Model theory of tame valued fields and beyond: recent developments and open questions

Franz-Viktor Kuhlmann
University of Szczecin, Poland

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Tame extensions

For a valued field (K, v), we denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$.

An algebraic extension $(L|K, v)$ of henselian valued fields is called tame if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by char Kv,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ is defectless, i.e.,

$$[K' : K] = (vK' : vK)[K'v : Kv] .$$

Tame fields

A henselian valued field (K, v) is called a tame field if $(K^{ac}|K, v)$ is a tame extension. All tame fields are perfect.

$\mathbb{F}_p((t))^{1/p^\infty}$ is perfect but not tame, as the extension generated by a root of the Artin-Schreier polynomial $X^p - X - 1/t$ is immediate, i.e., does not increase the value group or the residue field, and thus this extension does not satisfy (T3).

For the theory of tame fields, see

K: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43.

The results of this paper are now frequently applied in the model theory of valued fields. In particular:

Tame fields

Theorem (K (2016))

Tame fields (K, v) satisfy model completeness in the language $\mathcal{L}_{\mathrm{val}}$ of valued rings relative to the elementary theories of their value groups vK in the language of ordered groups and their residue fields Kv in the language of rings. If $\operatorname{char} K = \operatorname{char} Kv$, then also relative completeness and relative decidability hold.

However, there are still daunting questions about tame fields that have remained unanswered.

Open problem 1: Do tame fields admit quantifier elimination in a suitable language?

The problem is that we do not know enough about purely wild extensions, i.e., algebraic extensions of a valued field that are linearly disjoint from tame extensions.

Tame fields of mixed characteristic

There are also open problems about tame fields (K, v) that have mixed characteristic, i.e., char $K = 0$ while char $Kv = p > 0$. The question is whether (or under which additional conditions) they satisfy relative completeness and decidability. In

Sylvy Anscombe - K: *Notes on extremal and tame valued fields*, J. Symb. Logic **81** (2016), 400–416,

examples are given of two tame fields (K_1, v_1) and (K_2, v_2) with $v_1 K_1 \equiv v_1 K_2$ and $K_1 v_1 \equiv K_2 v_2$ such that $(K_1, v_1) \not\equiv (K_2, v_2)$.

The difference to the case of equal positive characteristic is that in this case the restriction of the valuations to the prime fields are trivial while in the mixed characteristic case they are p-adic and hence nontrivial.

Tame fields of mixed characteristic

Progress on this problem has been made in
Victor Lisinski: *Decidability of positive characteristic tame Hahn fields in $\mathcal{L}_{\text{val}}$*, preprint (2021); arXiv:2108.04132.
A valued field (K, v) is called algebraically maximal if it does not admit immediate algebraic extensions. Lisinski proves:

Theorem (Lisinski (2021))

Take tame fields (L, v) and (F, w) of mixed characteristic $(0, p)$ $(p > 0)$ such that $vL \equiv wF$ in the language of ordered groups with a constant symbol π interpreted as $v(p)$ and $w(p)$, respectively, and that $Lv \equiv Fw$ in the language of rings. Assume that the relative algebraic closure (K, v) of $\mathbb{Q}$ in (L, v) is algebraically maximal and that vL/vK is torsion fee. Assume further that every monic polynomial $f \in \mathbb{Z}[X]$ has a root in $\mathcal{O}_F$ if it has a root in $\mathcal{O}_L$. Then $(L, v) \equiv (F, w)$ in $\mathcal{L}_{\text{val}}$.

Tame fields: decidability

As mentioned already, in the cited article on tame fields, relative decidability is proved for tame fields of positive characteristic.
As an application, we obtain:

Theorem (K (2016))

Take $q = p^n$ for some prime p and some $n \in \mathbb{N}$, and an ordered abelian group Γ. Assume that Γ is divisible or elementarily equivalent to the p-divisible hull of $\mathbb{Z}$. Then the $\mathcal{L}_{\mathrm{val}}$-elementary theory of the power series field $\mathbb{F}_q((t^\Gamma))$ with coefficients in $\mathbb{F}_q$ and exponents in Γ, endowed with its canonical valuation v_t, is decidable.

Tame fields: decidability

Lisinski improves this result as follows:

Theorem (Lisinski (2021))

Take a perfect field $\mathbb{F}$ of characteristic $p > 0$ whose elementary theory in the language of rings is decidable, and a p-divisible group Γ whose elementary theory in the language of ordered groups with a constant symbol 1 is decidable. Then the $\mathcal{L}_{\mathrm{val}}(t)$-elementary theory of $\mathbb{F}((t^\Gamma))$ is decidable.

Here, $\mathcal{L}_{\mathrm{val}}(t)$ denotes the language $\mathcal{L}_{\mathrm{val}}$ with a constant symbol t.

Lisinski also proves a theorem giving a criterion for two tame fields containing $\mathbb{F}_p(t)$ to be equivalent in $\mathcal{L}_{\mathrm{val}}(t)$ that is analoguous to his already cited theorem for tame fields of mixed characteristic.

Decidability in mixed characteristic

Since Ax and Kochen, and independently Ershov, established in 1965 the decidability of the elementary theory of the field $\mathbb{Q}_p$ of p-adic numbers, several questions about the decidability of the elementary or the existential theory of local fields and their extensions have been answered, and several others have remained open. We have already seen some results in equal positive charateristic. In contrast, less is known in mixed characteristic, for instance about

- the totally ramified extension $\mathbb{Q}_p(\zeta_{p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining all p^n-th roots of unity, $n \in \mathbb{N}$,
- the totally ramified extension $\mathbb{Q}_p(p^{1/p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining a compatible system of p^n-th roots of p, $n \in \mathbb{N}$,
- the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$.

These are studied in

Konstantinos Kartas: *Decidability via the tilting correspondence*, Algebra and Number Theory **18** (2024), 209–248.

Decidability in mixed characteristic

Theorem (Kartas (2024))

The fields $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ equipped with their unique extensions v_p of the p-adic valuation admit maximal immediate extensions which have decidable elementary $\mathcal{L}_{\mathrm{val}}$-theories.

All of these maximal immediate extensions are tame fields. But the fields themselves are not Kaplansky fields, and Kartas shows that there are uncountably many maximal immediate extensions with distinct elementary $\mathcal{L}_{\mathrm{val}}$-theories. This implies that uncountably many are not decidable.

Decidability in mixed characteristic

Kartas proves a "perfectoid transfer theorem" which transfers the decidability of fields in equal positive characteristic in certain expansions of $\mathcal{L}_{\text{val}}$ to the decidability in $\mathcal{L}_{\text{val}}$ of suitable untilts.

By the result of Lisinski presented above, the perfectoid field $\mathbb{F}_p((t^\Gamma))$, where Γ is the p-divisible hull of $\mathbb{Z}$, is decidable in the language $\mathcal{L}_{\text{val}}(t)$. Kartas constructs a suitable untilt K of $\mathbb{F}_p((t^\Gamma))$ which by the perfectoid transfer theorem is decidable in the language $\mathcal{L}_{\text{val}}$. As $\mathbb{F}_p((t^\Gamma))$ is a maximal immediate extension of the completion of the perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$, which is the tilt of the completion of $\mathbb{Q}_p(p^{1/p^\infty})$, a theorem of Fargues and Fontaine can be used to show that K is a maximal immediate extension of the latter and hence also of $\mathbb{Q}_p(p^{1/p^\infty})$ itself. The case of $\mathbb{Q}_p(\zeta_{p^\infty})$ is similar. This finishes the proof of Kartas' theorem that we presented above.

Kartas notes that all tilts of the undecidable maximal immediate extensions of $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ are maximal immediate extensions of $\mathbb{F}_p((t))^{1/p^\infty}$. Being tame fields, they are decidable in the language $\mathcal{L}_{\mathrm{val}}$. But they are not decidable in the language $\mathcal{L}_{\mathrm{val}}(t)$.

Open problem 2: What is the structure of these extensions (apart from the fact that they are infinite)? What are the indications in their structure that distinguish the decidable from the undecidable extensions?

Kartas also notes that $\mathbb{Q}_p^{ab}$ admits a unique maximal immediate extension, and that it follows from the model theory of algebraically maximal Kaplansky fields that this extension is decidable in $\mathcal{L}_{\mathrm{val}}$.

Open problem 3: Are $\mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p^{ab}$ decidable in $\mathcal{L}_{\rm val}$? Are $\mathbb{F}_p((t))^{1/p^\infty}$ and $\bar{\mathbb{F}}_p((t))^{1/p^\infty}$ decidable in $\mathcal{L}_{\rm val}$ or even $\mathcal{L}_{\rm val}(t)$?

Here $\bar{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$.

We do not know the answers, but there are some conditional results connecting decidability of the mixed characteristic fields with those of the positive characteristic fields. Although these fields are not perfectoid, Kartas succeeds to deduce the following from the perfectoid transfer theorem.

Conditional decidability in mixed characteristic

Theorem (Kartas (2024))

(a) If $\mathbb{F}_p((t))^{1/p^\infty}$ *has a decidable elementary or existential* $\mathcal{L}_{\mathrm{val}}(t)$*-theory, then* $\mathbb{Q}_p(\zeta_{p^\infty})$ *and* $\mathbb{Q}_p(p^{1/p^\infty})$ *have decidable elementary or existential* $\mathcal{L}_{\mathrm{val}}$*-theories, respectively.*

(b) If $\bar{\mathbb{F}}_p((t))^{1/p^\infty}$ *has a decidable elementary or existential* $\mathcal{L}_{\mathrm{val}}(t)$*-theory, then* $\mathbb{Q}_p^{ab}$ *has a decidable elementary or existential* $\mathcal{L}_{\mathrm{val}}$*-theory, respectively.*

Open problem 4: What about the reverse direction?

Separably tame fields

A henselian field is called a separably tame field if every separable-algebraic extension is a tame extension. We let $\mathcal{L}_{\mathrm{val,Q}}$ denote the language $\mathcal{L}_{\mathrm{val}}$ enriched by m-ary predicates Q_m , $m \in \mathbb{N}$, for p-independence. That is, in a field K of characteristic $p > 0$, Q_m is interpreted by
$Q_m(x_1, \ldots, x_m) \Leftrightarrow$
the monomials of exponents $< p$ in the x_i's are linearly independent over the subfield K^p of p-th powers.

Field extensions $L|K$ as $\mathcal{L}_{\mathrm{val,Q}}$-structures are separable, i.e., linearly disjoint from the perfect hull of K.

The model theory of separably tame fields is studied in

K – Koushik Pal: *The model theory of separably tame fields*, J. Alg. **447** (2016), 74-108.

Separably tame fields

Theorem (K–Pal (2016))

Separably tame fields (K, v) of positive characteristic and finite degree of inseparability satisfy completeness and decidability in $\mathcal{L}_{\mathrm{val}}$ relative to the elementary theories of their value groups vK in the language of ordered groups and of their residue fields Kv in the language of rings. In the language $\mathcal{L}_{\mathrm{val,Q}}$, they also satisfy relative model completeness.

In the paper
Sylvy Anscombe: *On Lambda functions in henselian and separably tame valued fields*, arXiv:2505.07518,
Anscombe removes the condition of finite degree of inseparability from the relative decidability result in $\mathcal{L}_{\mathrm{val}}$ and from the other results in a language $\mathcal{L}_{\mathrm{val},\lambda}$ which is $\mathcal{L}_{\mathrm{val,Q}}$ with the predicates Q_m replaced by function symbols for Lambda functions.

Perfectoid and deeply ramified fields

Peter Scholze has defined a perfectoid field to be a complete nondiscrete rank 1 valued field of positive residue characteristic such that the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. Neither "complete" nor "rank 1" are elementary properties.

A suitable elementary class of valued fields containing the perfectoid fields is that of deeply ramified fields, studied in

K – Anna Rzepka: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738.

Deeply ramified fields

A nontrivially valued field (K, v) is deeply ramified if and only if the following conditions hold:
(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) if char $Kv = p > 0$, then the Frobenius is surjective on $\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}}$, where $\hat{K}$ denotes the completion of (K, v).

If (K, v) has rank 1 (i.e., its value group is embeddable in $\mathbb{R}$), then (DRvg) just means that (K, v) is nondiscrete. If (K, v) is complete, then (DRvr) means that the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. Hence every perfectoid field is deeply ramified.
Every perfect valued field of positive characteristic and every tame field is deeply ramified.

Roughly deeply ramified fields

Inspired by the notion "roughly p-divisible value group" introduced by Will Johnson, we call (K, v) a roughly deeply ramified field, or in short an rdr field, if it satisfies axiom (DRvr) together with:

(DRvp) if char $Kv = p > 0$, then $v(p)$ is not the smallest positive element in the value group vK.

The two axioms (DRvp) and (DRvr) together imply that the smallest convex subgroup of vK containing $v(p)$ (or equivalently, the interval $[-v(p), v(p)]$) is p-divisible.

Roughly tame fields

For the definition of roughly tame fields, we need some preparation. A valued field is called a defectless field if each finite extension is defectless.

Theorem (K (2016))

A henselian field (K, v) is a tame field if and only if the following conditions hold:

(TF1) if char $Kv = p > 0$*, then vK is p-divisible,*

(TF2) Kv is perfect,

(TF3) (K, v) is a defectless field.

Replacing (TF1) by

(TF1r) if char $Kv = p > 0$, then $[-v(p), v(p)]$ is p-divisible,

we obtain the definition of a roughly tame field.

Roughly tame fields

In the article

Anna Rzepka – Piotr Szewczyk: *Defect extensions and a characterization of tame fields*, J. Algebra **630** (2023), 68–91,

the following is proven:

Theorem (Rzepka – Szewczyk (2023))

A henselian field is roughly tame if and only if all of its algebraic extension fields are defectless fields.

We know that being defectless is an important property in the model theory of valued fields. For instance, if a valued field is existentially closed in its maximal immediate extensions, then it is henselian and defectless. In general, the property of being a defectless field is not preserved under infinite algebraic extensions.

Roughly tame fields and their application

In the article
Franziska Jahnke, Konstantinos Kartas: *Beyond the Fontaine-Wintenberger theorem*, to appear in the Journal of the AMS; arXiv:2304.05881,
Jahnke and Kartas generalize the model theoretic results about tame fields to the elementary class of roughly tame fields. They put this to work in their approach of "taming perfectoid fields". They work with an elementary class $\mathcal{C}$ of henselian fields (K,v) of residue characteristic $p>0$ with distinguished element $\pi\in K\setminus\{0\}$, $v\pi>0$, such that:

- the Frobenius on the ring $\mathcal{O}_K/(p)$ is surjective, and
- with the coarsening w of v associated with the valuation ring $\mathcal{O}_v[\pi^{-1}]$, (K,w) is algebraically maximal (which implies that it is roughly tame).

This class contains all henselian roughly deeply ramified fields of mixed characteristic.

Model theoretic results for the class $\mathcal{C}$

If (K',v') is an elementary (sufficiently saturated) extension of a perfectoid field (K,v) with distinguished element $\varpi \in K \setminus \{0\}$, $v\varpi > 0$, and w' is the coarsest coarsening of v' on K' such that $w'\varpi > 0$, then $(K',w') \in \mathcal{C}$ for any $\pi \in K \setminus \{0\}$, $v\pi > 0$, and the residue field $K'w'$ is an elementary extension of the tilt of K. This is used to show that certain model theoretic properties hold for perfectoid fields if and only if they hold for their tilts.

Jahnke and Kartas prove for the class $\mathcal{C}$ analogues of the model theoretic results for (roughly) tame fields, but with the residue fields Kv replaced by the residue rings $\mathcal{O}_K/(\pi)$. This "mods out" the non-tame part of the valued fields in $\mathcal{C}$.

So we are still left with the

Open problem 5: What can we say about the model theory of (roughly) deeply ramified fields (relative to their value groups and residue fields), and in particular of $\mathbb{F}_p((t))^{1/p^\infty}$?

It is well known that the henselization $\mathbb{F}_p(t)^h$ of $\mathbb{F}_p(t)$ is existentially closed in $\mathbb{F}_p((t))$. However, the following has remained a daunting

Open problem 6: Is $\mathbb{F}_p(t)^h$ an elementary substructure of $\mathbb{F}_p((t))$?

In contrast, Jahnke and Kartas prove that $\mathbb{F}_p(t^{1/p^\infty})^h$ is an elementary substructure of $\mathbb{F}_p((t))^{1/p^\infty}$. This positive result encourages us to ask:

Open problem 7: Is it possible to prove model theoretic results for henselian perfect valued fields of positive characteristic, analoguous to those for tame fields (but under mild additional conditions)?

The case of $\mathbb{F}_p((t))$

While model theoretic results about $\mathbb{Q}_p$ and in particular the decidability of $\mathbb{Q}_p$ are known since the work of Ax–Kochen and Ershov, we are still facing the

Open problem 8: What can we say about the model theory and in particular a complete axiomatization and the decidability of $\mathbb{F}_p((t))$?

In the article

K: *Elementary properties of power series fields over finite fields*, J. Symb. Logic **66** (2001), 771–791,

the following negative result is proven:

Theorem (K (2001))

The $\mathcal{L}_{\mathrm{val}}(t)$-elementary axiom system
(A_t) "henselian defectless valued field of positive characteristic with value group a $\mathbb{Z}$-group with smallest element $v(t)$ and residue field $\mathbb{F}_p$"
is not complete.

The case of $\mathbb{F}_p((t))$

This theorem is proven by constructing an extension (L, v) of $(\mathbb{F}_p((t)), v_t)$ with the following properties:

• (L, v) satisfies axiom system (A_t) and $v(t)$ is the smallest positive element of vL,

• $L|K$ is of trancendence degree 1 and regular (i.e., $L|K$ is separable and K is relatively algebraically closed in L),

• there is an elementary $\mathcal{L}_{\mathrm{val}}(t)$-sentence which holds in (K, v) but not in (L, v).

Moreover, it is shown that (L, v) is not $\mathcal{L}_{\mathrm{val}}$-existentially closed in its maximal immediate extensions. This proves:

Proposition (K (2001))

There are henselian defectless fields that are not $\mathcal{L}_{\mathrm{val}}$-existentially closed in their maximal immediate extensions.

Open problem 9: Is there a handy additional condition on the immediate extensions that remedies this situation?

In the cited paper, also the following is shown:

Theorem (K (2001))

The $\mathcal{L}_{\mathrm{val}}$-elementary axiom system
(A) "henselian defectless valued field of positive characteristic with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$"
is not complete.

The case of $\mathbb{F}_p((t))$

In the cited paper, a (not really handy) axiom scheme, called (PDOA), is suggested to be added to axiom system (A) or (A_t). A much more elegant axiom scheme was found after Yuri Ershov introduced the notion of "extremal field" and claimed that $\mathbb{F}_p((t))$ is extremal. However, his definition and proof were faulty. In the article
Salih Azgin – K – Florian Pop: *Characterization of extremal valued fields*, Proc. Amer. Math. Soc. **140** (2012), 1535–1547
it is shown that $\mathbb{F}_p((t))$ does not satisfy Ershov's definition, a corrected definition is given, and it is shown that $\mathbb{F}_p((t))$ satisfies this corrected definition, which we present now.

Extremal valued fields

A valued field (K, v) is called extremal if for every multi-variable polynomial $f(X_1, \ldots, X_n)$ over K, the set

$$\{v(f(a_1, \ldots, a_n)) \mid a_1, \ldots, a_n \in \mathcal{O}_K\} \subseteq vK \cup \{\infty\}$$

has a maximal element. This is an $\mathcal{L}_{\text{val}}$-elementary axiom scheme.

Theorem (Azgin – K – Pop (2012))

$\mathbb{F}_p((t))$ is an extremal field.

Open problem 10: Is (A) + "(K, v) is extremal" a complete axiom system?

There are many more open problems about extremal fields. See the already cited article with Anscombe on extremal and tame valued fields.

The following is shown in the article
Sylvy Anscombe – Arno Fehm: *The existential theory of equicharacteristic henselian valued fields*, Algebra and Number Theory **10** (2016), 665–683:

Theorem (Anscombe – Fehm (2016))

The existential $\mathcal{L}_{\text{val}}$-theory of $\mathbb{F}_p((t))$ is decidable.

However, as can be seen from our discussion of Kartas' work, we would like to have more. Jan Denef and Hans Schoutens proved in 2003 that the existential $\mathcal{L}_{\text{val}}(t)$-theory of $\mathbb{F}_p((t))$ is decidable, provided that resolution of singularities holds in positive characteristic. In order to discuss more recent improvements of this result, we need some preparations.

Rational place = existentially closed?

In the article
K: *On places of algebraic function fields in arbitrary characteristic*, Advances in Math. **188** (2004), 399–424,

the following question is studied: Take a field extension $F|K$ such that F admits a K-rational place, or in other words, a valuation with residue field K. Under which additional conditions does it follow that K is existentially closed in F?

Here a key role is played by large fields. While they are usually defined in a different way, one can also use the above approach: A field K is large if it is existentially closed in $K((t))$.

Rational place = existentially closed?

Theorem (K (2004))

Let K be a perfect field. Then the following conditions are equivalent:
1) K is a large field,
2) K is existentially closed in every power series field $K((t^\Gamma))$,
3) K is existentially closed in every extension field L which admits a K-rational place.

Local uniformization is a local form of resolution of singularities.

Theorem (K (2004))

If all rational places of arbitrary function fields admit local uniformization, then the three conditions of the previous theorem are equivalent, for arbitrary fields K.

A conditional result about $\mathbb{F}_p((t))$

In the paper
Sylvy Anscombe – Philip Dittmann – Arno Fehm: *Axiomatizing the existential theory of $F_q((t))$*, Algebra and Number Theory **17** (2023), 2013–2032,
the assumption that implication 1)⇒3) holds for *arbitrary* fields K is called hypothesis (R4). Hence local uniformization implies (R4). The authors prove:

Theorem (Anscombe – Dittmann – Fehm (2023))

If (R4) holds, then the existential $\mathcal{L}_{\mathrm{val}}(t)$-theory of $\mathbb{F}_p((t))$ is decidable.

Open problem 11: Does (R4) hold?

Alteration

De Jong has proved resolution by alteration. Alteration means that a finite extension of the function field of the algebraic variety under consideration is taken into the bargain. By valuation theoretical tools, Knaf and K have proved local uniformization by alteration.

Open problem 12: Does local uniformization by alteration imply a reasonable (and useful) hypothesis "(R4) by alteration"? Is there a "model theory by alteration"?

You cannot always get what you want

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You cannot always get what you want – but perhaps after a finite extension?

THE END

Thank you for your attention!

More detailed information

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and a lecture series on valued function fields and the defect can be found on the web page

https://www.fvkuhlmann.de/Fvkls.html.

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