# On local uniformization in arbitrary characteristic

Franz-Viktor Kuhlmann

1. 7. 1997

**Abstract**

We prove that every place of an algebraic function field of arbitrary characteristic admits a local uniformization in a finite extension of the function field. We give a valuation theoretical description of these extensions; in certain cases, they can be found in the henselization of the function field. For places satisfying the Abhyankar equality and for discrete rational places, no extension is needed if the base field is perfect. Our proof is based solely on valuation theoretical theorems which are of fundamental importance in positive characteristic. It provides additional assertions (which in general do not hold without extensions).

## 1 Introduction

In [Z], Zariski proved his famous Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In this paper, we will prove a weak version of the Local Uniformization Theorem, for function fields of arbitrary characteristic:

**Theorem 1.1** *Let $F|K$ be a function field of arbitrary characteristic, and $P$ a place of $F|K$. Then there exist a finite extension $\mathcal{F}$ of $F$, a finite purely inseparable extension $\mathcal{K}$ of $K$ within $\mathcal{F}$ and a projective model of $\mathcal{F}|\mathcal{K}$ on which $P$ is centered in a smooth point.*

Throughout this paper, "function field" will always mean "algebraic function field". By a place of $F|K$ we mean a place which is trivial on $K$, in the sense that its restriction to $K$ is the identity. Talking of an extension $\mathcal{F}$ of $F$, we will always tacitly assume that it is equipped with an extension of $P$ (which is again denoted by $P$). We write $\mathcal{O}_{\mathcal{F}}$ for the valuation ring of $P$ on $\mathcal{F}$. For an element $a$, its $P$-residue will be denoted by $aP$.

Theorem 1.1 follows from the results of de Jong [dJ]. However, we will give an entirely valuation theoretical proof which will provide important additional information.

We prove Theorem 1.1 by showing the following theorem, which gives a description of the special form in which the Jacobian condition for smoothness can be satisfied. Given polynomials $f_1,\ldots,f_n$ in variables $X_1,\ldots,X_n$, we write $f=(f_1,\ldots,f_n)$ and denote by $J_f$ the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}$ of their partial derivatives.

**Theorem 1.2** *Let $F|K$ be a function field of transcendence degree $s$, and $P$ a place of $F|K$. Take any elements $\zeta_1,\ldots,\zeta_m$ in the valuation ring $\mathcal{O}_F$ of $P$ on $F$. Then there are*

- *a finite extension $\mathcal{F}$ of $F$ and a finite purely inseparable extension $\mathcal{K}$ of $K$ within $\mathcal{F}$,*

- *a transcendence basis* $T = \{t_1, \ldots, t_s\} \subset \mathcal{O}_\mathcal{F}$ *of* $\mathcal{F}|\mathcal{K}$,
- *elements* $\eta_1, \ldots, \eta_n \in \mathcal{O}_\mathcal{F}$ , *with* $\zeta_1, \ldots, \zeta_m$ *among them,*
- *polynomials* $f_i(X_1, \ldots, X_n) \in \mathcal{K}[t_1, \ldots, t_s, X_1, \ldots, X_n]$, $1 \leq i \leq n$,

*such that* $\mathcal{F} = \mathcal{K}(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n)$, *and*

1) *for* $i < j$, $X_j$ *does not occur in* $f_i$ ,
2) $f_i(\eta_1, \ldots, \eta_n) = 0$ *for* $1 \leq i \leq n$, *and*
3) *the element* $\det J_f(\eta_1, \ldots, \eta_n) \in \mathcal{O}_\mathcal{F}$ *has non-zero* $P$*-residue.*

Assertion 1) implies that $J_f$ is triangular. Assertion 3) says that

$$\det J_{fP}(\eta_1 P, \ldots, \eta_n P) \;=\; \det J_f(\eta_1, \ldots, \eta_n)P \;\neq\; 0 \;, \tag{1}$$

where $fP = (f_1 P, \ldots, f_n P)$ and $f_i P$ denotes the $P$-reduction of $f_i$ , i.e., the polynomial obtained from $f_i$ through replacing every coefficient by its $P$-residue. In the present case, this just means that in every coefficient, $t_j$ is replaced by $t_j P$, for $1 \leq j \leq n$. Hence if we view the polynomials $f_i$ as polynomials in the variables $t_1, \ldots, t_s, X_1, \ldots, X_n$ , then (1) means that at the point $(t_1 P, \ldots, t_s P, \eta_1 P, \ldots, \eta_n P)$ the diagonal elements of $J_f$ and thus also its determinant do not vanish. This assertion says that on the variety defined over $\mathcal{K}$ by the generic point $(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n)$ (and having function field $\mathcal{F}$), the place $P$ is centered at the smooth point $(t_1 P, \ldots, t_s P, \eta_1 P, \ldots, \eta_n P)$. In particular, we obtain that the local ring of this point is regular.

By our additional assertion on the $\zeta$'s, one obtains the following important information: if we have already a model $V$ of $F|K$ with generic point $(z_1, \ldots, z_k)$, where $z_1, \ldots, z_k \in \mathcal{O}_F$ , then we can choose our new model $\mathcal{V}$ of $\mathcal{F}|\mathcal{K}$ in such a way that the local ring of the center of $P$ on $\mathcal{V}$ contains the local ring of the center $(z_1 P, \ldots, z_k P)$ of $P$ on $V$. For this, we only have to let $z_1, \ldots, z_k$ appear among the $\zeta$'s.

For the applications of these theorems (e.g., to the model theory of fields, cf. [J–R], [K3]), it is important to have a control on the finite extension $\mathcal{F}|F$ and the residue field extension $\mathcal{F}P|FP$. Before describing this, we introduce some basic notation and some useful definitions.

Let $P$ be a place on an arbitrary field $F$. We will call $(F, P)$ a **valued field**, keeping in mind its associated valuation, which we denote by $v_P$ . Its value group is denoted by $v_P F$, and its residue field by $FP$. When we write $(L|F, P)$ then we mean an extension of valued fields, that is, $P$ is a place on $L$ and $F$ is endowed with its restriction (which we will also denote by $P$). This restriction need not be trivial. A **tame field** is a henselian valued field $(L, P)$ for which the ramification field of the normal extension $(L^{\mathrm{sep}}|L, P)$ is algebraically closed. Here, $L^{\mathrm{sep}}$ denotes the separable algebraic closure. It follows from the definition that for a tame field $(L, P)$, $L^{\mathrm{sep}}$ is algebraically closed, i.e., $L$ is perfect. For further basic properties of tame fields, see Section 2.3.

An extension $(L|F, P)$ is called **immediate** if the canonical embedding of $v_P F$ in $v_P L$ and the canonical embedding of $FP$ in $LP$ are surjective (we then write $v_P F = v_P L$ and $FP = LP$). We wish to show how to obtain "minimal" extensions of a valued field which are tame fields. Let $(F, P)$ be an arbitrary valued field of equal characteristic (that is, $\mathrm{char}\, F = \mathrm{char}\, FP$). The place $P$ extends in a unique way to the perfect hull of $F$. Further, there is always a maximal immediate algebraic extension of a valued field. If we take $(F^{\mathrm{t}}, P)$ to be a maximal immediate algebraic extension of the perfect hull of

$(F,P)$, then $(F^{\mathrm{t}},P)$ is a tame field (cf. Corollary 2.12). On the other hand, no extension of $(F,P)$ can be tame if it is properly included in $(F^{\mathrm{t}},P)$ (since by Lemma 2.11, a tame field of equal characteristic is perfect and has no proper immediate algebraic extensions). Therefore, we will call $(F^{\mathrm{t}},P)$ a **tame hull** of $(F,P)$. Unfortunately, the tame hulls of $(F,P)$ are not unique up to valuation preserving isomorphism over $F$. They are if $(F,P)$ satisfies Kaplansky's hypothesis A (cf. [KA]), but this hypothesis is too restrictive for our purposes. However, the failure of uniqueness does not matter for our use of tame hulls. Every tame hull contains the henselization $(F^h,P)$ of $(F,P)$, which is always unique up to valuation preserving isomorphism over $F$.

**Theorem 1.3** *Choose any tame hull $(F^{\mathrm{t}},P)$ of $(F,P)$. Then in addition to the assertion of Theorem 1.2, $(\mathcal{F},P)$ can be chosen to lie in $(F^{\mathrm{t}},P)$. If* $\operatorname{char} K=0$, *then $(\mathcal{F},P)$ can be chosen to lie in the henselization of $(F,P)$.*

Let us draw some conclusions from this theorem. Assume first that $\operatorname{char} K=0$. Then also the residue field $FP$ has characteristic 0. In this case, already the henselization $(F^h,P)$ of $(F,P)$ has no more proper immediate algebraic extensions (cf. Lemma 2.10). Hence it is the unique tame hull of $(F,P)$. On the other hand, $(F^h,P)$ is an immediate extension of $(F,P)$. This proves:

**Corollary 1.4** *If* $\operatorname{char} K=0$ *then we can achieve that $v_P\mathcal{F}=v_PF$ and $\mathcal{F}P=FP$.*

Now assume that $\operatorname{char} K=p>0$. Then the value group of the perfect hull of $(F,P)$ is the $p$-divisible hull of $v_PF$. Similarly, the residue field of the perfect hull is the perfect hull of $FP$. The same holds for every tame hull of $(F,P)$ since it is an immediate extension of the perfect hull. By the fundamental inequality (10) (see below), we know that $(v_P\mathcal{F}:v_PF)$ and $[\mathcal{F}P:FP]$ are finite for every finite extension $(\mathcal{F}|F,P)$. This shows:

**Corollary 1.5** *If* $\operatorname{char} K=p>0$ *then we can achieve that $(v_P\mathcal{F}:v_PF)$ is a finite power of $p$, and that $\mathcal{F}P|FP$ is a finite purely inseparable extension. In particular, we obtain $\mathcal{F}P=FP$ if $FP$ is perfect.*

For places $P$ of $F|K$, we have the following inequality (it is a special case of the more general inequality (5) which we will introduce later):

$$\operatorname{trdeg} F|K \;\geq\; \operatorname{trdeg} FP|K \,+\, \dim_{\mathbb{Q}} \mathbb{Q}\otimes v_PF\;. \tag{2}$$

Note that $\dim_{\mathbb{Q}} \mathbb{Q}\otimes v_PF$ is the **rational rank** of the value group $v_PF$, i.e., the maximal number of rationally independent elements in $v_PF$. A place $P$ of $F|K$ is said to satisfy the **Abhyankar equality** if equality holds in (2). If $L$ is any subfield of the algebraic closure $\tilde{F}$ of $F$, then $F.L$ will denote the least subfield of $\tilde{F}$ containing both $F$ and $L$.

**Theorem 1.6** *Assume that the place $P$ of the function field $F|K$ satisfies the Abhyankar equality. Then in addition to the assertion of Theorem 1.2, we can choose $\mathcal{F}$ in the henselization of $F.\mathcal{K}$ and such that it contains $\mathcal{F}P$. If in addition $FP|K$ is separable, then we can choose $\mathcal{K}=K$ and $\mathcal{F}$ in the henselization of $F$.*

Almost the same result can be achieved under a drastically different assumption. A valued field $(F, P)$ is called **discretely valued** and $P$ is called **discrete** if $v_P F \simeq \mathbb{Z}$. A place $P$ of a function field $F|K$ is called **zero-dimensional** if $FP|K$ is algebraic. It is called **rational** if $FP = K$.

**Theorem 1.7** *Assume that the place $P$ of the function field $F|K$ is discrete and zero-dimensional, and that $K$ is perfect. Then in addition to the assertion of Theorem 1.2, we can choose $\mathcal{F}$ in the henselization of $(F, P)$.*

Next, we will specify some cases where we can uniformize without extending the function field. For the notion of **rank**, see Section 2.2.

**Theorem 1.8** *In the following cases, we can obtain the assertion of Theorem 1.2 already for $\mathcal{F} = F$ and $\mathcal{K} = K$:*
*a) $(F|K, P)$ satisfies the Abhyankar equality, $FP|K$ is separable, and $m = 0$, i.e., no elements $\zeta_i$ are given in advance,*
*b) $(F|K, P)$ satisfies the Abhyankar equality, has rank 1, and $FP = K$,*
*c) $(F, P)$ is discrete, $K$ is perfect, and $FP = K$.*

In [K3], we show that places satisfying the Abhyankar equality (as well as discrete zero-dimensional places) lie Zariski-dense in the Zariski–Riemann manifold of all places of $F|K$. Consequently, the foregoing theorem shows:

**Corollary 1.9** *If $K$ is perfect, then the places of $F|K$ which can be uniformized without extension of $F$ lie Zariski-dense in the Zariski–Riemann manifold of $F|K$. Similarly, the uniformizable rational places lie Zariski-dense in the submanifold of all rational places of $F|K$ (for arbitrary $K$).*

Let us note that if $FP$ can be embedded in a trivially valued subfield of $K$, then we may replace $K$ by the image of $FP$. In this way, this more general case is subsumed under the case of $FP = K$. For example, if $FP|K$ is a rational function field, then $FP$ can always be embedded in such a way. On the other hand, if $FP|K$ is separable, then $FP$ can always be embedded in the henselization of $(F, P)$: choose a separating transcendence basis, embed the rational function field generated by it in a trivially valued subfield of $F$, and extend the embedding by Hensel's Lemma. As an algebraic extension of a trivially valued field is again trivially valued, the same will hold for the image of $FP$ in $F$.

In characteristic 0, our result is obviously weaker than Zariski's original result. On the other hand, our proof will yield an interesting additional assertion. As we will see later, it is impossible to obtain it without taking into the bargain a finite extension of the function field. Let us consider a valued function field $(F|K, P)$ with $P$ trivial on $K$. We set $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes v_P F$ and $\tau = \operatorname{trdeg} FP|K$. We take elements $x_1, \ldots, x_\rho \in F$ such that $v_P x_1, \ldots, v_P x_\rho$ are rationally independent elements in $v_P F$. Further, we take elements $y_1, \ldots, y_\tau \in F$ such that $y_1 P, \ldots, y_\tau P$ are algebraically independent over $k$. Then $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ are algebraically independent over $K$ (cf. Lemma 2.1).

**Theorem 1.10** *Assume the situation as given in Theorem 1.2. Take elements $x_1, \ldots, x_\rho$, $y_1, \ldots, y_\tau \in F$ as described above. Then $\rho + \tau \leq s$, and in addition to the assertion of Theorem 1.2, $t_1, \ldots, t_{\rho+\tau}$ can be chosen algebraic over $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$.*

We may express the assertion of this theorem by saying that we can

*uniformize $(F,P)$ over any arbitrarily chosen rational subfunction field on which the restriction of $P$ satisfies the Abhyankar equality.*

To describe a necessary condition for $\mathcal{F}$ to be equal to $F$, we need some further definitions. For an arbitrary valued field $(F,P)$ and a given extension of $P$ from $F$ to $F^{\rm sep}$, the **absolute inertia field** is defined to be the inertia field of the normal extension $(F^{\rm sep}|F,P)$. A valued function field $(F|K,P)$ will be called **inertially generated** if it admits a transcendence basis $T$ such that $(F,P)$ lies in the absolute inertia field of $(K(T),P)$ (for some extension of $P$ from $K(T)$ to $K(T)^{\rm sep}$). If it admits a transcendence basis $T$ such that $(F,P)$ lies in the henselization of $(K(T),P)$, then we call it **henselian generated**. We call $(F|K,P)$ **henselian rational** if there is a transcendence basis $T\subset F^h$ such that $(F^h,P)=(K(T)^h,P)$, which is the same as to say that some finite extension of $(F,P)$ within its henselization is henselian generated.

**Theorem 1.11** *Assume that $(\mathcal{F}|\mathcal{K},P)$ satisfies the assertions of Theorem 1.2. Then it is inertially generated. In particular, $\mathcal{F}|\mathcal{K}$ and $\mathcal{F}P|\mathcal{K}$ are separable. If in addition $P$ is rational, i.e., $\mathcal{F}P=\mathcal{K}$, then $(\mathcal{F}|\mathcal{K},P)$ is even henselian generated.*

Now we can show that Theorem 1.10 is indeed a strong additional assertion. Let us give an example. Denote by $\mathbb{F}_p$ the field with $p$ elements. Consider the following function field of transcendence degree 3 over $\mathbb{F}_p$ :

$$F \;=\; \mathbb{F}_p(x_1,x_2,y,z) \quad \text{with} \quad z^p-z \;=\; x_1-x_2y^p\;. \tag{3}$$

Since $x_1\in\mathbb{F}_p(x_2,y,z)$, $F$ is a rational function field. However, in [K1] (cf. also [K2], [K4]) we have shown that there is a rational place $P$ of $F|\mathbb{F}_p$ such that the valued function field $(F|\mathbb{F}_p(x_1,x_2),P)$ is not henselian rational, but

$$(F(x_2^{1/p})|\mathbb{F}_p(x_1,x_2^{1/p}),P)$$

is a rational function field. (It is an interesting fact that there is also a separable extension of degree $p$ such that the function field becomes henselian rational.) It follows that $F$ cannot lie in the henselization of $(\mathbb{F}_p(t_1,t_2,t_3),P)$ if $t_1,t_2\in F$ are algebraic over (and hence lie in) $\mathbb{F}_p(x_1,x_2)$.

On the other hand, in our example $P$ is such that $v_PF=\mathbb{Z}v_Px_2\times\mathbb{Z}v_Px_1$ (ordered lexicographically). Therefore, the elements $x_1,x_2$ satisfy the requirements of Theorem 1.10 (there are no $y$'s since $FP=\mathbb{F}_p$). In view of Theorem 1.11, this shows that a function field $\mathcal{F}$ satisfying the assertions of Theorem 1.2 and Theorem 1.10 (with this choice of $x_1,x_2$) must have degree at least $p$ over $F$.

We will deduce Theorems 1.2, 1.3, 1.6, 1.8, and 1.10 from the following theorems which we proved in [K1] (cf. also [K2]).

The first theorem is a generalization of the "Grauert–Remmert Stability Theorem". It deals with an arbitrary **valued function field** $(F|K,P)$, that is, $P$ need not be trivial on $K$. To state it, we introduce a fundamental notion. Every finite extension $(L|K,P)$ of valued fields satisfies the **fundamental inequality** (cf. [EN], [R], [Z-S] or [K2]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{4}$$

where $n = [L:K]$ is the degree of the extension, $P_1,\ldots,P_{\mathrm{g}}$ are the distinct extensions of $P$ from $K$ to $L$, $\mathrm{e}_i = (v_{P_i}L : v_P K)$ are the respective ramification indices and $\mathrm{f}_i = [LP_i : KP]$ are the respective inertia degrees. Note that $\mathrm{g}=1$ if $(K,P)$ is henselian.

A valued field $(K,P)$ is called **defectless** (or **stable**) if for every finite extension $L|K$, the fundamental inequality is an equality. If this is true for $L|K$ and if $\mathrm{g}=1$, i.e., there is only one extension of $P$ from $K$ to $L$, then we will also say that $(L|K,P)$ is a **defectless extension**. If $\mathrm{char}\, KP = 0$, then $(K,P)$ is defectless (see Lemma 2.10 below).

Now let $(L|K,P)$ be any extension of valued fields (we do not require that $P$ be trivial on $K$). Assume that $L|K$ has finite transcendence degree. Then (by Corollary 2.2 below):

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, LP|KP \;+\; \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P L / v_P K)\;. \tag{5}$$

The non-negative integer $\mathrm{trdeg}\, L|K - \mathrm{trdeg}\, LP|KP - \dim_{\mathbb{Q}} \mathbb{Q}\otimes(v_P L/v_P K)$ will be called the **transcendence defect** of $(L|K,P)$. Note that (2) is the special version of (5) for the case of a valued function field with trivially valued base field. We will say that $(L|K,P)$ is **without transcendence defect** if equality holds in (5). Now we are able to state the Stability Theorem:

**Theorem 1.12** *Let $(F|K,P)$ be a valued function field without transcendence defect. If $(K,P)$ is a defectless field, then also $(F,P)$ is a defectless field.*

The second theorem is a structure theorem for immediate function fields over tame fields.

**Theorem 1.13** *Let $(K,P)$ be a tame field and $(F|K,P)$ an immediate function field of transcendence degree 1. Then*

$$\textit{there is } x\in F^h \textit{ such that } \;(F^h,P) \;=\; (K(x)^h,P)\;, \tag{6}$$

*that is, $(F|K,P)$ is henselian rational.*

For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless (in fact, every $x\in F\setminus K$ will then do the job). In contrast to this, the case of positive residue characteristic requires a much deeper structure theory of immediate algebraic extensions of henselian fields, in order to find suitable elements $x$.

In Chapter 2, we introduce some further valuation theoretical tools, including a part of Kaplansky's theory of immediate extensions, which also will play a crucial role in our proofs. In Chapter 3, we give criteria for valued function fields to be inertially generated, and prove Theorem 1.11. In Chapter 4, we introduce the notion of **relative uniformization property** for arbitrary extensions of valued fields and show its transitivity. This allows us to build up our function fields by various sorts of algebraic and transcendental extensions which all can be shown separately (in Chapter 4) to satisfy the relative uniformization property. Finally, in Chapter 5 we put everything together to prove our main theorems.

I would like to thank Peter Roquette, Mark Spivakovsky, Bernard Teissier and Hagen Knaf for support and inspiring conversations, and the staff of the Fields Institute for their hospitality. Very special thanks to Hans Schoutens for many pleasant and encouraging discussions.

# 2 Valuation theoretical preliminaries

For basic facts from valuation theory, see [EN], [R], [W], [K2]. A valued field is **henselian** if it satisfies Hensel's Lemma; see [R], [W], [K2]. Originally, Hensel's Lemma was proved for complete discretely valued fields. But it also holds for complete valued fields of rank 1; i.e., such fields are henselian (cf. [W], Theorem 32.11, or [K2]). The **henselization** of a valued field $(K,P)$ (inside of $(K^{\mathrm{sep}},P)$) is the **absolute decomposition field** of $(K,P)$, i.e., the decomposition field of the extension $(K^{\mathrm{sep}}|K,P)$. The henselization is the minimal henselian extension of $(K,P)$. That means, if $(L,P)$ is a henselian extension field of $(K,P)$, then the henselization of $(K,P)$ can be found inside of $(L,P)$.

## 2.1 Valuation independence

For the easy proof of the following lemma, see [B], Chapter VI, §10.3, Theorem 1, or [K2].

**Lemma 2.1** *Let $(L|K,P)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i\in I$, $j\in J$, such that the values $v_P x_i$, $i\in I$, are rationally independent over $v_P K$, and the residues $y_j P$, $i\in J$, are algebraically independent over $KP$. Then the elements $x_i, y_j$, $i\in I$, $j\in J$, are algebraically independent over $K$.*

*Moreover, if* $f = \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i,y_j \mid i\in I, j\in J]$, *then*

$$v_P f \;=\; \min_k \, v_P\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k \left( v_P\, c_k + \sum_{i\in I} \mu_{k,i} v_P x_i \right) .$$

*That is, the value of the polynomial $f$ is equal to the least of the values of its monomials. In particular, this implies:*

$$\begin{aligned} v_P K(x_i,y_j \mid i\in I, j\in J) &= v_P K \oplus \bigoplus_{i\in I} \mathbb{Z} v_P x_i \\ K(x_i,y_j \mid i\in I, j\in J)P &= KP\,(y_j P \mid j\in J)\,. \end{aligned}$$

*It also implies that the valuation $v_P$ on $K(x_i,y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to $K$, the values $v_P x_i$ and the residues $y_j P$.*

**Corollary 2.2** *Let $(L|K,P)$ be an extension of valued fields. Then (5) holds. If in addition $L|K$ is a function field, and if equality holds in (5), i.e., $(L|K,P)$ is a valued function field without transcendence defect, then the extensions $v_P L|v_P K$ and $LP|KP$ are finitely generated. In particular, if $P$ is trivial on $K$, then $v_P L$ is a product of finitely many copies of $\mathbb{Z}$, and $LP$ is again a function field over $K$.*

Proof: Choose elements $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\in L$ such that the values $v_Px_1,\ldots,v_Px_\rho$ are rationally independent over $v_PK$ and the residues $y_1P,\ldots,y_\tau P$ are algebraically independent over $KP$. Then by the foregoing lemma, $\rho+\tau\leq \operatorname{trdeg} L|K$. This proves that $\operatorname{trdeg} LP|KP$ and the rational rank of $v_PL/v_PK$ are finite. Therefore, we may choose the elements $x_i,y_j$ such that $\tau=\operatorname{trdeg} LP|KP$ and $\rho=\dim_{\mathbb{Q}}\mathbb{Q}\otimes(v_PL/v_PK)$ to obtain inequality (5).

Assume that this is an equality. This means that for $L_0:=K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$, the extension $L|L_0$ is algebraic. Since $L|K$ is finitely generated, it follows that $L|L_0$ is finite. By the fundamental inequality, this yields that $v_PL|v_PL_0$ and $LP|L_0P$ are finite extensions. Since already $v_PL_0|v_PK$ and $L_0P|KP$ are finitely generated by the foregoing lemma, it follows that also $v_PL|v_PK$ and $LP|KP$ are finitely generated. □

**Lemma 2.3** *Let $(F|L,P)$ and $(L|K,P)$ be extensions of valued fields. Then $(F|K,P)$ is without transcendence defect if and only if $(F|L,P)$ and $(L|K,P)$ are. In particular, if $F|L$ is algebraic, $K\subset L'\subset F$ and $(L|K,P)$ is without transcendence defect, then the same holds for $(L'|K,P)$.*

Proof: This follows from the more general fact that the transcendence defect is additive. This in turn holds since $\operatorname{trdeg} F|K=\operatorname{trdeg} F|L+\operatorname{trdeg} L|K$, and the transcendence degree of the corresponding residue field extension as well as the rational rank of the corresponding value group extension behave in the same way. □

The algebraic analogue to the transcendental case discussed in Lemma 2.1 is the following lemma:

**Lemma 2.4** *Let $(L|K,P)$ be an extension of valued fields. Suppose that $\eta_1,\ldots,\eta_k\in L$ such that $v_P\eta_1,\ldots,v_P\eta_k\in v_PL$ belong to distinct cosets modulo $v_PK$. Further, assume that $\theta_1,\ldots,\theta_\ell\in\mathcal{O}_L$ such that $\theta_1P,\ldots,\theta_\ell P$ are $KP$-linearly independent. Then the elements $\eta_i\cdot\theta_j$, $1\leq i\leq k$, $1\leq j\leq\ell$, are $K$-linearly independent, and for every choice of elements $c_{ij}\in K$, we have that*

$$v_P\sum_{i,j}c_{ij}\eta_i\theta_j\;=\;\min_{i,j}\,v_P\,c_{ij}\eta_i\theta_j\;=\;\min_{i,j}\,(v_Pc_{ij}\,+\,v_P\eta_i)\,.$$

(In this case, the elements $\eta_i\theta_j$ are said to be **valuation independent**.)

## 2.2 The rank

The **rank of an ordered abelian group** $\Gamma$ is the order type of the chain of its proper convex subgroups. By a theorem of Hölder, $\Gamma$ is embeddable in the ordered additive group of the reals if and only if its rank is 1, that is, the only proper convex subgroup is $\{0\}$.

The **rank of** a valued field $(F,P)$ is defined to be the rank of its value group $v_PF$. If this is finite, say $n$, then $P$ is the composition of $n$ places of rank 1: $P=P_1P_2\ldots P_n$ where all $P_i$ have value groups of rank 1 (cf. [Z–S], [K2]).

If $\Gamma_1$ is a subgroup of $\Gamma$, then $\mathbb{Q}\otimes\Gamma_1$ lies in the convex hull of $\Gamma_1$ in $\mathbb{Q}\otimes\Gamma$. Hence if $\Gamma_1$ is a proper convex subgroup of $\Gamma$, then $\mathbb{Q}\otimes\Gamma_1$ is a proper convex subgroup of $\mathbb{Q}\otimes\Gamma$

and thus, $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma_1 < \dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma$. It follows that if $\{0\} = \Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \ldots \subsetneqq \Gamma_n = \Gamma$ is a chain of convex subgroups of $\Gamma$, then $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma \geq n$. In view of (2), this proves that the rank of a place $P$ of a function field $F|K$ cannot exceed $\operatorname{trdeg} F|K$ and thus is finite.

If $(F'|F, P)$ is an algebraic extension of valued fields, then $v_P F'/v_P F$ is a torsion group and $F'P|FP$ is algebraic (this is a consequence of Lemma 2.1). In particular, the rank of $v_P F'$ is equal to that of $v_P F$. Hence, passing to an algebraic extension does not change the rank of a valued field. Trivially, the rank will not increase by passing to subfields.

## 2.3 Tame and defectless fields

We will now discuss basic properties of tame fields. We set $p = \operatorname{char} KP$ if this is positive, and $p = 1$ if $\operatorname{char} KP = 0$. For a given valued field $(K, P)$, we choose an extension of $P$ to the algebraic closure $\tilde{K}$ of $K$. We denote by $K^r$ the **absolute ramification field** of $(K, P)$, that is, the ramification field of the normal extension $(K^{\mathrm{sep}}|K, P)$. By ramification theory, $K^{\mathrm{sep}}|K^r$ is a $p$-extension (cf. [EN], [K2]). Hence if $(K, P)$ is a henselian field of residue characteristic $\operatorname{char} KP = 0$, then this extension is trivial. Since then also $\operatorname{char} K = 0$, it follows that $K^r = K^{\mathrm{sep}} = \tilde{K}$. Therefore,

**Lemma 2.5** *Every henselian field of residue characteristic 0 is a tame field.*

Suppose that $K_1|K$ is a subextension of $K^r|K$. Then $K_1^r = K^r$ (cf. [EN], [K2]). This proves:

**Lemma 2.6** *Every algebraic extension of a tame field is again a tame field.*

If in addition $K_1|K$ is finite, then it satisfies the fundamental equality (cf. [EN], [K2]). Thus, if $K^r = \tilde{K}$, then $(K, v)$ is a defectless field. So we note:

**Lemma 2.7** *Every tame field is henselian defectless and perfect.*

The following lemma is proved in [K2] (and partially also in [EN]):

**Lemma 2.8** *A valued field $(K, P)$ is defectless if and only if its henselization $(K^h, P)$ is.*

A valued field $(K, v)$ is called **algebraically maximal** if it admits no proper immediate algebraic extension. Since the henselization is an immediate algebraic extension (cf. [R] or [K2]), every algebraically maximal field is henselian. For a finite extension $(L|K, P)$ of a henselian field $(K, P)$, the fundamental inequality reads as $[L : K] \geq \mathrm{e} \cdot \mathrm{f}$. If in addition $(K, P)$ is defectless, then we have equality. If in this situation the extension is immediate, that is, if $\mathrm{e} = \mathrm{f} = 1$, then $[L : K] = 1$, i.e., the extension is trivial. Therefore,

**Lemma 2.9** *Every henselian defectless field is algebraically maximal.*

The next lemma follows from Lemma 2.5, Lemma 2.7 and Lemma 2.9.

**Lemma 2.10** *Assume that $(K, P)$ is a valued field of residue characteristic 0, that is,* $\operatorname{char} KP = 0$. *Then $(K, P)$ is a defectless field. If in addition it is henselian, then it is algebraically maximal.*

We give a characterization for tame fields (for the proof, see [K2]):

**Lemma 2.11** *The valued field $(K,P)$ is tame if and only if it is algebraically maximal, $v_PK$ is p-divisible and $KP$ is perfect. If* $\mathrm{char}\, K = \mathrm{char}\, KP$ *then $(K,P)$ is tame if and only if it is algebraically maximal and perfect.*

Every henselian defectless field is algebraically maximal, but the converse does not hold in general. However, this lemma together with Lemma 2.7 shows that for perfect valued fields $(K,P)$ with $\mathrm{char}\, K = \mathrm{char}\, KP$, the properties "algebraically maximal" and "henselian defectless" are equivalent.

**Corollary 2.12** *Assume that* $\mathrm{char}\, K = \mathrm{char}\, KP$. *Then every maximal immediate algebraic extension of the perfect hull of $(K,P)$ is a tame field. If* $\mathrm{char}\, KP = 0$ *then already the henselization $(K^h, P)$ is a tame field.*

**Corollary 2.13** *If $P$ is trivial on $K$, then the perfect hull of $K$ (with trivial $P$) is the unique tame hull of $(K,P)$.*

Proof: By definition of the tame hull it is clear that every tame hull $K^{\mathrm{t}}$ contains the perfect hull of $K$. As a consequence of Lemma 2.1, an algebraic extension of a trivially valued field is again trivially valued. Further, an algebraic extension of a trivially valued field is immediate only if it is trivial. This shows that the perfect hull of a trivially valued field is its uniquely determined tame hull. $\square$

The following is a crucial lemma in the theory of tame fields. For its proof, see [K1] or [K2].

**Lemma 2.14** *Let $(L,P)$ be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $LP|KP$ is an algebraic extension, then $(K,P)$ is also a tame field and moreover, $v_PK$ is pure in $v_PL$ and $KP = LP$.*

We also note the following fact, which is easy to prove:

**Lemma 2.15** *Every finite extension field of a defectless field is again a defectless field.*

## 2.4 Kaplansky approximation

The material of this section was developed by Ostrowski and Kaplansky [KA] (cf. also [K2]). The proof of the first lemma is an easy excercise.

**Lemma 2.16** *The extension $(L|K,P)$ is immediate if and only if for every $z \in L$, the set $\{v_P(z-a) \mid a \in K\}$ has no maximal element.*

Note that if $(K,P)$ is an algebraically maximal field and $(L|K,P)$ is an immediate extension, then every $z \in L \setminus K$ must be transcendental over $K$.

**Lemma 2.17** *Let $(K(z)|K,P)$ be an immediate transcendental extension. Assume that $(K,P)$ is an algebraically maximal field or that $(K(z),P)$ lies in the completion of $(K,P)$. Then for every polunomial $f \in K[X]$, the value $v_Pf(a)$ is fixed for all $a \in K$ sufficiently close to $z$, i.e.,*

$$\begin{gathered} \forall f \in K[X]\ \exists \alpha \in v_PK\ \exists \beta \in \{v_P(z-b) \mid b \in K\}\ \forall a \in K : \\ v_P(z-a) \geq \beta \ \Rightarrow\ v_Pf(a) = \alpha\ . \end{gathered} \tag{7}$$

Kaplansky proves that if (7) does not hold, then there is a proper immediate algebraic extension of $(K,P)$. This is excluded if we require that $(K,P)$ be algebraically maximal. If on the other hand $(K(z),P)$ lies in the completion of $(K,P)$, then one can show that if $f$ does not satisfy (7), then $v_P f(z)=\infty$. But this means that $f(z)=0$, contradicting the assumption that $K(z)|K$ is transcendental.

For a polynomial $f$ in one variable over a field of arbitrary characteristic, we denote by $f^{\langle i\rangle}$ its $i$-th formal derivative (cf. [KA], [K2]). These polynomials are defined such that the following Taylor expansion holds (in arbitrary characteristic):

$$f(z) \;=\; f(a)+\sum_{i=1}^{\deg f} f^{\langle i\rangle}(a)(z-a)^i\;.$$

**Lemma 2.18** *Assume that (7) holds, and take any polynomial $f\in K[z]$. Then there are $a,b\in K$ such that for $\tilde{z}:=\frac{z-a}{b}\in K[z]$, we have that $v_P\tilde{z}=0$ and the values of the nonzero among the elements $f^{\langle i\rangle}(a)b^i\in K$ are all distinct. Then*

$$v_P f(z) \;=\; v_P\sum_{i=0}^{\deg f} f^{\langle i\rangle}(a)(z-a)^i \;=\; v_P\sum_{i=0}^{\deg f} f^{\langle i\rangle}(a)b^i\tilde{z}^i \;=\; \min_i v_P f^{\langle i\rangle}(a)b^i\;. \qquad (8)$$

*If finitely many polynomials in $K[z]$ are given, then $a,b$ can be chosen such that (8) holds simultaneously for all of them.*

Proof: Take finitely many polynomials $f_1,\ldots,f_n\in K[z]$. From Lemma 2.17 we know that for all $a\in K$ close enough to $z$, the values $v_P f_j^{\langle i\rangle}(a)$ of the nonzero among the polynomials $f_j^{\langle i\rangle}$, $i,j\in\mathbb{N}$, are fixed. Since by Lemma 2.16 the set $\{v_P(z-a)\mid a\in K\}$ has no maximal element, we can then take $a$ so close to $z$ that for every fixed $j$, the values of all nonzero elements $f_j^{\langle i\rangle}(a)(z-a)^i$, $i\in\mathbb{N}$, are distinct. Having picked such an element $a\in K$, we choose an element $b\in K$ such that $v_P b=v_P(z-a)$. Then (8) holds by the ultrametric triangle law. □

There is also a version of this lemma for immediate algebraic extensions. It is a corollary to the preceding proof.

**Lemma 2.19** *Let $(K(z)|K,P)$ be an immediate extension. Assume that for every polynomial $f$ of degree smaller than $[K(z):K]$, the value $vf(a)$ is fixed for all $a\in K$ sufficiently close to $z$. Then the assertion of Lemma 2.18 also holds in this case, provided that all of the given polynomials $f_1,\ldots,f_n$ have degree less than $[K(z):K]$.*

# 3 Inertially generated and henselian rational function fields

We let $F|K$ be a function field and $P$ a place of $F|K$. Recall that the extension of a place to every purely inseparable extension field is uniquely determined.

**Theorem 3.1** *Assume that $(F|K,P)$ satisfies the Abhyankar equality and that $FP|K$ is a separable extension. Then $(F|K,P)$ is inertially generated. If in addition $FP=K$ or $FP|K$ is a rational function field, then $(F|K,P)$ is henselian generated. In all cases, the generating transcendence basis can be chosen of the form $\{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$, where $v_PF=\mathbb{Z}v_Px_1\oplus\ldots\oplus\mathbb{Z}v_Px_\rho$ and $y_1P,\ldots,y_\tau P$ is a separating transcendence basis of $FP|K$.*

Proof: By Corollary 2.2, value group and residue field of $(F,P)$ are finitely generated. We choose $x_1,\ldots,x_\rho\in F$ such that $v_PF=\mathbb{Z}v_Px_1\oplus\ldots\oplus\mathbb{Z}v_Px_\rho$. Since $FP|K$ is a finitely generated separable extension, it is separably generated. Therefore, we can choose $y_1,\ldots,y_\tau\in F$ such that $FP|K(y_1P,\ldots,y_\tau P)$ is separable-algebraic ($\tau=\operatorname{trdeg} FP|K$). Now we can choose some $a\in FP$ such that $FP=K(y_1P,\ldots,y_\tau P,a)$. Since $a$ is separable-algebraic over $K(y_1P,\ldots,y_\tau P)$, by Hensel's Lemma there exists an element $\eta$ in the henselization of $(F,P)$ such that $\eta P=a$ and that the reduction of the minimal polynomial of $\eta$ over $F_0:=K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$ is the minimal polynomial of $a$ over $KP(y_1P,\ldots,y_\tau P)$. Then $\eta$ lies in the absolute inertia field of $F_0$. Now the field $F_0(\eta)$ has the same value group and residue field as $F$, and it is contained in the henselization of $F$. Hence, $(F^h|F_0(\eta)^h,P)$ is an immediate algebraic extension. On the other hand, we know by Theorem 1.12, Lemma 2.8 and Lemma 2.15 that $(F_0(\eta)^h,P)$ is a defectless field. Thus, the extension must be trivial. Therefore, $F$ is contained in $F_0(\eta)^h$, which in turn is a subfield of the absolute inertia field of $F_0$. This shows that $(F|K,P)$ is inertially generated.

If $FP=K$, then we do not need the elements $y_j$ and $a$. If $FP|K$ is a rational function field, then we can choose $y_1,\ldots,y_\tau\in F$ such that $FP=K(y_1P,\ldots,y_\tau P)$, and we do not need $a$. In both cases, we find that $F^h=F_0^h$, which yields that $(F|K,P)$ is henselian generated. □

Let $K^{\mathrm{t}}$ denote the perfect hull of $K$ (this notation is justified by Corollary 2.13).

**Lemma 3.2** *Assume that $F.K^{\mathrm{t}}|K^{\mathrm{t}}$ is inertially generated. Then there is a finite subextension $\mathcal{K}|K$ of $K^{\mathrm{t}}|K$ such that $F.\mathcal{K}|\mathcal{K}$ is inertially generated (by the same generators). The same assertion holds with "henselian generated" in the place of "inertially generated".*

Proof: We recall that the absolute Galois group of a field $L$ is defined to be $G(L):=\operatorname{Aut}(L^{\mathrm{sep}}|L)$. The absolute inertia group of a valued field $(L,v)$ (with respect to a fixed extension of the valuation to $L^{\mathrm{sep}}$) is defined to be

$$G^{\mathrm{i}}(L)\;:=\;\{\sigma\in G(L)\mid\forall x\in\mathcal{O}_{L^{\mathrm{sep}}}:v(\sigma x-x)>0\}\;,$$

and the absolute decomposition group is defined to be

$$G^{\mathrm{d}}(L)\;:=\;\{\sigma\in G(L)\mid v\sigma=v\text{ on }L^{\mathrm{sep}}\}\;.$$

Now let $L_0$ be a subfield of $L$ such that $L|L_0$ is algebraic and purely inseparable. Then $L^{\mathrm{sep}}=L.L_0^{\mathrm{sep}}$ is a purely inseparable algebraic extension of $L_0^{\mathrm{sep}}$, and restriction to $L_0^{\mathrm{sep}}$ induces an isomorphism of $G(L)$ onto $G(L_0)$. It is clear from the above definitions that the $L_0^{\mathrm{sep}}$-restriction of an automorphism in $G^{\mathrm{i}}(L)$ will lie in $G^{\mathrm{i}}(L_0)$, and similarly for the decomposition groups. For the converse, take $\sigma\in G(L)$ such that $\sigma|_{L_0^{\mathrm{sep}}}\in G^{\mathrm{i}}(L_0)$. Set $p:=\operatorname{char}L$. As the extension $L^{\mathrm{sep}}|L_0^{\mathrm{sep}}$ is purely inseparable, for every $x\in L^{\mathrm{sep}}$ there is some $n \in \mathbb{N}$ such that $x^{p^n} \in L_0^{\mathrm{sep}}$, and then, $x \in \mathcal{O}_{L^{\mathrm{sep}}}$ if and only if $x^{p^n} \in \mathcal{O}_{L_0^{\mathrm{sep}}}$. If the latter holds, we have that $v(\sigma x^{p^n} - x^{p^n}) > 0$ by assumption. This yields that $v(\sigma x - x) = \frac{1}{p^n} v(\sigma x - x)^{p^n} = \frac{1}{p^n} v(\sigma x^{p^n} - x^{p^n}) > 0$. We have thus proved that restriction to $L_0^{\mathrm{sep}}$ induces an isomorphism from $G^{\mathrm{i}}(L)$ onto $G^{\mathrm{i}}(L_0)$. The same holds for the decomposition groups by a similar argument.

By hypothesis, we can choose a transcendence basis $T$ of $F.K^{\mathrm{t}}|K^{\mathrm{t}}$ such that $F.K^{\mathrm{t}}$ lies in the absolute inertia field of $K^{\mathrm{t}}(T)$. Since $T$ is a finite set, there is a finite subextension $K_1|K$ of $K^{\mathrm{t}}|K$ such that $T \subset F.K_1$. As $F.K_1|K_1(T)$ is finite and $F.K^{\mathrm{t}}|K^{\mathrm{t}}(T)$ is separable-algebraic, there is a finite subextension $\mathcal{K}|K_1$ of $K^{\mathrm{t}}|K_1$ such that $F.\mathcal{K}|\mathcal{K}(T)$ is separable and generated by the same elements as $F.K^{\mathrm{t}}|K^{\mathrm{t}}(T)$.

Restriction to $\mathcal{K}(T)^{\mathrm{sep}}$ induces an isomorphism of $G(K^{\mathrm{t}}(T))$ onto $G(\mathcal{K}(T))$. Under this isomorphism, the subgroup $G(F.K^{\mathrm{t}})$ of $G(K^{\mathrm{t}}(T))$ is sent onto the subgroup $G(F.\mathcal{K})$ of $G(\mathcal{K}(T))$. As $F.K^{\mathrm{t}}$ lies in the absolute inertia field of $K^{\mathrm{t}}(T)$, $G(F.K^{\mathrm{t}})$ contains the absolute inertia group $G^{\mathrm{i}}(K^{\mathrm{t}}(T))$. By what we have shown in the beginning, the restriction map sends $G^{\mathrm{i}}(K^{\mathrm{t}}(T))$ onto $G^{\mathrm{i}}(\mathcal{K}(T))$. Hence, $G(F.\mathcal{K})$ contains $G^{\mathrm{i}}(\mathcal{K}(T))$, which shows that $F.\mathcal{K}$ lies in the absolute inertia field of $\mathcal{K}(T)$. We have thus proved that $F.\mathcal{K}|\mathcal{K}$ is inertially generated, by the same generators. A similar argument works for "henselian generated" in the place of "inertially generated". $\square$

- **Proof of Theorem 1.11**

Assume the assertion of Theorem 1.2. Denote by $(L, P)$ the absolute inertia field of $(\mathcal{K}(t_1, \ldots, t_s), P)$.

First, $\det J_{fP}(\eta_1 P, \ldots, \eta_n P) \neq 0$ implies that $\eta_1 P, \ldots, \eta_n P$ are separable algebraic over $KP(t_1 P, \ldots, t_s P)$ (cf. [L], Chapter X, §7, Proposition 8). Therefore, there are elements $\eta'_1, \ldots, \eta'_n$ in $L$ such that $\eta'_i P = \eta_i P$. Since $(L, P)$ is henselian, the multidimensional Hensel's Lemma (cf. [K2]) now shows that $\eta_1, \ldots, \eta_n$ lie in $L$. This proves that $(\mathcal{F}|\mathcal{K}, P)$ is inertially generated, provided that $\mathcal{K}(t_1, \ldots, t_s)|\mathcal{K}$ is purely transcendental. This is always the case in the situation of Theorem 1.2.

If we have in addition that $P$ is a rational place, then $\eta_1 P, \ldots, \eta_n P \in \mathcal{K}$. In this case, we can choose $\eta'_1, \ldots, \eta'_n$ already in the henselization of $(\mathcal{K}(t_1, \ldots, t_s), P)$, which implies that also $\eta_1, \ldots, \eta_n$ lie in this henselization. $\square$

# 4 The relative uniformization property

Let $(L|K, P)$ be an arbitrary extension of valued fields. We will say that $(L|K, P)$ has the **relative uniformization property** if for every choice of elements $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_L$ there are

- elements $t_1, \ldots, t_s \in \mathcal{O}_L$, algebraically independent over $K$ $(s \geq 0)$,
- elements $\eta_1, \ldots, \eta_n \in \mathcal{O}_L$, with $\zeta_1, \ldots, \zeta_m$ among them,
- polynomials $f_i(X_1, \ldots, X_n) \in \mathcal{O}_K[t_1, \ldots, t_s, X_1, \ldots, X_n]$, $1 \leq i \leq n$,

such that

1) for $i < j$, $X_j$ does not occur in $f_i$,

2) $f_i(\eta_1,\dots,\eta_n)=0$ for $1\leq i\leq n$, and
3) $\det J_f(\eta_1,\dots,\eta_n)P\neq 0$ for $f=(f_1,\dots,f_n)$.

The relative uniformization property is transitive:

**Lemma 4.1** *Assume that $(M|L,P)$ and $(L|K,P)$ are extensions of valued fields. If $(M|L,P)$ and $(L|K,P)$ have the relative uniformization property, then $(M|K,P)$ has the relative uniformization property.*

Proof: Take any elements $\zeta_1,\dots,\zeta_m\in\mathcal{O}_M$. As $(M|L,P)$ has the relative uniformization property, there are elements $t_1,\dots,t_{\tilde{s}}\in\mathcal{O}_M$, algebraically independent over $L$ $(\tilde{s}\geq 0)$, elements $\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}}\in\mathcal{O}_M$, with $\zeta_1,\dots,\zeta_m$ among them, and polynomials $\tilde{f}_i(X_1,\dots,X_{\tilde{n}})\in\mathcal{O}_L[t_1,\dots,t_{\tilde{s}},X_1,\dots,X_{\tilde{n}}]$, $1\leq i\leq\tilde{n}$, such that for $i<j$, $X_j$ does not occur in $\tilde{f}_i$, $\tilde{f}_i(\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}})=0$ for $1\leq i\leq\tilde{n}$, and $\det J_{(\tilde{f}_1,\dots,\tilde{f}_{\tilde{n}})}(\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}})P\neq 0$.

Now take $\zeta'_1,\dots,\zeta'_{m'}\in\mathcal{O}_L$ to be all the coefficients of the $\tilde{f}_i$ as polynomials in $t_1,\dots,t_{\tilde{s}},X_1,\dots,X_{\tilde{n}}$. Since $(L|K,P)$ has the relative uniformization property, there are elements $t_{\tilde{s}+1},\dots,t_s\in\mathcal{O}_L$, algebraically independent over $K$ $(s\geq\tilde{s})$, elements $\eta_1,\dots,\eta_{n'}\in\mathcal{O}_L$, with the elements $\zeta'_1,\dots,\zeta'_{m'}$ among them, and polynomials $f_i(X_1,\dots,X_{n'})\in\mathcal{O}_K[t_{\tilde{s}+1},\dots,t_s,X_1,\dots,X_{n'}]$, $1\leq i\leq n'$, such that for $i<j$, $X_j$ does not occur in $f_i$, $f_i(\eta_1,\dots,\eta_{n'})=0$ for $1\leq i\leq n'$, and $\det J_{(f_1,\dots,f_{n'})}(\eta_1,\dots,\eta_{n'})P\neq 0$.

We observe that the elements $t_1,\dots,t_s$ are algebraically independent over $K$. We set $n:=n'+\tilde{n}$. Trivially, the polynomials $f_i$, $1\leq i\leq n'$, can be viewed as polynomials in $\mathcal{O}_K[t_1,\dots,t_s,X_1,\dots,X_n]$. For $1\leq i\leq\tilde{n}$, we define the polynomial $f_{n'+i}\in\mathcal{O}_K[t_1,\dots,t_s,X_1,\dots,X_n]$ to be the polynomial $\tilde{f}_i$ with

i) $X_j$ replaced by $X_{n'+j}$ for $1\leq j\leq\tilde{n}$,
ii) every appearance of $\eta_j$ (which is then equal to some $\zeta'_\ell$) replaced by $X_j$, $1\leq j\leq n'$.

Accordingly, we set $\eta_{n'+i}:=\tilde{\eta}_i$. Then 1) and 2) hold. For $1\leq i\leq\tilde{n}$, we have that

$$\frac{\partial f_{n'+i}}{\partial X_{n'+i}}(\eta_1,\dots,\eta_n)=\frac{\partial\tilde{f}_i}{\partial X_i}(\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}})\;.$$

This shows that the diagonal elements of the lower triangular matrix $J_{(f_1,\dots,f_n)}(\eta_1,\dots,\eta_n)$ are precisely the diagonal elements of the lower triangular matrices $J_{(\tilde{f}_1,\dots,\tilde{f}_{\tilde{n}})}(\tilde{\eta}_1,\dots,\tilde{\eta}_{\tilde{n}})$ and $J_{(f_1,\dots,f_{n'})}(\eta_1,\dots,\eta_{n'})$. Consequently, $\det J_{(f_1,\dots,f_n)}(\eta_1,\dots,\eta_n)P\neq 0$. □

## 4.1 Rational function fields with Abhyankar equality

The following lemma was proved (but not explicitly stated) by Zariski in [Z] for subgroups of $\mathbb{R}$, using the algorithm of Perron. We leave it as an easy excercise to the reader to prove the general case by induction on the rank of the ordered abelian group. However, an instant proof of the lemma can also be found in [EL] (Theorem 2.2).

**Lemma 4.2** *Let $\Gamma$ be a finitely generated ordered abelian group. Take any non-negative elements $\alpha_1,\dots,\alpha_\ell\in\Gamma$. Then there exist positive elements $\gamma_1,\dots,\gamma_\rho\in\Gamma$ such that $\Gamma=\mathbb{Z}\gamma_1\oplus\ldots\oplus\mathbb{Z}\gamma_\rho$ and every $\alpha_i$ can be written as a sum $\sum_j n_{ij}\gamma_j$ with non-negative integers $n_{ij}$.*

The foregoing lemma and Lemma 2.1 are the main ingredients in the proof of the next proposition. A **cross-section** of a valued field $(F,P)$ is an embedding $\iota$ of the value group $v_PF$ in the multiplicative group $F^\times$ of $F$ such that $v_P(\iota\alpha)=\alpha$ for every $\alpha\in v_PF$.

We consider a function field $F|K$ and a place $P$ of $F|K$. We take elements $x_1,\ldots,x_\rho$ in $F$ such that $v_Px_1,\ldots,v_Px_\rho$ form a maximal set of rationally independent elements in $v_PF$. Further, we take elements $y_1,\ldots,y_\tau$ in $F$ such that $y_1P,\ldots,y_\tau P$ form a transcendence basis of $FP$ over $K$. In this situation, we have:

**Proposition 4.3** *The rational function field $(K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)|K,P)$ has the relative uniformization property. More precisely, the transcendence basis $T=\{t_1,\ldots,t_s\}$ can be chosen in the form $\{x'_1,\ldots,x'_\rho,y_1,\ldots,y_\tau\}$, where $x'_1,\ldots,x'_\rho\in\mathcal{O}_F$ generate the same multiplicative subgroup of $F^\times$ as $x_1,\ldots,x_\rho$.*

Proof: For the proof, we can assume that $F=K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$. By Lemma 2.1 we know that

$$v_PF\;=\;\mathbb{Z}v_Px_1\oplus\ldots\oplus\mathbb{Z}v_Px_\rho\;.$$

Thus,

$$v_PF\ni\mu_1v_Px_1+\ldots+\mu_\rho v_Px_\rho\;\mapsto\;x_1^{\mu_1}\cdot\ldots\cdot x_\rho^{\mu_\rho}\qquad\qquad(\mu_1,\ldots,\mu_\rho\in\mathbb{Z})$$

is a cross-section from $v_PF$ onto the multiplicative subgroup of $F^\times$ generated by $x_1,\ldots,x_\rho$. We denote this group by $\mathcal{X}$.

Now let $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_F$. Take $\zeta$ to be any of these elements and write $\zeta=f/g$ with polynomials $f,g\in K[x_1,\ldots,x_\rho,y_1,\ldots,y_\tau]$. Write

$$f\;=\;\sum_i c_i\,x_1^{\mu_{1,i}}\cdot\ldots\cdot x_\rho^{\mu_{\rho,i}}\cdot y_1^{\nu_{1,i}}\cdot\ldots\cdot y_\tau^{\nu_{\tau,i}}\quad\text{and}\quad g\;=\;\sum_i c'_i\,x_1^{\mu'_{1,i}}\cdot\ldots\cdot x_\rho^{\mu'_{\rho,i}}\cdot y_1^{\nu'_{1,i}}\cdot\ldots\cdot y_\tau^{\nu'_{\tau,i}}\;.$$

By Lemma 2.1, the value of $f$ is equal to the least of the values of its monomials, say of the first. Similarly for $g$. Note that the coefficients $c_i,c'_i\in K$ have value 0 since $P$ is trivial on $K$. So we can write

$$\zeta\;=\;x_1^{\mu_{1,1}-\mu'_{1,1}}\cdot\ldots\cdot x_\rho^{\mu_{\rho,1}-\mu'_{\rho,1}}\cdot\frac{\sum_i c_i\,x_1^{\mu_{1,i}-\mu_{1,1}}\cdot\ldots\cdot x_\rho^{\mu_{\rho,i}-\mu_{\rho,1}}\cdot y_1^{\nu_{1,i}}\cdot\ldots\cdot y_\tau^{\nu_{\tau,i}}}{\sum_i c'_i\,x_1^{\mu'_{1,i}-\mu'_{1,1}}\cdot\ldots\cdot x_\rho^{\mu'_{\rho,i}-\mu'_{\rho,1}}\cdot y_1^{\nu'_{1,i}}\cdot\ldots\cdot y_\tau^{\nu'_{\tau,i}}}\tag{9}$$

where the two sums both have value 0 and the summands appearing in them all have value $\geq 0$. Since $\zeta\in\mathcal{O}_F$, also the first factor has value $\geq 0$. The only obstruction is that some of the $x_i$'s may appear with negative exponents in some summands.

We collect all products of the form $x_1^{\mu_1}\cdot\ldots\cdot x_\rho^{\mu_\rho}\in\mathcal{X}$ which appear in (9). We do the same for all elements $\zeta_1,\ldots,\zeta_m$. In this way, we obtain finitely many elements $\xi_1,\ldots,\xi_\ell\in\mathcal{X}$, all of them having non-negative value. By adding some of the $x_i$ to this list if necessary, we can achieve that $\xi_1,\ldots,\xi_\ell$ generate $\mathcal{X}$. At this point, we apply Lemma 4.2 to the values $v_P\xi_1,\ldots,v_P\xi_\ell$. Pulling the result back through the cross-section, we find generators $x'_1,\ldots,x'_\rho$ of $\mathcal{X}$ of positive value such that every $\xi_i$ can be written as a (unique) product of the $x'_j$ *with non-negative exponents.*

As we can read off from (9), every $\zeta_j$ can be written as $\zeta'_j/\zeta''_j$, where $\zeta'_j,\zeta''_j$ lie in $K[x'_1,\ldots,x'_\rho,y_1,\ldots,y_\tau]$, with $v_P\zeta''_j=0$. We set $s=\rho+\tau$, $t_i:=x'_i$ for $1\leq i\leq\rho$ and $t_{i+\rho}:=y_i$ for $1\leq i\leq\tau$. Next, we set $n:=m$ and put $\eta_j:=\zeta_j$ and $f_j(X_1,\ldots,X_n):=\zeta''_jX_j-\zeta'_j$, and we are done. (Note that $K=\mathcal{O}_K$ since $P$ is trivial on $K$.) □

## 4.2 Immediate simple transcendental extensions

**Lemma 4.4** *Let $(K(z)|K,P)$ be an immediate transcendental extension. If (7) holds, then $(K(z)|K,P)$ has the relative uniformization property.*

Proof: Let $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_{K(z)}$ and write $\zeta_j = f_j(z)/g_j(z)$ with polynomials $f_j(z)$, $g_j(z) \in K[z]$. We apply Lemma 2.18 to these finitely many polynomials and choose $\tilde{z} = \frac{z-a}{b}$ according to this lemma. Then by (8), for every $j$ we can find $i_j, k_j$ such that $v_P f_j(z) = v_P f_j^{\langle i_j\rangle}(a)\, b^{i_j} = \min_i v_P f_j^{\langle i\rangle}(a)\, b^i$ and $v_P g_j(z) = v_P g_j^{\langle k_j\rangle}(a)\, b^{k_j} = \min_i v_P g_j^{\langle i\rangle}(a)\, b^i$. Thus, we can write

$$\zeta_j \;=\; \frac{f_j^{\langle i_j\rangle}(a)\, b^{i_j}}{g_j^{\langle k_j\rangle}(a)\, b^{k_j}} \cdot \frac{\tilde{f}_j(\tilde{z})}{\tilde{g}_j(\tilde{z})}$$

where $\tilde{f}_j, \tilde{g}_j$ are polynomials with coefficients in $\mathcal{O}_K$ and $v_P\tilde{f}_j(\tilde{z}) = 0 = v_P\tilde{g}_j(\tilde{z})$. Note that also the first fraction is an element of $\mathcal{O}_K$ since its value is equal to $v_P\zeta_j$ and $\zeta_j \in \mathcal{O}_{K(z)}$ by assumption. Now we set $t_1 := \tilde{z}$, $n := m$, $\eta_j := \zeta_j$ and

$$f_j(X_1,\ldots,X_n) \;:=\; \tilde{g}_j(t_1)\, X_j \;-\; \frac{f_j^{\langle i_j\rangle}(a)\, b^{i_j}}{g_j^{\langle k_j\rangle}(a)\, b^{k_j}} \cdot \tilde{f}_j(t_1)$$

for $1 \le j \le m$, and we are done. $\square$

## 4.3 Inert extensions

For the next lemma, we assume that $(K,P)$ is an arbitrary valued field and $(L|K,P)$ a finite extension within the absolute inertia field $(K^i,P)$ of $(K,P)$ (for some extension of $P$ from $K$ to $\tilde{K}$). Then $LP|KP$ is a finite separable extension. Hence, there is some $\vartheta \in LP$ such that $LP = KP(\vartheta)$. We can assume that $\vartheta \neq 0$.

**Lemma 4.5** *Let $g(X) \in KP[X]$ be the minimal polynomial of $\vartheta$ over $KP$. Then there exists some $\eta \in L$ with minimal polynomial $h$ over $K$ such that $L = K(\eta)$, $\eta P = \vartheta$ and $hP(X) = X^\ell g(X)$, where $\ell = [L:K] - [LP:KP]$. Consequently, $\frac{\partial h}{\partial X}(\eta)P \neq 0$.*

Proof: See [K2] or [EN] for the facts from ramification theory and about henselizations that we will freely use in this proof. If $v$ is any valuation of a field $K$ and $\sigma$ is an automorphism of $K$, then $v\sigma$ denotes the **conjugate valuation** given by $a \mapsto v(\sigma a)$.

We assume $P$ to be extended to $\tilde{L} = \tilde{K}$. Let $(K^h,P)$ and $(L^h,P)$ be the henselizations of $(K,P)$ and $(L,P)$ inside of $(K^i,P)$. Then the henselization $L^h$ of $L$ is the field $L.K^h$. This lies in $K^i$ since $K^h \subset K^i$ and by assumption, $L \subset K^i$. Since the henselization is an immediate extension, we have that $K^hP = KP$ and $L^hP = LP$. As $L.K^h|K^h$ is a finite subextension of $K^i|K^h$, $[L.K^h : K^h] = [L^hP : K^hP] = [LP : KP]$. Therefore, there are $k := [LP : KP]$ many automorphisms $\sigma_1,\ldots,\sigma_k \in \mathrm{Aut}(K^{\mathrm{sep}}|K^h)$ which induce distinct embeddings $\overline{\sigma}_1,\ldots,\overline{\sigma}_k$ of $LP$ in $(KP)^{\mathrm{sep}}$ over $KP$ (note that $LP|KP$ is separable since $L \subset K^i$). Their restrictions to $L$ are $k$ distinct embeddings of $L$ over $K$ in $K^{\mathrm{sep}}$.

Now we choose further automorphisms $\sigma_{k+1},\ldots,\sigma_{k+\ell} \in \mathrm{Aut}(K^{\mathrm{sep}}|K)$ such that the restrictions of $\sigma_1,\ldots,\sigma_{k+\ell}$ to $L$ are precisely all distinct embeddings of $L$ over $K$ in $K^{\mathrm{sep}}$. As a subextension of $K^i|K$, also $L|K$ is separable; so $k+\ell = [L:K]$.

We claim that $v_P\sigma_i \neq v_P$ on $L$ for $k < i \leq k+\ell$. Assume that $v_P\sigma = v_P$ on $L$ for some $\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}|K)$. As all extensions of a valuation to an algebraic extension are conjugate, there is some $\sigma' \in \mathrm{Aut}(K^{\mathrm{sep}}|L)$ such that $v_P\sigma\sigma' = v_P$ on $K^{\mathrm{sep}}$. By definition, $\sigma\sigma'$ is thus an element of the decomposition group of the extension $(K^{\mathrm{sep}}|K, v_P)$. On the other hand, this decomposition group is the fixed group (in $K^{\mathrm{sep}}$) of the henselization of $(K,P)$ in $(K^{\mathrm{sep}},P)$. Therefore, $\sigma\sigma'$ is trivial on $K^h$ and must consequently coincide on $L.K^h$ with $\sigma_j$ for some $j \in \{1,\ldots,k\}$. It follows that $\sigma$ coincides with $\sigma_j$ on $L$. This proves our claim.

Observe that the place associated to the valuation $v_P\sigma$ is given by $a \mapsto (\sigma a)P$. Since $v_P\sigma_i \neq v_P$ for $k < i \leq k+\ell$, we can employ the Chinese Remainder Theorem to find an element $\eta \in L$ such that $\eta P = \vartheta$ and $(\sigma_i\eta)P = 0$ for $k < i \leq k+\ell$. Then we have that for $1 \leq i \leq k$, $(\sigma_i\eta)P = \overline{\sigma}_i(\eta P)$ are precisely all conjugates of $\vartheta$. In particular, we find that $\sigma_i\eta \neq \eta$ for $1 < i \leq k+\ell = [L:K]$, which proves that $L = K(\eta)$. For the minimal polynomial $h = \prod_{i=1}^{k+\ell}(X-\sigma_i\eta)$ of $\eta$ over $K$, we obtain that $hP = \prod_{i=1}^{k}(X-\overline{\sigma}_i\vartheta)\cdot\prod_{j=1}^{\ell}(X-(\sigma_{k+j}\eta)P) = g(X)\cdot X^\ell$. This also shows that $\vartheta$ is a simple root of $hP$ and hence, $\frac{\partial h}{\partial X}(\eta)P = \frac{\partial hP}{\partial X}(\vartheta) \neq 0$. □

In view of our definition of the relative uniformization property, the assertion of the foregoing lemma is too weak. Although we find a good generator for every given finite extension within the absolute inertia field, we are not able to add arbitrary elements from the valuation ring to the list of $\eta$'s. The obstruction is that possibly no lifting of $g$ to a polynomial in $\mathcal{O}_K[X]$ of the same degree admits a root in $L$. We do not know of a concrete example, but we believe that for this reason, a finite extension within the absolute inertia field will in general not have the relative uniformization property. We can do much better if $(K,P)$ is already henselian, as follows from the next lemma (with $K = K^h$):

**Lemma 4.6** *Let $(K,P)$ be an arbitrary valued field and $(L|K^h,P)$ an extension within the absolute inertia field of $(K,P)$. Then $(L|K^h,P)$ has the relative uniformization property.*

Proof: The given elements $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_L$ generate a finite extension of $K^h$, so we can assume that $L|K^h$ itself is finite. Then $LP|K^hP$ is a finite separable extension. Hence it is generated by one element $\vartheta$. We take $g$ to be its minimal polynomial over $K^hP$. We choose a monic polynomial $h \in \mathcal{O}_{K^h}[X]$ such that $hP = g$. By Hensel's Lemma, there is a root $\theta$ of $h$ in the henselian field $(L,P)$. Since

$$[L:K^h] \;=\; [LP:K^hP] \;=\; [K^hP(\vartheta):K^hP] \;=\; [K^h(\theta):K^h]\,,$$

we find that $\theta$ generates $L$ over $K^h$. Since $g = hP$ has $\vartheta = \theta P$ as a simple root, we can put $\eta_1 := \theta$ and $f_1 := h(X_1)$.

By Lemma 2.4, the value of every polynomial in $K^h[\theta]$ of degree less than $[K^h(\theta):K^h]$ is equal to the least of the values of its monomials. Since every element of $L$ can be written as such a polynomial, we find that $\mathcal{O}_L = \mathcal{O}_{K^h}[\theta]$. Given , we put $n := m+1$ and $\eta_{j+1} := \zeta_j$. Then we write out $\zeta_j$ as a polynomial $g_j(\theta)$ with coefficients in $\mathcal{O}_{K^h}$ and set $f_{j+1} := X_{j+1} - g_j(X_1)$. □

## 4.4 Extensions within the henselization

**Lemma 4.7** *Every separable-algebraic extension of a valued field within its completion has the relative uniformization property.*

Proof: Take any separable-algebraic extension $(L|K,P)$ such that $(L,P)$ lies in the completion of $(K,P)$. Further, take $\zeta_1,\ldots,\zeta_m \in \mathcal{O}_L$. Let $\zeta$ be any of these elements. We extend $v_P$ to the algebraic closure of the completion. Since $L|K$ is separable, we can write the minimal polynomial of $\zeta$ in the form $(X-\zeta)(X-\sigma_1\zeta)\ldots(X-\sigma_k\zeta)$ where $\sigma_j$ are automorphisms in $\mathrm{Aut}(K^{\mathrm{sep}}|K)$ and all $\sigma_j\zeta$ are distinct from $\zeta$. Since $\zeta$ lies in the completion of $(K,P)$, it follows that there is some $a\in K$ such that $v_P(\zeta-a)>v_P(\sigma_j\zeta-a)$ for all $j$. We can choose $a$ such that $v_P(\zeta-a)>0$. On the other hand, $v_P(\zeta-a)\in v_PK$ and $K(\zeta)P=KP$, so we can choose $b\in\mathcal{O}_K$ such that $v_P(\frac{\zeta-a}{b}-1)>0$. Thus, $v_P\frac{b}{\zeta-a}=0$ and $v_P\sigma_j\frac{b}{\zeta-a}=v_P\frac{b}{\sigma_j\zeta-a}>0$. Therefore, the reduction $hP$ of the minimal polynomial $h$ of $\frac{b}{\zeta-a}$ over $K$ is the polynomial $X^{k+1}-X^k$ which has $1=\frac{b}{\zeta-a}P$ as a simple root. We set $n:=3m$. If $\zeta$ was $\zeta_j$ then we set $\eta_{3j-2}:=\frac{b}{\zeta-a}$, $f_{3j-2}:=h(X_{3j-2})$, $\eta_{3j-1}:=\frac{\zeta-a}{b}$, $f_{3j-1}:=X_{3j-2}X_{3j-1}-1$, $\eta_{3j}:=\zeta=b\frac{\zeta-a}{b}+a\in\mathcal{O}_K[\frac{\zeta-a}{b}]$, $f_{3j}:=X_{3j}-bX_{3j-1}-a$, and we are done. $\square$

The henselization of a valued field $(K,P)$ is always a separable-algebraic extension. If $(K,P)$ has rank 1, then moreover, the henselization lies in the completion of $(K,P)$ (since in this case the completion is henselian). Therefore, the foregoing lemma yields:

**Corollary 4.8** *Assume that $(K,P)$ has rank 1. If $(L|K,P)$ is a subextension of the henselization of $(K,P)$, then it has the relative uniformization property.*

We leave it as an exercise to the reader to use the foregoing corollary together with Lemma 4.1 and an adaptation of the proof of Lemma 4.6 in order to show:

**Corollary 4.9** *Assume that $(K,P)$ has rank 1, and let $(L|K,P)$ be a finite extension within the absolute inertia field of $(K,P)$. If there is a subextension $L_0|K$ of $L|K$ such that $L_0P=LP$ and $[L_0:K]=[LP:KP]$, then $(L|K,P)$ has the relative uniformization property.*

To treat the case of a rank higher than 1, we use a well known lemma about composite places (cf. [R] or [K2]).

**Lemma 4.10** *Suppose that the place $P$ of $K$ is composite: $P=Q\overline{Q}$. Then $(K,P)$ is henselian if and only if $(K,Q)$ and $(KQ,\overline{Q})$ are. If $(KQ,\overline{Q})$ is henselian, then the henselization of $K$ with respect to $P$ is equal to the henselization of $K$ with respect to $Q$ (as fields).*

We need a further technical lemma which relates the present situation to Lemma 2.19.

**Lemma 4.11** *Suppose again that $P=Q\overline{Q}$. Take $\theta$ in the henselization of $(K,P)$ such that $\vartheta=\theta Q\neq 0$ lies in the completion of $(KQ,\overline{Q})$. Then for every $f\in K[X]$ of degree smaller than $[KQ(\vartheta):KQ]$, the value is fixed for all $a\in K$ sufficiently close to $\theta$.*

Proof: Multiplication of $f$ by elements from $K$ has no influence on our assertion. Hence, by multiplication with a suitable element from $K$ we can assume that all coefficients of $f$ lie in the valuation ring of $Q$ on $K$ and at least one of them is a unit in this ring. Then the reduction $fQ$ is a nonzero polynomial in $KQ[X]$, again of degree smaller than $[KQ(\vartheta):KQ]$. Let $b_1,\ldots,b_\ell$ be its roots in the algebraic closure of $KQ$. Because of our assumption on the degree, they are different from $\vartheta$. Since $\vartheta$ lies in the completion of $(KQ,\overline{Q})$, there is some $\alpha\in\{v_{\overline{Q}}(\vartheta-c)\mid c\in KQ\}=v_{\overline{Q}}(KQ)$ such that $\alpha>v_{\overline{Q}}(\vartheta-b_i)$ for all $i$. Then for all $c\in KQ$ with $v_{\overline{Q}}(\vartheta-c)\geq\alpha$, we have that

$$v_{\overline{Q}}(c-b_i) \;=\; \min\{v_{\overline{Q}}(c-\vartheta),v_{\overline{Q}}(\vartheta-b_i)\} \;=\; v_{\overline{Q}}(\vartheta-b_i)\;,$$

implying that

$$v_{\overline{Q}}\,fQ(c) \;=\; \sum_{i=1}^{\ell} v_{\overline{Q}}(c-b_i) \;=\; \sum_{i=1}^{\ell} v_{\overline{Q}}(\vartheta-b_i) \;=\; v_{\overline{Q}}\,fQ(\vartheta)$$

is fixed.

Observe that the assertion of the lemma is trivial if $\vartheta\in KQ$. Assume that $\vartheta\notin KQ$. Then for every $a\in K$, $v_Q(\theta-a)\leq 0$. This implies that all $a\in K$ sufficiently close to $\theta$ (with respect to $v_P$) satisfy that $v_Qa=v_Q\theta=0$, that is, $\infty\neq aQ\in KQ$. Observe that $a$ is the closer to $\theta$, the closer $aQ$ is to $\vartheta=\theta Q$. On the other hand, the value $v_Pf(a)$ is determined by the value $v_{\overline{Q}}\,fQ(aQ)$. Since this is fixed for $aQ$ close enough to $\vartheta$, it follows that $v_Pf(a)$ is fixed for $a$ close enough to $\theta$. $\square$

Now we are able to prove the relative uniformization property for the henselization (in the place of finite subextensions). That means, we take finite extensions of our function fields into the bargain.

**Lemma 4.12** *For every valued field $(K,P)$ of finite rank, $(K^h|K,P)$ has the relative uniformization property.*

Proof: Since $K(\frac{b}{\zeta-a})=K(\zeta)$, the proof of Lemma 4.7 shows that in the case of rank 1, we can build up the henselization by succesively adjoining elements $\vartheta$ with residue 1 whose minimal polynomials reduce to polynomials of the form $X^{k+1}-X^k$ (if the field is uncountable, we use transfinite induction). Now if $P=Q\overline{Q}$ with $Q$ of rank 1 and $(KQ,\overline{Q})$ henselian, then the henselization of $K$ with respect to $P$ is equal to that with respect to $Q$. This gives us the idea of henselizing $K$ in steps of rank 1, starting from the "bottommost" place. This is done as follows.

Suppose that $P=Q\overline{Q}$, where $\overline{Q}=\overline{Q}_1\overline{Q}_2$ with $\overline{Q}_1$ a place of rank 1 and $(LQ\overline{Q}_1,\overline{Q}_2)$ henselian for some algebraic extension $L$ of $K$; here, $Q$ and $\overline{Q}_2$ are allowed to be trivial. Now we apply the above procedure to the field $(LQ,\overline{Q})$ whose henselization lies in its completion. Everytime we adjoin an element $\vartheta$ of the above form, we lift it to an extension of $L$ in the following way. Assume we have already obtained a subextension $L'|L$ of $K^h|L$ such that $L'Q|LQ$ is the extension that we have built up so far by adjoining the elements $\vartheta$. Now take a new $\vartheta$ of the above form, and let $g$ be its minimal polynomial over $L'Q$. Choose a monic polynomial $f$ with coefficients in the valuation ring of $Q$ on $L'$ such that $fQ=g$. Since $\vartheta$ is separable over $L'Q$ and $(K^h,Q)$ is henselian by Lemma 4.10, Hensel's

Lemma shows that there is a root $\theta \in K^h$ of $f$ such that $\theta Q = \vartheta$. Since $fP = fQ\overline{Q} = g\overline{Q}$ is of the form $X^{k+1} - X^k$ and $\theta P = \theta Q\overline{Q} = \vartheta\overline{Q} = 1$, we find that $\theta$ is a suitable element to be adjoined to $L'$. In this way, we build up a field $L^Q$ such that $L^Q Q$ is the henselization of $LQ$ with respect to $\overline{Q}$.

Now we write $P = P_1 P_2 \ldots P_r$ where all $P_i$ are places of rank 1. We apply the above procedure successively to construct a chain of intermediate fields $K = K_{r+1} \subset K_r \subset K_{r-1} \subset \ldots \subset K_1 = K^h$ where for $1 \leq i < r$ the field $(K_{i+1}P_1 \ldots P_i\,,\, P_{i+1} \ldots P_r)$ is already henselian and $K_i P_1 \ldots P_i$ is the henselization of $K_{i+1}P_1 \ldots P_i$ with respect to $P_{i+1} \ldots P_r$.

By the transitivity (Lemma 4.1), we have to show that every extension $(K_i|K_{i+1}, P)$, $1 \leq i \leq r$, has the relative uniformization property. Given elements $\zeta_1, \ldots, \zeta_m \in K_i$ generate a finite subextension of $K_i|K_{i+1}$. Hence, there are finitely many elements $\theta_1, \ldots, \theta_\ell$ obtained by the above procedure such that

- for every $i \in \{1, \ldots, \ell\}$, every $\theta$ appearing in the coefficients of the minimal polynomial of $\theta_i$ is already among the elements $\theta_1, \ldots, \theta_{i-1}$,
- $\zeta_1, \ldots, \zeta_m \in K[\theta_1, \ldots, \theta_\ell]$.

So again by transitivity, our proof is reduced to showing that an extension of the form $(L(\theta)|L, P)$ as described above has the relative uniformization property.

Take $\zeta_1, \ldots, \zeta_m \in \mathcal{O}_{L(\theta)}$. Set $d = [L(\theta) : L]$; by our construction, $d$ is equal to $[LQ(\vartheta) : LQ]$ for $\vartheta = \theta P$. We write every $\zeta_j$ as a polynomial $g_j$ in $\theta$ of degree less than $d$ and with coefficients in $L$. By Lemma 4.11 and Lemma 2.19, we find $a, b \in L$ such that for all $j$,

$$\min_i v_P\, g_j^{\langle i \rangle}(a) b^i \;=\; v_P\, g_j(\theta) \;\geq\; 0\,.$$

So if we write $g_j$ as a polynomial $\tilde{g}_j$ in $\tilde{\theta} := \frac{\theta - a}{b}$, then $\tilde{g}_j$ has coefficients in $\mathcal{O}_L$.

At the same time, we can choose $a$ so close to $\theta$ that (as in the proof of Lemma 4.7 for $\zeta$), $v_P(\theta - a) > v_P(\sigma_j\theta - a)$ for all conjugates $\sigma_j\theta \neq \theta$. (In fact, in the present case this is already true as soon as $v_P(\theta - a) > 0$ since then, $v_P a = 0$ and $v_P(\sigma_j\theta - a) = 0$, as $v_P \sigma_j a > 0$.) Also, we can choose $b$ such that $v_P(\frac{\theta - a}{b} - 1) > 0$. Thus, the minimal polynomial $h$ of $\frac{b}{\theta - a}$ over $L$ has the same reduction property as in the proof of Lemma 4.7.

Now we set $n := 2 + m$. We define $\eta_1 := \frac{b}{\theta - a}$, $f_1 := h(X_1)$. Further, $\eta_2 := \frac{\theta - a}{b}$, $f_2 := X_1 X_2 - 1$. Finally, we set $\eta_{j+2} := \zeta_j$ and $f_{j+2} := X_{j+2} - \tilde{g}_j(X_2)$ for $1 \leq j \leq m$, and we are done. $\square$

**Question:** Does the relative uniformization property for finite extensions in the henselization also hold in the case of arbitrary finite rank?

**Remark:** In the equal characteristic 0 case, our proof yields Zariski's original result (without extension of the function field $F$) if $P$ on $F$ is of rank 1 and satisfies the Abhyankar equality and $FP = K$ (Theorem 1.8). If the above question has a positive answer, then the restriction on the rank can be dropped.

# 5 Proof of the main theorems

• **Proof of Theorem 1.2, Theorem 1.3 and Theorem 1.10**

Assume that $F|K$ is a function field of transcendence degree $s$ and that $P$ is a place of $F|K$. As we have shown in Section 2.2, $(F,P)$ has finite rank.

As in the introduction, we choose $x_1,\ldots,x_\rho \in F$ such that $v_Px_1,\ldots,v_Px_\rho$ are rationally independent elements in $v_PF$, and $y_1,\ldots,y_\tau \in F$ such that $y_1P,\ldots,y_\tau P$ are algebraically independent over $KP$. By Lemma 2.1, $\rho+\tau \leq s$. We choose an arbitrary tame hull $(F^{\mathrm{t}},P)$ of $(F,P)$. Then we take $L$ to be the relative algebraic closure of $K' := K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$ in $F^{\mathrm{t}}$. By the following (slightly more general) proposition, the extension $(F^{\mathrm{t}}|L,P)$ has the relative uniformization property.

**Proposition 5.1** *Let $(F|K',P)$ be an extension of finite transcendence degree such that $v_PF/v_PK'$ is a torsion group and $FP|K'P$ is algebraic. Assume that $(F,P)$ is of finite rank. Choose any tame hull $(F^{\mathrm{t}},P)$ of $(F,P)$ and take $L$ to be the relative algebraic closure of $K'$ in $F^{\mathrm{t}}$. Then $(F^{\mathrm{t}}|L,P)$ has the relative uniformization property.*

Proof: Choose a transcendence basis $\{t_1,\ldots,t_k\}$ of $F|K'$. Let $L_i$ be the relative algebraic closure of $K'(t_1,\ldots,t_i)$ in $F^{\mathrm{t}}$, and $L_0=L$. Note that $L_k=F^{\mathrm{t}}$. By Lemma 2.14, the $(L_i,P)$ are tame fields and the extensions $(L_{i+1}|L_i,P)$ are immediate of transcendence degree 1 $(0\leq i<k)$. By the transitivity (Lemma 4.1), this shows that it suffices to prove the proposition in the case of transcendence degree 1.

In this case, the given elements $\zeta_1,\ldots,\zeta_m$ generate an immediate function field $(F_0|L,P)$ of transcendence degree 1. By Theorem 1.13, it is henselian rational. That is, there is some $z\in F_0^h$ such that $F_0\subset L(z)^h$. By Lemma 2.11, the tame field $(L,P)$ is algebraically maximal. Hence by Lemma 2.17, condition (7) holds for $L$ in the place of $K$. So we can apply Lemma 4.4 to obtain that $(L(z)|L,P)$ has the relative uniformization property.

Since $(F,P)$ is of finite rank by assumption, so is $(L(z),P)$. Hence by Lemma 4.12 and the transitivity, $(L(z)^h|L,P)$ has the relative uniformization property. Since $\zeta_1,\ldots,\zeta_m$ were arbitrarily chosen and $L(z)^h$ is contained in the henselian field $F^{\mathrm{t}}$, this proves that $(F^{\mathrm{t}}|L,P)$ has the relative uniformization property. □

Note that $(L|K,P)$ satisfies the Abhyankar equality. As we have already noted in the foregoing proof, $(L,P)$ is a tame field. Hence by Corollary 2.13, it contains the tame hull $K^{\mathrm{t}}$ of the trivially valued field $K$, which is just the perfect hull of $K$.

**Proposition 5.2** *The extension $(L|K^{\mathrm{t}},P)$ has the relative uniformization property.*

Proof: Take arbitrary elements $\zeta_1,\ldots,\zeta_m\in\mathcal{O}_L$. These elements generate a function field $(F_1|K^{\mathrm{t}},P)$ inside of $L$, which by Lemma 2.3 satisfies the Abhyankar equality. As $K^{\mathrm{t}}$ is perfect, the extension $F_1P|K^{\mathrm{t}}$ is separable. Hence by Theorem 3.1, there are $\tilde\rho\leq\rho$, $\tilde\tau\leq\tau$ and elements $\tilde x_1,\ldots,\tilde x_{\tilde\rho},\tilde y_1,\ldots,\tilde y_{\tilde\tau}\in F_1$ with $v_P\tilde x_1,\ldots,v_P\tilde x_{\tilde\rho}$ rationally independent and $\tilde y_1P,\ldots,\tilde y_{\tilde\tau}P$ algebraically independent over $K^{\mathrm{t}}$ such that $F_1^h$ is a finite extension of the henselization of $(K^{\mathrm{t}}(\tilde x_1,\ldots,\tilde x_{\tilde\rho},\tilde y_1,\ldots,\tilde y_{\tilde\tau}),P)$ within its absolute inertia field. By our remarks at the end of Section 2.2, the subfield $(F_1^h,P)$ of $(F^{\mathrm{t}},P)$ has finite rank.

From Proposition 4.3, Lemma 4.12, Lemma 4.6 and the transitivity (Lemma 4.1) we infer that $(F_1^h|K^{\mathrm{t}},P)$ has the relative uniformization property. Since $F_1^h$ lies in the henselian field $L$ and $\zeta_1, \ldots, \zeta_m$ were arbitrary, this proves that $(L|K^{\mathrm{t}}, P)$ has the relative uniformization property. □

By the transitivity, we have now proved that $(F^{\mathrm{t}}|K^{\mathrm{t}}, P)$ has the relative uniformization property. This means that for given elements $\zeta_1, \ldots, \zeta_m$, there is a finite extension $\mathcal{F}'|F$ within $F^{\mathrm{t}}$ such that the assertion of Theorem 1.2 holds for the function field $\mathcal{F}'.K^{\mathrm{t}}|K^{\mathrm{t}}$ in the place of $\mathcal{F}|\mathcal{K}$. After adding new elements $\zeta_i \in \mathcal{O}_F$ if necessary, we can assume from the start that among $\zeta_1, \ldots, \zeta_m$ there are generators of the function field $F|K$. We collect all elements $c_1, \ldots, c_\ell \in K^{\mathrm{t}}$ appearing in the polynomials from $K^{\mathrm{t}}[t_1, \ldots, t_s]$ which are the coefficients of the polynomials $f_i(X_1, \ldots, X_n) \in K^{\mathrm{t}}[t_1, \ldots, t_s][X_1, \ldots, X_n]$. Then $\mathcal{K} := K(c_1, \ldots, c_\ell)$ is a finite purely inseparable extension of $K$, and we have that $F \subset \mathcal{F} := \mathcal{K}(t_1, \ldots, t_s, \eta_1, \ldots, \eta_n) \subset F^{\mathrm{t}}$ and that the assertion of Theorem 1.2 holds for $\mathcal{F}|\mathcal{K}$.

As we started our proof with an arbitrarily chosen tame hull $F^{\mathrm{t}}$ of $F$, we have also proved Theorem 1.3. For the deduction of Theorem 1.10 we note that by the proof of the transitivity of the relative uniformization property, the above transcendence basis $t_1, \ldots, t_s$ can be taken to be the union of a transcendence basis $t_1, \ldots, t_{\rho+\tau}$ of $L|K$ and a transcendence basis $t_{\rho+\tau+1}, \ldots, t_s$ of $F^{\mathrm{t}}|L$. As $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ is also a transcendence basis of $L|K$, the elements $t_1, \ldots, t_{\rho+\tau}$ will then be algebraic over $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$. This proves Theorem 1.10.

- **Proof of Theorem 1.6**

In addition to the assumptions for the foregoing proof, let $P$ satisfy the Abhyankar equality. As before, let $K^{\mathrm{t}}$ denote the perfect hull of $K$. Then also $(F.K^{\mathrm{t}}|K^{\mathrm{t}}, P)$ satisfies the Abhyankar equality, and $(F.K^{\mathrm{t}})P|K^{\mathrm{t}}$ is separable. Hence by Theorem 3.1, $(F.K^{\mathrm{t}}|K^{\mathrm{t}}, P)$ is inertially generated, and we choose the generating transcendence basis $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ according to that theorem. By Lemma 3.2 there is a finite subextension $\mathcal{K}|K$ of $K^{\mathrm{t}}|K$ such that $F_1 := F.\mathcal{K}$ is contained in the absolute inertia field of $F_0 := \mathcal{K}(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$. Note that consequently, $F_1P|\mathcal{K}$ is separable.

By Proposition 4.3, the extension $(F_0|\mathcal{K}, P)$ has the relative uniformization property. By Lemma 4.6 and Lemma 4.12, the same holds for the extensions $(F_1^h|F_0^h, P)$ and $(F_0^h|F_0, P)$. By the transitivity it follows that $(F_1^h|\mathcal{K}, P)$ has the relative uniformization property. Hence, to satisfy the assertion of Theorem 1.2, $\mathcal{F}$ need only be chosen in the henselization $F_1^h$ of $F.\mathcal{K}$.

As $F_1P|\mathcal{K}$ is separable, Hensel's Lemma shows that $F_1P \subset F_1^h$. Hence to get $\mathcal{F}P$ included in $\mathcal{F}$, we just have to add generators of the extension $F_1P|\mathcal{K}$ to our list of $\zeta$'s.

If in addition to all previous assumptions $FP|K$ is separable, then by Theorem 3.1, already $(F|K, P)$ is inertially generated. In this case, our proof works with $\mathcal{K} = K$.

- **Modifications for the proof of Theorem 1.8 a), b)**

a): If $(F|K, P)$ satisfies the Abhyankar equality, $FP|K$ is separable, and $m = 0$, then we modify the foregoing proof as follows. We take $\mathcal{K} = K$ and define $F_0$ as before. As $F$ is a finite extension of $F_0$ within the absolute inertia field of $(F_0, P)$, we can apply Lemma 4.5 to find that $(F|F_0, P)$ satisfies a relativized version of the assertion of Theorem 1.2, with $m = 0$. As $(F_0|K, P)$ has the relative uniformization property, one can now apply the proof of the transitivity to show that $(F|K,P)$ satisfies the assertion of Theorem 1.2, with $m = 0$, $\mathcal{F} = F$, $\mathcal{K} = K$.

b): If $(F|K,P)$ satisfies the Abhyankar equality, has rank 1, and $FP = K$, then by Lemma 3.1, $(F|K,P)$ is henselian generated. Hence in this case, $(F|F_0,P)$ is a finite extension within the henselization of $(F_0,P)$. Since the rank of $(F,P)$ is 1, we can apply Corollary 4.8 to find that $(F|F_0,P)$ has the relative uniformization property. By the transitivity, the same holds for $(F|K,P)$. That is, the assertion of Theorem 1.2 holds with $\mathcal{F} = F$ and $\mathcal{K} = K$.

- **Proof of Theorem 1.7 and Theorem 1.8 c)**

Assume that $(F|K,P)$ is discrete and zero-dimensional, and that $K$ is perfect. The two last conditions tell us that $FP|K$ is separable-algebraic. Hence by Hensel's Lemma, we can embed $FP$ over $K$ in $F^h$. We identify $FP$ with its image in $F$ and set $F' := F.(FP) \subset F^h$. As a trivially valued field, $(K,P)$ is henselian, and as a separable-algebraic extension, $(FP,P)$ lies in the absolute inertia field of $(K,P)$. Hence by Lemma 4.6, the extension $(FP|K,P)$ has the relative uniformization property.

We may write $v_PF = v_PF' = \mathbb{Z}$. We choose a set of generators of $F'$ over $FP$ such that each of these generators has $v_P$-value 1. Since $K$ is assumed to be perfect, the same holds for $FP$ and we can choose a separating transcendence basis $t_1,\ldots,t_s$ for $F'|FP$ from these generators (cf. [L], Ch. X, §6, Prop. 5).

For every $i$, the extension $(FP(t_1,\ldots,t_{i+1})|FP(t_1,\ldots,t_i),P)$ is immediate since the value group of both fields is $v_PF$ and the residue field of both fields is $FP$. This shows that $(FP(t_1,\ldots,t_{i+1}),P)$ lies in the completion of $(FP(t_1,\ldots,t_i),P)$. Hence by Lemma 2.17, (7) holds (for $t_{i+1}$ in the place of $z$ and $FP(t_1,\ldots,t_i)$ in the place of $K$). From Lemma 4.4 we can now infer that the extension $(FP(t_1,\ldots,t_{i+1})|FP(t_1,\ldots,t_i),P)$ has the relative uniformization property. By transitivity, the same holds for $(FP(t_1,\ldots,t_s)|K,P)$.

By construction, $(F'|FP(t_1,\ldots,t_s),P)$ is an immediate separable-algebraic extension. Since it is discretely valued, it lies in the completion of $(FP(t_1,\ldots,t_s),P)$. Hence by Lemma 4.7, the extension $(F'|FP(t_1,\ldots,t_s),P)$ has the relative uniformization property. By transitivity, now the same holds for $(F'|K,P)$. As $F' \subset F^h$, this proves Theorem 1.7.

If $FP = K$, then $F' = F$ and we obtain the assertion of part c) of Theorem 1.8.

## References


[B] Bourbaki, N.: *Commutative algebra*, Paris (1972)

[dJ] de Jong, A. J.: *Smoothness, semi-stability and alterations*, preprint

[EL] Elliott, G. A.: *On totally ordered groups, and* $K_0$, in: Ring Theory Waterloo 1978, eds. D. Handelman and J. Lawrence, Lecture Notes Math. **734**, 1–49

[EN] Endler, O.: *Valuation theory*, Berlin (1972)

[J–R] Jarden, M. – Roquette, P.: *The Nullstellensatz over* $\wp$*–adically closed fields*, J. Math. Soc. Japan **32** (1980), 425–460

[KA] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. J. **9** (1942), 303–321

[K1] Kuhlmann, F.-V.: *Henselian function fields and tame fields*, preprint (extended version of Ph.D. thesis), Heidelberg (1990)

[K2] Kuhlmann, F.-V.: *Valuation theory of fields, abelian groups and modules*, preprint, Heidelberg (1996), to appear in the "Algebra, Logic and Applications" series (Gordon and Breach), eds. A. Macintyre and R. Göbel

[K3] Kuhlmann, F.–V.: *On places of algebraic function fields in arbitrary characteristic*, to appear in the Fields Institute Preprint Series, Toronto (1997)

[K4] Kuhlmann, F.–V.: *Elementary properties of power series fields over finite fields*, prepublication in: Structures Algébriques Ordonnées, Séminaire Paris VII (1997); also to appear in the Fields Institute Preprint Series, Toronto (1997)

[L] Lang, S.: *Algebra*, New York (1965)

[R] Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal (1964)

[S] Spivakovsky, M.: *Resolution of singularities I: local uniformization*, manuscript, Toronto (1996)

[WA3] Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam (1989)

[Z] Zariski, O.: *Local uniformization on algebraic varieties*, Ann. Math. **41** (1940), 852–896

[Z–S] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin (1960)

The Fields Institute, 222 College Street, Toronto, Ontario M5T 3J1, Canada
email: fkuhlman@fields.utoronto.ca

# Appendix: Introduction to ramification theory

(Conference hand-out for the Valuation Theory Workshop, The Fields Institute, April 1997)

*Franz-Viktor Kuhlmann*

Throughout, we assume that $L|K$ is an algebraic extension, not necessarily finite, and that $v$ is a valuation on $K$. If $w$ is a valuation on $L$ which extends $v$, then there is a natural embedding of the value group $vK$ of $v$ in the value group $wL$ of $w$. Similarly, there is a natural embedding of the residue field $Kv$ of $v$ in the residue field $Lw$ of $w$. If both embeddings are onto (which we just express by writing $vK = wL$ and $Kv = Lw$), then the extension $(L,w)|(K,v)$ is called **immediate**. WARNING: It may happen that $vK \simeq wL$ or $Kv \simeq Lw$ although the corresponding embedding is not onto and therefore, the extension is not immediate. For example, every finite extension of the $p$-adics $(\mathbb{Q}_p, v_p)$ will again have a value group isomorphic to $\mathbb{Z}$, but $v_p p$ may not be anymore the least positive element in this value group.

We choose an arbitrary extension of $v$ to the algebraic closure $K^a$ of $K$. Then for every $\sigma \in \mathrm{Aut}\,(K^a|K)$, the map

$$\tilde{v}\sigma = \tilde{v} \circ \sigma : \; L \ni a \;\mapsto\; \tilde{v}(\sigma a) \in \tilde{v}K^a$$

is a valuation of $L$ which extends $v$.

**Theorem 5.3** *The set of all extensions of $v$ from $K$ to $L$ is*

$$\{\tilde{v}\sigma \mid \sigma \text{ an embedding of } L \text{ in } K^a \text{ over } K\}\;.$$

(We say that "all extensions of $v$ from $K$ to $L$ are **conjugate**".)

**Corollary 5.4** *If $L|K$ is finite, then the number $\mathrm{g}$ of distinct extensions of $v$ from $K$ to $L$ is smaller or equal to the extension degree $[L:K]$. More precisely, $\mathrm{g}$ is smaller or equal to the degree of the maximal separable subextension of $L|K$. In particular, if $L|K$ is purely inseparable, then $v$ has a unique extension from $K$ to $L$.*

Let us denote the extensions by $v_1,\ldots,v_{\mathrm{g}}$. We set $n := [L:K]$.

**Theorem 5.5** *For every $i \in \{1,\ldots,\mathrm{g}\}$, the* **ramification index** $\mathrm{e}_i = (v_iL : vK)$ *and the* **inertia degree** $\mathrm{f}_i = [Lv_i : Kv]$ *are finite, and we have the* **fundamental inequality**

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i\mathrm{f}_i \tag{10}$$

From now on, let us assume that $L|K$ is normal. Hence, the set of all extensions of $v$ from $K$ to $L$ is given by $\{\tilde{v}\sigma \mid \sigma \in \mathrm{Aut}\,(L|K)\}$. For simplicity, we denote the restriction of $\tilde{v}$ to $L$ again by $v$. The valuation ring of $v$ on $L$ will be denoted by $\mathcal{O}_L$. We define distinguished subgroups of $G := \mathrm{Aut}\,(L|K)$. The subgroup

$$G^d := G^d(L|K,v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L :\; v\sigma x \geq 0\} \tag{11}$$

is called the **decomposition group of** $(L|K,v)$. It is easy to show that $\sigma$ sends $\mathcal{O}_L$ into itself if and only if the valuations $v$ and $v\sigma$ agree on $L$. Thus,

$$G^d = \{\sigma \in G \mid v\sigma = v \text{ on } L\}\;. \tag{12}$$

Further, the **inertia group** is defined to be

$$G^i := G^i(L|K,v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > 0\}\ , \tag{13}$$

and the **ramification group** is

$$G^r := G^r(L|K,v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > vx\}\ . \tag{14}$$

Let S denote the maximal separable extension of $K$ in $L$ (we call it the **separable closure of $K$ in $L$**). The fixed fields of $G^d$, $G^i$ and $G^r$ in S are called **decomposition field**, **inertia field** and **ramification field of** $(L|K,v)$. For simplicity, let us abbreviate them by Z, T and V. (These letters refer to the german words "Zerlegungskörper", "Trägheitskörper" and "Verzweigungskörper".)

WARNING: In contrast to the classical definition, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the maximal separable subextension.* The reason for this will become clear later.

By our definition, V, T and Z are separable-algebraic extensions of $K$, and S|V, S|T, S|Z are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \ \text{ and thus, } \ \mathrm{S} \supset \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z} \supset K\ .$$

(For the inclusion $G^i \subset G^d$ note that $vx \geq 0$ and $v(\sigma x - x) > 0$ implies that $v\sigma x \geq 0$.)

**Theorem 5.6** *$G^i$ and $G^r$ are normal subgroups of $G^d$, and $G^r$ is a normal subgroup of $G^i$. Therefore,* T|Z, V|Z *and* V|T *are Galois extensions.*

First, we consider the decomposition field Z. In some sense, it represents all extensions of $v$ from $K$ to $L$.

**Theorem 5.7** *a) $v\sigma = v\tau$ on $L$ if and only if $\sigma\tau^{-1}$ is trivial on Z.*
*b) $v\sigma = v$ on Z if and only if $\sigma$ is trivial on Z.*
*c) The extension of $v$ from Z to $L$ is unique.*
*d) The extension* (Z$|K,v$) *is immediate.*

WARNING: It is in general not true that $v\sigma \neq v\tau$ holds already on Z if it holds on $L$.

a) and b) are easy consequences of the definition of $G^d$. c) follows from b) by Theorem 5.3. For d), there is a simple proof using a trick which is mentioned in a paper of J. Ax.

Now we turn to the inertia field T. Let $\mathcal{M}_L$ denote the valuation ideal of $v$ on $L$ (the unique maximal ideal of $\mathcal{O}_L$). For every $\sigma \in G^d(L|K,v)$ we have that $\sigma\mathcal{O}_L = \mathcal{O}_L$, and it follows that $\sigma\mathcal{M}_L = \mathcal{M}_L$. Hence, every such $\sigma$ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_L/\mathcal{M}_L = Lv$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. Since $\sigma$ fixes $K$, it follows that $\overline{\sigma}$ fixes $Kv$.

**Lemma 5.8** *Since $L|K$ is normal, the same is true for $Lv|Kv$. Moreover, the map*

$$G^d(L|K,v) \ni \sigma \ \mapsto\ \overline{\sigma} \in \mathrm{Aut}\,(Lv|Kv) \tag{15}$$

*is a group homomorphism.*

**Theorem 5.9** *a) The homomorphism (15) is onto and induces an isomorphism*

$$\mathrm{Aut}\,(\mathrm{T}|\mathrm{Z}) \;=\; G^d/G^i \;\simeq\; \mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v)\;.$$

*b) For every finite subextension* $F|\mathrm{Z}$ *of* $\mathrm{T}|\mathrm{Z}$*,*

$$[F:\mathrm{Z}] \;=\; [Fv:\mathrm{Z}v]\;.$$

*c) We have that* $v\mathrm{T} = v\mathrm{Z} = vK$*. Further,* $\mathrm{T}v$ *is the separable closure of* $Kv$ *in* $Lv$*, and therefore,*

$$\mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v) \;=\; \mathrm{Aut}\,(Lv|Kv)\;.$$

If $F|\mathrm{Z}$ is normal, then b) is an easy consequence of a). From this, the general assertion of b) follows by passing from $F$ to the normal hull of the extension $F|\mathrm{Z}$ and then using the multiplicativity of the extension degree. c) follows from b) by use of the fundamental inequality.

We set $p := \mathrm{char}\,Kv$ if this is positive, and $p := 1$ if $\mathrm{char}\,Kv = 0$. Given any extension $\Delta \subset \Delta'$ of abelian groups, the **$p'$-divisible closure of $\Delta$ in $\Delta'$** is defined to be the subgroup $\{\alpha \in \Delta' \mid \exists n:\ (p,n) = 1 \wedge n\alpha \in \Delta\}$ of all elements in $\Delta'$ whose order modulo $\Delta$ is prime to $p$.

**Theorem 5.10** *a) There is an isomorphism*

$$\mathrm{Aut}\,(\mathrm{V}|\mathrm{T}) \;=\; G^i/G^r \;\simeq\; \mathrm{Hom}\,(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times)\;\;,$$

*where the character group on the right hand side is the full character group of the abelian group* $v\mathrm{V}/v\mathrm{T}$*. Since this group is abelian,* $\mathrm{V}|\mathrm{T}$ *is an abelian Galois extension.*
*b) For every finite subextension* $F|\mathrm{T}$ *of* $\mathrm{V}|\mathrm{T}$*,*

$$[F:\mathrm{T}] \;=\; (vF:v\mathrm{T})\;.$$

*c)* $\mathrm{V}v = \mathrm{T}v$*, and* $v\mathrm{V}$ *is the $p'$-divisible closure of* $vK$ *in* $vL$*.*

b) follows from a) since for a finite extension $F|\mathrm{T}$, the group $vF/v\mathrm{T}$ is finite and therefore, there exists an isomorphism of $vF/v\mathrm{T}$ onto its full character group. The equality $\mathrm{V}v = \mathrm{T}v$ follows from b) by the fundamental inequality. The second assertion of c) follows from the fact that $(\mathrm{T}v)^\times$ and thus also $\mathrm{Hom}\,(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times)$ has no elements of order divisible by $p$, together with the next theorem.

**Theorem 5.11** *The ramification group* $G^r$ *is a $p$-group and therefore,* $\mathrm{S}|\mathrm{V}$ *is a $p$-extension. Further,* $vL/v\mathrm{V}$ *is a $p$-group, and the residue field extension* $Lv|\mathrm{V}v$ *is purely inseparable. If* $\mathrm{char}\,Kv = 0$*, then* $\mathrm{V} = \mathrm{S} = L$*.*

**Remark:** Every $p$-extension is a tower of Galois extensions of degree $p$. In characteristic $p$, Galois extensions of degree $p$ are **Artin–Schreier–extensions**, that is, they are generated by elements with minimal polynomials of the form $X^p - X - a$.

From the last theorem it follows that there is a canonical isomorphism

$$\mathrm{Hom}\,(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times) \;\simeq\; \mathrm{Hom}\,(vL/vK\,,\,(Lv)^\times)\;. \tag{16}$$

We summarize our main results in the following table:

| **Galois group** | **field** | | **value group** | **residue field** |
|---|---|---|---|---|
| | $L$ | | $vL$ | $Lv$ |
| | purely inseparable | | division by $p$ | purely inseparable |
| 1 | S | maximal separable subextension | | |
| | Galois $p$-extension | | | |
| $G^r(L\|K,v)$ | V | ramification field | $(vL\|vK)^{p'}$ | $(Lv\|Kv)^{\mathrm{sep}}$ |
| Char | abelian Galois $p'$-extension | | division prime to $p$ | $\Vert$ |
| $G^i(L\|K,v)$ | T | inertia field | $vK$ | $(Lv\|Kv)^{\mathrm{sep}}$ |
| Aut $(Lv\|Kv)$ | Galois unramified | | $\Vert$ | Galois |
| $G^d(L\|K,v)$ | Z | decomposition field | $vK$ | $Kv$ |
| | immediate | | $\Vert$ | $\Vert$ |
| Aut $(L\|K)$ | $K$ | | $vK$ | $Kv$ |

where $(vL|vK)^{p'}$ denotes the $p'$-divisible closure of $vK$ in $vL$, $(Lv|Kv)^{\mathrm{sep}}$ denotes the separable closure of $Kv$ in $Lv$, and Char denotes the character group (16).

If we take for $L|K$ the normal extension $K^a|K$, then we speak of **absolute ramification theory**. The fixed fields $K^d$, $K^i$ and $K^r$ of $G^d(K^a|K,v)$, $G^i(K^a|K,v)$ and $G^r(K^a|K,v)$ in the separable-algebraic closure of $K$ are called **absolute decomposition field**, **absolute inertia field** and **absolute ramification field of** $(K,v)$ (with respect to the given extension of $v$ from $K$ to its algebraic closure $K^a$).

From part c) of Theorem 5.7 we infer that the extension of $v$ from $K^d$ to $K^a$ is unique. On the other hand, if $L$ is any extension field of $K$ within $K^d$, then one shows that $K^d = L^d$. Thus, if $L \neq K^d$, then it follows from part b) of Theorem 5.7 that there are at least two distinct extensions of $v$ from $L$ to $K^d$ and thus also to $K^a = L^a$. This proves that the absolute decomposition field $K^d$ is a minimal algebraic extension of $K$ admitting a unique extension of $v$ to its algebraic closure. We call it the **henselization of** $(K,v)$ **in** $(K^a,v)$. Instead of $K^d$, we also write $K^h$. A valued field $(K,v)$ is called **henselian** if the extension of $v$ from $K$ to $K^a$ is unique, or equivalently, if $K = K^h$. From this definition, one easily obtains that every algebraic extension of a henselian field is again henselian. Henselizations have the following universal property:

**Theorem 5.12** *Let $(K,v)$ be an arbitrary valued field and $(L,v)$ any henselian extension field of $(K,v)$. Then there is a unique embedding of $(K^h,v)$ in $(L,v)$ over $K$.*

From the definition of the henselization as a decomposition field, together with part d) of Theorem 5.7, we obtain another very important property of the henselization:

**Theorem 5.13** *The henselization $(K^h,v)$ is an immediate extension of $(K,v)$.*

There are many properties of valued fields that are equivalent to the property of being henselian. The most important is commonly called "Hensel's Lemma", or more precisely, "satisfying Hensel's Lemma". The original Hensel's Lemma states that the $p$-adic valuation $v_p$ on the $p$-adic field $\mathbb{Q}_p$ has this property. To introduce it, we need some notation. For $a \in \mathcal{O}_K$, we denote by $av$ its residue in $Kv$. For a polynomial $f \in \mathcal{O}_K[X]$, we denote by $fv$ the polynomial in $Kv[X]$ obtained from $f$ by replacing its coefficients by their residues.

**Theorem 5.14** *A valued field $(K,v)$ is henselian if and only if it satisfies the following property: if $f \in \mathcal{O}_K[X]$ and $fv$ has a simple root $bv \in Kv$, then $f$ admits a root $a \in \mathcal{O}_K$ such that $av = bv$.*

From this characterization one easily deduces that every relatively algebraically closed subfield of a henselian field is again henselian.

Assume that $(K,v)$ is henselian. If $n = [L:K] < \infty$, then we have to deal only with a single ramification index e and a single inertia degree f. Hence, the fundamental inequality now reads as

$$n \geq \mathrm{e}\,\mathrm{f}\;.$$

If $L$ is contained in $K^i$ then by Theorem 5.9, $n = \mathrm{f}$. If $K = K^i$ and $L$ is contained in $K^r$, then by Theorem 5.10, $n = \mathrm{e}$. Putting these observations together (using the fact that extension degree, ramification index and inertia degree are multiplicative), one finds that if $L$ is contained in $K^r$, then the **fundamental equality** $n = \mathrm{e}\,\mathrm{f}$ holds. Hence, an inequality can only result from some part of the extension which lies beyond the absolute ramification field. So Theorem 5.11 shows that the missing factor can only be a power of $p$. In this way, one proves the important **Lemma of Ostrowski**:

**Theorem 5.15** *Set $p := \mathrm{char}\, Kv$ if this is positive, and $p := 1$ if $\mathrm{char}\, Kv = 0$. If $(K,v)$ is henselian and $L|K$ is finite, then*

$$n = \mathrm{d}\,\mathrm{e}\,\mathrm{f}\;,$$

*where* d *is a power of $p$. In particular, if $\mathrm{char}\, Kv = 0$, then we always have the fundamental equality $n = \mathrm{e}\,\mathrm{f}$.*

The integer $\mathrm{d} \geq 1$ is called the **defect** of the extension $(L|K,v)$. We call $(K,v)$ **defectless** if every finite extension has trivial defect $\mathrm{d} = 1$. In rigid analysis, this is also called **stable**. By the Lemma of Ostrowski, every henselian field with a residue field of characteristic 0 is a defectless field. Also, every subfield of $(\mathbb{Q}_p, v_p)$ and every power series field with its canonical valuation is defectless. But not every henselian field is defectless. We give an important example.

**Example:** We denote by $\mathbb{F}_p$ the field with $p$ elements. Take $t$ to be transcendental over $\mathbb{F}_p$. On $\mathbb{F}_p(t)$ there is the $t$-adic valuation $v = v_t$, given by $v_t t = 1$. The field $\mathbb{F}_p(t)$ is not perfect. Its perfect hull is the field $K := \mathbb{F}_p(t^{1/p^k} \mid k \in \mathbb{N})$. By Corollary 5.4, $v$ has a unique extension to $K$. It is easy to show that $vK$ is the $p$-divisible hull $\frac{1}{p^\infty}\mathbb{Z}$ of $v\mathbb{F}_p(t) = \mathbb{Z}$, and that $Kv = \mathbb{F}_p(t)v = \mathbb{F}_p$ since this is already perfect. Now we consider the extension $K(\eta)|K$ generated by a root $\eta$ of the Artin–Schreier–polynomial $X^p - X - \frac{1}{t}$. It can be shown that this extension is of degree $p$, admits a unique extension of $v$ and is immediate under this extension. So we have $n = p$, $\mathrm{e} = \mathrm{f} = 1$, and we find that its defect is $p$. Further, $K(\eta)|K$ is disjoint from $K^h|K$, and lifting it up, we find that $K^h(\eta)|K^h$ is again of degree $p$. But it follows from Theorem 5.13 that $vK^h = vK = vK(\eta) = vK^h(\eta)$, and similarly for the residue fields. Hence, we have found an immediate extension of degree $p$ and defect $p$ of a henselian field.

In our example, we may replace $\mathbb{F}_p(t)$ by $\mathbb{F}_p((t))$, taking $K$ to be its perfect hull $\mathbb{F}_p((t))(t^{1/p^k} \mid k \in \mathbb{N})$. Then $(K, v)$ is already henselian, and $(K(\eta)|K, v)$ gives an instant example of an immediate extension of a henselian field. This example also shows that an infinite extension of a defectless field is in general not again defectless. The example can also be transferred into the $p$-adic situation, showing that there are infinite extensions of $(Q_p, v_p)$ which are not defectless fields.

All of these examples show that extensions with defect need not be inseparable. Certainly, there are also purely inseparable extensions with non-trivial defect. For instance, one can interchange the role of purely inseparable extensions (going to the perfect hull) and Artin–Schreier–extensions in our above examples.

It is interesting to note that the element $\eta$ has the power series expansion

$$\eta = t^{-1/p} + t^{-1/p^2} + \ldots + t^{-1/p^n} + \ldots . \tag{17}$$

Note that the values $vt^{-1/p^n} = -1/p^n$ converge from below to 0. This shows that $\eta$ does not lie in the Puiseux series field

$$\bigcup_{m\in\mathbb{N}} \mathbb{F}_p((t^{1/m}))$$

(where the notation "$t^{1/m}$" means that we take a compatible system of roots of $t$). This proves that in positive characteristic, Puiseux series fields are not algebraically closed. Also, the expansion given in (17) is ambiguous since it does not characterize an element in any completion. In fact, the summands with positive value in this expansion can be arbitrary. This observation can be used to show that in positive characteristic, a valued field cannot always be embedded in a power series field with the same value group and residue field. This is related to the fact that the field may have non-isomorphic maximal immediate extensions.

We have seen that the extensions beyond the absolute ramification field are responsible for non-trivial defects. To get this picture, we have shifted the purely inseparable extensions to the top by taking the fixed fields in the maximal separable subextension. In fact, that is where the purely inseparable extensions belong, because from the ramification theoretical point of view, they have much in common with the extension S|V.

The defect, appearing only for positive residue characteristic, is essentially the cause of the problems that we have in algebraic geometry as well as in the model theory of valued fields, in positive characteristic.