TOPICS IN HIGHER RAMIFICATION THEORY: RAMIFICATION IDEALS

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ABSTRACT. We introduce and study the notion of ramification ideals in higher ramification theory. After general results on their computation, we discuss their connection with defect and compute them for Artin-Schreier extensions and Kummer extensions of prime degree equal to the residue characteristic, with or without defect. We present an example that shows that nontrivial defect in an extension of degree not a prime may not imply the existence of a nonprincipal ramification ideal.

1. INTRODUCTION

Higher ramification theory is the theory of valued field extensions $\mathcal{E} = (L|K, v)$ where (K, v) has positive residue characteristic p and is its own **absolute ramification field** (see Section 2.3). The latter means that (K, v) is henselian, its **value group** vK is divisible by all primes different from p, and its **residue field** Kv is separable-algebraically closed. The **absolute Galois group** $\operatorname{Gal} K := \operatorname{Gal} K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K, is then a p-group. This implies that every finite Galois extension of K is a tower of Galois extensions of degree p. In **equal characteristic**, i.e., if $\operatorname{char} K = \operatorname{char} Kv = p$, the latter are **Artin-Schreier extensions**, and in **mixed characteristic**, i.e., if $\operatorname{char} K = 0$ and $\operatorname{char} Kv = p$, they are **Kummer extensions** because K contains all p-th roots of unity (see Section 3.3).

Our interest in higher ramification theory originated from the following well known deep open valuation theoretical problems in positive characteristic:

1) local uniformization, the local form of resolution of singularities in arbitrary dimension,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, and of its perfect hull, where q is a power of a prime p.

Both problems are connected with the structure theory of valued function fields of positive characteristic p. The main obstruction here is the phenomenon of the **defect**, which we define in Section 2.2. For background on the defect and its impact on the above problems, see [13, 14, 11, 15, 17, 19, 21, 20].

Date: 30. 4. 2026.

2010 *Mathematics Subject Classification.* 12J10, 12J25.

Key words and phrases. Higher ramification theory, ramification ideal, defect, Artin-Schreier extension, Kummer extension.

The author would like to thank Josnei Novacoski for helpful discussions and for pointing out a mistake in the example given in Section 3.5. He also thanks Konstantinos Kartas for helping to develop the last example in Section 3.4.

Via ramification theory, the study of defect in extensions of arbitrary finite degree can be reduced to the investigation of purely inseparable extensions and of Galois extensions of degree $p = \mathrm{char}\, Kv > 0$ with nontrivial defect. This is explained e.g. in [4, Section 2.1]. Defects of Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree have been classified ("dependent" vs. "independent" defect) first in [16] for the equal characteristic case and then in [25] in general. Theorem 1.4 of [25] presents various criteria for independent defect.

One of them uses the ramification ideal $I_{\mathcal{E}}$, which we define in Section 2.5. Section 3 is then devoted to the computation of ramification ideals. Starting with a first approach described by Ribenboim in [32] we develop more elaborate computations. Of particular interest is the case of extensions that have valuation bases; for this notion, see Section 2.6. Based on this, we treat towers of two Galois extensions where the upper one has a valuation basis, which we will need in Section 3.5.

In Section 3.2 we discuss the correlation between defect and the existence of nonprincipal ramification ideals. While it is true that a finite Galois extension without defect has only principal ramification ideals, the converse does not hold. We give an example for this phenomenon in Section 3.5 for the equal characteristic case.

In Section 3.3 we first compute the unique ramification ideals $I_{\mathcal{E}}$ for Galois extensions $\mathcal{E} = (L|K, v)$ of degree $p = \mathrm{char}\, Kv$ without defect; the results are applied in [5]. We then take a closer look at the unique ramification ideals $I_{\mathcal{E}}$ for Galois extensions $\mathcal{E} = (L|K, v)$ of degree $p = \mathrm{char}\, Kv$ with defect which are computed in [4].

2. Preliminaries

For basic facts from valuation theory, see [7, 8, 30, 34, 36].

Take a valued field (K, v). We denote its value group by vK, its residue field by Kv, its valuation ring by $\mathcal{O}_K$, with its maximal ideal $\mathcal{M}_K$. For $a \in K$, we write va for its value and av for its residue. By $(L|K, v)$ we denote an extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K, v)$ is called **immediate** if these embeddings are onto.

By $\tilde{K}$ we denote the algebraic closure of K. Every valuation on K^{sep} has a unique extension to $\tilde{K}$; this allows us to identify the absolute Galois group $\mathrm{Gal}\, K$ with the automorphism group $\mathrm{Aut}\, \tilde{K}|K$.

2.1. **Final segments and ideals.** A subset S of a totally ordered set Γ is an **initial segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \leq \alpha$, it follows that $\gamma \in S$. Symmetrically, S is a **final segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \geq \alpha$, it follows that $\gamma \in S$. *Throughout, we will assume initial and final segments to be nonempty and not equal to* Γ.

Take a valued field (L, v). A subset $I \subset L$ is a **fractional $\mathcal{O}_L$-ideal** if there is some $a \in L$ such that aI is an $\mathcal{O}_L$-ideal (contained in $\mathcal{O}_L$). In particular, every fractional $\mathcal{O}_L$-ideal is a proper subset of L.

The function

$$v : \ I \ \mapsto \ \Sigma_I \ := \ \{vb \mid 0 \neq b \in I\} \tag{1}$$

is an order preserving bijection from the set of all nonzero fractional $\mathcal{O}_L$-ideals onto the set of all final segments of vL. This set is again linearly ordered by inclusion, and the function (1) is order preserving: $J \subseteq I$ holds if and only if $\Sigma_J \subseteq \Sigma_I$ holds. The inverse of the above function is the order preserving function

$$\Sigma \mapsto I_\Sigma := \{a \in L \mid va \in \Sigma\} \cup \{0\} = \{a \in L \mid va \in \Sigma\} \cup \{\infty\} \tag{2}$$

from the set of all final segments of vL onto the set of all nonzero fractional $\mathcal{O}_L$-ideals.

2.2. **The defect.**

A valued field extension $(L|K, v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation. A valued field (K, v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, if all of its algebraic extensions are unibranched. A **henselization** of (K, v) is an algebraic extension of (K, v) which admits a valuation preserving embedding in every other henselian extension of (K, v). Henselizations always exist and are unique up to valuation preserving isomorphism over K; therefore we will talk of *the* henselization of (K, v) and denote it by $(K, v)^h = (K^h, v^h)$. The henselization of (K, v) is an immediate separable-algebraic extension. The valued field (K, v) is henselian if and only if it is equal to its henselization.

If $(L|K, v)$ is a finite unibranched extension, then by the Lemma of Ostrowski [36, Corollary to Theorem 25, Section G, p. 78]),

$$[L : K] = \tilde{p}^\nu \cdot (vL : vK)[Lv : Kv] , \tag{3}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K, v) := \tilde{p}^\nu$ is the **defect** of the extension $(L|K, v)$. We call $(L|K, v)$ a **defect extension** if $d(L|K, v) > 1$, and a **defectless extension** if $d(L|K, v) = 1$. Throughout this paper, when we talk of a **defect extension** $(L|K, v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. Then it follows from (3) that $[L : K] = p = \operatorname{char} Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$, that is, $(L|K, v)$ is an immediate extension.

Nontrivial defect only appears when $\operatorname{char} Kv = p > 0$, in which case $\tilde{p} = p$. A henselian field (K, v) is called a **defectless field** if all of its finite extensions are defectless.

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 2.1. *Take a valued field (K, v). If $L|K$ and $M|L$ are finite extensions and the extension of v from K to M is unique, then*

$$\mathrm{d}(M|K, v) = \mathrm{d}(M|L, v) \cdot \mathrm{d}(L|K, v) \tag{4}$$

In particular, $(M|K, v)$ is defectless if and only if $(M|L, v)$ and $(L|K, v)$ are defectless.

For every extension $(L|K, v)$ of valued fields and $a \in L$ we define

$$v(a - K) := \{v(a - c) \mid c \in K\} .$$

The set $v(a-K)\cap vK$ is an initial segment of vK. For more information on its properties, see [22].

Lemma 2.2. *Take a unibranched algebraic extension $(K(a)|K,v)$ and an extension of v from $K(a)$ to $\tilde{K}$. Denote by (K^h,v) the henselization of (K,v) in $(\tilde{K},v)$. Then:*
a) $K(a)|K$ is linearly disjoint from $K^h|K$,
b) $(K^h(a)|K^h,v)$ is a defect extension if and only if $(K(a)|K,v)$ is, and
c) $v(a-K^h)=v(a-K)$.

Proof. Our first assertion follows from [2, Lemma 2.1]. For the proof of the second assertion, recall that henselizations are immediate extensions, so we have $vK^h = vK$ and $K^hv = Kv$. Further, we have $K^h(a) = K(a)^h$ since on the one hand, $K^h(a)$ is henselain, being an algebraic extension of K^h, and on the other hand, it contains $K(a)$. Hence, $vK^h(a) = vK(a)$ and $K^h(a)v = K(a)v$. Since $K(a)|K$ is linearly disjoint from $K^h|K$, we also have $[K^h(a):K^h] = [K(a):K]$. As an algebraic extension of a henselian field, $(K^h(a)|K^h,v)$ is unibranched. It follows that

$$\begin{aligned} d(K^h(a)|K^h,v) &= [K^h(a):K^h]/(vK^h(a):vK^h)[K^h(a)v:K^hv] \\ &= [K(a):K]/(vK(a):vK)[K(a)v:Kv] \\ &= d(K(a)|K,v)\,. \end{aligned}$$

This proves our second assertion.

Suppose that $v(a-K^h)\neq v(a-K)$. Since $v(a-K)$ is an initial segment of $vK = vK^h$, this means that there must be some $z\in K^h$ such that $v(a-z) > v(a-K)$. However, as $K(a)|K$ is linearly disjoint from $K^h|K$, we know from [18, Theorem 2] that this cannot be true. This proves our third assertion. □

2.3. **The ramification field.**

In order to reduce the study of arbitrary finite defect extensions to purely inseparable extensions and Galois extensions of degree $p = \operatorname{char} Kv > 0$, we fix an extension of v from K to $\tilde{K}$. The **absolute ramification field of** (K,v) (with respect to the chosen extension of v), denoted by (K^r,v), is the ramification field of the Galois extension $(K^{\rm sep}|K,v)$. The **ramification field** of a Galois extension $(L|K,v)$ with Galois group $G = \operatorname{Gal}(L|K)$ is the fixed field in L of the **ramification group**

$$G^r := \left\{\sigma\in G \;\middle|\; \frac{\sigma b - b}{b}\in\mathcal{M}_L \text{ for all } b\in L^\times\right\}. \tag{5}$$

Take a finite defect extension $(L|K,v)$. Then $(L.K^r|K^r,v)$ is a defect extension with the same defect (see [25, Proposition 2.12]). On the other hand, $K^{\rm sep}|K^r$ is a p-extension (see [16, Lemma 2.7]), so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p. Note that by general ramification theory, $(K,v) = (K^r,v)$ if and only if (K,v) is henselian, vK is divisible by all primes different from $\operatorname{char} Kv$, and Kv is separable-algebraically closed.

2.4. **Immediate extensions.**

Let us give more details about immediate extensions.

Lemma 2.3. *Take an arbitrary extension* $(L|K,v)$ *and* $b\in L$. *Then there is* $c\in K$ *such that* $v(b-c)>vb$ *if and only if* $vb\in vK$ *and* $c'bv\in Kv$ *for every* $c'\in K$ *such that* $vc'b=0$.

Proof. Assume first that $v(b-c)>vb$. Then $vb=vc\in vK$ and for any $c'\in K$ such that $vc'b=0$ we have $v(c'b-c'c)>0$ so that $c'bv=c'cv\in Kv$. Now assume that $vb\in vK$ and $c'bv\in Kv$ for every $c'\in K$ such that $vc'b=0$. Take $c_1\in K$ such that $vc_1=vb$ and set $c'=c_1^{-1}$. Then $vc'b=0$, hence by assumption, $c'bv\in Kv$. Take $c_2\in K$ such that $c'bv=c_2v$, so that $v(c'b-c_2)>0$. Multiplying with c_1 we obtain $v(b-c_1c_2)>vc_1=vb$. □

It follows that an extension $(L|K,v)$ is immediate if and only if for all $b\in L$ there is $c\in K$ such that $v(b-c)>vb$. This lays the basis for the proof of the following theorem; see [9, Theorem 1] and [22, Lemma 2.29].

Theorem 2.4. *If* $(L|K,v)$ *is an immediate extension of valued fields, then for every element* $a\in L\setminus K$ *the set* $v(a-K)$ *is an initial segment of* vK *without maximal element.*

The following partial converse of this theorem also holds (see [1, Lemma 4.1], cf. also [16, Lemma 2.21]):

Lemma 2.5. *Assume that* $(K(a)|K,v)$ *is a unibranched extension of prime degree such that* $v(a-K)$ *has no maximal element. Then the extension* $(K(a)|K,v)$ *is immediate and hence a defect extension.*

2.5. **Higher ramification groups and ramification ideals.**

Take a valued field extension $\mathcal{E}=(L|K,v)$. Assume that $L|K$ is a Galois extension, and let $G=\mathrm{Gal}\,L|K$ denote its Galois group. We define the **series of upper ramification groups**

$$G_I\;:=\;\left\{\sigma\in G\;\middle|\;\frac{\sigma b-b}{b}\in I\;\text{ for all } b\in L^\times\right\}\,, \tag{6}$$

where I runs through all $\mathcal{O}_L$-ideals (cf. [36, §12]). Note that $G_{\mathcal{M}_L}=G^r$ is the ramification group and $G_{\mathcal{O}_L}$ is the decomposition group of $(L|K,v)$. Every G_I is a normal subgroup of G ([36, (d) on p.79]). We call G_I a **higher ramification group** if it is a nontrivial subgroup of $G_{\mathcal{M}_L}$. We call $\mathcal{E}$ a **purely wild extension** if $G=G_{\mathcal{M}_L}$; this matches the (more general) definition of "purely wild extension" in [23].

The function

$$\varphi:\;I\;\mapsto\;G_I \tag{7}$$

from the set of $\mathcal{O}_L$-ideals to the set of upper ramification groups preserves $\subseteq$, that is, if $I\subseteq J$, then $G_I\subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the series of upper ramification groups is linearly ordered by inclusion. Note that in general, φ will neither be injective nor surjective as a function to the set of normal subgroups of G. This gives rise to the task to determine the smallest ideal that is sent by φ to a group G_I in its image. To this end, for each subgroup H of $G_{\mathcal{O}_L}$ we define the $\mathcal{O}_L$-ideal

$$(8)\qquad I_H := \left(\frac{\sigma b - b}{b} \mid \sigma \in H\,,\, b \in L^\times\right) = \left(\frac{\sigma b}{b} - 1 \mid \sigma \in H\,,\, b \in L^\times\right)$$

and consider the function

$$(9)\qquad \psi:\ H \mapsto I_H$$

from the set of all subgroups H of $G_{\mathcal{O}_L}$ to the set of all $\mathcal{O}_L$-ideals. Also ψ preserves $\subseteq$ and is in general neither injective nor surjective. However, it is easy to see that $G_{(0)} = \{\mathrm{id}\}$ and $I_{\{\mathrm{id}\}} = (0)$. If I_H is nonzero and contained in $\mathcal{M}_L$, then we call it a **ramification ideal**. We note:

Proposition 2.6. *1) For every $\mathcal{O}_L$-ideal I, the group G_I is the largest of all subgroups H' of G such that $I_{H'} \subseteq I$.*

2) For every subgroup H of G, the ideal I_H is the smallest of all $\mathcal{O}_L$-ideals I' such that $H \subseteq G_{I'}$.

3) If $I = I_H$ for some subgroup H of G, then $I_{G_I} = I$. If $H = G_I$ for some $\mathcal{O}_L$-ideal I, then $G_{I_H} = H$. Hence φ is an inclusion preserving bijection from the set of all $\mathcal{O}_L$-ideals onto the set of all upper ramification groups, with ψ its inverse.

4) The function φ induces an inclusion preserving bijection from the set of all ramification ideals onto the set of all higher ramification groups, with its inverse induced by ψ.

5) A subgroup H of G is an upper ramification group if and only if it is a subgroup of $G_{\mathcal{O}_L}$ and for every subgroup H' of $G_{\mathcal{O}_L}$ we have $H \subsetneq H' \Rightarrow I_H \subsetneq I_{H'}$.

6) An $\mathcal{O}_L$-ideal I is a ramification ideal if and only if it is nonzero and contained in $\mathcal{M}_L$ and for every $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I \Rightarrow G_{I'} \subsetneq G_I$.

7) If $\mathcal{E} = (L|K,v)$ is a nontrivial purely wild Galois extension, then I_G is its largest ramification ideal. If in addition $\mathcal{E}$ is of prime degree, then I_G is its unique ramification ideal.

Proof. 1) and 2) follow directly from the definitions of G_I and I_H.

3): If $I = I_H$, then it follows from part 1) that $H \subseteq G_I$. Thus $I = I_H \subseteq I_{G_I} \subseteq I$, so $I_{G_I} = I$. If $H = G_I$, then it follows from part 2) that $I_H \subseteq I$. Thus $H \subseteq G_{I_H} \subseteq G_I = H$, so $G_{I_H} = H$.

4): If I_H is a ramification ideal, then I_H is nonzero and contained in $\mathcal{M}_L$, hence H is nontrivial and by part 3), $H = G_{I_H} \subseteq G_{\mathcal{M}_L}$, so G_{I_H} is a higher ramification group. This shows that φ sends ramification ideals to higher ramification groups.

If G_I is a higher ramification group, then $G_I \subseteq G_{\mathcal{M}_L}$, hence again by part 3), $I = I_{G_I} \subseteq I_{G_{\mathcal{M}_L}} = \mathcal{M}_L$, and since G_I is nontrivial, $I = I_{G_I}$ is nonzero. This shows that ψ sends higher ramification groups to ramification ideals. Now the assertion of part 4) follows from part 3).

5): Assume first that H is an upper ramification group, and take an $\mathcal{O}_L$-ideal I such that $H = G_I$. Take a subgroup H' of $G_{\mathcal{O}_L}$ which properly contains G_I. Then by part 1), $I_H = I_{G_I} \subsetneq I_{H'}$.

For the converse, assume that H is a subgroup of $G_{\mathcal{O}_L}$ such that for every subgroup H' of $G_{\mathcal{O}_L}$ we have $H \subsetneq H' \Rightarrow I_H \subsetneq I_{H'}$. By part 1), G_{I_H} is the largest of all subgroups H' of $G_{\mathcal{O}_L}$ such that $I_{H'} \subseteq I_H$. Therefore $G_{I_H} = H$, which shows that H is an upper ramification group.

6): It suffices to show that there is a subgroup H of $G_{\mathcal{O}_L}$ such that $I = I_H$ if and only if for every $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I \Rightarrow G_{I'} \subsetneq G_I$.

Assume first that $I = I_H$. Take an $\mathcal{O}_L$-ideal I' properly contained in I_H. Then by parts 2) and 3), $G_{I'} \subsetneq H = G_{I_H} = G_I$.

For the converse, assume that I is an $\mathcal{O}_L$-ideal such that for every $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I \Rightarrow G_{I'} \subsetneq G_I$. By part 2), I_{G_I} is the smallest of all $\mathcal{O}_L$-ideals I' such that $G_I \subseteq G_{I'}$. Therefore $I = I_{G_I}$.

7): Since $\mathcal{E}$ is a nontrivial purely wild extension, also $G = G_{\mathcal{M}_L}$ is nontrivial, which by definition of I_G implies that I_G is nonzero. Since $G = G_{\mathcal{M}_L}$, we have $I_G \subseteq \mathcal{M}_L$. Thus I_G is a ramification ideal. As ψ preserves inclusion, I_G is the largest ramification ideal of $\mathcal{E}$.

If in addition $\mathcal{E}$ is of prime degree, then the only subgroups of G are G and $\{\mathrm{id}\}$. Since $I_{\{\mathrm{id}\}} = (0)$ is not a ramification ideal, I_G is then the unique ramification ideal of $\mathcal{E}$. □

The upper ramification groups can be represented as

$$G_\Sigma := G_{I_\Sigma} = \left\{ \sigma \in G \;\middle|\; v\,\frac{\sigma b - b}{b} \in \Sigma \cup \{\infty\} \text{ for all } b \in L^\times \right\},$$

where Σ runs through all final segments of $(vL)^{>0}$ and $\emptyset$.

Like the function (7), also the function $\Sigma \mapsto G_\Sigma$ is in general neither injective nor surjective. We call a final segment Σ of $(vL)^{>0}$ a **ramification jump** if and only if

$$\Sigma' \subsetneq \Sigma \;\Rightarrow\; G_{\Sigma'} \subsetneq G_\Sigma$$

for every final segment Σ' of $(vL)^{>0}$.

Proposition 2.7. *Take a nontrivial purely wild Galois extension $\mathcal{E} = (L|K,v)$. Then a nonempty final segment Σ of $(vL)^{>0}$ is a ramification jump if and only if I_Σ is a ramification ideal.*

Proof. First note that for every nonempty final segment Σ of $(vL)^{>0}$ the ideal I_Σ is nonzero, and contained in $\mathcal{M}_L$ by our assumption on $\mathcal{E}$. Now a nonempty final segment Σ of $(vL)^{>0}$ is a ramification jump if and only if for every nonempty final segment Σ' of $(vL)^{>0}$ we have $\Sigma' \subsetneq \Sigma \Rightarrow G_{I_{\Sigma'}} \subsetneq G_{I_\Sigma}$. This holds if and only if for every nonzero $\mathcal{O}_L$-ideal I' we have $I' \subsetneq I_\Sigma \Rightarrow G_{I'} \subsetneq G_{I_\Sigma}$. By Proposition 2.6, this in turn holds if and only if I_Σ is a ramification ideal. □

By Propositions 2.6 and 2.7, the number of ramification ideals and ramification jumps in a purely wild Galois extension is bounded by the number of nontrivial subgroups of its Galois group. It may not always be equal to this number, as an example given in Section 3.5 below will show. For computations of the number of ramification ideals in finite Galois extensions, see [6].

In this paper we are particularly interested in the case where $\mathcal{E} = (L|K,v)$ is a purely wild Galois extension of prime degree p. Then by Lemma 2.6, $\mathcal{E}$ has the unique ramification ideal I_G, and we denote it by $I_\mathcal{E}$. Hence $\Sigma_\mathcal{E} := \Sigma_{I_\mathcal{E}}$ is the unique ramification jump of $\mathcal{E}$. As we will show in the next section, ramification jump and ramification ideal carry important information about the extension $(L|K,v)$.

Remark 2.8. In [25, Section 2.1] we also included zero ideals and empty segments in the definitions of the functions (1) and (2). However, classically ramification jumps have always been defined as integers in the case of discrete valuations, and as real numbers in the case of valuations of rank one, and the intended meaning of "jump" does not fit well with the value $v0 = \infty$. #

Further, we want to quickly discuss the **series of lower ramification groups**

$$G_I^l \;:=\; \{\sigma \in G \mid \sigma b - b \in I \text{ for all } b \in \mathcal{O}_L\}\;, \tag{10}$$

where I runs through all $\mathcal{O}_L$-ideals (see [36, $12]). Note that $G^l_{\mathcal{M}_L}$ is the inertia group of $(L|K,v)$. Again, for every $\mathcal{O}_L$-ideal I, G_I^l is a normal subgroup of G (see [36, (d) on p.79]), and $G_I \subseteq G_I^l$ (see [36, (a) on p.78]). But in the case of an immediate extension $(L|K,v)$, the two groups coincide, as follows from the next, more general, result:

Lemma 2.9. *If $vL = vK$, then $G_I = G_I^l$ for all nonzero ideals I of $\mathcal{O}_L$ contained in $\mathcal{M}_L$.*

Proof. It suffices to show that $G_I^l \subseteq G_I$. Take $\sigma \in G_I^l$ and $b \in L^\times$. Since $vL = vK$, we can pick some $c \in K$ such that $vcb = 0$. As $\sigma \in G_I^l$, we have that $\sigma(cb) - cb \in I$. Since $vcb = 0$, it follows that

$$\frac{\sigma b - b}{b} \;=\; \frac{\sigma(cb) - cb}{cb} \;\in\; I\,.$$

This shows that $\sigma \in G_I$. □

2.6. Valuation bases.

Take an extension $(L|K,v)$. The elements $b_1,\dots,b_n \in L$ are called **valuation independent** (over K) if for all choices of $c_1,\dots,c_n \in K$,

$$v\sum_{i=1}^{n} c_i b_i \;=\; \min_i v c_i b_i\;.$$

If in addition these elements form a basis of $L|K$, then they are called a **valuation basis** of $(L|K,v)$. If the valuation basis contains 1, we will speak of a **valuation basis with 1**.

Recall that a unibranched extension $(L|K,v)$ is defectless if it satisfies the fundamental equality $[L:K] = \mathrm{e}\cdot\mathrm{f}$, where $\mathrm{e} = (vL : vK)$ is the ramification index and $\mathrm{f} = [Lv : Kv]$ is the inertia degree. In this case, $(L|K,v)$ admits a **standard valuation basis**, which we construct as follows: we take $y_1,\dots,y_{\mathrm{e}} \in L$ such that $vy_1 + vK,\dots,vy_{\mathrm{e}} + vK$ are the cosets of vK in vL, and $z_1,\dots,z_{\mathrm{f}} \in L$ such that $z_1v,\dots,z_{\mathrm{f}}v$ are a basis of $Lv|Kv$. Then the products y_iz_j, $1 \le i \le \mathrm{e}$, $1 \le j \le \mathrm{f}$, form a valuation basis of $(L|K,v)$ (see [8, Lemma 3.2.2]). Note that we can always choose $y_1 = z_1 = 1$ so that $y_1z_1 = 1$. We will then speak of a **standard valuation basis with 1**.

The next result has been shown in the proof of [16, Lemma 2.1].

Lemma 2.10. *Take an extension $(L|K,v)$ of prime degree p. If for $b \in L$, either $vb \notin vK$ or there is some $c \in K$ such that $vcb = 0$ and $cbv \notin Kv$, then $1, b, \dots, b^{p-1}$ forms a standard valuation basis with 1 of $(L|K,v)$.*

For the following, cf. [3, Proposition 3.4].

Lemma 2.11. *Take a finite unibranched extension $(L|K,v)$. Then the following are equivalent:*

a) is defectless,
b) $(L|K,v)$ admits a valuation basis,
c) $(L|K,v)$ admits a standard valuation basis,
d) $(L|K,v)$ admits a standard valuation basis with 1.

Proof. Implication a)⇒d) has just been shown above. Implications d)⇒c) and c)⇒b) are trivial. For the implication b)⇒a), see the proof of [3, Proposition 3.4]. □

In particular, for a finite unibranched defectless extension there is always a valuation basis with 1.

Lemma 2.12. *Take a finite unibranched defectless extension $(L|K,v)$ and $a \in L$. Then the set $\{v(a-c) \mid c \in K\}$ has a maximum. More precisely, if we choose a valuation basis $b_1,\ldots,b_n$ for $(L|K,v)$ with $b_1 = 1$ and write*

$$a \;=\; \sum_{i=1}^{n} c_i b_i \;,$$

then $v(a-c_1)$ is the maximum of $\{v(a-c) \mid c \in K\}$.

Proof. For every $c \in K$,

$$\begin{aligned} v(a-c_1) \;&=\; v\sum_{i=2}^{n} c_i b_i \;=\; \min_{2\le i\le n} vc_i b_i \;\ge\; \min\{v(c_1-c)\,,\, vc_i b_i \mid 2 \le i \le n\} \\ &=\; v\left(c_1 - c + \sum_{i=2}^{n} c_i b_i\right) \;=\; v(a-c)\,. \end{aligned}$$

□

Corollary 2.13. *Take a unibranched defectless extension and $a_0 \in L$. Then there is some $c \in K$ such that for $a = a_0 - c$, the elements $1, a, \ldots, a^{p-1}$ form a valuation basis.*

Proof. By Lemma 2.12 there is some $c \in K$ such that $v(a_0 - c) = \max\{v(a_0 - c) \mid c \in K\}$. By Lemma 2.3 this can only happen if either $v(a_0 - c) \notin vK$ or there is some $d \in K$ such that $vd(a_0 - c) = 0$ and $d(a_0 - c)v \notin Kv$. We set $a = a_0 - c$; then in both cases, the elements $1, a, \ldots, a^{p-1}$ form a valuation basis by Lemma 2.10. □

For a more general setting, see Lemma 2.10 and Corollary 2.11 of [3].

3. Computation of ramification ideals

3.1. Basic computations.

Proposition 3.1. *Take a finite unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ with Galois group G. Then every ramification ideal is principal.*

Take a nontrivial subgroup H of G. We will prove the proposition by giving an algorithm for the computation of an element $b_{\min}$ such that for some $\sigma \in H$,

$$v\left(\frac{\sigma b_{\min}}{b_{\min}} - 1\right) \;=\; \min\left\{ v\left(\frac{\sigma b}{b} - 1\right) \;\middle|\; b \in L^\times \,,\, \sigma \in H \right\}\,, \tag{11}$$

which means that $\frac{\sigma b_{\min}}{b_{\min}} - 1$ generates the ramification ideal (8).

Remark 3.2. This proposition was proven in 1970 by P. Ribenboim in [32]. Ribenboim assumes that (L,v) has rank 1, that is, vL is archimedean ordered. Our computations presented below are inspired by his. As they will show, the assumption of rank 1 is not necessary.

A different version of the computation was presented by M. Marshall in [27]. He does not assume that (L,v) has rank 1, but that (K,v) is maximally complete and that the extension $Lv|Kv$ is separable. The assumption that (K,v) is maximally complete means that it has no nontrivial immediate extensions, and this implies that (K,v) is defectless and henselian.

In [31] Ribenboim attempts to prove Proposition 3.1 for all non-discrete valuations and all finite unibranched Galois extensions, but this is false. (We will present counterexamples below.) Ribenboim's mistake was noticed by J. L. Chabert. In [32] Ribenboim then gives a correct proof of Proposition 3.1 for all finite defectless unibranched Galois extensions in the case of rank one valuations. #

We shall now present computations that will not only prove the above proposition, but will also be used later for more advanced results. Let us start with some useful basic principles.

Lemma 3.3. *Let K be any field and take and $\sigma \in \operatorname{Gal} K$.*
1) For all $a,b \in \tilde{K}$ and $c \in K$,

$$\frac{\sigma cab}{cab} - 1 \;=\; \frac{\sigma ab}{ab} - 1 \;=\; \left(\frac{\sigma a}{a} - 1\right)\left(\frac{\sigma b}{b} - 1\right) \,+\, \left(\frac{\sigma a}{a} - 1\right) \,+\, \left(\frac{\sigma b}{b} - 1\right)\,. \tag{12}$$

2) Assume that v is a valuation on $\tilde{K}$ and that $a \in \tilde{K}$ is such that $v\left(\frac{\sigma a}{a} - 1\right) > 0$. Take $i \in \mathbb{N}$ and assume that $i < \operatorname{char} K$ if $\operatorname{char} K > 0$. Then

$$v\left(\frac{\sigma a^i}{a^i} - 1\right) \;=\; v\left(\frac{\sigma a}{a} - 1\right)\,. \tag{13}$$

Proof. 1): We leave the straightforward proof to the reader.
2): By our assumption on i, we have $vi = 0$. Using this together with equation (12), one proves equation (13) by induction on i. □

Further, we will need the following fact.

Lemma 3.4. *Take a normal unibranched extension $(L|L_0,v)$ and pick elements $a_1,\ldots,a_n \in K$. Assume that the elements $b_1,\ldots,b_n \in L$ are valuation independent over K and set*

$$b \;=\; \sum_{i=1}^{n} a_i b_i\,. \tag{14}$$

Then for each automorphism σ of L,

$$v\left(\frac{\sigma b}{b}-1\right)\;\geq\;\min_i v\left(\frac{\sigma a_ib_i}{a_ib_i}-1\right)\,. \tag{15}$$

If in addition σ is trivial on all b_i, then $\frac{\sigma b}{b}-1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_i}{a_i}-1$.

Proof. We have

$$\frac{\sigma b}{b}-1\;=\;\sum_i\left(\frac{\sigma a_ib_i}{a_ib_i}-1\right)\cdot\frac{a_ib_i}{b}\,. \tag{16}$$

Since $vb\leq va_ib_i=v\sigma a_ib_i$ for $1\leq i\leq n$, this implies (15) and that $\frac{\sigma b}{b}-1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_ib_i}{a_ib_i}-1$. If in addition $\sigma b_i=b_i$, then $\frac{\sigma a_ib_i}{a_ib_i}-1=\frac{\sigma a_i}{a_i}-1$. □

We note that if $(L|K,v)$ is a unibranched Galois extension, then for every $\sigma\in\operatorname{Gal}L|K$ and $b\in L^\times$,

$$\frac{\sigma b}{b}-1\;\in\;\mathcal{O}_L\,. \tag{17}$$

Lemma 3.5. *Assume that $(L|K,v)$ is a purely wild Galois extension. Then for every $\sigma\in\operatorname{Gal}L|K$ and all $a,b\in L^\times$,*

$$\frac{\sigma b}{b}-1\;\in\;\mathcal{M}_L \tag{18}$$

and

$$v\left(\frac{\sigma cab}{cab}-1\right)\;\geq\;\min\left\{v\left(\frac{\sigma a}{a}-1\right),\,v\left(\frac{\sigma b}{b}-1\right)\right\}\,, \tag{19}$$

with equality holding if $v\left(\frac{\sigma a}{a}-1\right)\neq v\left(\frac{\sigma b}{b}-1\right)$.

Proof. Equation (18) holds since by the definition of "purely wild extension", $\operatorname{Gal}L|K=G_{\mathcal{M}_L}$. Equation (19) follows from equation (18). □

Proposition 3.6. *Assume that $\mathcal{E}=(L|K,v)$ is a finite unibranched Galois extension with Galois group G.*

1) Assume that $\mathcal{E}$ is defectless and choose a valuation basis b_i, $1\leq i\leq n$. Set

$$\gamma_{\mathcal{E}}\;:=\;\min\left\{v\left(\frac{\sigma b_i}{b_i}-1\right)\,\middle|\,1\leq i\leq n\,,\,\sigma\in G\right\}\,. \tag{20}$$

Then

$$\gamma_{\mathcal{E}}\;=\;\min\left\{v\left(\frac{\sigma b}{b}-1\right)\,\middle|\,b\in L^\times\,,\,\sigma\in G\right\}\;\geq\;0\,. \tag{21}$$

Hence $b_{\min}$ can be chosen to be b_i for suitable i.

2) Assume in addition that $\mathcal{E}$ is purely wild and choose a standard valuation basis y_iz_j, $1\leq i\leq e$, $1\leq j\leq f$ of $(L|K,v)$ as described in Section 2.6. Then

$$\gamma_{\mathcal{E}}\;=\;\min\left\{v\left(\frac{\sigma y_i}{y_i}-1\right),\,v\left(\frac{\sigma z_j}{z_j}-1\right)\,\middle|\,1\leq i\leq \mathrm{e}\,,\,1\leq j\leq \mathrm{f}\,,\,\sigma\in G\right\}\,. \tag{22}$$

3) Assume in addition that $\mathcal{E}$ is purely wild and that L_0 is an intermediate field of $\mathcal{E}$ such that $\mathcal{E}_1 = (L|L_0, v)$ is defectless and that b_i, $1 \leq i \leq n$ is a valuation basis of $(L|L_0, v)$. With respect to this valuation basis, define γ as in (20). Assume further that there is $\gamma_0 \in vL$ such that

$$v\left(\frac{\sigma a}{a} - 1\right) \;\geq\; \gamma_0 \quad \textit{for all } a \in L_0^\times \textit{ and } \sigma \in G\,. \tag{23}$$

Then

$$v\left(\frac{\sigma b}{b} - 1\right) \;\geq\; \min\{\gamma_0\,,\,\gamma\} \quad \textit{for all } b \in L^\times \textit{ and } \sigma \in G\,. \tag{24}$$

If "$>$" holds in (23) and $\gamma \leq \gamma_0$, then

$$\gamma_{\mathcal{E}} \;=\; \gamma\,. \tag{25}$$

Proof. 1): It follows from (17) that $\gamma_{\mathcal{E}} \geq 0$. We apply Lemma 3.4, so $\sigma a = a$ for $a \in L_0$. Take $b \in L$ and write it in the form (14). Then (15) reads as

$$v\left(\frac{\sigma b}{b} - 1\right) \;\geq\; \min_i v\left(\frac{\sigma b_i}{b_i} - 1\right)\,.$$

This proves (21).

2): Take $b \in L$ and write it in the form $b = \sum_{i,j} c_{i\,j} y_i z_j$. Part 1) together with (19) shows that for all $b \in K$,

$$\begin{aligned} v\left(\frac{\sigma b}{b} - 1\right) \;&\geq\; \min\left\{ v\left(\frac{\sigma y_i z_j}{y_i z_j} - 1\right) \;\middle|\; 1 \leq i \leq \mathrm{e}\,,\, 1 \leq j \leq \mathrm{f}\,,\, \sigma \in G \right\} \\ &=\; \min\left\{ v\left(\frac{\sigma y_i}{y_i} - 1\right),\, v\left(\frac{\sigma z_j}{z_j} - 1\right) \;\middle|\; 1 \leq i \leq \mathrm{e}\,,\, 1 \leq j \leq \mathrm{f}\,,\, \sigma \in G \right\}, \end{aligned}$$

which proves our assertion.

3): Take $b \in L$ and write it in the form (14). Using (15) together with (19), we obtain:

$$\begin{aligned} v\left(\frac{\sigma b}{b} - 1\right) \;&\geq\; \min\left\{ v\left(\frac{\sigma a_i b_i}{a_i b_i} - 1\right) \;\middle|\; a_i \in L_0^\times\,,\, 1 \leq i \leq n,\, \sigma \in G \right\} \\ &=\; \min\left\{ v\left(\frac{\sigma a_i}{a_i} - 1\right),\, v\left(\frac{\sigma b_i}{b_i} - 1\right) \;\middle|\; a_i \in L_0^\times\,,\, 1 \leq i \leq n,\, \sigma \in G \right\} \\ &=\; \min\{\gamma_0\,,\,\gamma\}\,, \end{aligned}$$

which proves (24).

Now assume in addition that "$>$" holds in (23) and that $\gamma \leq \gamma_0$. Then

$$v\left(\frac{\sigma a}{a} - 1\right) \;>\; \gamma$$

for all $\sigma \in G$ and $a \in L_0^\times$. Together with (24) and the definition of γ, this implies (25). □

Proof of Proposition 3.1:
Take any nontrivial subgroup H of $G_{\mathcal{M}_L}$ and denote its fixed field in L by K_H. Then also $\mathcal{E}_H := (L|K_H, v)$ is a finite unibranched Galois extension, by Lemma 2.1 it is again defectless, and its Galois group is H. Applying part 1) of Proposition 3.6

with $\mathcal{E}_H$ in place of $\mathcal{E}$, equation (21) yields that $I_H = (a \in L \mid va \geq \gamma_{\mathcal{E}_H})$, which is principal. This proves our proposition. □

Finally, we prove a generalization of a fact that has been used in [33, Section 7.1]. For information on tame and purely wild extensions, see [19, 23].

Proposition 3.7. *Take a henselian field (K, v), a finite purely wild Galois extension $(L|K, v)$ and a tame Galois extension $(K'|K, v)$. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K', v)$ is a purely wild Galois extension of degree $[L : K]$, and*

$$I \mapsto I\mathcal{O}_{L'} \tag{26}$$

is a bijection between the ramification ideals of $(L|K, v)$ and those of $(L'|K', v)$. Its inverse is the function

$$I' \mapsto I' \cap \mathcal{O}_L \tag{27}$$

from the ramification ideals of $(L'|K', v)$ and those of $(L|K, v)$.

Proof. The extensions $L|K$ and $K'|K$ are linearly disjoint and therefore, $L'|K'$ is a Galois extension with its Galois group G isomorphic to $\mathrm{Gal}\, L|K$ via the restriction of its elements to L. Take a finite Galois subextension $(K'_0|K, v)$ of $(K'|K, v)$. It is again tame, and so $(L'_0|L, v)$ is a finite tame Galois extension, where L'_0 is the field compositum $L.K'_0$. In particular, every valuation basis $b_1, \ldots, b_n$ of $(K'_0|K, v)$ is also a valuation basis of $(L'_0|L, v)$.

Each element $b \in L'$ already lies in such a compositum $L'_0 = L.K'_0$, so it can be written as $b = \sum_{1 \leq i \leq n} a_i b_i$ with $b_1, \ldots, b_n$ a valuation basis of $(K'_0|K, v)$ and suitable elements $a_i \in L$.

Now take a ramification ideal $I = I_{H'}$ of $(L'|K', v)$ where H is a nontrivial subgroup of G. Take $\sigma \in H'$ and $0 \neq b \in L'$ such that $\frac{\sigma b}{b} - 1 \in I_{H'}$ and write b in the form as indicated above. Since σ is trivial on K', Lemma 3.4 with L'_0 in place of L and L in place of K shows that $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_{L'_0}$-ideal generated by the elements

$$\frac{\sigma a_i}{a_i} - 1 = \frac{\sigma|_L\, a_i}{a_i} - 1 \in I_H\,,$$

where H is the subgroup $\{\sigma|_L \mid \sigma \in H'\}$ of $\mathrm{Gal}\, L|K$. Therefore,

$$\frac{\sigma b}{b} - 1 \in I_H\mathcal{O}_{L'_0} \subseteq I_H\mathcal{O}_{L'}\,, \tag{28}$$

which shows that the ramification ideal I'_H of $(L'|K', v)$ is a subset of $I_H\mathcal{O}_{L'}$. To prove the reverse inclusion, take an element $\frac{\tau a}{a} - 1 \in I_H$, where $\tau \in H$ and $0 \neq a \in L$. We write $\tau = \sigma|_L$ with $\sigma \in H'$. Then

$$\frac{\tau a}{a} - 1 = \frac{\sigma a}{a} - 1 \in I_{H'}\,.$$

Thus $I_H \subseteq I'_H$, and we obtain

$$I_H\mathcal{O}_{L'} = I'_H\,.$$

This proves that the function (26) sends ramification ideals of $(L|K, v)$ to ramification ideals of $(L'|K', v)$. It also shows that I'_H is the collection of all elements in L' whose value is not less than the value of some element in I_H. This implies that $I'_H \cap \mathcal{O}_L$ is the collection of all elements in L whose value is not less than the value of some element in I_H. In other words, $I'_H \cap \mathcal{O}_L = I_H\mathcal{O}_L = I_H$. Hence, $I_H\mathcal{O}_{L'} \cap \mathcal{O}_L = I_H$, which proves that the function (26) is a bijection, with the function (27) its inverse. □

Remark 3.8. In [33, Section 7.1] only the special case is considered where (K,v) is a henselian field of mixed characteristic, $L|K$ has prime degree p and $K' = K(\zeta_p)$ where ζ_p is a p-th root of unity. The latter implies that $(K'|K,v)$ is a tame extension. This case is of interest when $L|K$, though being Galois, is not a Kummer extension, since $L'|K'$ will be a Kummer extension. #

With a proof adapted from the one of the previous proposition, the following can be shown:

Proposition 3.9. *Take a henselian field (K,v), a finite immediate Galois extension $(L|K,v)$ and a Galois extension $(K'|K,v)$ for which every finite subextension is defectless. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K',v)$ is an immediate Galois extension of degree $[L:K]$, and (26) is again a bijection between the ramification ideals of $(L|K,v)$ and those of $(L'|K',v)$.* □

3.2. **Ramification ideals and defect.**

Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p with Galois group G. For every $\sigma \in G \setminus \{\mathrm{id}\}$ we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma b - b}{b}\right) \,\middle|\, b \in L^\times \right\}. \tag{29}$$

The next theorem follows from [25, Theorems 3.4 and 3.5] together with Theorem 2.4.

Theorem 3.10. *For every generator $a \in L$ of $\mathcal{E}$ and every $\sigma \in G \setminus \{\mathrm{id}\}$,*

$$\Sigma_\sigma = -v(a-K) + v(a-\sigma a), \tag{30}$$

and this set is a final segment of $vK^{>0} = \{\alpha \in vK \mid \alpha > 0\}$ without a smallest element. Moreover, Σ_σ does not depend on the choice of $\sigma \in G \setminus \{\mathrm{id}\}$, and G is the unique ramification group of $\mathcal{E}$.

Our theorem shows that for every Galois defect extension of prime degree, the set (30) is independent of the choice of a and σ, so we denote it by $\Sigma_{\mathcal{E}}$.

Corollary 3.11. *In the situation of Theorem 3.10, the unique ramification ideal of $\mathcal{E} = (L|K,v)$ is the nonprincipal ideal*

$$I_{\mathcal{E}} := I_{\Sigma_{\mathcal{E}}} = \left(\frac{\sigma_0\, a - a}{a-c} \mid c \in K\right) = \left(\frac{\sigma_0(a-c)}{a-c} - 1 \mid c \in K\right), \tag{31}$$

where σ_0 is any generator of G and a is any generator of $L|K$.

Proof. This follows from Theorem 3.10. Since $\Sigma_{\mathcal{E}}$ has no smallest element, $I_{\Sigma_{\mathcal{E}}}$ does not contain an element of smallest value and is thus nonprincipal. □

In what follows, let $(L|K,v)$ be a finite unibranched Galois extension. Denote its ramification field ("Verzweigungskörper" in German) by V. Assuming that $V \neq L$, we wish to investigate the ramification ideals of the Galois extension $(L|V,v)$. Since $\operatorname{Gal} L|V$ is a p-group, $L|V$ is a tower

$$V = K_0 \subset \ldots \subset K_n = L \tag{32}$$

of Galois extensions of degree p such that each extension $K_i|V$ is again a p-extension, $1 \leq i \leq n$. By Lemma 2.1, $(L|K,v)$ is a defect extension if and only if at least one extension of degree p in the tower is a defect extension.

Proposition 3.12. *If the extension $(L|K,v)$ is such that (32) holds with $n \geq 1$ and $(K_n|K_{n-1},v)$ is not defectless, then the smallest ramification ideal of $(L|K,v)$ is nonprincipal.*

Proof. Set $H = \operatorname{Gal} K_n|K_{n-1} \subseteq \operatorname{Gal} L|K$. Since $\subsetneq K_n$, H is a higher ramification group of $(L|K,v)$ and by part 4) of Proposition 2.6, I_H is a ramification ideal of $(L|K,v)$. It is the smallest since H has no nontrivial subgroup. As it is at the same time the unique ramification ideal of the extension $(K_n|K_{n-1},v)$ by part 7) of Proposition 2.6, we know from Corollary 3.11 that it is nonprincipal. □

Corollary 3.13. *Take a finite unibranched Galois extension $(L|K,v)$ and assume that (32) holds with every extension $K_i|K_0$ being Galois. Then $(L|K,v)$ is defectless if and only if for every subextension $(K_i|K,v)$ every ramification ideal is principal.*

Proof. First assume that $(L|K,v)$ is defectless. Then by Lemma 2.1, also every Galois subextension is defectless, and it is again unibranched. Hence by Proposition 3.1, each of its ramification ideals is principal.

Now assume that $(L|K,v)$ is not defectless. Then at least one of the extensions $(K_i|K_{i-1},v)$ in the tower (32) and hence also $(K_i|K,v)$ is not defectless. With K_i in place of L, Proposition 3.12 then shows that the smallest ramification ideal of $(K_i|K,v)$ is nonprincipal. □

3.3. Unibranched Galois extensions of prime degree.

A Galois extension of degree p of a field K of characteristic $p > 0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** $X^p - X - c$ with $c \in K$. A Galois extension of degree p of a field K of characteristic 0 which contains all p-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^p \in K$. For these facts, see [26, Chapter VIII, §8].

If $(L|K,v)$ is a Galois defect extension of degree p of fields of characteristic 0, then a Kummer generator of $L|K$ can be chosen to be a 1-unit. Indeed, choose any Kummer generator η. Since $(L|K,v)$ is immediate, we have that $v\eta \in vK(\eta) = vK$, so there is $c \in K$ such that $vc = -v\eta$. Then $v\eta c = 0$, and since $\eta cv \in K(\eta)v = Kv$, there is $d \in K$ such that $dv = (\eta cv)^{-1}$. Then $v(\eta cd) = 0$ and $(\eta cd)v = 1$. Hence ηcd is a 1-unit. Furthermore, $K(\eta cd) = K(\eta)$ and $(\eta cd)^p = \eta^p c^p d^p \in K$. Thus we can replace η by ηcd and assume from the start that η is a 1-unit. It follows that also $\eta^p \in K$ is a 1-unit.

Throughout this article, whenever we speak of "Artin-Schreier extension" we refer to fields of positive characteristic, and with "Kummer extension" we refer to fields of characteristic 0.

3.3.1. *The defectless case.*

The following proposition is taken from [4]. For the convenience of the reader, and as an illustration of the usefulness of Lemma 2.12, we include its proof here.

Proposition 3.14. *1) Take a valued field (K,v) of equal positive characteristic p and a unibranched defectless Artin-Schreier extension $(L|K,v)$.*

If $\mathrm{f}\,(L|K,v)=p$, then the extension has an Artin-Schreier generator ϑ of value $v\vartheta\leq 0$ such that $Lv=Kv(\tilde{c}\vartheta v)$ for every $\tilde{c}\in K$ with $v\tilde{c}\vartheta=0$; the extension $Lv|Kv$ is separable if and only if $v\vartheta=0$.

If $\mathrm{e}\,(L|K,v)=p$, then the extension has an Artin-Schreier generator ϑ such that $vL=vK+\mathbb{Z}v\vartheta$. Every such ϑ satisfies $v\vartheta<0$.

2) Take a valued field (K,v) of mixed characteristic and a unibranched defectless Kummer extension $(L|K,v)$ of degree $p=\mathrm{char}\,Kv$. Then the extension has a Kummer generator η such that:

a) if $\mathrm{f}\,(L|K,v)=p$, then either ηv generates the residue field extension, in which case it is inseparable, or η is a 1-unit and for some $\tilde{c}\in K$, $\tilde{c}(\eta-1)v$ generates the residue field extension;

b) if $\mathrm{e}\,(L|K,v)=p$, then either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta-1)$ generates the value group extension.

Proof. 1): Take any Artin-Schreier generator y of $(L|K,v)$. Then by Lemma 2.12 there is $c\in K$ such that either $v(y-c)\notin vK$, or for every $\tilde{c}\in K$ such that $v\tilde{c}(x-c)=0$ we have $\tilde{c}(y-c)v\notin Kv$. Since p is prime, in the first case it follows that $\mathrm{e}\,(L|K,v)=p$ and that $v(y-c)$ generates the value group extension. In the second case it follows that $\mathrm{f}\,(L|K,v)=p$ and that $\tilde{c}(y-c)v$ generates the residue field extension. In both cases, $\vartheta=y-c$ is an Artin-Schreier generator. Let $\vartheta^p-\vartheta=b\in K$.

Assume that $\mathrm{f}\,(L|K,v)=p$. If $v\vartheta<0$, then $v(\vartheta^p-b)=v\vartheta>pv\vartheta=v\vartheta^p$, whence $v((\tilde{c}\vartheta)^p-\tilde{c}^pb)=v\tilde{c}^p\vartheta>v(\tilde{c}\vartheta)^p$ for $\tilde{c}\in K$ with $v\tilde{c}\vartheta=0$ and therefore, $(\tilde{c}\vartheta)^pv=\tilde{c}^pbv\in Kv$. In this case, the residue field extension is inseparable. Now assume that $v\vartheta\geq 0$ and hence also $vb\geq 0$. The reduction of X^p-X-b to $Kv[X]$ is a separable polynomial, so $Lv|Kv$ is separable. The polynomial X^p-X-bv cannot have a zero in Kv, since otherwise the p distinct roots of this polynomial give rise to p distinct extensions of v from K to L, contradicting our assumption that $(L|K,v)$ is unibranched. Consequently, $bv\neq 0$, whence $vb=0$ and $v\vartheta=0$.

Assume that $\mathrm{e}\,(L|K,v)=p$. If $v\vartheta\geq 0$, then $vb\geq 0$ and ϑv is a root of X^p-X-bv. If this polynomial does not have a zero in Kv, then ϑv generates a nontrivial residue field extension, contradicting our assumption that $\mathrm{e}\,(L|K,v)=p$. If the polynomial has a zero in Kv, then similarly as before one deduces that $(L|K,v)$ is not unibranched, contradiction. Hence $v\vartheta<0$.

2): Take any Kummer generator y of $(L|K,v)$. If there is a Kummer generator η such that $v\eta\notin vK$, then it follows as before that $\mathrm{e}\,(L|K,v)=p$ and that $v\eta$ generates the value group extension. Now assume that there is no such η.

If there is a Kummer generator y and some $\tilde{c} \in K$ such that $v\tilde{c}y = 0$ and $\tilde{c}yv \notin Kv$, then it follows as before that $\mathrm{f}(L|K,v) = p$ and that $\tilde{c}yv$ generates the residue field extension. We set $\eta = \tilde{c}y$ and observe that also η is a Kummer generator. Since $(\eta v)^p \in Kv$, $Lv|Kv$ is purely inseparable in this case.

Now assume that the above cases do not appear, and choose an arbitrary Kummer generator y of $(L|K,v)$. Consequently, we have that $vy \in vK$ and $\tilde{c}yv \in Kv$ for all $\tilde{c} \in K$ with $v\tilde{c}y = 0$. Then as described at the start of this section, there are $c_1, c_2 \in K$ such that c_2c_1y is a Kummer generator of $(L|K,v)$ which is a 1-unit. We replace y by c_2c_1y.

By Lemma 2.12 there is $c \in K$ such that $v(y-c)$ is maximal in $v(y-K)$ and either $v(y-c) \notin vK$ or there is some $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since y is a 1-unit, we know that $v(y-1) > 0$, hence also $v(y-c) > 0 = vy$, showing that also c is a 1-unit. Then $\eta := c^{-1}y$ is again a Kummer generator of $(L|K,v)$ which is a 1-unit. Since $vc = 0$, we know that $v(\eta - 1) = vc(\eta - 1) = v(y-c)$. Hence if $v(y-c) \notin vK$, then $v(\eta - 1)$ generates the value group extension.

Now assume that there is $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since c is a 1-unit, it follows that $v\tilde{c}(\eta-1) = v\tilde{c}c(\eta-1) = v\tilde{c}(y-c) = 0$ and $\tilde{c}(\eta-1)v = \tilde{c}c(\eta-1)v = \tilde{c}(y-c)v$. We find that $\tilde{c}(\eta-1)v$ generates the residue field extension. □

From this proposition we deduce:

Theorem 3.15. *Take a unibranched defectless Galois extension $(L|K,v)$ of prime degree p.*

1) If $\mathcal{E} = (L|K,v)$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis for $(L|K,v)$. The element $b_{\min}$ as in (11) can be chosen to be ϑ, so that

$$I_{\mathcal{E}} = \left(\frac{1}{\vartheta}\right) . \tag{33}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v\vartheta = 0$, and this holds if and only if $Lv|Kv$ is separable of degree p.

2) Let $\mathcal{E} = (L|K,v)$ be a Kummer extension. Then there are two cases:

a) $(L|K,v)$ admits a Kummer generator η such that $v\eta \geq 0$ and $1, \eta, \ldots, \eta^{p-1}$ form a valuation basis for $(L|K,v)$. In this case, $b_{\min}$ can be chosen to be η and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$ and

$$I_{\mathcal{E}} = (\zeta_p - 1) . \tag{34}$$

b) $(L|K,v)$ admits a Kummer generator η such that η is a 1-unit with $v(\eta-1) \leq v(\zeta_p - 1)$ and $1, \eta-1, \ldots, (\eta-1)^{p-1}$ is a valuation basis for $(L|K,v)$. In this case, $b_{\min}$ can be chosen to be $\eta - 1$ and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1) - v(\eta - 1)$ and

$$I_{\mathcal{E}} = \left(\frac{\zeta_p - 1}{\eta - 1}\right) . \tag{35}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v(\eta-1) = v(\zeta_p - 1)$, and this holds if and only if $Lv|Kv$ is separable of degree p.

Proof. Throughout the proof we use part 1) of Proposition 3.6,

1): By part 1) of Proposition 3.14 there exists an Artin-Schreier generator ϑ of value $v\vartheta \le 0$ such that $v\vartheta$ generates the value group extension, or $v\tilde{c}\vartheta = 0$ and $Lv = Kv(\tilde{c}\vartheta v)$ for some $\tilde{c} \in K$. By Lemma 2.10, it follows that $1, \vartheta, \ldots, \vartheta^{p-1}$ is a valuation basis for $(L|K, v)$.

If $v\vartheta < 0$, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{\sigma\vartheta - \vartheta}{\vartheta}\right) = -v\vartheta = v\left(\frac{1}{\vartheta}\right) > 0 \tag{36}$$

for every $\sigma \in \operatorname{Gal} L|K \setminus \{\mathrm{id}\}$ since then $\sigma\vartheta - \vartheta \in \mathbb{F}_p \setminus \{0\}$. Hence by Lemma 3.3, for $1 \le j \le p-1$ we have

$$v\left(\frac{\sigma\vartheta^j}{\vartheta^j} - 1\right) = v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{1}{\vartheta}\right) .$$

This proves that $b_{\min}$ can be chosen to be ϑ in this case.

If $v\vartheta = 0$, which by part 1) of Proposition 3.14 holds if and only if $Lv|Kv$ is separable of degree p, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) = v\left(\frac{1}{\vartheta}\right) = 0 ,$$

and as the value $\gamma_{\mathcal{E}}$ defined in (20) is non-negative, this is equivalent to $I_{\mathcal{E}} = \mathcal{O}_L$.

2): By part 2) of Proposition 3.14 there exists a Kummer generator η such that either

a) $v\eta$ generates the value group extension, or ηv generates the residue field extension, or

b) η is a 1-unit and $v(\eta - 1)$ generates the value group extension or for some $\tilde{c} \in K$, $\tilde{c}(\eta - 1)v$ generates the residue field extension.

We first consider case a). By Lemma 2.10, it follows that $1, \eta, \ldots, \eta^{p-1}$ is a valuation basis for $(L|K, v)$. If $v\eta$ generates the value group extension, then so does $v\eta^{-1}$. Therefore, we can assume that $v\eta \ge 0$. For $1 \le j \le p-1$,

$$v\left(\frac{\sigma\eta^j}{\eta^j} - 1\right) = v\left(\frac{\sigma\eta^j - \eta^j}{\eta^j}\right) = v\left(\frac{\zeta_p^k\eta^j - \eta^j}{\eta^j}\right) = v(\zeta_p^k - 1) = v(\zeta_p - 1)$$

for some $k \in \mathbb{N}$; the last equation holds since $v(\zeta - 1) = vp/(p-1)$ for every primitive p-th root of unity ζ (cf. [4, Lemma 2.5]). This proves that in case a), $b_{\min}$ can be chosen to be η and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$.

Now we consider case b). Again by Lemma 2.10, $1, \eta - 1, \ldots, (\eta - 1)^{p-1}$ is a valuation basis for $(L|K, v)$. Since $v\eta = 0$, we have

$$v\left(\frac{\sigma\eta - 1}{\eta - 1} - 1\right) = v\left(\frac{\sigma\eta - \eta}{\eta - 1}\right) = v(\zeta_p - 1) - v(\eta - 1) .$$

This value must be non-negative since it is not less than $\gamma_{\mathcal{E}}$. If it is equal to 0, then it must be equal to $\gamma_{\mathcal{E}}$. If it is positive, then we can apply Lemma 3.3, obtaining that for $1 \le j \le p-1$,

$$v\left(\frac{\sigma(\eta - 1)^j}{(\eta - 1)^j} - 1\right) = v\left(\frac{\sigma\eta - 1}{\eta - 1} - 1\right)$$

and consequently, this value is again equal to $\gamma_{\mathcal{E}}$. Hence in case b), $b_{\min}$ can be chosen to be $\eta-1$ and we have $\gamma_{\mathcal{E}} = v(\zeta_p-1)-v(\eta-1)$. We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if the ramification field of $(L|K,v)$ is equal to L, which means that p does not divide $\mathrm{e}\,(L|K,v)$ and $Lv|Kv$ must be separable. Since $(L|K,v)$ is assumed to be unibranched and defectless of degree p, this can only hold if and only if $Lv|Kv$ is separable of degree p. □

Remark 3.16. Equation (35) also holds in case 2 a) of the previous theorem since in this case, $v(\eta-1)=0$. Indeed, in that case we have $v\eta\geq 0$, and $1,\eta,\ldots,\eta^{p-1}$ form a valuation basis for $(L|K,v)$. If $v\eta>0$, then $v(\eta-1)=0$. If $v\eta=0$, then $1,\eta v,\ldots,(\eta v)^{p-1}$ form a basis of $Lv|Kv$, so $\eta v\neq 1$, whence $v(\eta-1)=0$ again. #

3.3.2. *The defect case.*

The next results follow from Theorem 3.10 and are part of [25, Theorems 3.4 and 3.5].

Theorem 3.17. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree with Galois group G. If $(L|K,v)$ is an Artin-Schreier defect extension with any Artin-Schreier generator ϑ, then*

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta-K)\,. \tag{37}$$

If K contains a primitive root of unity ζ_p and $(L|K,v)$ is a Kummer extension with Kummer generator η of value 0, then

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p-1)\;-\;v(\eta-K)\;=\;\frac{vp}{p-1}\;-\;v(\eta-K)\,. \tag{38}$$

Theorem 3.18. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree p.*
1) If $(L|K,v)$ is an Artin-Schreier extension with Artin-Schreier generator ϑ, then

$$\begin{aligned} I_{\mathcal{E}} &= \left(\frac{1}{\vartheta-c}\;\mid\; c\in K\right)\\ &= \left(\frac{1}{b}\;\mid\; b\;\text{ an Artin-Schreier generator of } L|K\right)\,. \end{aligned}$$

2) Let $(L|K,v)$ be a Kummer extension with a Kummer generator η which is a 1-unit, and ζ_p a primitive p-th root of unity. Then

$$\begin{aligned} I_{\mathcal{E}} &= \left(\frac{\zeta_p-1}{\eta-c}\;\mid\; c\in K\;\text{ a } 1\text{-unit}\right)\\ &= \left(\frac{\zeta_p-1}{b-1}\;\mid\; b\;\text{ a Kummer generator of } L|K \text{ which is a } 1\text{-unit}\right)\,. \end{aligned}$$

Proof. 1): The first equation follows from equation (31) of Corollary 3.11, where we take σ_0 such that $\sigma_0\vartheta=\vartheta+1$. The ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation because $\vartheta-c$ is again an Artin-Schreier generator for every $c\in K$. Further, by Corollary 3.11 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the second equation follows from the first.

2): The first equation follows from equation (31) of Corollary 3.11, where we take σ_0 such that $\sigma_0\eta = \zeta_p\eta$, because then $\sigma_0(\eta - c) - (\eta - c) = (\zeta_p - 1)\eta$ and we can drop η since it is a unit. Further, we can restrict c to 1-units since if c is not a 1-unit, then $v(\eta - c) \leq 0 < v(\eta - 1)$ and $\frac{\zeta_p-1}{\eta-c} \in \left(\frac{\zeta_p-1}{\eta-1}\right)$.

When c is a 1-unit, then $\eta - c = c(\frac{\eta}{c} - 1)$, the quotient $b = \frac{\eta}{c}$ is again a Kummer generator which is a 1-unit, and we can drop the unit factor c. This shows that the ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation. Further, by Corollary 3.11 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the second equation again follows from the first. □

3.4. **When does the equality $I_{\mathcal{E}} = \mathcal{M}_L$ hold?**

Throughout, we assume that $\mathcal{E} = (L|K, v)$ is a purely wild Galois extension of degree $p = \operatorname{char} Kv$. If $\operatorname{char} K = 0$, we assume in addition that K contains a primitive p-th roots of unity ζ_p, so that $L|K$ is a Kummer extension. Under these assumptions, we will determine the cases where $I_{\mathcal{E}} = \mathcal{M}_L$.

3.4.1. *The defectless case.*

We assume $\mathcal{E}$ to be defectless. Then we know from Proposition 3.1 that $I_{\mathcal{E}}$ is principal. Hence for it to be equal to $\mathcal{M}_L$, the latter and also $\mathcal{M}_K$ must be principal. As $\mathcal{E}$ is purely wild, the extension $Lv|Kv$ cannot be separable of degree p.

Proposition 3.19. *Let the assumptions be as described above.*

1) Assume that $\mathcal{E}$ is an Artin-Schreier extension. If $(vL : vK) = p$, then $I_{\mathcal{E}} = \mathcal{M}_L$ if and only if $\mathcal{E}$ admits an Artin-Schreier generator ϑ such that $-v\vartheta$ is the smallest positive element of vL and $-pv\vartheta$ is the smallest positive element of vK. If $[Lv : Kv] = p$, then $I_{\mathcal{E}} = \mathcal{M}_L$ if and only if $\mathcal{E}$ admits an Artin-Schreier generator ϑ such that $-v\vartheta$ is the smallest positive element of $vL = vK$.

2) Assume that $\mathcal{E}$ is a Kummer extension. Then $I_{\mathcal{E}} = \mathcal{M}_L$ holds if and only if $\mathcal{E}$ admits a Kummer generator η such that one of the following cases holds:

(a) $\mathrm{e}(L|K, v) = p$ and $v\eta$ generates the value group extension or $\mathrm{f}(L|K, v) = p$, $v\eta = 0$ and ηv generates the residue field extension, and $\mathcal{M}_L = (\zeta_p - 1)\mathcal{O}_L$ as well as $\mathcal{M}_K = (\zeta_p - 1)\mathcal{O}_K$,

(b) $\mathrm{e}(L|K, v) = p$, η a 1-unit, $v(\eta - 1)$ generates the value group extension or $\mathrm{f}(L|K, v) = p$ and $v\tilde{c}(\eta - 1)$ generates the residue field extension for some $\tilde{c} \in K$, and $\mathcal{M}_L = \frac{\zeta_p-1}{\eta-1}\mathcal{O}_L$.

Proof. Our statements follow from Proposition 3.14 together with Theorem 3.15. In case 2(a) note that $\mathcal{M}_L = (\zeta_p - 1)\mathcal{O}_L$ means that $v(\zeta_p - 1)$ is the smallest positive element in vL, which implies that it also is the smallest positive element in vK. This is clear if $\mathrm{f}(L|K, v) = p$, whence $vL = vK$. On the other hand, if $\mathrm{e}(L|K, v) = p$, then since $v(\zeta_p - 1) = \frac{vp}{p-1}$ and $vp \in vK$, we again must have that $v(\zeta_p - 1)$ is the smallest positive element in vK. □

Note that if case 2(a) holds with $\mathrm{e}\,(L|K,v) = p$, then vK cannot be archimedean, because $v(\zeta_p - 1)$ is the smallest positive element of both vL and vK.

Let us give examples for the different types of extensions appearing in the proposition.

• Artin-Schreier extension with $(vL : vK) = p$: take a valued field (K, v) of characteristic $p > 0$ such that vK has a smallest positive element vc, $c \in K$. Let ϑ be a root of $X^p - X - c^{-1}$. Then $v\vartheta = -vc/p$ (cf. [12, Lemma 2.12]). Thus $v\vartheta^{-1}$ is the smallest positive element of $vK(\vartheta)$, whence $I_{\mathcal{E}} = \mathcal{M}_L$ for $L = K(\vartheta)$.

• Artin-Schreier extension with $[Lv : Kv] = p$: take (K, v) and $c \in K$ as before. Further, assume that Kv contains an element dv, $d \in \mathcal{O}_K^\times$, which does not have a p-th root in Kv. Let ϑ be a root of $X^p - X - c^{-p}d$. Then $v\vartheta = -vc$ and $v(c\vartheta - d) > 0$, so that $c\vartheta v$ is a p-th root of dv (cf. [12, Lemma 2.13]). We obtain $[Lv : Kv] = p$ for $L = K(\vartheta)$, so $vL = vK$. As $v\vartheta^{-1} = vc$ is the smallest positive element of $vK = vL$, it follows that $I_{\mathcal{E}} = \mathcal{M}_L$.

• Kummer extension with Kummer generator η such that $v\eta$ generates the value group extension, with $(vL : vK) = p$: take $K = \mathbb{Q}_p(\zeta_p, t)$, where t is transcendental over $\mathbb{Q}_p(\zeta_p)$ and extend the p-adic valuation to a valuation v of K in such a way that vK is the lexicographic product $vt\,\mathbb{Z} \times v(\zeta_p - 1)\mathbb{Z}$. Let η be a root of $X^p - t$ and set $L := K(\eta)$. Then $(vL : vK) = p$, $v\eta$ generates the value group extension, and vL is the lexicographic product $\frac{vt}{p}\mathbb{Z} \times v(\zeta_p - 1)\mathbb{Z}$. Consequently, $v(\zeta_p - 1)$ is still the smallest positive element of vL, showing that $I_{\mathcal{E}} = \mathcal{M}_L$.

• Kummer extension with Kummer generator η such that $v\eta = 0$ and ηv generates the residue field extension, with $[Lv : Kv] = p$: take again $K = \mathbb{Q}_p(\zeta_p, t)$, but now extend the p-adic valuation to a valuation v of K in such a way that v is the Gauß vakuation of the rational function field $K = \mathbb{Q}_p(\zeta_p)(t)$. Then tv is transcendental over $\mathbb{Q}(\zeta_p)v = \mathbb{F}_p$ and does not have a p-th root in Kv. Let η be a root of $X^p - t$ and set $L := K(\eta)$. Then $\eta v = (tv)^{1/p}$ and ηv generates the residue field extension. Since $[Lv : Kv] = p$ implies $vL = vK$, $v(\zeta_p - 1)$ is still the smallest positive element of vL, showing that $I_{\mathcal{E}} = \mathcal{M}_L$.

• To construct extensions described in case 2(b), take a valued field (K, v) of characteristic 0 with residue characteristic $p > 0$ and assume that $\zeta_p \in K$. Take $c \in \mathcal{O}_K$ and a root a of the polynomial

$$f(X) \;=\; (X+1)^p - c \;=\; X^p + \sum_{0<i<p} \binom{p}{i} X^{p-i} + 1 - c\,. \tag{39}$$

If $a \notin K$, then $\eta := a + 1$ is a Kummer generator of $K(a)|K$, while $a = \eta - 1$ is not.

For $0 < i < p$, each binomial coefficient in (39) has value $vp > 0$ and we have $v\binom{p}{i}a^{p-i} \geq vp + va$. It follows that $va^p = pva$ is smaller than the value of all the terms $\binom{p}{i}a^{p-i}$, and consequently must be equal to $v(1-c)$, if and only if $pva < vp + va$, that is, $va < \frac{vp}{p-1}$. If this is the case, then $v(a^p - (1-c)) > va^p$ and thus $v(1-c) = pva < \frac{pvp}{p-1}$. Conversely, if $v(1-c) < \frac{pvp}{p-1}$, then $va < \frac{vp}{p-1}$ since otherwise, $v\binom{p}{i}a^{p-i} \geq vp + va \geq \frac{pvp}{p-1} > v(1-c)$ so that $va^p = v(1-c) < \frac{pvp}{p-1}$, whence $va < \frac{vp}{p-1}$, a contradiction.

Let us give an example of an extension with $(vL : vK) = p$. Take $K = \mathbb{Q}(\zeta_p)$ with the extension v of the p-adic extension. Then $vK = \frac{1}{p-1}\mathbb{Z}$, so vp is not p-divisible in vK. Set $c = 1 - p$ so that $1 - c = p$. Thus $v(1-c) = vp < \frac{pvp}{p-1}$ and we obtain from our above computations that $v(\eta - 1) = va = \frac{vp}{p}$, which generates the value group extension. Further,

$$v\,\frac{\zeta_p - 1}{\eta - 1} \;=\; \frac{vp}{p-1} - \frac{vp}{p} \;=\; \frac{vp}{p(p-1)}\,, \tag{40}$$

which is the smallest positive element in vL. This shows that $I_{\mathcal{E}} = \mathcal{M}_L$.

Now we give an example of an extension with $[Lv : Kv] = p$. Choose $\tilde{c}$ such that $\tilde{c}^{-p} = p$ and extend the p-adic valuation to $\mathbb{Q}(\zeta_p, \tilde{c})$. Further, take a transcendental element t and v to be the Gauß valuation on the rational function field $K := \mathbb{Q}(\zeta_p, \tilde{c})(t)$. Then tv is transcendental over $\mathbb{Q}(\zeta_p, \tilde{c})v = \mathbb{F}_p$ and does not have a p-th root in Kv. Set $c = 1 - pt$ so that $1 - c = pt$. Thus $v(1-c) = vpt = vp < \frac{pvp}{p-1}$ and again we obtain from our above computations that $v(a^p - \tilde{c}^{-p}t) = v(a^p - pt) > va^p$ and $v(\eta - 1) = va = \frac{vp}{p}$. The former implies

$$v(\tilde{c}^p(\eta - 1)^p - t) \;=\; v(\tilde{c}^p a^p - t) \;>\; v\tilde{c}^p a^p \;=\; 0\,.$$

This implies $\tilde{c}(\eta - 1)v = (tv)^{1/p}$. We set $L := K(\eta) = K(a)$ and observe that $[Lv : Kv] = p$, so that $vL = vK$. Further, (40) again holds, so that $v\frac{\zeta_p - 1}{\eta - 1}$ is the smallest positive element in $vK = vL$. This shows that $I_{\mathcal{E}} = \mathcal{M}_L$.

3.4.2. *The defect case.*
We assume $\mathcal{E}$ to be a defect extension. Then $\mathcal{E}$ is immediate and we know from Theorem 2.5 and Theorem 3.18 that $vI_{\mathcal{E}} = \Sigma_{\mathcal{E}}$ has no minimal element, so $I_{\mathcal{E}}$ is nonprincipal. Hence for it to be equal to $\mathcal{M}_L$, both $\mathcal{M}_L$ and $\mathcal{M}_K$ must be nonprincipal.

We note:

Lemma 3.20. *Let the assumptions be as described above. Then $I_{\mathcal{E}} = \mathcal{M}_L$ holds if and only if $\Sigma_{\mathcal{E}} = \{\alpha \in vL \mid \alpha > 0\}$.*

This shows that if $I_{\mathcal{E}} = \mathcal{M}_L$ holds, then $\mathcal{E}$ has independent defect in the sense of [25]. However, the converse does only hold if vL is archimedean, because otherwise $\mathcal{E}$ has independent defect if and only if $I_{\mathcal{E}}$ is equal to the maximal ideal of any coarsening of $\mathcal{O}_{\mathcal{E}}$ other than L itself.

For a famous example of an Artin-Schreier extension with independent defect satisfying the equation $I_{\mathcal{E}} = \mathcal{M}_L$ originally due to Shreeram Abhyankar, see [17, Example 3.12]. For other examples, see [29, Proposition 6.14]. In that paper, also Galois extensions $(L|K, v)$ with higher powers of p as their degrees are constructed whose unique ramification ideal is equal to $\mathcal{M}_L$: see Corollary 6.10 and the interesting system of Artin-Schreier extensions in Section 6.3.

Finally, we present an example of a Kummer extension with independent defect satisfying the equation $I_{\mathcal{E}} = \mathcal{M}_L$; it has been developed with the help of Konstantinos Kartas. We work over $\mathbb{Q}_p$ and assume $p \neq 2$ (for simplicity). Let K be the p-cyclotomic field which is obtained by adjoining a primitive p-th root of unity ζ_p to $\mathbb{Q}_p$ and then closing under a compatible system of p^n-th roots of ζ_p. The

p-adic valuation of $\mathbb{Q}_p$ extends (uniquely) to a valuation v on K, which again is henselian.

We consider the unibranched extension $\mathcal{E} = (K(p^{1/p})|K, v)$. First, we note that $p^{1/p} \notin K$; otherwise $Q_p(p^{1/p})$ would be a subfield of the abelian extension $K|\mathbb{Q}_p$ and hence would be Galois over $\mathbb{Q}_p$ (since every subgroup of an abelian group is normal), which is not the case. Therefore, $K(p^{1/p})|K$ is a proper extension and thus must be of degree p. As $\zeta_p \in K$, it is a Kummer extension. We claim that $(K(p^{1/p})|K, v)$ is immediate. Since vK is p-divisible, it suffices to show that $K(p^{1/p})v = Kv$. But if this were not true, then $K(p^{1/p})$ would be contained in the maximal unramified extension of K, which is equal to $\mathbb{Q}_p^{ab}$. Again, this is a contradiction because this would imply that $\mathbb{Q}_p(p^{1/p})|\mathbb{Q}_p$ is Galois. Therefore $(K(p^{1/p})|K, v)$ must be immediate. As the extension is unibranched, it has defect p.

It is known that (K, v) is a deeply ramified field (this follows e.g. from computations in [10]). Hence by [25, part (1) of Theorem 1.10], $(K(p^{1/p})|K, v)$ has independent defect. We set $L := K(p^{1/p})$; as an algebraic extension of $\mathbb{Q}_p$, it has archimedean value group. Therefore, $I_{\mathcal{E}} = \mathcal{M}_L$.

3.5. **Defect does not always imply nonprincipality of ramification ideals.** We are going to give an example of a Galois defect extension $(L|K, v)$ of degree p^2, $p = \operatorname{char} K > 0$, which is a tower of two Galois extensions of degree p, the upper one defectless and the lower a defect extension, but has only one ramification ideal, this being principal.

We will construct a tower of two Galois extensions $L|L_0$ and $L_0|K$ of degree $p = \operatorname{char} K$. We need a criterion for $L|K$ to be Galois. We set $\wp(X) := X^p - X$. The following is Lemma 2.9 in [28]:

Lemma 3.21. *Take Artin-Schreier extensions $L|L_0$ and $L_0|K$, and an Artin-Schreier generator ϑ of $L|L_0$ with $\vartheta^p - \vartheta = b \in L_0$. Then $L|K$ is a Galois extension if and only if $\sigma_0 b - b \in \wp(L_0)$ for some generator σ_0 of* $\operatorname{Gal} L_0|K$*.*

Consider the rational function field $\widetilde{F_p}(t)$ with the t-adic valuation $v = v_t$. Extend v to its algebraic closure and let $K_0 = \widetilde{F_p}(t)^r$ be the respective ramification field. Then vK_0 is a subgroup of $\mathbb{Q}$ divisible by each prime other than p, but vt is not divisible by p in vK_0. Choose a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ in vK_0 with upper bound $-1/p$ and starting with $q_1 = -1$. Define

$$s \;:=\; \sum_{i\in\mathbb{N}} t^{pq_i} \;\in\; \widetilde{F_p}((t^{\mathbb{Q}}))\;.$$

Take (K, v) to be the henselization of $(K_0(s), v)$.

Let ϑ_0 be a root of the Artin-Schreier polynomial $X^p - X - s$. Define

$$c_k \;:=\; \sum_{i=1}^{k} t^{q_i} \;\in\; K\;.$$

We compute:

$$v(\vartheta_0 - c_k)^p \;=\; v(\vartheta_0^p - c_k^p) \;=\; v(\vartheta_0 + s - c_k^p) \;\geq\; \min\{v\vartheta_0, v(s - c_k^p)\}\;.$$

Since $vs = -pvt < 0$, we have $v\vartheta_0 = -vt$. Further, $v(s - c_k^p) = pq_{k+1}vt < -vt$ since $q_{k+1} < -1/p$. It follows that $v(\vartheta_0 - c_k)^p = pq_{k+1}vt$, so that $v(\vartheta_0 - c_k) = q_{k+1}vt$. This increasing sequence of values is contained in $v(\vartheta_0 - K)$. It must be cofinal, showing that $v(\vartheta_0 - K)$ has no maximal element, because the pseudo Cauchy sequence $(c_k)_{k\in\mathbb{N}}$ has no limit in (K, v). It thus follows from Lemma 2.5 that for $L_0 := K(\vartheta_0)$, the extension $\mathcal{E}_0 := (L_0|K, v)$ is immediate and thus a defect extension. From Theorem 3.17 we know that

$$vI_{\mathcal{E}_0} \;=\; -v(\vartheta_0 - K)\,,$$

which has no minimal element and lower bound $\gamma := vt/p \notin vI_{\mathcal{E}_0}$. Hence $I_{\mathcal{E}_0}$ is nonprincipal. However, we will construct the extension $(L|K, v)$ such that $I_{\mathcal{E}_0}$ is not a ramification ideal of it.

Let ϑ be a root of the Artin-Schreier polynomial $X^p - X - \vartheta_0$, and set $L := L_0(\vartheta) = K(\vartheta_0, \vartheta)$. Since $v\vartheta_0 = -vt < 0$, We have $v\vartheta = -vt/p \notin vK = vL_0$. Hence by Corollary 2.13, the elements $1, \vartheta, \dots, \vartheta^{p-1}$ form a valuation basis of $\mathcal{E}_1 := (L|K(\vartheta_0), v)$, showing that this extension is defectless. By part 1) of Theorem 3.15,

$$I_{\mathcal{E}_1} \;=\; \left(\frac{1}{\vartheta}\right)\,,$$

so the minimum of $vI_{\mathcal{E}_1}$ is $-v\vartheta = vt/p = \gamma$, which is smaller than the values of all elements of $vI_{\mathcal{E}_0}$.

Since $\vartheta^p - \vartheta = \vartheta_0$, we have $L = K(\vartheta)$. To show that $L|K$ is a Galois extension, take some generator σ_0 of $\mathrm{Gal}\, L_0|K$. Since $\sigma_0\vartheta_0$ is also a root of $X^p - X - s$, we have $\sigma_0\vartheta_0 - \vartheta_0 = i$ for some $i \in \mathbb{F}_p$. As K contains $\widetilde{\mathbb{F}_p}$, it contains the Artin-Schreier roots of i, i.e., $i \in \wp(K) \subseteq \wp(L_0)$. Now Lemma 3.21 shows that $L|K$ is a Galois extension. However, by Corollary 2.10 of [28] it is not cyclic, and the discussion leading up to this corollary shows the following. Take $\sigma \in G = \mathrm{Gal}\, L|K$ such that $\sigma\vartheta_0 - \vartheta_0 = 1$. Then $\zeta := \sigma\vartheta - \vartheta$ satisfies $\zeta^p - \zeta = 1$ and is therefore an element of $\widetilde{\mathbb{F}_p} \subset K_0$. Note that for every $n \in \mathbb{N}$, $\sigma^n\vartheta = n\zeta + \vartheta$.

Further, take $\tau \in G$ such that $\tau\vartheta - \vartheta = 1$. Then τ is trivial on L_0 and σ and τ commute. Thus the subgroups of G of order p are generated by the automorphisms τ and $\sigma\tau^i$, $0 \le i \le p-1$. Note that for every $n \in \mathbb{N}$, $\tau^n\vartheta = \vartheta + n$.

Let us first consider the subgroup $\langle\tau\rangle$ of G. Since $\langle\tau\rangle = \mathrm{Gal}\, L|L_0$, the ramification ideal $I_{\langle\tau\rangle}$ is the ramification ideal $I_{\mathcal{E}_1}$ of the extension $\mathcal{E}_1$.

Let us now consider the subgroups $\langle\sigma\tau^i\rangle$ of G, for $0 \le i \le p-1$. Since τ is trivial on L_0, the restrictions of all elements of each subgroup $\langle\sigma\tau^i\rangle$ form the Galois group of $\mathcal{E}_0$. Therefore,

$$v\left(\frac{\rho a}{a} - 1\right) \;>\; \gamma \quad \text{for all } a \in L_0^\times \text{ and } \rho \in \mathrm{Gal}\,\mathcal{E}_0\,. \tag{41}$$

For $1 \le k \le p-1$ we have $(\sigma\tau^i)^k = \sigma^k\tau^{ik}$ and

$$\sigma^k\tau^{ik}\vartheta - \vartheta \;=\; \sigma^k(\vartheta + ik) - \vartheta \;=\; k\zeta + \vartheta + ik - \vartheta \;=\; k\zeta + ik \;\in\; \widetilde{\mathbb{F}_p}\,,$$

hence $v(\sigma^k\tau^{ik}\vartheta - \vartheta) = 0$ and

$$v\left(\frac{\sigma^k\tau^{ik}\vartheta}{\vartheta} - 1\right) \;=\; -v\vartheta \;=\; \gamma\,.$$

Applying part 2) of Lemma 3.3, we find that for $1 \leq \ell \leq p-1$,

$$v\left(\frac{\sigma^k \tau^{ik} \vartheta^\ell}{\vartheta^\ell} - 1\right) \;=\; \gamma\,. \tag{42}$$

Now we can apply part 3) of Proposition 3.6 to deduce that (25) holds with $\langle \sigma\tau^i \rangle$ in place of G. This shows that also the ramification ideals $I_{\langle \sigma\tau^i \rangle}$ are equal to $I_{\mathcal{E}_1}$.

Finally, since $\mathrm{Gal}\, L|K$ is the union of all subgroups listed above, it follows that (25) also holds for $G = \mathrm{Gal}\, L|K$. Hence, $I_G = I_{\mathcal{E}_1}$. We have now proved:

Proposition 3.22. *There are Galois extensions of degree p^2 of valued fields in equal characteristic $p > 0$ that have only one ramification ideal, and this ramification ideal is principal although the extension is not defectless.*

The connection between the number of ramification ideals in a finite Galois extension and its depth is studied in [29]. For the notion of depth, see [28].

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