Abhyankar places admit local uniformization in any characteristic

— *preliminary version* —

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Abstract

We prove that every place of an algebraic function field $F|K$ of arbitrary characteristic admits local uniformization, provided that the sum of the rational rank of its value group and the transcendence degree of its residue field over K is equal to the transcendence degree of $F|K$. Further, we show that finitely many such places admit simultaneous local uniformization if they have isomorphic value groups. We also prove the arithmetic analogues for extensions of p-adic places, and a generalization.

1 Introduction

In [Z], Zariski proved the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In [Z3], he uses this theorem to prove resolution of singularities for surfaces in characteristic 0. As the resolution of singularities for arbitrary dimensions in positive characteristic is still an open problem, one is interested in generalizations of the Local Uniformization Theorem to positive characteristic.

Throughout this paper, "function field" will always mean "algebraic function field". By a "place of $F|K$" we mean a place whose restriction to K is the identity. The valuation associated with P will be denoted by v_P, its value group on any field F by v_PF and its residue field on F by Fv_P or FP. The v_P-value of an element a will be denoted by v_Pa and its v_P-residue by av_P or aP.

For every place P of a function field $F|K$, we have the following inequality:

$$\operatorname{trdeg} F|K \;\geq\; \operatorname{trdeg} FP|K \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_PF\;. \tag{1}$$

This is a special case of the Abhyankar inequality. Note that $\dim_{\mathbb{Q}} \mathbb{Q}\otimes v_PF$ is the **rational rank** of the value group v_PF, i.e., the maximal number of rationally independent elements in v_PF. We call P an **Abhyankar place** if equality holds in (1).

In this paper, we will prove the Local Uniformization Theorem for Abhyankar places in arbitrary characteristic:

Theorem 1.1 *Let $F|K$ be a function field of arbitrary characteristic, and P an Abhyankar place of $F|K$ such that $FP|K$ is separable. Further, take any elements $\zeta_1,\ldots,\zeta_m$*

in the valuation ring $\mathcal{O}_F$ of P on F. Then there exists a model of $F|K$ on which P is centered at a smooth point whose local ring contains $\zeta_1, \ldots, \zeta_m$.

Theorem 1.1 follows from the following structure theorem for function fields with Abhyankar places.

Theorem 1.2 *Under the assumptions of Theorem 1.1, there exist a transcendence basis $\{t_1, \ldots, t_s\} \subset \mathcal{O}_F$ of $F|K$, elements $z \in K[t_1, \ldots, t_s]$ and $\eta_1, \ldots, \eta_n \in \mathcal{O}_F$, and polynomials $h_i(X) \in K[t_1, \ldots, t_s, z^{-1}][X]$, $1 \leq i \leq n$, such that*
a) $v_P z = 0$, hence $K[t_1, \ldots, t_s, z^{-1}, \eta_1, \ldots, \eta_n] \subset \mathcal{O}_F$,
b) $h_i(\eta_i) = 0$ and $h_i'(\eta_i)P \neq 0$ for $1 \leq i \leq n$,
c) $F = K(t_1, \ldots, t_s, \eta_1)$,
d) $\zeta_1, \ldots, \zeta_m \in K[t_1, \ldots, t_s, z^{-1}, \eta_1, \ldots, \eta_n]$.

From condition b) one can deduce

$$h_iP(\eta_iP) \;=\; h_i(\eta_i)P \;=\; 0 \qquad \text{and} \qquad (h_iP)'(\eta_iP) \;=\; h_i'(\eta_i)P \;\neq\; 0\,,$$

where h_iP denotes the P-reduction of h_i, i.e., the polynomial obtained from h_i through replacing every coefficient by its P-residue. This shows that η_iP is a simple root of h_iP. Since this is the condition of Hensel's Lemma, such an element η_i is called v_P-henselian. We will discuss henselian elements in Section 5.

Suppose that the assertion of Theorem 1.2 holds for the place P of $F|K$. Choose polynomials $f_i(X_1, \ldots, X_{n+s+1}) \in K[X_1, \ldots, X_{n+s+1}]$, $1 \leq i \leq n+1$, such that
1) $f_1(t_1, \ldots, t_s, X_{s+1}, \ldots, X_{n+s+1}) = zX_{s+1} - 1$,
2) $f_i(t_1, \ldots, t_s, z^{-1}, X_{s+2}, \ldots, X_{n+s+1}) = h_{i-1}(X_{s+i})$ for $2 \leq i \leq n+1$.

Then

$$J \;:=\; \left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1\leq i\leq n+1\\ s+1\leq j\leq s+n+1}} \;=\; \begin{pmatrix} z & & & 0 \\ \star & \frac{\partial f_2}{\partial X_{s+2}} & & \\ \vdots & & \ddots & \\ \star & 0 & & \frac{\partial f_{n+1}}{\partial X_{s+n+1}} \end{pmatrix}.$$

Therefore,

$$\det J(t_1P, \ldots, t_sP, z^{-1}P, \eta_1P, \ldots, \eta_nP) =$$

$$\det \begin{pmatrix} zP & & & 0 \\ \star & h_1'(\eta_1)P & & \\ \vdots & & \ddots & \\ \star & 0 & & h_n'(\eta_n)P \end{pmatrix} \;=\; zP \cdot h_1'(\eta_1)P \cdot \ldots \cdot h_n'(\eta_n)P \;\neq\; 0\,.$$

This means that the place P is centered on $K[t_1, \ldots, t_s, z^{-1}, \eta_1, \ldots, \eta_n]$ at the smooth point $(t_1P, \ldots, t_sP, z^{-1}P, \eta_1P, \ldots, \eta_nP)$. Therefore, Theorem 1.1 follows from Theorem 1.2.

In [K3], we show that the Abhyankar places lie dense in the Zariski space of all places of $F|K$, with respect to a "Zariski patch topology". This topology is finer than the Zariski topology (but still compact); its basic open sets are the sets of the form

$$\{P \mid P \text{ a place of } F|K \text{ such that } a_1P \neq 0, \ldots, a_kP \neq 0\,;\, b_1P = 0, \ldots, b_\ell P = 0\}$$

with $a_1,\ldots,a_k,b_1,\ldots,b_\ell \in F\setminus\{0\}$. (The density actually holds for even stronger topologies; cf. [K3].)

Corollary 1.3 *The uniformizable places of $F|K$ lie dense in the Zariski space of $F|K$, with respect to the Zariski patch topology.*

In order to formulate an arithmetic analogue of the foregoing results, we introduce the following notion. Assume that F is a field extension of $\mathbb{Q}$ of finite transcendence degree. A place P of F will be called an **arithmetic Abhyankar place of** F if it extends the p-adic place of $\mathbb{Q}$ and satisfies

$$\operatorname{trdeg} F|\mathbb{Q} + 1 \;=\; \operatorname{trdeg} FP|\mathbb{F}_p \;+\; \dim_{\mathbb{Q}} \mathbb{Q}\otimes v_P F\;, \tag{2}$$

for some prime p. We will prove the following arithmetic analogue of Theorem 1.2:

Theorem 1.4 *Let F be a function field over $\mathbb{Q}$ and P an arithmetic Abhyankar place of $F|\mathbb{Q}$. Further, take any elements $\zeta_1,\ldots,\zeta_m$ in the valuation ring $\mathcal{O}_F$ of P on F. Set $s = \operatorname{trdeg} F|\mathbb{Q}$. Then there exist a transcendence basis $\{t_1,\ldots,t_s\}\subset\mathcal{O}_F$ of $F|K$, elements $t_{s+1}\in\mathcal{O}_F$, $z\in\mathbb{Z}[t_1,\ldots,t_{s+1}]$ and $\eta_1,\ldots,\eta_n\in\mathcal{O}_F$, and polynomials $h_i(X)\in\mathbb{Z}[t_1,\ldots,t_{s+1},z^{-1}][X]$, $1\le i\le n$, such that*

a) $v_P z = 0$, hence $\mathbb{Z}[t_1,\ldots,t_{s+1},z^{-1},\eta_1,\ldots,\eta_n]\subset\mathcal{O}_F$,
b) $h_i(\eta_i)=0$ and $h_i'(\eta_i)P\neq 0$ for $1\le i\le n$,
c) $F=\mathbb{Q}(t_1,\ldots,t_{s+1},\eta_1)$,
d) $\zeta_1,\ldots,\zeta_m\in\mathbb{Z}[t_1,\ldots,t_{s+1},z^{-1},\eta_1,\ldots,\eta_n]$.

Both Theorem 1.2 and Theorem 1.4 follow from a more general structure theorem which we will state now. See Section 3 for the definition of "defectless".

Theorem 1.5 *Assume that R is an integral domain with quotient field K, and that F is a function field over K. Let P be a place of F such that $FP|KP$ is separable, (K,P) is a defectless field with $R\subset\mathcal{O}_K$, and*

$$\operatorname{trdeg} F|K \;=\; \operatorname{trdeg} FP|KP \;+\; \dim_{\mathbb{Q}} \mathbb{Q}\otimes(v_PF/v_PK)\;. \tag{3}$$

Further, take any elements $\zeta_1,\ldots,\zeta_m$ in the valuation ring $\mathcal{O}_F$ of P on F. Then there exist a transcendence basis $\{t_1,\ldots,t_s\}\subset\mathcal{O}_F$ of $F|K$, elements $t_{s+1},\ldots,t_{s+k}\in\mathcal{O}_F$, $z\in R[t_1,\ldots,t_{s+k}]$ and $\eta_1,\ldots,\eta_n\in\mathcal{O}_F$, and polynomials $h_i(X)\in R[t_1,\ldots,t_{s+k},z^{-1}][X]$, $1\le i\le n$, such that

a) $v_P z = 0$, hence $R[t_1,\ldots,t_{s+k},z^{-1},\eta_1,\ldots,\eta_n]\subset\mathcal{O}_F$,
b) $h_i(\eta_i)=0$ and $h_i'(\eta_i)P\neq 0$ for $1\le i\le n$,
c) $F=K(t_1,\ldots,t_{s+k},\eta_1)$,
d) $\zeta_1,\ldots,\zeta_m\in R[t_1,\ldots,t_{s+k},z^{-1},\eta_1,\ldots,\eta_n]$.

The number k is bounded from above by the maximal number of rationally independent elements in v_PK which are bigger than some positive element in $v_PF\setminus v_PK$, if this number is finite. If K is of finite transcendence degree k_0 over its prime field, then $k\le k_0$ if P is trivial on the prime field, and $k\le k_0+1$ otherwise.

Theorem 1.2 follows from this theorem with $R = K$ and $k = 0$; note that in this case (K, P) is defectless since P is trivial on K.

Theorem 1.4 follows with $R = \mathbb{Z}$ and $k = 1$; in fact, one can take $k = 0$ and set $t_{s+1} = 1$, say, if each positive element of $v_P F \setminus v_P \mathbb{Q}$ is bigger than every positive multiple of $v_P p$ (P extending the p-adic valuation of $\mathbb{Q}$). The latter is equivalent to saying that $\mathbb{Z} v_P p$ is a convex subgroup of $v_P F$. Note that FP is automatically separable over $\mathbb{Q}P = \mathbb{F}_p$, and that $\mathbb{Q}$ together with any p-adic valuation is a defectless field.

Remark 1.6 Here are two criteria for (K, P) to be a defectless field (they follow from Theorem 3.1 below):

a) Suppose that K is a finitely generated field extension of $\mathbb{Q}$. If P extends the p-adic place of $\mathbb{Q}$ for some prime p and satisfies

$$\operatorname{trdeg} K|\mathbb{Q} + 1 \;=\; \operatorname{trdeg} KP|\mathbb{F}_p \;+\; \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_P K \;, \tag{4}$$

then (K, P) is a defectless field.

b) Suppose that $K|K_0$ is a finitely generated field extension. If P is trivial on K_0 and satisfies

$$\operatorname{trdeg} K|K_0 \;=\; \operatorname{trdeg} KP|K_0 \;+\; \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_P K \;, \tag{5}$$

then (K, P) is a defectless field.

Remark 1.7 Assume that P is a place of F and Q is a **coarsening of** P, i.e., a place of F whose valuation ring contains that of P. Then for every element $a \in F$, $aP \neq 0$ implies $aQ \neq 0$. Thus, assertions a), b), c), d) carry over to every coarsening of P. This shows that if P admits local uniformization, then so does Q.

We conclude with a generalization of Theorem 1.5. It gives simultaneous uniformization for finitely many places which satisfy the assumptions of Theorem 1.5 and have isomorphic value groups.

Theorem 1.8 *Assume that R is an integral domain with quotient field K, and that F is a function field over K. Let P be a place of K such that (K, P) is a defectless field with $R \subset \mathcal{O}_K$. Further, let $P_1, \ldots, P_\ell$ be extensions of P to F such that $FP_j|KP$ is separable and*

$$\operatorname{trdeg} F|K \;=\; \operatorname{trdeg} FP_j|KP \;+\; \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_{P_j} F / v_P K) \tag{6}$$

for every j. Assume also that all value groups $v_{P_j} F$ are isomorphic over $v_P K$ (as ordered groups). Then there exist a transcendence basis $\{t_1, \ldots, t_s\} \subset \mathcal{O}$ of $F|K$, elements $t_{s+1}, \ldots, t_{s+k} \in \mathcal{O} := \bigcap_{j=1}^{\ell} \mathcal{O}_{P_j}$, $z \in R[t_1, \ldots, t_{s+k}]$ and $\eta \in \mathcal{O}$, and a polynomial $h(X) \in R[t_1, \ldots, t_{s+k}, z^{-1}][X]$, such that

a) z is a unit in $\mathcal{O}$, hence $R[t_1, \ldots, t_{s+k}, z^{-1}, \eta] \subset \mathcal{O}$,

b) $h(\eta) = 0$ and $h'(\eta)P_j \neq 0$ for $1 \leq j \leq \ell$,

c) $F = K(t_1, \ldots, t_{s+k}, \eta)$.

The number k is bounded from above as in Theorem 1.5.

Using this theorem and results from [K3], we deduce that every algebraic variety admits "simultaneous local resolution of singularities at finitely many points":

Theorem 1.9 *Let K be a perfect field and V an affine variety defined over K, with function field F and coordinate ring A. Let $\mathfrak{P}_1,\ldots,\mathfrak{P}_\ell$ be prime ideals of A of equal dimension d. Suppose that* $\operatorname{trdeg} F|K > d' \geq d$. *Then there are places $P_1,\ldots,P_\ell$ of F of dimension d' centered on A at $\mathfrak{P}_1,\ldots,\mathfrak{P}_\ell$ respectively, and a model of F on which $P_1,\ldots,P_\ell$ all are centered at smooth points.*

Here, the **dimension of** P is the transcendence degree of FP over K. The places $P_1,\ldots,P_\ell$ can be chosen of "maximal rank" $\operatorname{trdeg} F|K - d'$.

2 Valuation independence

For the easy proof of the following theorem, see [B], Chapter VI, §10.3, Theorem 1.

Theorem 2.1 *Let $(F|K,P)$ be an extension of valued fields. Take elements $x_i, y_j \in F$, $i\in I$, $j\in J$, such that the values $v_P x_i$, $i\in I$, are rationally independent over v_PK, and the residues y_jP, $j\in J$, are algebraically independent over KP. Then the elements x_i, y_j, $i\in I$, $j\in J$, are algebraically independent over K.*

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i,y_j \mid i\in I, j\in J]$$

in such a way that for every $k\neq\ell$ there is some i s.t. $\mu_{k,i}\neq\mu_{\ell,i}$ or some j s.t. $\nu_{k,j}\neq\nu_{\ell,j}$, then

$$v_P f \;=\; \min_k v_P\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \left(v_P\, c_k + \sum_{i\in I} \mu_{k,i} v_P x_i \right) .$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} v_P K(x_i,y_j \mid i\in I, j\in J) &= v_PK \oplus \bigoplus_{i\in I} \mathbb{Z} v_P x_i \\ K(x_i,y_j \mid i\in I, j\in J)P &= KP\,(y_jP \mid j\in J)\,. \end{aligned}$$

It also implies that the valuation v_P and the place P on $K(x_i,y_j \mid i\in I, j\in J)$ are uniquely determined by their restrictions to K, the values v_Px_i and the residues y_jP.

Every finite extension $(L|K,P)$ of valued fields satisfies the **fundamental inequality** (cf. [EN]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{7}$$

where $n=[L:K]$ is the degree of the extension, $P_1,\ldots,P_{\mathrm{g}}$ are the distinct extensions of P from K to L, $\mathrm{e}_i = (v_{P_i}L : v_PK)$ are the respective ramification indices and $\mathrm{f}_i = [LP_i : KP]$ are the respective inertia degrees. Note that $\mathrm{g}=1$ if (K,P) is henselian.

Corollary 2.2 *Let $(F|K,P)$ be an extension of valued fields of finite transcendence degree. Then the following inequality holds:*

$$\operatorname{trdeg} F|K \;\geq\; \operatorname{trdeg} FP|KP \,+\, \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P F/v_P K)\;. \tag{8}$$

If in addition $F|K$ is a function field, and if equality holds in (8), then the extensions $v_P F|v_P K$ and $FP|KP$ are finitely generated. In particular, if P is trivial on K, then $v_P F$ is a product of finitely many copies of $\mathbb{Z}$, and FP is again a function field over K.

Proof: Choose elements $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau \in F$ such that the values $v_P x_1,\ldots,v_P x_\rho$ are rationally independent over $v_P K$ and the residues $y_1 P,\ldots,y_\tau P$ are algebraically independent over KP. Then by the foregoing lemma, $\rho+\tau \leq \operatorname{trdeg} F|K$. This proves that $\operatorname{trdeg} FP|KP$ and the rational rank of $v_P F/v_P K$ are finite. Therefore, we may choose the elements x_i, y_j such that $\tau = \operatorname{trdeg} FP|KP$ and $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P F/v_P K)$ to obtain inequality (8).

Assume that this is an equality. This means that for $F_0 := K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$, the extension $F|F_0$ is algebraic. Since $F|K$ is finitely generated, it follows that $F|F_0$ is finite. By the fundamental inequality, this yields that $v_P F|v_P F_0$ and $FP|F_0 P$ are finite extensions. Since already $v_P F_0|v_P K$ and $F_0 P|KP$ are finitely generated by the foregoing theorem, it follows that also $v_P F|v_P K$ and $FP|KP$ are finitely generated. □

Note that (1) is the special version of (8) for the case of a valued function field with trivially valued base field. We will say that $(F|K,P)$ is **without transcendence defect** if equality holds in (8).

3 Inertially generated function fields

We will make essential use of a theorem which we proved in [K1] (cf. also [K2]). It is a generalization of the "Grauert–Remmert Stability Theorem". To state it, we introduce a fundamental notion. A valued field (K,P) is called **defectless** (or **stable**) if equality holds in the fundamental inequality (7) for every finite extension $L|K$. If $\operatorname{char} KP = 0$, then (K,P) is defectless (this is a consequence of the "Lemma of Ostrowski", cf. [EN], [R]).

Now we are able to state the **Stability Theorem**, which deals with an arbitrary valued function field $(F|K,P)$, where P need not be the identity on K.

Theorem 3.1 *Let $(F|K,P)$ be a valued function field without transcendence defect. If (K,P) is a defectless field, then also (F,P) is a defectless field.*

Whenever we talk of a valued field K, we will implicitly assume the valuation and place to be extended to the algebraic closure. Therefore, we will talk of *the* henselization K^h, and of *the* **absolute inertia field of** K, which we define to be the inertia field of the normal separable extension $K^{\text{sep}}|K$ with respect to the given valuation; here, K^{sep} denotes the separable-algebraic closure of K.

The following lemma is proved in [K2] (and partially also in [EN]):

Lemma 3.2 *A valued field (K,P) is defectless if and only if its henselization (K^h,P) is.*

An extension $(L|K,P)$ of valued fields is called **immediate** if the canonical embeddings $KP \to LP$ and $v_P K \to v_P L$ are onto.

Corollary 3.3 *If (K,P) is defectless, then (K^h,P) does not admit proper immediate algebraic extensions.*

Proof: If (K,P) is defectless, then so is (K^h,P), by the foregoing lemma. Suppose that $(L|K^h,P)$ is a finite immediate algebraic extension. Hence, $(v_P L : v_P K^h) = 1 = [LP : K^h P]$. Since (K^h,P) is a henselian field, there is a unique extension of v_P from K^h to L. Since (K^h,P) is defectless, we have that $[L : K^h] = (v_P L : v_P K^h)[LP : K^h P] = 1$, showing that $L = K^h$. As every proper immediate extension would contain a proper finite immediate extension, it follows that (K^h,P) does not admit any proper immediate algebraic extension. □

From these facts, we deduce:

Theorem 3.4 *Assume that $F|K$ is a function field and P is a place of F such that $FP|KP$ is a separable extension, (K,P) is a defectless field and equality holds in (8). Then $(F|K,P)$ is* **inertially generated***, by which we mean that there is a transcendence basis $T = \{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$ such that*

a) $v_P F = v_P K \oplus \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$,

b) $y_1 P,\ldots,y_\tau P$ is a separating transcendence basis of $FP|KP$,

c) (F,P) lies in the absolute inertia field of $(K(T),P)$.

Assertion c) holds for each transcendence basis T which satisfies assertions a) and b).

Proof: By Corollary 2.2, value group and residue field of (F,P) are finitely generated. We choose $x_1,\ldots,x_\rho \in F$ such that $v_P F = \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$ (where $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P F / v_P K)$. Since $FP|KP$ is a finitely generated separable extension, it is separably generated. Therefore, we can choose $y_1,\ldots,y_\tau \in F$ such that $FP|KP(y_1 P,\ldots,y_\tau P)$ is separable-algebraic (where $\tau = \operatorname{trdeg} FP|KP$). We set $T := \{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$ and $F_0 := K(T)$.

Now we can choose some $a \in FP$ such that $FP = KP(y_1 P,\ldots,y_\tau P, a)$. Since a is separable-algebraic over $KP(y_1 P,\ldots,y_\tau P)$, by Hensel's Lemma there exists an element η in the henselization of (F,P) such that $\eta P = a$ and that the reduction of the minimal polynomial of η over F_0 is the minimal polynomial of a over $KP(y_1 P,\ldots,y_\tau P)$. Then η lies in the absolute inertia field of F_0. Now the field $F_0(\eta)$ has the same value group and residue field as F, and it is contained in the henselization F^h of F. As henselizations are immediate extensions and the henselization $F_0(\eta)^h$ of $F_0(\eta)$ can be chosen inside of F^h, we obtain an immediate algebraic extension $(F^h|F_0(\eta)^h,P)$. On the other hand, (K,P) is assumed to be a defectless field. By construction, $(F_0|K,P)$ is without transcendence defect, and the same is true for $(F_0(\eta)|K,P)$ since this property is preserved by algebraic extensions. Hence we know from Theorem 3.1 that $(F_0(\eta),P)$ is a defectless field. Now Corollary 3.3 shows that the extension $F^h|F_0(\eta)^h$ must be trivial. Therefore, F is contained in $F_0(\eta)^h$, which in turn is a subfield of the absolute inertia field of F_0. This proves our theorem. □

4 Abhyankar places on rational function fields

The following lemma was proved (but not explicitly stated) by Zariski in [Z] for subgroups of $\mathbb{R}$, using the algorithm of Perron. We leave it as an easy exercise to the reader to prove the general case by induction on the rank of the ordered abelian group. However, an instant proof of the lemma can also be found in [EL] (Theorem 2.2).

Lemma 4.1 *Let Γ be a finitely generated ordered abelian group. Take any non-negative elements $\alpha_1,\ldots,\alpha_\ell \in \Gamma$. Then there exist positive elements $\gamma_1,\ldots,\gamma_\rho \in \Gamma$ such that $\Gamma = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_\rho$ and every α_i can be written as a sum $\sum_j n_{ij}\gamma_j$ with non-negative integers n_{ij} .*

The foregoing lemma and Theorem 2.1 are the main ingredients in the proof of the next proposition. We consider the following situation:

$$\left.\begin{array}{l} F|K \ \text{ a function field,}\\ P \ \text{ a place of } F,\\ x_1,\ldots,x_\rho,y_1,\ldots,y_\tau \in \mathcal{O}_F \ \text{ such that}\\ \quad 1)\ v_Px_1,\ldots,v_Px_\rho \ \text{ form a maximal set of elements in } v_PF\\ \qquad \text{rationally independent over } v_PK,\\ \quad 2)\ y_1P,\ldots,y_\tau P \text{ form a transcendence basis of } FP \text{ over } KP,\\ E := K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)\,,\\ \zeta_1,\ldots,\zeta_m \in \mathcal{O}_E\ . \end{array}\right\} \tag{9}$$

Proposition 4.2 *In the situation described in (9), assume in addition that v_PK is a convex subgroup of v_PE (this always holds if P is trivial on K.) Then there are $x'_1,\ldots,x'_\rho \in \mathcal{O}_E$ and $z \in \mathcal{O}_K[x'_1,\ldots,x'_\rho,y_1,\ldots,y_\tau]$ of value $vz=0$ such that*

a) $\mathcal{O}_K[x_1,\ldots,x_\rho] \subseteq \mathcal{O}_K[x'_1,\ldots,x'_\rho]$ and $K(x'_1,\ldots,x'_\rho,y_1,\ldots,y_\tau) = E$,
b) $v_PE = v_PK \oplus \bigoplus_{i=1}^{\rho} \mathbb{Z}v_Px'_i$,
c) $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_K[x'_1,\ldots,x'_\rho,y_1,\ldots,y_\tau,z^{-1}]$.

The number k is bounded from above as in Theorem 1.5.

Proof: By Theorem 2.1, we know that

$$v_PE \;=\; v_PK \oplus \mathbb{Z}v_Px_1 \oplus \ldots \oplus \mathbb{Z}v_Px_\rho\ .$$

We may assume that $v_Px_i > 0$ for $1 \le i \le \rho$. The map

$$v_PE/v_PK \ni \mu_1(v_Px_1+v_PK)+\ldots+\mu_\rho(v_Px_\rho+v_PK) \mapsto x_1^{\mu_1}\cdot\ldots\cdot x_\rho^{\mu_\rho} \qquad (\mu_1,\ldots,\mu_\rho \in \mathbb{Z})$$

is an isomorphism from v_PE/v_PK onto the multiplicative subgroup of $E^\times$ generated by $x_1,\ldots,x_\rho$. We denote this group by $\mathcal{X}$.

Take ζ to be any of the elements $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_E$ and write $\zeta = f/g$ with polynomials $f,g \in K[x_1,\ldots,x_\rho,y_1,\ldots,y_\tau]$. Write

$$f \;=\; \sum_{i=1}^{d} c_i\, x_1^{\mu_{1,i}}\cdot\ldots\cdot x_\rho^{\mu_{\rho,i}}\cdot y_1^{\nu_{1,i}}\cdot\ldots\cdot y_\tau^{\nu_{\tau,i}} \quad\text{and}\quad g \;=\; \sum_{i=1}^{d'} c'_i\, x_1^{\mu'_{1,i}}\cdot\ldots\cdot x_\rho^{\mu'_{\rho,i}}\cdot y_1^{\nu'_{1,i}}\cdot\ldots\cdot y_\tau^{\nu'_{\tau,i}}$$

as sums of monomials (in such a way that in either polynomial, two different monomials differ in at least one exponent). Then by Theorem 2.1, the value of f is equal to the least of the values of its monomials, say, to the value of the first. Similarly for g. So we can write

$$\zeta \;=\; \frac{\sum_i \frac{c_i}{c'_1}\, x_1^{\mu_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_\rho^{\mu_{\rho,i}-\mu'_{\rho,1}} \cdot y_1^{\nu_{1,i}} \cdot \ldots \cdot y_\tau^{\nu_{\tau,i}}}{\sum_i \frac{c'_i}{c'_1}\, x_1^{\mu'_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_\rho^{\mu'_{\rho,i}-\mu'_{\rho,1}} \cdot y_1^{\nu'_{1,i}} \cdot \ldots \cdot y_\tau^{\nu'_{\tau,i}}} \tag{10}$$

where the denominator has value 0 and the summands appearing in it all have value ≥ 0. Since $\zeta \in \mathcal{O}_E$, the numerator has value ≥ 0, and the same must thus be true for all its summands. The only obstruction is that some of the x_i's may appear with negative exponents in some summands, and that, if P is non-trivial on K, some of the c_i/c'_1 or c'_i/c'_1 may have negative value.

We collect all summands of the form $h = cx_1^{\mu_1} \cdot \ldots \cdot x_\rho^{\mu_\rho} \cdot y_1^{\nu_1} \cdot \ldots \cdot y_\tau^{\nu_\tau}$ which appear in the numerator or denominator of (10), and all products of the form $\xi = x_1^{\mu_1} \cdot \ldots \cdot x_\rho^{\mu_\rho} \in \mathcal{X}$ which appear in these summands. We do the same for all elements $\zeta_1, \ldots, \zeta_m$. We also add the elements $x_1, \ldots, x_\rho$ to the list of elements so obtained. In this way, we get finitely many elements $h_1, \ldots, h_l$, all of them having non-negative value, and corresponding elements $\xi_1, \ldots, \xi_l \in \mathcal{X}$. Since $v_P x_1, \ldots, v_P x_\rho$ are among $\xi_1, \ldots, \xi_l$, these elements generate $\mathcal{X}$. We note that $v_P \xi_j \in v_P h_j + v_P K$, and as $v_P K$ is a convex subgroup of $v_P E$ and $v_P h_j$ is non-negative, it follows that $v_P \xi_j + v_P K$ is non-negative in $v_P E / v_P K$, for the induced order. At this point, we apply Lemma 4.1 to $v_P \xi_1 + v_P K, \ldots, v_P \xi_\ell + v_P K$. Pulling the result back through the above isomorphism, we find generators $x''_1, \ldots, x''_\rho$ of $\mathcal{X}$ for which the values $v_P x''_j + v_P K$ in $v_P E / v_P K$ are positive, and such that each of the ξ's can be written as a (unique) product of the x''_j *with non-negative exponents.*

Let $\alpha \in v_P K$ be the minimum of the values of the coefficients $c \in K$ appearing in the monomials $h_1, \ldots, h_k$. We take $\tilde{c} \in K$ such that $v_P \tilde{c} = \alpha$ if $\alpha < 0$, and $\tilde{c} = 1$ if $\alpha \geq 0$. We set $x'_j := \tilde{c} x''_j$. It follows that $x_1, \ldots, x_\rho \in \mathcal{O}_K[x'_1, \ldots, x'_\rho]$ and therefore, assertion a) holds. It also follows that $h_1, \ldots, h_k \in \mathcal{O}_K[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$. Since the values $v_P x''_j + v_P K$ in $v_P E / v_P K$ were positive, all values $v_P x'_j$ are positive.

As we can read off from (10), every ζ_j can be written as ζ'_j / ζ''_j, where ζ'_j, ζ''_j lie in $\mathcal{O}_K[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$, with $v_P \zeta''_j = 0$. We set $z = \prod_{j=1}^m \zeta''_j$, and we are done. □

Proposition 4.3 *In the situation described in (9), assume in addition that K is the quotient field of the ring R and that $R \subset \mathcal{O}_E$. Then there are $x'_1, \ldots, x'_{\rho+k} \in \mathcal{O}_E$ and $z \in R[x'_1, \ldots, x'_{\rho+k}, y_1, \ldots, y_\tau]$ of value $vz = 0$ such that*

a) $K(x'_1, \ldots, x'_{\rho+k}, y_1, \ldots, y_\tau) = E$,

b) $v_P E = v_P K + \bigoplus_{i=1}^{\rho+k} \mathbb{Z} v_P x'_i$,

c) $\zeta_1, \ldots, \zeta_m \in R[x'_1, \ldots, x'_{\rho+k}, y_1, \ldots, y_\tau, z^{-1}]$.

The number k is bounded from above as described in Theorem 1.5.

Proof: We start as in the foregoing proof and write $\zeta = f/g$. As K is the quotient field of R, we can choose $c_i, c'_i \in R$. We write down such a representation for each ζ_i, $1 \leq i \leq m$. We collect all coefficients c_i, c'_i from these representations. The values of these finitely many elements generate a finitely generated subgroup Γ of $v_P K$. This subgroup can be written in the form

$$\mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_k \oplus \mathbb{Z}\gamma_{k+1} \oplus \ldots \oplus \mathbb{Z}\gamma_{k'}$$

where

i) $\gamma_j > 0$ for all j,

ii) there is some $N \in \mathbb{N}$ and a positive element δ in

$$\Delta := \mathbb{Z}v_P x_1 \oplus \ldots \oplus \mathbb{Z}v_P x_\rho$$

such that $N\gamma_j > \delta$ for $1 \leq j \leq k$, and

iii) $N\gamma_j < \delta$ for all $N \in \mathbb{N}$ and all positive $\delta \in \Delta$.

In other words, $\gamma_1, \ldots, \gamma_k$ are archimedean comparable to some elements in Δ, and $\gamma_{k+1}, \ldots, \gamma_{k'}$ are infinitely smaller than all positive elements in Δ. As $\delta \in v_P F \setminus v_P K$ and $N\gamma_1, \ldots, N\gamma_k$ are rationally independent elements in $v_P K$, we find that k is bounded from above by the maximal number of rationally independent elements in $v_P K$ which are bigger than some positive element in $v_P F \setminus v_P K$.

Now we can rewrite the representations using polynomials of the form

$$f = \sum_{i=1}^{d} r_i \, x_1^{\mu_{1,i}} \cdot \ldots \cdot x_{\rho+k}^{\mu_{\rho+k,i}} \cdot y_1^{\nu_{1,i}} \cdot \ldots \cdot y_\tau^{\nu_{\tau,i}} \quad \text{and} \quad g = \sum_{i=1}^{d'} r'_i \, x_1^{\mu'_{1,i}} \cdot \ldots \cdot x_{\rho+k}^{\mu'_{\rho+k,i}} \cdot y_1^{\nu'_{1,i}} \cdot \ldots \cdot y_\tau^{\nu'_{\tau,i}}$$

where

$$r_i \, , r'_i \in R \quad \text{with} \quad v_P r_i \, , v_P r'_i \in \Gamma_0 := \mathbb{Z}\gamma_{k+1} \oplus \ldots \oplus \mathbb{Z}\gamma_{k'} \; .$$

We obtain

$$\zeta = \frac{\sum_i \frac{r_i}{r'_1} x_1^{\mu_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_{\rho+k}^{\mu_{\rho+k,i}-\mu'_{\rho,1}} \cdot y_1^{\nu_{1,i}} \cdot \ldots \cdot y_\tau^{\nu_{\tau,i}}}{\sum_i \frac{r'_i}{r'_1} x_1^{\mu'_{1,i}-\mu'_{1,1}} \cdot \ldots \cdot x_{\rho+k}^{\mu'_{\rho,i}-\mu'_{\rho+k,1}} \cdot y_1^{\nu'_{1,i}} \cdot \ldots \cdot y_\tau^{\nu'_{\tau,i}}} \tag{11}$$

From now on, we proceed as in the foregoing proof, with the elements $x_1, \ldots, x_{\rho+k}$ in the place of $x_1, \ldots, x_\rho$. Also, we replace $v_P E$ by $\Delta + \Gamma$, and $v_P K$ by Γ_0 and note that by construction, Γ_0 is a convex subgroup of $\Delta + \Gamma$. In this way, we obtain the elements x''_j, $1 \leq i \leq \rho + k$.

As before, let α be the minimum of the values of the coefficients $c \in R$ appearing in the monomials $h_1, \ldots, h_k$. Then $\alpha \in \Gamma_0$. We take $\tilde{c} \in R$ such that $v_P \tilde{c} = \alpha$ if $\alpha < 0$, and $\tilde{c} = 1$ if $\alpha \geq 0$. We set $x'_j := \tilde{c} x''_j$, $1 \leq i \leq \rho + k$. It follows that $x_1, \ldots, x_{\rho+k} \in R[x'_1, \ldots, x'_{\rho+k}]$ and therefore, assertion a) holds. It also follows that $h_1, \ldots, h_k \in R[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$. Since the values $v_P x''_j + \Gamma_0$ in $\Delta + \Gamma$ were positive, all values $v_P x'_j$ are positive. Every ζ_j can be written as ζ'_j / ζ''_j with $\zeta'_j, \zeta''_j \in R[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$ and $v_P \zeta''_j = 0$. We set $z = \prod_{j=1}^m \zeta''_j$, and we are done. $\square$

5 Henselian elements

Throughout this section, let $L|K$ be an algebraic extension and v a valuation on the algebraic closure $\tilde{L}$ of L. We will always assume that (L,v) is contained in the absolute inertia field of (K,v). By $\mathcal{O}_L$ we denote the valuation ring of v on L, and by $\mathcal{O}_K$ the valuation ring of v on K. An element $a \in \mathcal{O}_L$ is called **v-henselian** over K if there is a (not necessarily monic) polynomial $h(X) \in \mathcal{O}_K[X]$ such that

$$h(a) = 0 \qquad \text{and} \qquad vh'(a) = 0\,,$$

or in other words, the residue av is a simple root of the reduction hv of h modulo v. We will say that h is an **associated polynomial** for a. It may not be monic, and it also may not be irreducible over K.

5.1 The basic results

We show:

Lemma 5.1 *Assume that $L|K$ is an algebraic extension and v a valuation on $\tilde{L}$ such that (L,v) is contained in the absolute inertia field of (K,v). Then $\mathcal{O}_L$ is generated as an $\mathcal{O}_K$-algebra by v-henselian elements.*

Clearly, we only have to prove this assertion for the case where $L|K$ is finite. The proof for this case uses two results on the construction of henselian elements.

Lemma 5.2 *Let $a \in \mathcal{O}_L$ be v-henselian and $v(a) = 0$. Then a^{-1} is v-henselian too.*

Proof: Let $h(X) \in \mathcal{O}_K[X]$ and $h(a) = 0$. Put $g(X) = X^n h(X^{-1})$ where n is the degree of $h(X)$. Then $g(a^{-1}) = 0$. Moreover

$$g'(X) = nX^{n-1}h(X^{-1}) - X^{n-2}h'(X^{-1})$$

and hence

$$g'(a^{-1}) = -a^{-(n-2)}h'(a)\,.$$

□

In what follows, we denote by W the finite set of all valuations $w \neq v$ of L which coincide with v on K. The letter w will always denote a valuation from W. Since $L|K$ is algebraic, we know that the valuations in the set $W \cup \{v\}$ are pairwise incomparable.

Lemma 5.3 *Assume that $(L|K,v)$ is a finite extension of valued fields such that (L,v) is contained in the absolute inertia field of (K,v). Let ϑ be a non-zero generator of the residue field extension, i.e., $Lv = Kv(\vartheta)$. Assume that $z \in L$ is such that $zv = \vartheta$ and $w(z) \neq 0$ for every $w \in W$. Then z is v-henselian over K. Moreover, $L = K(z)$.*

Proof: Let $\sigma_1, \ldots, \sigma_n$ denote the isomorphic mappings of $L|K$ into the algebraic closure $\widetilde{K}$ of K, where $n = [L : K]$. We regard the valuation v of L to be extended to $\widetilde{K}$. The mappings $x \mapsto v_i(x) := v(\sigma_i x)$ are all the valuations of L which coincide with v on K, but they are not necessarily distinct. After changing the enumeration we may assume that $v_1 = \ldots = v_k = v$ but $v_i \neq v$ for $i > k$. Then $k = [Lv : Kv] = [LK^h : K^h]$. Each of these σ_i for $i \leq k$ induces an isomorphism of $Lv|Kv$ into its algebraic closure (which we also denote by σ_i), and we have

$$(\sigma_i(z))v \;=\; \sigma_i(zv) \qquad \text{for} \qquad 1 \leq i \leq k\,.$$

These are the conjugates of ϑ over Kv, and they are all distinct since ϑ generates the separable extension $Lv|Kv$.

We enumerate the other isomorphisms $\sigma_{k+1}, \ldots, \sigma_n$ in such a way that

$$\begin{array}{llll} v_\nu z > 0 & \text{hence} \quad (\sigma_\nu z)v = 0 & \text{for} & k < \nu \leq r \\ v_\mu z < 0 & \text{hence} \quad (\sigma_\mu z)v = \infty & \text{for} & r < \mu \leq n\,. \end{array}$$

The monic minimal polynomial of z over K is

$$h_0(X) = \prod_{1\leq i\leq k} (X - \sigma_i z) \prod_{k<\nu\leq r} (X - \sigma_\nu z) \prod_{r<\mu\leq n} (X - \sigma_\mu z)\,.$$

Dividing $h_0(X)$ by a coefficient c of minimal v-value, we obtain a polynomial $h(X) \in K[X]$ which is v-primitive, i.e., every coeffcient has v-value ≥ 0 and there is at least one coefficient of v-value $= 0$. Now consider the polynomial

$$h_1(X) = \prod_{1\leq i\leq k} (X - \sigma_i z) \prod_{k<\nu\leq r} (X - \sigma_\nu z) \prod_{r<\mu\leq n} ((\sigma_\mu z)^{-1} X - 1)$$

which is obtained by dividing $h_0(X)$ by the factor $\prod_\mu (\sigma_\mu z)^{-1} \in \widetilde{K}$. The two polynomials $h(X)$, $h_1(X)$ differ by a constant factor $d \in \widetilde{K}$, and since both are v-primitive we have $vd = 0$. (Observe that $h_1(X)$ is v-primitive because by construction, each linear factor is v-primitive; use Gauss' lemma.) Thus we have

$$h(X) = d \cdot h_1(X) \qquad \text{with} \qquad d \in \widetilde{K}, \quad dv \neq 0, \infty$$

and thus

$$hv(X) = \pm dv \cdot g(X) \cdot X^\ell\,.$$

This shows that ϑ is a simple root of $hv(X)$ and thus $h'(z)v \neq 0$.

Finally, we see that $\sigma_i z \neq z$ for every $\sigma_i \neq \mathrm{id}$, which shows that h_0 is irreducible and therefore, $L = K(z)$. □

By Theorem 11.14 of [EN], an element z as in the foregoing lemma will always exist. Hence, we have proved:

Corollary 5.4 *Assume that $(L|K, v)$ is a finite extension of valued fields such that (L, v) is contained in the absolute inertia field of (K, v). Then there is $z \in L$ such that z is v-henselian over K, $L = K(z)$ and $Lv = Kv(zv)$.*

Now we give a **Proof of Lemma 5.1**:

Let $a \in \mathcal{O}_L$ be arbitrary. We are going to show that a is contained in a ring which is generated over $\mathcal{O}_K$ by finitely many henselian elements.

We start with a henselian element $z \in \mathcal{O}_L$ as in Corollary 5.4. The residue $av \in Lv$ can be written as a polynomial in ϑ with coefficients in Kv. Consider an inverse image in $\mathcal{O}_K[X]$ of that polynomial, say $f(X)$. Then $av = f(z)v$. We put

$$b = a - f(z) + z \qquad \text{so that} \qquad bv = zv\,.$$

We have

$$a \in \mathcal{O}_K[z, b]\,.$$

Hence it suffices to show that b is contained in a ring generated over $\mathcal{O}_K$ by finitely many henselian elements.

Employing again Theorem 11.14 of [EN], we can find an element $c \in L$ with the following properties:

$$\begin{array}{rcll} v(c-1) & > & 0 & \text{so that } cv = 1 \\ wc & > & 0 & \text{if} \quad wb \geq 0 \\ wc & = & 0 & \text{if} \quad wb < 0 \end{array}$$

for all $w \in W$. Now, if we put $d = bc$ then we have:

$$\begin{array}{rcll} dv & = & bv \;=\; zv & \\ wd & > & 0 & \text{if} \quad wb \geq 0 \\ wd & < & 0 & \text{if} \quad wb < 0 \end{array}$$

and hence $wd \neq 0$ for all $w \in W$. According to Lemma 5.3 we conclude that d is henselian. On the other hand, the element $e = cd$ also satisfies these relations, hence e is henselian too. Therefore, e^{-1} is henselian by Lemma 5.2. Since

$$b = d \cdot c^{-1} = d^2 \cdot e^{-1} \in \mathcal{O}_K[d, e^{-1}]$$

we see that b and hence a is contained in a ring generated over $\mathcal{O}_K$ by henselian elements. □

5.2 A simultaneous version for finitely many places

Lemma 5.5 *Suppose that $L|K$ is a finite extension, v a valuation of K and $v_1, \ldots, v_\ell$ distinct extensions of v to L. Assume that Kv is an infinite field. Further, assume that for each i, (L, v_i) is contained in the absolute inertia field of (K, v). Then there exists an element $\eta \in \mathcal{O} := \bigcap_{i=1}^{\ell} \mathcal{O}_{(L,v_i)}$ which is v_i-henselian over (K, v) for all i and such that $L = K(\eta)$.*

Proof: We adapt the proof of Lemma 5.3. We take an extension of v to $\tilde{K}$ and $\sigma_1, \ldots, \sigma_n$ as in that proof, this time choosing the enumeration in such a way that for $1 \leq i \leq \ell$,

$v_i = v\circ\sigma_j$ for $k_{i-1} < j \le k_i$ (where $k_0 = 0$). Then we denote by $v_{\ell+1},\dots,v_k$ the remaining distinct extensions of v to L which are of the form $v\circ\sigma_j$ for $\ell < j \le n$. For $1 \le i \le \ell$, we choose non-zero generators ϑ_i of the finite separable extensions $Lv_i|Kv$. After multiplying them with suitable elements from Kv if necessary, we may assume that ϑ_i is not conjugate to ϑ_j over Kv if $i \ne j$; this is possible since Kv is infinite by assumption. Being distinct extensions of v to the algebraic extension L of K, the valuations $v_1,\dots,v_k$ are pairwise incomparable. By Theorem 11.14 of [EN], there is an element $z \in \mathcal{O}$ such that $zv_i = \vartheta_i$ for $1 \le i \le \ell$ and $v_i z > 0$ for $\ell < i \le k$. The monic minimal polynomial of z over K is

$$h(X) = \left(\prod_{1\le i\le \ell}\ \prod_{k_{i-1}<j\le k_i} (X - \sigma_j z)\right) \prod_{\ell<\nu\le k} (X - \sigma_\nu z)\ .$$

The reduction of this polynomial modulo v is

$$hv(X) = \left(\prod_{1\le i\le \ell}\ \prod_{k_{i-1}<j\le k_i} (X - \sigma_j \vartheta_i)\right) \cdot X^{k-\ell}\ .$$

By our choice of the elements ϑ_i, the elements $\sigma_j\vartheta_i$ are distinct. Since the coefficients of h are fixed under all σ_j, we have that $hv_i = hv$ for all i. Hence for $1 \le i \le \ell$, the residue $zv_i = \vartheta_i$ is a simple root of hv_i. As before, $\sigma_i z \ne z$ for every $\sigma_i \ne \mathrm{id}$, which shows that $L = K(z)$. □

For the sake of completeness, we include the following result.

Lemma 5.6 *Suppose that $L|K$ is a finite extension and that $v_1,\dots,v_\ell$ are valuations of L whose restrictions to K are pairwise incomparable. Further, assume that for each i, v_i is extended to L such that (L,v_i) is contained in the absolute inertia field of (K,v_i). Take any elements $\zeta_1,\dots,\zeta_m \in \mathcal{O} := \bigcap_{i=1}^{\ell} \mathcal{O}_{(L,v_i)}$. Then there are $\eta_1,\dots,\eta_{2m+1} \in \mathcal{O}$ which are henselian over all (K,v_i) and such that*

a) $L = K(\eta_1)$,

b) $\zeta_1,\dots,\zeta_m \in \mathcal{O}_{(K,v_i)}[\eta_1,\dots,\eta_{2m+1}]$ *for all i.*

Proof: We denote by W_i the finite set of valuations of L which differ from v_i on L but coincide with v_i on K. Since $v_1,\dots,v_\ell$ are pairwise incomparable on K, the valuations in $\{v_1,\dots,v_\ell\} \cup \bigcup_{i=1}^m W_i$ are pairwise incomparable on L. We will now copy the proof of Lemma 5.1, simultaneously for $v_1,\dots,v_\ell$.

Applying Theorem 11.14 of [EN], for every i we choose some $z_i \in L$ such that $Lv_i = Kv_i(z_iv_i)$ and $wz_i > 0$ for all $w \in W_i$. By Lemma 5.3, z_i is v_i-henselian over K. Again by Theorem 11.14 of [EN], we find an element $\eta_1 \in L$ such that for all i, $v_i(\eta_1 - z_i) > 0$ and $w\eta_1 > 0$ for $w \in W_i$. It follows from Lemma 5.3 that η_1 is v_i-henselian over K for all i, and that $L = K(\eta_1)$.

Now take $j \in \{1,\dots,m\}$ and set $a = \zeta_j$. For $1 \le i \le \ell$, choose $f_i(X) \in \mathcal{O}_{(K,v_i)}[X]$ such that $av_i = f_i(\eta_1)v_i$. After adding coefficients of value > 0 if necessary we may assume that all f_i have the same degree. Applying Theorem 11.14 of [EN] to the coefficients of the

f_i (using that $v_1, \dots, v_\ell$ are pairwise incomparable on the field K), we find a polynomial $f(X) \in \mathcal{O}[X]$ such that $av_i = f(\eta_1)v_i$ for all i. We set $b = a - f(z) + z$.

Using Theorem 11.14 of [EN] once more, we find an element $c \in L$ with the following properties:

$$\begin{array}{rcll} v_i(c-1) & > & 0 & \text{for } 1 \le i \le \ell \\ wc & > & 0 & \text{if} \quad wb \ge 0 \\ wc & = & 0 & \text{if} \quad wb < 0 \end{array}$$

for all $w \in \bigcup_{i=1}^{\ell} W_i$. We put $d = bc$. According to Lemma 5.3 we obtain that d and $e = cd$ are v_i-henselian over K for all i. Therefore, e^{-1} is also v_i-henselian by Lemma 5.2, for all i. By construction, we have that $\zeta_j \in \mathcal{O}_K[\eta_1, d, e^{-1}]$. We set $\eta_{2j} = d$ and $\eta_{2j+1} = e^{-1}$. Repeating this for all $j \in \{1, \dots, m\}$, we obtain elements $\eta_1, \dots, \eta_{2m+1}$ which are v_i-henselian over K for all i and such that $\zeta_1, \dots, \zeta_m \in \mathcal{O}_K[\eta_1 \dots, \eta_{2m+1}]$. □

6 Proof of Theorem 1.5

Assume that $F|K$, R, P and $\zeta_1, \dots, \zeta_m$ satisfy the assumptions of Theorem 1.5. By Theorem 3.4, there is a transcendence basis $x_1, \dots, x_\rho, y_1, \dots, y_\tau$ of $F|K$ such that assertions a), b), c) of the theorem are satisfied. We set $E := K(x_1, \dots, x_\rho, y_1, \dots, y_\tau)$. Applying Lemma 5.1 to the finite extension $F|E$, we find v_P-henselian elements $\eta_1, \dots, \eta_n \in \mathcal{O}_F$ with associated polynomials $h_i \in \mathcal{O}_E[X]$, such that $\zeta_1, \dots, \zeta_m \in \mathcal{O}_E[\eta_1, \dots, \eta_n]$. By Lemma 5.3, we can choose η_1 such that $F = E(\eta_1)$. We collect all coefficients of the polynomials $h_1, \dots, h_n$ and all elements from $\mathcal{O}_E$ which are necessary to generate $\zeta_1, \dots, \zeta_m$ from $\eta_1, \dots, \eta_n$. The finitely many elements so-obtained will be called $\zeta_1^*, \dots, \zeta_{m^*}^*$, say.

Now we set $s = \rho + \tau$ and apply Proposition 4.3 with $E = K(T)$ to these elements. We obtain a set $\{t_1, \dots, t_{s+k}\} := \{x'_1, \dots, x'_{\rho+k}, y_1, \dots, y_\tau\} \subset \mathcal{O}_F$ and an element $z \in R[t_1, \dots, t_{s+k}]$ of value $v_P z = 0$ such that $E = K(t_1, \dots, t_{s+k})$ and $\zeta_1^*, \dots, \zeta_{m^*}^* \in R[t_1, \dots, t_s, z^{-1}]$. It follows that $h_i(X) \in R[t_1, \dots, t_{s+k}, z^{-1}][X]$, $1 \le i \le n$, and that $\zeta_1, \dots, \zeta_m \in R[t_1, \dots, t_s, z^{-1}, \eta_1, \dots, \eta_n]$. Since $\{t_1, \dots, t_{s+k}\}$ contains a transcendence basis of $F|K$, we can choose the enumeration such that $t_1, \dots, t_s$ form a transcendence basis. This completes the proof of Theorem 1.5. □

7 Proof of Theorems 1.8 and 1.9

Throughout this section, we assume that the hypotheses of Theorem 1.8 hold. We set $v_j = v_{P_j}$. In view of Remark 1.7, we may assume that the valuations $v_1, \dots, v_\ell$ are pairwise incomparable. Since their value groups are assumed to be isomorphic over $v_P K$, ρ and consequently also τ are the same for all v_j. Moreover, passing to equivalent valuations if necessary, we may assume that $v_1 F = \ldots = v_\ell F$. Since this group is finitely generated over vK by Corollary 2.2 we can write $v_1 F = v_P K \oplus \bigoplus_{i=1}^{\rho} \mathbb{Z}\alpha_i$ with positive α_i.

If $\tau > 0$, we proceed as follows. We choose a separating transcendence basis $\xi_1, \dots, \xi_\tau$ of the separable extension $FP_1|KP$. By passing to equivalent places if necessary, we may assume that $\xi_1, \ldots, \xi_\tau$ is also the separating transcendence basis $\xi_1, \ldots, \xi_\tau$ of the separable extensions $FP_j|KP$, $0 < j \leq \ell$.

For this situation, we prove:

Proposition 7.1 *There are elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau \in \mathcal{O}$ such that for $1 \leq j \leq \ell$,*
i) $v_j x_i = \alpha_i$ for $1 \leq i \leq \rho$,
ii) $v_j y_i = 0$ and $y_i P_j = \xi_i$ for $1 \leq i \leq \tau$.
Moreover, the valuations $v_1, \ldots, v_\ell$ and the places $P_1, \ldots, P_\ell$ coincide on the rational function field $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$.

Proof: For each $i \in \{1, \ldots, \rho\}$, we employ the Approximation Theorem for pairwise incomparable valuations ([R], Théorème 1, p. 135) to find elements $x_i \in \mathcal{O}$ such that $v_j x_i = \alpha_i$ for $1 \leq j \leq \ell$. Since α_i does not depend on the valuation v_i, the compatibility condition of the theorem is satisfied.

For each $i \in \{1, \ldots, \tau\}$, we employ Theorem 11.14 of [EN] to find an element $y_i \in \mathcal{O}$ such that $y_i P_j = \xi_i$ for $1 \leq j \leq \ell$.

The last assertion is a direct consequence of Theorem 2.1. □

Remark 7.2 The following more general assertion can be proved for pairwise incomparable valuations with not necessarily equal value groups (using the [R], Théorème 3, p. 136):
There are elements $z_1, \ldots, z_s \in \mathcal{O}$ which simultaneously form a transcendence basis of $(F|K, P_i)$ as described in Theorem 3.4, for $1 \leq i \leq \ell$.

- **Proof of Theorem 1.8:**

Choose $T = \{x_1, \ldots, x_\rho, y_1, \ldots, y_\tau\}$ according to Proposition 7.1. Let P denote the common restriction of $P_1, \ldots, P_\ell$ to $K(T)$. By Theorem 3.4, for each j, (F, P_j) lies in the absolute inertia field of $(K(T), P)$.

We choose an element η for the extension $F|K(T)$ according to Lemma 5.5. We collect all coefficients of its associated polynomial h and call them $\zeta_1^*, \ldots, \zeta_k^*$, say. Now the rest of the proof can be taken over from the proof of Theorem 1.5. □

- **Proof of Theorem 1.9:**

We show in [K3] that there are places $P_1, \ldots, P_\ell$ of F of dimension d' centered on A at $\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell$ respectively, and having value group $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, ordered lexicograpically and of rational rank $\rho = \operatorname{trdeg} F|K - d'$. Since $d' = \tau$, these places are Abhyankar places, and they have equal value group. Now our assertion follows from Theorem 1.8. □

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