showed that the **theory of algebraically maximal Kaplansky fields of fixed p-degree** is relatively complete. Adding predicates to the language of valued fields which guarantee that every extension is separable, she also obtained relative model completeness. Similar results can be derived for separably tame fields of fixed p-degree.

14.7 Classical results

In this section, we deduce some results about valued fields that can be found in the literature, as far as they can be subsumed under the case of tame fields. The formally $\wp$-adic and the finitely ramified fields will be treated in the next section.

The following result can be viewed as the starting point of the model theory of valued fields. It is due to A. Robinson [ROB].

Theorem 14.29 *The elementary class of algebraically closed nontrivially valued fields is model complete and admits quantifier elimination. The elementary class of algebraically closed valued fields of fixed characteristic and fixed residue characteristic is complete and decidable.*

Proof: Trivially, every algebraically closed field is a tame field. By Lemma 7.59, the value groups of algebraically closed fields are divisible, and the residue fields are algebraically closed. Since the elementary classes of nontrivial divisible ordered abelian groups and of algebraically closed fields are model complete by Corollary 13.43, the model completeness assertion follows from Theorem 14.16. The quantifier elimination follows from part f) of Lemma 13.56 together with Lemma 13.57.

If we fix characteristic and residue characteristic, then we have prime structures. Indeed, if both characteristics are 0, then the prime structure is $\mathbb{Q}$ with the trivial valuation, and if both are $p > 0$, then the prime structure is $\mathbb{F}_p$ with the trivial valuation. If the characteristic is 0 and the residue characteristic is $p > 0$, then the prime structure is $\mathbb{Q}$ with the p-adic valuation. The decidability follows from the completeness and the fact that the axioms for algebraically closed fields of fixed characteristic and fixed residue characteristic are recursive. □

In view of the fact that the theory of algebraically closed valued fields is a hull extension of the theory of valued fields, one can avoid to use the results on tame fields and directly prove the quantifier elimination by use of Lemma 13.59. By this lemma, it suffices to show that an algebraically closed nontrivially valued field (K, v) is existentially closed in every extension which is generated by just one element. If $(K(t), v)$ is such an extension, then there are two possibilities: Either, $(K(t)|K, v)$ admits a valuation transcendence basis (and then we can even assume that t itself is valuation transcendental), or $(K(t)|K, v)$ is immediate and appr (t, K) is transcendental (or trivial). We leave the details of the further proof as an exercise to the reader.

The following results are due to J. Ax and S. Kochen [AX–KOC1] and independently to Y. Ershov [ER1,3]:

Theorem 14.30 *The elementary class of henselian nontrivially valued fields of residue characteristic 0 is relatively model complete and relatively complete. If* $\mathbf{C}$ *is an elementary class consisting of henselian nontrivially valued fields of residue characteristic 0, then* $\mathbf{C}$ *is model complete resp. complete if* $v\mathbf{C}$ *and* $\overline{\mathbf{C}}$ *are model complete resp. complete elementary classes. If* (K, v) *is a henselian nontrivially valued fields of residue characteristic 0 and* $\mathrm{Th}(vK)$ *and* $\mathrm{Th}(\overline{K})$ *are decidable, then also* $\mathrm{Th}(K, v)$ *is decidable.*

Since henselian valued fields of residue characteristic 0 are tame fields, all this follows readily from Theorem 14.13, Theorem 14.16 and Corollary 14.17.

Finally, let us mention the Kaplansky fields. Here, the results are due to Y. Ershov [ER4] and independently to M. Ziegler [ZIE]. Basically, for Kaplansky fields we have the same situation as for tame fields, and we do not want to restate the results. F. Delon has given a quantifier elimination for Kaplansky fields of equal characteristic in an enlarged language ([DEL1], Théorème 2.1). It is related to the quantifier elimination with respect to amc-structures that we will develop later. Delon also gave a converse: If a valued field of equal characteristic admits quantifier elimination in a "fairly small" language, then it is a Kaplansky field ([DEL1], Théorème 6.1).

For quantifier elimination in the case of residue characteristic 0, see G. Cherlin and M. Dickmann [CHER-DI], and the survey [DEL6] of Delon. For valuations compatible with an order, see Dickmann [DI]. For further information on the model theory and quantifier elimination for valued fields, see V. Weispfenning [WEI1,2,7,8], S. Basarab [BAS5], and [KU3].

14.8 Formally $\wp$-adic and finitely ramified fields

Recall that a finitely ramified field is a valued field of mixed characteristic such that its value group has a least positive element of the form $\frac{vp}{e}$ for some $e \in \mathbb{N}$. (The number e is called the **absolute ramification index**.) We can fix it by the axiom

(ABSR$_e$) $\quad \exists x : \, vx^e = vp \, \wedge \, \forall y \, \neg(0 < vy < vx) \; .$

By taking the disjunction over these axioms for $e \leq n$, one can axiomatize the class $\mathbf{C}_n$ of all finitely ramified fields of absolute ramification index at most n. The class of all finitely ramified fields is not elementary (see Exercise 14.4). But it is the union over the elementary classes $\mathbf{C}_n$, $n \in \mathbb{N}$.

A finitely ramified field is formally $\wp$-adic if its residue field is finite. Then it must be of the form $\mathbb{F}_{p^f}$ for some $f \in \mathbb{N}$. (The number f is the **absolute inertia degree**.) This residue field can be fixed by the axiom

(RESF$_f$) $\quad \exists x_1 \ldots \exists x_{p^f} : \, \bigwedge_{1 \leq i < j \leq p^f} v(x_i - x_j) = 0 \, \wedge \, \forall y \, \bigvee_{1 \leq i \leq p^f} v(x_i - y) > 0$

(where $\bigvee$ denotes the disjunction of the formulas $v(x_i - y) > 0$). Like for the absolute ramification index, one can limit the size of the residue fields by taking finite disjunctions over these axioms. Hence we obtain that the class of all formally $\wp$-adic fields is a union over elementary classes. Such classes have the property to be closed under elementary equivalence. We leave it to the reader to check that Lemma 14.12 also holds for such classes.

Let $\mathbf{C}$ be the class of all henselian finitely ramified fields. By Theorem 10.13, all finitely ramified fields are defectless fields, so $\mathbf{C}$ satisfies condition (CHDL) of Lemma 14.12. Further, $\mathbf{C}$ satisfies (CIMM) by virtue of Lemma 12.44. By Lemma 9.27, the formally $\wp$-adic fields also satisfy condition (CRAC), because their residue fields are finite and extensions of finite fields are always separable algebraic. But we can not prove this for finitely ramified fields in general, because their residue fields may not be perfect. Hence, we have to generalize Lemma 14.12 before we can apply it here. But let us first treat the formally $\wp$-adic fields. From Lemma 14.12, we infer:

Theorem 14.31 *The class of henselian formally $\wp$-adic fields has the Relative Embedding Property and is relatively subcomplete and relatively model complete.*

Let (K, v) be a finitely ramified field and $(L|K, v)$ be an extension such that $vK \prec_\exists vL$ and $\overline{K} \prec_\exists \overline{L}$. Because of $vK \prec_\exists vL$, the least positive element in vK remains the least positive element in vL, and this shows that also (L, v) is finitely ramified. Then $(L, v)^h$ is a henselian finitely ramified field with $vL^h = vL$ and $\overline{L^h} = \overline{L}$, so we have that $vK \prec_\exists vL^h$ and $\overline{K} \prec_\exists \overline{L^h}$.

Now assume in addition that (K, v) is formally $\wp$-adic and henselian. Then by the foregoing theorem, $(K, v) \prec_\exists (L, v)^h$, hence also $(K, v) \prec_\exists (L, v)$. We have proved:

Theorem 14.32 *Every henselian formally $\wp$-adic field is an AKE field.*

By Theorem 10.14, the value groups of $\wp$-adically closed fields are $\mathbb{Z}$-groups. By Theorem 13.70, the $\mathbb{Z}$-groups form a complete elementary class in the language of ordered groups, and by Theorem 13.69 they form a model complete elementary class in the language of $\mathbb{Z}$-groups. To axiomatize that a valued field has a value group which is a $\mathbb{Z}$-group, we just have to translate the axioms (ZG1) and (ZG2) to the language of valued fields. The residue field k of a $\wp$-adically closed field is finite, so the class of all fields which are isomorphic to k is an elementary class (see Exercise 13.2). This class is trivially complete and model complete. It is fixed by an axiom of the form (RESF$_f$).

To obtain model completeness for suitable classes of $\wp$-adically closed fields, we have to make sure that for every extension of models of this class, the corresponding extension of their value groups preserves the least positive element. Whenever $(L|K, v)$ is an extension of fields of characteristic 0 and residue characteristic p such that both (K, v) and (L, v) satisfy axiom (ABSR$_e$), then both will be finitely ramified with the same prime element, having value $\frac{vp}{e}$, and the extension of their value groups will preserve the least positive element. So we have to use one single axiom (ABSR$_e$), that is, we have to fix the absolute ramification index. Similarly, if both (K, v) and (L, v) satisfy axiom (RESF$_f$), then they will have the same residue field, consisting of p^f elements.

Now we obtain from Theorem 14.31:

Theorem 14.33 *The elementary class of all $\wp$-adically closed fields of absolute ramification index e and absolute inertia degree f is model complete.*

Prime models exist for the classes with $e = 1$. Indeed, if also $f = 1$, then $(\mathbb{Q}, v_p)^h$ is the prime model. Now assume that $f > 1$. There is, up to isomorphism, just one extension of $\mathbb{F}_p$ of degree d. Now one shows by use of Hensel's Lemma (or of ramification theory) that there is, again up to isomorphism, just one extension of $(\mathbb{Q}, v_p)^h$ with $e = 1$ and residue field $\mathbb{F}_p$. This extension is a prime model for the elementary class of all $\wp$-adically closed fields of absolute ramification index 1 and absolute inertia degree f. By the Prime Model Test (Lemma 13.48), we obtain:

Theorem 14.34 *The elementary class of all $\wp$-adically closed fields of absolute ramification index* 1 *and absolute inertia degree f is complete.*

For the p-adically closed fields (that is, $\wp$-adically closed fields with $e = 1 = f$), Theorems 14.33 and 14.34 were proved by J. Ax and S. Kochen in their second paper [AX–KOC2]. In their third paper, they proved quantifier elimination for p-adically closed fields in a language of valued fields which contains an additional symbol for a cross-section.

In full generality, Theorems 14.33 and 14.34 were proved by A. Prestel and P. Roquette in [PR–ROQ]. Note that they use the **p-rank** $e \cdot f$ in the place of the absolute ramification

index e and absolute inertia degree f. The model completeness was used by M. Jarden and P. Roquette to prove a **$\wp$-adic Nullstellensatz** for algebraic function fields over $\wp$-adically closed fields (cf. [JA–ROQ]).

Moreover, A. Macintyre [MAC2] showed that the p-adically closed fields admit quantifier elimination (without cross-section) if one adds the **power predicates** P_n, for every $n \in \mathbb{N}$, which are to be interpreted by

$$P_n(x) \leftrightarrow \exists y\, y^n = x\ .$$

In other words, the predicate P_n is interpreted in (K, v) by the power set K^n. The comprehensive theorem for $\wp$-adically closed fields was given by Prestel and Roquette ([PR–ROQ], Theorem 5.6):

Theorem 14.35 *In the language of valued fields enriched by a constant for the prime element and by the predicates P_n, $n \in \mathbb{N}$, the theory of $\wp$-adically closed fields of fixed p-rank d for which P_n is interpreted by the power set K^n, admits elimination of quantifiers.*

A proof of this theorem can be given by use of the fact that the relative algebraic closure of a valued field in a $\wp$-adically closed field is again a $\wp$-adically closed field, if both have the same prime element (cf. Corollary 10.15). If (K, v) is the common subfield of two $\wp$-adically closed fields (K_1, v_1) and (K_2, v_2), then one shows that the relative algebraic closures in these fields are isomorphic if for every $n \in \mathbb{N}$, $K_1^n \cap K = K_2^n \cap K$. This in turn is the case if (K, v) is a common substructure of (K_1, v_1) and (K_2, v_2) with respect to the language extended by the power predicates. In [KU3], this theorem is put this into a more general framework by studying quantifier elimination with respect to "amc-structures".

The quantifier elimination result of the theorem is used e.g. in p-adic semialgebraic geometry (Robinson [RO1], [RO2]; L. Belair [BEL1]; J. Schinke [SCHK]; L. Bröcker and Schinke [BR–SCHK]). J. Denef used the quantifier elimination in [DEN] to show the rationality of the Poincaré series associated to the p-adic points on a variety.

A nice survey on the model theory of $\wp$-adic fields is given by Macintyre in [MAC5]. Let us also recommend P. Roquette's paper [ROQ8], which gives a survey on model theoretic field theory and its applications.

Let us return to the finitely ramified fields. Condition (CRAC) of Lemma 14.12 can be reasonably weakened, so that also finitely ramified fields have a chance. To show this, we need the following lemma:

Lemma 14.36 *Let (K, v) be a valued field and (L, v) a henselian nontrivially valued extension field. Then for every strong limit extension (L^*, v) of basis $\kappa \geq |L|$, there is a henselian intermediate field (K', v) such that K' is relatively algebraically closed in L^*, the extension $(K'|K, v)$ admits a valuation transcendence basis, and $(L^*|K', v)$ is immediate.*

Proof: The fields in the chain of L^* will be denoted by L_ν, and we have that $L_0 = L$. We construct valued fields (K_ν, v) by induction on ν as follows. By Theorem 7.57 there exists an extension $(K_0|K, v)$ admitting a valuation transcendence basis $\mathcal{T}_0$ and having value group vL and residue field $\overline{L}$. Note that $|K_0| \leq \max\{\aleph_0\,, |K|\,, |vL|\,, |\overline{L}|\} \leq |L| \leq \kappa$ (cf. Exercise 7.6), because (L, v) is infinite, being assumed to be nontrivially valued. From Embedding Lemma II (Lemma 14.8) we know that (K_0, v) can be embedded over K in the κ^+-saturated extension field (L_1, v) of (L, v). We identify K_0 with its image in L_1.

Assume that we have found a subextension $(K_i|K, v)$ of $(L_{i+1}|K, v)$, admitting a valuation transcendence basis $\mathcal{T}_i$ and such that $vK_i = vL_i$ and $\overline{K}_i = \overline{L}_i$. Again by Theorem 7.57, we choose an extension $(K_{i+1}|K_i, v)$ admitting a valuation transcendence basis $\mathcal{T}'_{i+1}$ and such that $vK_{i+1} = vL_{i+1}$ and $\overline{K}_{i+1} = \overline{L}_{i+1}$. Again, we apply Embedding Lemma II to embed it in (L_{i+2}, v) over K_i. Now $\mathcal{T}_{i+1} = \mathcal{T}'_{i+1} \cup \mathcal{T}_i$ is a valuation transcendence basis of $(K_{i+1}|K, v)$.

We set $(K'', v) := (\bigcup_{\nu<\omega} K_\nu, v)$. We find that $vK'' = \bigcup_{\nu<\omega} vK_\nu = \bigcup_{\nu<\omega} vL_\nu = vL^*$, and similarly, $\overline{K''} = \overline{L^*}$. Moreover, $(K''|K, v)$ has valuation transcendence basis $\bigcup_{\nu<\omega} \mathcal{T}_\nu$. All this remains true for the relative algebraic closure K' of K'' in L^*. Since (L^*, v) is an elementary extension of (L, v) it is a henselian field. Hence, (K', v) is a henselian field by Theorem 7.107. □

Lemma 14.37 *The assertion of Lemma 14.12 remains true if condition* (CRAC) *is restricted to immediate extensions:*

(CRAC)$_i$ *given $(L, v) \in \mathbf{C}$ and K' relatively algebraically closed in L such that $\overline{L} = \overline{K'}$ and $vL = vK'$, then $(K', v) \in \mathbf{C}$.*

Proof: The restricted version (CRAC)$_i$ is still sufficient for the slicing of immediate extensions into extensions of transcendence degree 1. In the proof of Lemma 14.12, the original version was also used to find the intermediate field (K', v) splitting up the extension into an immediate part and a part without transcendence defect. To find such an intermediate field, we may enlarge (L, v) as long as the side conditions are preserved. But by Corollary 14.2, $(L, v) \prec (L^*, v)$ implies that $vL \prec vL^*$ and $\overline{L} \prec \overline{L^*}$, so that the original side conditions imply that also $vK \prec_\exists vL^*$ and $\overline{K} \prec_\exists \overline{L^*}$. Now if (L^*, v) and (K', v) are as in the foregoing lemma, then (CRAC)$_i$ yields that $(K', v) \in \mathbf{C}$, and we can construct an embedding of (L', v) like we did for (L, v) in the proof of Lemma 14.12. This embedding will induce the required embedding of (L, v). Hence, (CRAC) can indeed be replaced by (CRAC)$_i$. □

If (L, v) is a finitely ramified field with subfield (K', v) such that $vK' = vL$, then also (K', v) is a finitely ramified field. In view of the fact that a relatively algebraically closed subfield of a henselian field is again henselian (Theorem 7.107), this proves that (CRAC)$_i$ holds for the class of henselian finitely ramified fields.

From Lemma 14.37 and the argument succeeding to Theorem 14.31, we now obtain:

Theorem 14.38 *The elementary class of finitely ramified fields has the Relative Embedding Property and is relatively subcomplete and relatively model complete. Every finitely ramified field is an AKE field.*

The model theoretic content of this theorem is due to Y. Ershov [ER5] and independently to M. Ziegler [ZIE].

Exercise 14.4 *Show that the class of all finitely ramified fields, every class of all finitely ramified fields with fixed residue characteristic and every class of all formally ℘-adic fields with fixed residue field are not elementary. (Hint: compactness.)*

14.9 An application to number theory

The aim of the work of J. Ax and S. Kochen [AK1] was to prove a deep number theoretic result by use of model theory. There had been a common feeling that the p-adic fields $\mathbb{Q}_p$ resemble very much the power series fields $\mathbb{F}_p((t))$, although $\mathbb{Q}_p$ has characteristic 0 and $\mathbb{F}_p((t))$ has characteristic $p > 0$. Recall that

$$v_p\mathbb{Q}_p = v_p\mathbb{Q} = \mathbb{Z} = v_t\mathbb{F}_p(t) = v_t\mathbb{F}_p((t))$$

and

$$\mathbb{Q}_p v_p = \mathbb{Q}v_p = \mathbb{F}_p = \mathbb{F}_p(t)v_t = \mathbb{F}_p((t))v_t \ .$$

According to their value groups and their residue fields, $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ indeed seem to be similar. The similarity is made more precise by the following **Theorem of Ax and Kochen** (cf. [AK1]):

Theorem 14.39 *Let φ be any sentence in the language of valued fields. Then for all but finitely many primes p (which depend on φ),*

$$\mathbb{Q}_p \models \varphi \iff \mathbb{F}_p((t)) \models \varphi \ .$$

Proof: Let $\mathcal{D}$ be any nonprincipal ultrafilter over the set of all prime numbers. The fields

$$(K, v) = \prod_{p \text{ prime}} (\mathbb{Q}_p, v_p) \,/\, \mathcal{D} \quad \text{and} \quad (L, w) = \prod_{p \text{ prime}} (\mathbb{F}_p((t)), v_t) \,/\, \mathcal{D}$$

are henselian fields by Theorem 13.74, because all $(\mathbb{Q}_p, v_p)$ and all $(\mathbb{F}_p((t)), v_t)$ are henselian and this is an elementary property. Both fields have value group $\prod_{p \text{ prime}} \mathbb{Z} \,/\, \mathcal{D}$ and residue field $\prod_{p \text{ prime}} \mathbb{F}_p \,/\, \mathcal{D}$ (cf. Exercise 14.5). For every prime q, the set $\{p \text{ prime } \mid \mathbb{F}_p \models \forall x\, qx = 0\}$ contains only q and can thus not be contained in $\mathcal{D}$ because $\mathcal{D}$ was assumed to be nonprincipal. This proves that the field $\prod_{p \text{ prime}} \mathbb{F}_p \,/\, \mathcal{D}$ can not have positive characteristic. Hence, both (K, v) and (L, w) are henselian fields of residue characteristic 0. Now Theorem 14.30 yields that

$$\prod_{p \text{ prime}} \mathbb{Q}_p \,/\, \mathcal{D} \ \equiv \ \prod_{p \text{ prime}} \mathbb{F}_p((t)) \,/\, \mathcal{D}$$

and consequently, by (13.8),

$$\begin{aligned} J_\varphi \ &:= \ \{p \text{ prime } \mid (\mathbb{Q}_p, v_p) \models \varphi \text{ if and only if } (\mathbb{F}_p((t)), v_t) \models \varphi\} \\ &= \ \{p \text{ prime } \mid (\mathbb{Q}_p, v_p) \models \varphi \text{ if and only if } (K, v) \models \varphi\} \\ &\quad \cap \{p \text{ prime } \mid (L, w) \models \varphi \text{ if and only if } (\mathbb{F}_p((t)), v_t) \models \varphi\} \in \mathcal{D} \ . \end{aligned}$$

Let J denote the complement of J_φ in the set of primes. If J_φ were not cofinite in the set of primes, then J would be infinite and by Corollary 13.73 there would exist an ultrafilter $\mathcal{D}$ on the set of primes such that $J \in \mathcal{D}$. But then by $J_\varphi \in \mathcal{D}$ and (FI2), $\emptyset = J \cap J_\varphi \in \mathcal{D}$, in contradiction to (FI1). This proves that J_φ must contain all but finitely many primes. □

Let us now describe the application to number theory that Ax and Kochen gave in [AX–KOC1]. Let $i \geq 0$ and $d \geq 1$ be integers. A field K is called $\mathrm{C}_i(d)$ if every form

(that is, homogeneous polynomial) of degree d in $n > d^i$ variables has a nontrivial zero. Further, K is called C_i if it is $\mathrm{C}_i(d)$ for every $d \geq 1$. For detailed information about these notions and their background, see S. Lang [LANG1] and M. J. Greenberg [GRB2]. A nice exposition by F. Lorenz can be found in §27 of [LOR2].

The notion originates from work of Tsen [TSEN] in 1933. Dealing with division algebras, he proved that every function field in one variable over an algebraically closed field is a C_1 field. C. Chevalley [CHEV] proved that every $\mathbb{F}_p$ is a C_1 field. Lang (who introduced the notation "C_i") proved that the power series fields $\mathbb{F}_p((t))$ are C_2 fields. Greenberg generalized this result by showing that the power series field $K((t))$ over a C_i field is a C_{i+1} field. The notion becomes interesting for number theory by **Artin's conjecture** which claims that every $\mathbb{Q}_p$ is a C_2 field (like $\mathbb{F}_p((t))$). This was already known to be true for quadratic forms, i.e., all $\mathbb{Q}_p$ are $C_2(2)$ fields. Then it was proved by D. J. Lewis [LEW] that all $\mathbb{Q}_p$ are $C_2(3)$ fields. But it turned out that Artin's conjecture is not true: For $d \geq 4$, not all $\mathbb{Q}_p$ are $C_2(d)$ fields. Indeed, Terjanian [TER] showed that the form

$$f(X_1, X_2, X_3)+f(Y_1, Y_2, Y_3)+f(Z_1, Z_2, Z_3)+4f(U_1, U_2, U_3)+4f(V_1, V_2, V_3)+4f(W_1, W_2, W_3)$$

with

$$f(X_1, X_2, X_3) = X_1^4 + X_2^4 + X_3^4 - X_1^2X_2^2 - X_1^2X_3^2 - X_2^2X_3^2 - X_1^2X_2X_3 - X_1X_2^2X_3 - X_1X_2X_3^2$$

does not admit a nontrivial zero in $\mathbb{Q}_2$.

However, Ax and Kochen proved by use of Theorem 14.39 that Artin's conjecture is "almost true":

Theorem 14.40 *For every positive integer d there exists a finite set of primes $A = A(d)$ such that for every prime $p \notin A$, every $\mathbb{Q}_p$ is a $C_2(d)$ field.*

Proof: We leave it to the reader to write down a sentence $\varphi_{d,n}$ in the language of rings which states that every form of degree d in n variables admits a nontrivial zero. Note that if φ_{d,d^2+1} holds in a field, then also $\varphi_{d,n}$ holds, for all $n > d^2$. Indeed, in a form in $n > d^2 + 1$ variables, we can just set the "superfluous" $n - d^2 - 1$ variables equal to zero. If the result is zero, then we can choose the other variables to be nonzero and thus have found a nontrivial zero of the form. Otherwise, the result will be a form of degree d in $d^2 + 1$ variables.

By the result of Lang, the sentence φ_{d,d^2+1} holds in $\mathbb{F}_p((t))$ for every prime p. Now we apply Theorem 14.39 to the sentence φ_{d,d^2+1}, which tells us that there is a finite set of primes $A = A(d)$ such that the sentence φ_{d,d^2+1} holds in $\mathbb{Q}_p$ for every prime $p \notin A$. It follows that $\mathbb{Q}_p$ is a $C_2(d)$ field for every prime $p \notin A$. □

The same theorem was proved independently and at about the same time by M. Greenleaf [GRL], using methods from algebraic geometry. A constructive proof was given by P. Cohen [CO]. In this connection, cf. also A. Nerode [NER].

For the conclusion of this section, let us remark that like the notions introduced at the end of Section 5.4, also the $C_i(d)$ property is transferred downwards through places. Let v be any valuation of K. Suppose that we are given a form $\overline{f} = \sum_{\underline{\nu}} \overline{c}_{\underline{\nu}} X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} \in \overline{K}[X_1, \ldots, X_n]$, where $\underline{\nu} = (\nu_1, \ldots, \nu_n)$ runs over all n-tuples of nonnegative integers whose sum is equal to d, and $c_{\underline{\nu}} \in \mathcal{O}_v$. Suppose that the corresponding form $f = \sum_{\underline{\nu}} c_{\underline{\nu}} X_1^{\nu_1} \cdot \ldots \cdot$

$X_n^{\nu_n}$ in $\mathcal{O}_v[X_1,\ldots,X_n]$ has a nontrivial zero $(a_1,\ldots,a_n)\in K^n$. Let a_j be an element of minimal value and divide by a_j^d to find that $f(X)$ has a nontrivial zero $(a_1/a_j,\ldots,a_n/a_j)$ in $\mathcal{O}_v^n$. Since the residue map is a homomorphism, we find that $\overline{f}(\overline{a_1/a_j},\ldots,\overline{a_n/a_j}) = \sum_{\underline{\nu}}\overline{c}_{\underline{\nu}}\,\overline{a_1/a_j}^{\nu_1}\cdot\ldots\cdot\overline{a_n/a_j}^{\nu_n} = \overline{\sum_{\underline{\nu}} c_{\underline{\nu}}\,a_1/a_j^{\nu_1}\cdot\ldots\cdot a_n/a_j^{\nu_n}} = \overline{0}=0$ with $\overline{a_j/a_j}=\overline{1}\neq 0$. Hence also $\overline{f}$ has a nontrivial zero. This proves:

If K is $\mathrm{C}_i(d)$ (resp. C_i) and v is any valuation of K, then $\overline{K}$ is $\mathrm{C}_i(d)$ (resp. C_i).

It seems that for the $C_i(d)$ property to be transferred upwards through places, it is not known whether there exist general conditions like those given for some other notions in Section 9.6.

Exercise 14.5 *Let (K_i,v_i), $i\in I$, be valued fields and $\mathcal{D}$ an ultrafilter on $\mathcal{D}$. Prove that $\prod_{i\in I}(K_i,v_i)$ is a valued field with value group $\prod_{i\in I}v_iK_i$ and residue field $\prod_{i\in I}K_iv_i$.*

14.10 Residue field embeddings and cross-sections

In our approach to the model theory of valued fields, we have not worked with embeddings of the residue field or with cross-sections. It can be shown that the main results remain valid if one introduces predicates whose range is the embedded residue field resp. the image of the cross-section. In most applications of the model theory of valued field, they do not play a role. Nevertheless, it can be helpful in certain cases to have them at hand for special constructions. However, it is known that there are henselian valued fields of equal characteristic which do not admit an embedding of the residue field or a cross-section. In this section, we show that this does not affect their elementary theory. More precisely: every henselian valued field has an elementary extension which admits an embedding of the residue field and a cross-section.

For henselian fields of residue characteristic 0, we have already shown in Lemma 9.25 that their residue fields can always be embedded. The proof uses Hensel's Lemma and can be adapted to show:

Lemma 14.41 *Let (K,v) be an infinite henselian field of equal characteristic. Then every strong limit extension $(K,v)^*$ of (K,v) of basis $\kappa\geq|K|$ admits a field embedding ι of its residue field $\overline{K^*}$ such that*

$$\overline{\iota\zeta}=\zeta \tag{14.3}$$

for all $\zeta\in\overline{K^*}$.

Proof: As usual, the fields in the chain of $(K,v)^*$ will be denoted by (K_ν,v), and we have that $K_0=K$. Let k be the prime field of K. It is also the prime field of $\overline{K}$. Since k is perfect, every finitely generated subextension $k'|k$ of $\overline{K}|k$ is separably generated. So by Lemma 9.25, k' can be embedded over k in K such that (14.3) holds for all $\zeta\in k'$. Fixing the latter by an additional predicate in a similar way as in the proof of Embedding Lemma II (Lemma 14.8), we can conclude that $\overline{K}$ admits an embedding over k in the $|K|^+$-saturated field (K_1,v) which satisfies (14.3) for all $\zeta\in\overline{K}$.

Now we proceed by induction over ν. Having embedded $\overline{K}_\nu$ in $(K_{\nu+1},v)$ such that (14.3) holds, we now have to embed $\overline{K}_{\nu+1}$. Since $(K_\nu,v)\prec(K_{\nu+1},v)$, we know from Corollary 14.2 that $\overline{K}_\nu\prec\overline{K}_{\nu+1}$. Hence by Lemma 13.37, the extension $\overline{K}_{\nu+1}|\overline{K}_\nu$ is separable. Thus, the embedding is found as before. By induction, we obtain embeddings

of all $\overline{K}_\nu$, hence also an embedding of their union in $(K,v)^*$. But this union is just the residue field of $(K,v)^*$. □

Note that if (K,v) is a finite field, then it is trivially valued and the embedding of its residue field is just the identity (if the residue map is taken to be the identity).

The case of cross-sections does not even require that the field be henselian:

Lemma 14.42 *Let (K,v) be an infinite valued field of equal characteristic. Then every strong limit extension $(K,v)^*$ of (K,v) of basis $\kappa \geq |K|$ admits a group embedding ι of its value group vK in the multiplicative group $K^\times$ such that*

$$v\iota\alpha = \alpha \tag{14.4}$$

for all $\alpha \in vK$.

Proof: Let $(K,v)^*$ and (K_ν,v) be as in the last proof. Let k be the prime field of K. Since (K,v) is assumed to be of equal characteristic, v is trivial on k. Every finite subextension of $vK|vk$, that is, every finitely generated subgroup of vK, is of the form $\Gamma = \mathbb{Z}\alpha_1 \oplus \ldots \mathbb{Z}\alpha_n$ for suitable $\alpha_i \in vK$. Now choose $t_1,\ldots,t_n \in K$ such that $vt_i = \alpha_i$. Then the assignment $\alpha_i \mapsto t_i$ induces a group embedding of Γ in $K^\times$ which satisfies (14.4). As in the proof of the last lemma one deduce the existence of an embedding of vK in $K_1^\times$ which satisfies (14.4). For the induction, we note that by Corollary 14.2, $(K_\nu,v) \prec (K_{\nu+1},v)$ implies that $vK_\nu \prec vK_{\nu+1}$. Hence, $vK_{\nu+1}/vK_\nu$ is torsion free by virtue of Lemma 13.37. So every finitely generated subextension of $vK_{\nu+1}|vK_\nu$ is of the form $\mathbb{Z}\alpha_1 \oplus \ldots \mathbb{Z}\alpha_n \oplus vK_\nu$ for suitable $\alpha_i \in vK_{\nu+1}$. So one has to choose t_i as before in order to extend the embedding of vK_ν in $K_{\nu+1}^\times$ to this larger group.

By induction, we obtain an embedding of $vK^* = \bigcup_{\nu<\omega} vK_\nu$ in $(K^*)^\times$ which satisfies (14.4), as required. □

Exercise 14.6 *For the embedding of the value group, discuss the case of mixed characteristic.*

14.11 Beyond perfect fields

In this section we will show the existence of henselian defectless fields of positive characteristic which are not AKE fields and even admit an immediate function field of transcendence degree 1 in which they are not existentially closed. In view of Corollary 14.11, we see that Theorem 12.45 does not remain true if we omit the condition that the ground field K is perfect, not even in the case of a function field of rank 1. However, this does not prove that a good structure theory for immediate henselian function fields like the one given in Theorem 12.45 is really necessary for an elementary class of valued fields in order to consist of AKE fields. But it will tell us that in order to know more about the model theory of power series fields in positive characteristic, we have to consider axioms that have not been considered in the literature before.

At this point, we want to use the occasion to put our considerations in a more systematic setting. We define the following six basic properties of a valued field (K,v), three of them being of algebraic nature and elementary, the other three being of model theoretic nature:

- **(Alg 1)**: (K, v) is henselian
- **(Alg 2)**: (K, v) is algebraically maximal
- **(Alg 3)**: (K, v) is algebraically complete
- **(Mod 1)**: there exists at least one maximal immediate extension (M, v) of (K, v) such that $(K, v) \prec_\exists (M, v)$
- **(Mod 2)**: (K, v) is existentially closed in every immediate extension
- **(Mod 3)**: (K, v) is an AKE field

There is the following general connection between these properties:

Lemma 14.43 *For every valued field K, the following holds:*

$$\textbf{(Mod 3)} \Longrightarrow \textbf{(Mod 2)} \Longrightarrow \textbf{(Mod 1)} \Longrightarrow \textbf{(Alg 3)} \Longrightarrow \textbf{(Alg 2)} \Longrightarrow \textbf{(Alg 1)} .$$

Proof: **(Mod 3)** $\Longrightarrow$ **(Mod 2)**: If $(L|K, v)$ is an immediate extension, then $vL = vK$ and $\overline{L} = \overline{K}$, so the side conditions of the AKE Principle are trivially fulfilled.
(Mod 2) $\Longrightarrow$ **(Mod 1)**: Take any maximal immediate extension of K (which exists by Theorem 5.7).
(Mod 1) $\Longrightarrow$ **(Alg 3)**: Lemma 14.5.
(Alg 3) $\Longrightarrow$ **(Alg 2)** and **(Alg 2)** $\Longrightarrow$ **(Alg 1)**: Corollary 8.30. □

Note that the implication **(Mod 2)** $\Longrightarrow$ **(Alg 3)** was already shown by F. Delon in [DEL1].

From Theorem 14.15 and Theorem 14.38, we know the following converses:
a) For every valued field of residue characteristic 0 and for every finitely ramified field, the implication

$$\textbf{(Alg 1)} \Longrightarrow \textbf{(Mod 3)}$$

holds. Hence, all stated properties are equivalent for such fields.
b) For every perfect field K of positive characteristic and more generally, for every valued field whose value group is divisible by its residue characteristic and whose residue field is perfect and thus also for every Kaplansky-field), the implication

$$\textbf{(Alg 2)} \Longrightarrow \textbf{(Mod 3)}$$

holds. Hence, all stated properties with the exception of **(Alg 1)** are equivalent for such fields.
But we will show in this section that the implication

$$\textbf{(Alg 3)} \Longrightarrow \textbf{(Mod1)}$$

does not hold in general.

Let (L, v) be a spherically complete field such that $1, t, \ldots, t^{p-1}$ form a valuation p-basis for $L|L^p$. Recall the additive decomposition that we have deduced for this case:

$$L = \wp(L) + \mathcal{O}_\mathbf{L} + tL^p + \ldots + t^{p-1}L^p \tag{14.5}$$

(cf. Lemma 11.20). This can be transformed into an elementary $\mathcal{L}_{\mathrm{VF}}(\{t\})$-sentence:

$$\forall x \exists y \exists x_0 \ldots \exists x_{p-1} : \; x = y^p - y + x_0 + t x_1^p + \ldots + t^{p-1} x_{p-1}^p \; \wedge \; v(x_0) = 0 \; . \tag{14.6}$$

In Chapter 11 of [KU1] (Chapter 10 of [KU2]) we have constructed a rather simple henselian defectless field which does not satisfy this sentence:

Theorem 14.44 *There exists a henselian defectless extension (L, v) of $(\mathbb{F}_p(t), v_t)^h$ of transcendence degree 1 without transcendence defect such that $\mathbb{Z} = v_t(\mathbb{F}_p(t)) \prec_\exists vL = \mathbb{Q} \amalg \mathbb{Z}$ and $\overline{\mathbb{F}_p(t)} = \overline{L}$ and:*
(L, v) has valuation p-basis $1, t, \ldots, t^{p-1}$ but does not satisfy (14.6).

The constructed example instantly produces a second interesting example. Taking w to be the coarsening of v, which is of rank 1, we have obtained:

Theorem 14.45 *There exists a henselian defectless field (L, w) of transcendence degree 1 over its embedded residue field, having value group $wL = \mathbb{Q}$, and an element $t \in L$ such that the residue field Lw has p-basis $1, tw, \ldots, (tw)^{p-1}$ and $(L|L^p, w)$ has valuation basis $1, t, \ldots, t^{p-1}$, but (L, w) does not satisfy property (14.6).*

Every maximal immediate extension of our examples (L, v) or (L, w) is a spherically complete field by Theorem 7.124 and thus satisfies (14.6) according to Lemma 11.20. Note that it will have the same valuation p-basis: On the one hand, the valuation p-basis remains valuation independent since the extension is immediate. On the other hand, Corollary 8.63 shows that the maximal immediate extension has the same p-degree as L.

Further, L is relatively algebraically closed in every such extension since (L, v) and (L, w) are henselian defectless. This shows that $(\mathrm{CRAC})_i$ does not hold in this situation. Since every maximal immediate extension has the same valuation p-basis, it does not even help to fix the finite valuation p-basis by an elementary sentence. Hence:

Corollary 14.46 *The elementary class of henselian defectless fields with fixed valuation p-basis does not satisfy $(CRAC)_i$.*

The model theoretic consequences of these examples are:

Corollary 14.47 *The henselian defectless fields (L, v) and (L, w) do not have property* **(Mod 1)**. *Hence,*

$$\textbf{(Alg 3)} \quad \not\Longrightarrow \textbf{(Mod 1)} \; .$$

In particular, there exist henselian defectless fields that are not AKE fields. Moreover, there are henselian defectless fields (L, v) such that the theory

$$\{(L, v) \text{ is henselian defectless}\} \cup \{\mathrm{char}\, L = p\} \cup \mathrm{Th}(vL) \cup \mathrm{Th}(\overline{L}) \tag{14.7}$$

(with respect to the language $\mathcal{L}_{\mathrm{VF}}$ of valued fields) is not complete. In particular, the theory

$$\{(L, v) \text{ is henselian defectless}\} \cup \{\mathrm{char}\, L = p\} \cup \{vL \equiv \mathbb{Z}\} \cup \{\overline{L} = \mathbb{F}_p\} \; ,$$

which is satisfied by the valued power series field $\mathbb{F}_p((t))$, is not complete.

Furthermore, the example (L,v) together with $(K,v) = (\mathbb{F}_p(t), v_t)^h$ shows that if $(L|K,v)$ is an extension of henselian defectless fields, then the side conditions $vK \prec vL$ and $\overline{K} \prec \overline{L}$ do in general not imply that $(K,v) \prec (L,v)$, even if the extension has no transcendence defect. This shows:

Corollary 14.48 *In the AKE Principle for extensions without transcendence defect (cf. Theorem 14.7), $\prec_\exists$ can not be replaced by $\prec$, even if* both *fields are henselian defectless.*

For (L,v) and (L,w) as above, we have constructed in [KU1,2] immediate function fields $(F|L,v)$ and $(F|L,w)$ of transcendence degree 1 and generated by two elements over L, such that (L,v) (resp. (L,w)) is not existentially closed in (F,v) (resp. (F,w)). This contrasts Corollary 14.11 and shows:

Corollary 14.49 *The AKE Principle does not hold for all immediate extensions of transcendence degree 1 of henselian defectless fields.*

In view of Corollary 14.11, this also shows:

Corollary 14.50 *Theorem 12.45 is not true in general if the condition "(K,v) is tame" is replaced by "(K,v) is henselian defectless".*

This bunch of negative results shows that the methods developed in this chapter to treat tame and finitely ramified fields have to be reasonably refined in order to consider the model theory of non-perfect fields in positive characteristic, and in particular of the power series field $\mathbb{F}_p((t))$. As we have seen in Corollary 14.47, the straightforward axioms for the power series field do not give a complete axiomatization. We have also seen that the structure as stated in Theorem 11.19 is essentially independent of these axioms. We leave it to the reader to show that the following assertion for a valued field (L,v) of characteristic $p>0$ can be put into a scheme of elementary $\mathcal{L}_{\mathrm{VF}}$-sentences:

(NICE) If $f_1,\ldots,f_n$ are additive polynomials and $G_1,\ldots,G_n$ subgroups of (L,v) of the form L, $\mathcal{O}_\mathbf{L}$ or $\mathcal{M}_\mathbf{L}$ such that assertion (11.3) holds for every element $a \in f_1(G_1)+\ldots+f_n(G_n)$, then $f_1(G_1)+\ldots+f_n(G_n)$ is nice in (L,v).

By Theorem 11.19, every spherically complete field (L,v) satisfies (NICE) since its valued subgroups $(\mathcal{O}_\mathbf{L},v)$ and $(\mathcal{M}_\mathbf{L},v)$ are also spherically complete.

There is some hope that Theorem 12.45 can also be proved for henselian defectless fields of positive characteristic which satisfy (NICE). Let us call them **nice fields**. So it seems reasonable to ask:

Open Problem 14.1 Are all nice fields AKE fields? Are the elementary classes of nice fields with fixed characteristic relatively model complete or relatively complete? Is

"(L,v) is a nice field of characteristic p with vL a $\mathbb{Z}$-group and $\overline{K} = \mathbb{F}_p$"

a complete axiomatization of the theory of $(\mathbb{F}_p((t)), v_t)$?

Note that the $f_i(G_i)$ are *definable* subgroups of L. So perhaps one has to talk about other existing definable subgroups of spherically complete fields and study their properties. We are not able to exclude the possibility that there are other definable subgroups having properties which are independent of the proposed axiomatization. So we ask:

Open Problem 14.2 Which are the definable subgroups in spherically complete fields, and what are their elementary properties?

However, there is some evidence that the definable subgroups considered in (NICE) play a key role for our open questions. For this reason, L. v. d. Dries has suggested to study the model theory of $\mathbb{F}_p((t))$ considered as a module with multiplication by t and application of the Frobenius. In our terminology, this reads as follows:

Open Problem 14.3 Study the model theory of $(\mathbb{F}_p((t)), v_t)$ as a valued $K[\varphi]$-module (with K a suitably chosen subfield containing t).

As soon as $K \subset \mathbb{F}_p((t))$ contains t, it is not perfect and the ring $K[\varphi]$ has all the bad properties that we listed in Section 11.1 for this case. They show how complex the problem is. In contrast to this, a perfect valued field (L, v) is a valued $L[\varphi]$-module with $L[\varphi]$ a well-behaved ring (both-sided euclidean). This seems to be the reason why the model theory of tame fields is more accessible than that of the power series fields. However, the model theory of valued $K[\varphi]$-modules for a perfect field K has still to be worked out. But we hope that every necessary ingredient for that can be found in this thesis.

Part IV

Appendix

Chapter 15

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